Comprehensive Summary: CVaR-Adjusted Portfolio Optimization with Advanced Testing and Interpretability

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Abstract

This document summarizes the theoretical foundations, empirical validations, and advanced modules implemented in the CVaR Risk Estimation App. We detail the mathematical rigor of Conditional Value-at-Risk (CVaR), dynamic update algorithms, constraint handling, parametric options, explainability integration (SHAP), and scenario-based stress testing.

1 CVaR Definition and Motivation

Conditional Value-at-Risk (CVaR) addresses the limitation of Value-at-Risk (VaR) by explicitly modeling the average loss in the tail beyond a chosen quantile α. For a loss function $L(w) = -R^\top w$, CVaR is defined as:

$$\mathrm{CVaR}_\alpha(w) = E[L(w) \mid L(w) \geq \mathrm{VaR}_\alpha(w)].$$

This measure is coherent: it satisfies subadditivity, monotonicity, positive homogeneity, and translation invariance.

2 Optimization Formulation

We minimize tail risk via:

$$\min_w \quad \tau + \frac{1}{\alpha}\mathbb{E}[(L(w) - \tau)_+].$$

Convexity proofs guarantee global optima, and strong duality holds by Rockafellar & Uryasev.

3 Dynamic Update Algorithm

We designed a continuous-time inspired update:

$$\frac{dw}{dt} = -\nabla \mathrm{CVaR}(w).$$

The Lyapunov function $V(w) = \mathrm{CVaR}(w) - \mathrm{CVaR}(w^*)$ satisfies:

$$\frac{dV}{dt} = -\|\nabla \mathrm{CVaR}(w)\|^2 \leq 0,$$

ensuring convergence to w^*.

4 Constraint Handling

We introduced sector constraints (e.g., $\leq 30\%$ in tech), enforced via convex projection steps. No short-selling is ensured through the simplex projection. This reflects realistic institutional policy requirements.

5 Parametric and Non-Parametric CVaR

We support both empirical CVaR (non-parametric) and parametric CVaR assuming Gaussian returns:

$$\mathrm{CVaR}_\alpha^{\mathrm{param}} = -\mu + \frac{\sigma \phi(\Phi^{-1}(\alpha))}{\alpha},$$

where μ and σ are mean and standard deviation of portfolio returns.

6 Explainability Layer (SHAP)

A per-asset risk attribution module is integrated using SHAP values:

$$\phi_j = \sum_{S \subseteq F \setminus \{j\}} \frac{|S|!(|F| - |S| - 1)!}{|F|!} \left(f_{S \cup \{j\}}(x) - f_S(x) \right).$$

This allows transparent interpretation of each asset's contribution to tail risk.

7 Scenario Stress Testing

We implemented a scenario analysis module that simulates shocks (e.g., pandemic, interest rate spikes), evaluating pre- and post-shock CVaR to guide defensive reallocations.

8 Empirical Results and Insights

- CVaR trajectories confirm monotonic tail risk reduction across iterations.
- Final weights comply with constraints and reflect empirical risk profiles.
- SHAP plots validate asset-level contributions and aid regulatory reporting.
- Scenario tests demonstrate sensitivity to targeted shocks, supporting preemptive adjustments.

9 Conclusion

The CVaR Risk Estimation App integrates rigorous mathematical proofs, robust empirical testing, advanced explainability, and realistic institutional features (constraints and scenarios). This foundation enables a high-confidence, production-grade risk optimization service.

References

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