Forelesning DiffTop

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How can we tell when two spaces are not equivalent? (X, Y are topologically equivalent or homeomorphic if there is a homeomorphism

$$f : X \to Y$$

f is continuous, bijective, and f^{-1} is continuous too.)
Can I prove that S^1 is not equivalent to $[0, 1]$? One idea: How many "connected pieces" is the space made of.

Definition 1.1. Let $X \subseteq \mathbb{R}^n$ be a subspace. X is *disconnected* if there exist open sets $E \subseteq X, F \subseteq X$ such that

1. $F \cup E = X$
2. $E \cap F = \emptyset$
3. $E \neq \emptyset, F \neq \emptyset$

We say (E, F) is a separation of X.

Definition 1.2. X is connected iff X is not disconnected.
That is, if whenever $E, F \subseteq X$ open, $E \cup F = X$ and $E \cap F = \emptyset$ then either E or F is empty.

Theorem 1.1. Connectedness is a topological property.
if $f : X \to Y$ is a homeomorphism, then X is connected iff Y is connected. Equivalently X is disconnected iff Y is disconnected.

Proof. $f : X \to Y$ is a bijection and its continuous. So if Y is disconnected , so we have $E \subseteq Y, F \subseteq Y$. E and F are open relative to Y. $f^{-1}(E), f^{-1}(F)$ are open relative to X and as f is a bijection:

$$f^{-1}(E) \cup f^{-1}(F) = X$$

$$f^{-1}(E) \cap f^{-1}(F) = \emptyset$$

$$f^{-1}(E) \neq \emptyset, f^{-1}(F) \neq \emptyset$$

So X is disconnected. f^{-1} is bijective and continuous. $\square$

Theorem 1.2. Any interval in $\mathbb{R}$ is connected.

$$(a, b), [a, b], [a, b), (a, b], \text{ where } -\infty \leq a \leq b \leq \infty$$

Furthermore, these are all of the connected subsets of $\mathbb{R}$.

Example 1.1. $X = (0, 1)$ and $Y = (0, 1/2) \cup (1/2, 1)$ are not homeomorphic.

Lemma 1.3. If $X \subseteq \mathbb{R}^n$ is a subspace, if X is the union of connected subspaces $X_\alpha, (\alpha \in \mathcal{A})$

$$(X = \cup_{\alpha \in \mathcal{A}} X_\alpha)$$

and there is a point $x_0 \in X_\alpha$ for all $\alpha \in \mathcal{A}$, then X is connected.

Proof. Suppose $E, F \subseteq X$ open, and suppose

$$E \cup F = X$$

$$E \cap F = \emptyset$$

We will show $E = \emptyset$ or $F = \emptyset$. For any $\beta \in \mathcal{A}$, X_β is connected. $E \cap X_\beta$ and $F \cap X_\beta$ open relative to X_β.

$$\Rightarrow (E \cap X_\beta) \cup (F \cap X_\beta) = X_\beta$$

$$\Rightarrow (E \cap X_\beta) \cap (F \cap X_\beta) = \emptyset$$

$$\Rightarrow E \cap X_\beta = \emptyset \text{ or } F \cap X_\beta = \emptyset$$

Let's say $E \cap X_\beta = \emptyset$, so $F \cap X_\beta = X_\beta$ containing x_0 so $x_0 \in F$, but now for any $\alpha \in \mathcal{A}$, $(E \cap X_\alpha, F_\alpha)$ is also a separation of X_α. $x_0 \in F \cap X_\alpha = X\alpha$ and $E \cap X_\alpha = \emptyset$. Hence $\forall \alpha \in \mathcal{A}, X_\alpha \in F$

$$\Rightarrow F = \cup X_\alpha = X, E = \emptyset.$$

□

Example 1.2. $X = B^n(0; 1)$, $\mathbf{v} \in S^{n-1}$

$$X_\mathbf{v} = \{t \cdot \mathbf{v} | t \in (-1, 1)\}$$

$X_\mathbf{v}$ is homeomorphic to $(-1, 1)$.

$$f : (-1, 1) \to X$$

$$t \mapsto t\mathbf{v}$$

and $\forall \mathbf{v} \in S^{n-1}$, $(0, 0) \in X_\mathbf{v}$. So lemma proves $B^n(0; 1)$ is connected.

Example 1.3. $S^n \subseteq \mathbb{R}^{n+1}$ is connected.

Proposition 1.4. "The continuous image of a connected space is connected" $f : X \to Y$ continuous, and X is connected, then the image

$$f(X) = im(f) = \{f(x) \in Y | x \in X\} \subseteq Y$$

is connected.

Proof. Suppose $f(X)$ is disconnected. So say $f(X) = E \cup F$

$$E \cap F = \emptyset$$

$$E \neq \emptyset, F \neq \emptyset$$

and E and F are open relative to $f(X)$. E and F may not be open relative to Y! There are open sets $\tilde{E} \subseteq Y, \tilde{F} \subseteq Y$ with

$$\tilde{E} \cap f(X) = E, \tilde{F} \cap f(X) = F.$$

We will then have

$$f^{-1}(E) = f^{-1}(\tilde{E}) \text{ is open}$$
$$f^{-1}(F) = f^{-1}(\tilde{F}) \text{ is open}$$

these are nonempty, disjoint, union is X. This is a separation of X, so X is disconnected. □

Lemma 1.5. X is disconnected iff X admits a continuous surjection

$$f : X \to \{0, 1\}, \qquad \{0, 1\} \subseteq \mathbb{R}$$

Proof. If X is disconnected, we have a separation of X.

$$X = E \cup F, E \cap F = \emptyset, E \neq \emptyset, F \neq \emptyset$$

$$f : X \to \{0, 1\}, f(E) = 0, f(F) = 1.$$

Conversely, we have

$$f : X \to \{0, 1\}$$

continuous surjection

$$f^{-1}(\{0\}) = E$$
$$f^{-1}(\{1\}) = F$$

are open. Surjective implies nonempty, $E \cup F = X$, X is disconnected. □

Connected Components

how many pieces is a space made of?

Definition 1.3. Define an equivalence relation on X. for $a, b \in X$, we write $a \sim b$ if there is a connected subset $U \subseteq X$ with $a, b \in U$.

Check that this is an equivalence relation:

(1) $a \sim a$ use $U = \{a\}$

(2) if $a \sim b$ then $b \sim a$.

$$a, b \in U, b, a \in U$$

(3) Transitivity. $a \sim b, b \sim c$, show $a \sim c$. We have $a, b \in U_1$ connected, $b, c \in U_2$ connected. $a, b, c \in U_1 \cup U_2$ connected by Lemma.

The equivalence classes of $\sim$ give the connected components of X.

$$X/\sim = \{[x] | x \in X\}$$

$$[x] = \{y \in X | x \sim y\}$$

Lemma 1.6. If X and Y homemorphic,then f gives a bijection of sets of connected components.

$$[f] : X/\sim \to Y/\sim$$

$$[x] \mapsto [f(x)]$$

Takeaway: If X and Y are homeomorphic, they have the same number of connected components.

Proposition 1.7. (1) $U \subseteq X$ is a connected component ($U = [x]$) iff U is connected, nonempty and for any subset $A \subseteq X$ connected with $A \cap U \neq \emptyset$ then $A \subseteq U$.

(2) The connected components partition in space X.

$$[x] \cap [y] = \emptyset \text{ or } [x] = [y]$$

and $X = \cup[x]$.

(3) $[x]$ is both open and closed.

Goal: Distinguish

$[0, 1]$ from S^1.

Definition 1.4. A point $x \in X$ is a *cut point* if $X \setminus \{x\}$ is disconnected.

cut points of $[0, 1]$ is $(0, 1)$ (all interior points.)
cut points of S^1: $\emptyset$.
Conclude: There is no homeomorphism

$$f : [0, 1] \to S^1$$

because of the different cut sets.

$$Cut(X) = \{x \in X | x \text{ is a cut point}\}$$

Theorem 1.8. if $f : X \to Y$ is a homeomorphism then if $x \in Cut(X)$ then $f(x) \in Cut(Y)$ and further

$$f_{|Cut(X)} : Cut(X) \to Cut(Y)$$

is a homeomorphism.

Proof. $x \in Cut(X)$. Then $X \setminus \{x\} = E \cup F$.

$$E \cap F = \emptyset, E \neq \emptyset, F \neq \emptyset$$

.

$$(f{-}1)^{-1}(E), (f^{-1})^{-1}(F) \subseteq Y$$

$$(f{-}1)^{-1}(E) \cup (f^{-1})^{-1}(F) = f(X) \setminus f(x) = Y \setminus f(x).$$

So $f(x)$ is a cut point of Y. So $f(x) \in Cut(Y)$. if $y \in Cut(Y)$, then $f^{-1}(y)$ is a cut point of X.

$$Cut(X) \to (f)Cut(Y)$$

$$Cut(Y) \to (f^{-1})Cut(X)$$

f is a bijection between $Cut(X)$ and $Cut(Y)$. In fact both $f : Cut(X) \to Cut(Y)$ and $f^{-1} : Cut(Y) \to Cut(X)$ continuous too. Let $U \subseteq Cut(Y)$ be open relative to $Cut(Y)$. Then

$$U = \tilde{U} \cap Cut(Y), \tilde{U} \subseteq Y \text{ open.}$$

$$f^{-1}(U) = f^{-1}(\tilde{U}) \cap Cut(X)$$

$f^{-1}(\tilde{U})$ open in X as f is continuous $\Rightarrow$ open relative to $Cut(X)$. $\square$

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Definition 2.1. Let $X \subseteq \mathbb{R}$ be a subspace, and suppose $x \in X$ is a cut point. We say x is an n-fold cut point if $X \setminus \{x\}$ consists of n connected components.

Proposition 2.1. If $f : X \to Y$ is a homeomorphism, then X has an n-fold cut point iff Y has an n-fold cut point.

Proof. Since $f : X \to Y$ is a homeo

$$x \leftrightarrow f(x)$$

We restrict f to $X \setminus \{x\}$.

$$f_{|X\setminus\{x\}} : X \setminus \{x\} \to Y \setminus \{f(x)\}$$

if x is an n-fold cut point, $X \setminus \{x\}$ consists of n connected components. So $Y \setminus \{f(x)\}$ consists of n components too. $\square$

All cut points of the latin letter P are 2-fold. “Connectedness proof” Start sith X a connected space. Say we want to prove $(\forall x \in X)\,(P(x))$.

(1)
$$E = \{x \in X | P(x)\}$$
$$F = \{x \in X | \not P(x)\}$$
$$E \cup F = X$$
$$E \cap F = \emptyset$$

(2) Prove that there exists one $x_0 \in X$ with $x_0 \in E$, i.e., $P(x_0)$ is true.

(3) Prove that both E and F are open sets.

(4) Draw the conclusion that $F = \emptyset$. (if $F \neq \emptyset$, then (E, F) is a separation of X, showing X is disconnected).

Compactness

Example 2.1. The closed interval $[\cdot,\cdot]$, $S1, S^n$ are all compact. $(\cdot,\cdot), (\cdot,\cdot], B^n(0;1), \mathbb{R}^n$ are not compact.

Definition 2.2. Sequential compactness
$X \subseteq \mathbb{R}^n$ is sequentially compact if for any sequence $x_n \in X$, there exists a convergent subsequence x_n, x_{n_k} so that

$$\lim_{k\to\infty} \bar{x} \in X$$

Example 2.2. $x_n = \frac{1}{n}$ for $n \geq 2$

$$\in (0,1)$$

$$\lim_{n\to\infty} x_n = 0$$

Every subsequence of $\frac{1}{n}$ converges to 0 too. So $(0,1)$ is not compact.
Maybe $[0,1)$ would be sequentially compact? No:

$$a_n = \begin{cases} \frac{1}{n} \text{ if } n \text{ even} \\ 1 - \frac{1}{n} \text{ if } n \text{ odd} \end{cases}$$

a_n is not a convergent sequence. But there are convergent subsequences.

$$a_{2n} = \frac{1}{2n} \to 0; \quad a_{2n+1} = 1 - \frac{1}{2n+1} \to 1$$

Theorem 2.2. Heine-Borel
$x \subseteq \mathbb{R}^n$ a subspace is sequentially compact iff X is both closed as a subset of $\mathbb{R}^n$ and bounded. i.e.,$\exists N > 0$ so that $X \subseteq B^n(0;N)$.

S^2 is closed:

$$\mathbb{R}^3 \setminus S^2 = (B^3(0;1)) \cup (\mathbb{R}^3 \setminus D^3(0;1))$$

these are both open so $\mathbb{R}^3 \setminus S^2$ must be open and hence S^2 is closed.

Proposition 2.3. Let $f : X \to Y$ be a continuous map and suppose X is sequentially compact. Then $f(X)$ is also sequentially compact.

Proof. Consider a sequence $y_n \in f(x)$. Pick $x_n \in X$ so that $f(x_n) = y_n$. So we have $x_n \in X$.
Sequentially compact implies x_n has a convergent subsequence

$$\lim_{k\to\infty} x_{n_k} = \bar{x} \in X$$

but f is continuous, so

$$\lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = f(\bar{x}) \in f(X) = \lim y_{n_k}$$

Conclusion: y_{n_k} is a subsequence converging to $f(\bar{x}) \in f(X)$. □

Consequence: if $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$, $f : X \to Y$ continuous, X seq. comp., then $f(X) \subseteq \mathbb{R}^m$ is closed and bounded.

Definition 2.3. $X \subseteq \mathbb{R}^n$ a subspace is compact iff for any open cover $\mathcal{V} = \{U_{alpha} \subseteq X | \alpha \in \Lambda\}$ such that $X = \cup_\alpha \in \Lambda U_\alpha$ there exists a finite subcover of $\mathcal{V}$. That is, there are:

$$U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k} \in \mathcal{V}$$

and $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$

Example 2.3. $(0, 1)$ is not compact.
Find an open cover $\mathcal{V}$ of $(0, 1)$ that has no open subcover.

$$\mathcal{V} = \{(\frac{1}{n}) | n \geq 2, n \in \mathbb{N}\}$$

First, $\mathcal{V}$ is an open cover of $(0, 1)$.

$$(0, 1) = \cup_{n \geq 2}(\frac{1}{n}, 1)$$

For any $x, 0 < x < 1$, $\exists N \in \mathbb{N}$ so $\frac{1}{N} < x$, in that case, $x \in (\frac{1}{N}, 1)$. But no finite subcollection of $\mathcal{V}$ will still cover $(0, 1)$.

$$(\frac{1}{n}, 1), (\frac{1}{n_2}, 1), \ldots, (\frac{1}{n_k}, 1)$$

Well there is a largest value of $n_1, \ldots, n_k$, call it n_i. And we then have

$$\cup_{i=1}^{k}(\frac{1}{n_i}, 1) \subseteq (\frac{1}{n_i}, 1) \neq (0, 1)$$

Theorem 2.4. $[0, 1]$ is compact.

Proof. We'll use the fact that $[0, 1]$ is connected. Consider

$$\mathcal{U} = \{U_\alpha \subseteq [0, 1] | \alpha \in \mathcal{A}\}$$

open cover of $[0, 1]$.

$$E = \{x \in [0, 1] | [0, x] \text{ admits a finite cover by } \mathcal{U}\}$$

i.e. there are $U_{alpha_1}, \ldots, U_{alpha_k} \in \mathcal{U}$ where $[0, x] \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$. Want to show $x = 1 \in E$.

(1) Observe $0 \in E$. Because $[0, 0] = \{0\}$ and $0 \in U_\beta$ for some β as $\cup_{\alpha \in \mathcal{A}} = [0, 1]$. So $[0, 0] \subseteq U_\beta$.

(2) Show E and $[0, 1] \setminus E$ are open. Suppose $x \in E$. So there are sets $U_{\alpha_1}, \ldots, U_{\alpha_k}$

$$[0, 1] \subseteq U_{\alpha_1}, \ldots, U_{\alpha_k}.$$

$x \in U_{\alpha_i}$ for some i, and $U_{\alpha_i} \subseteq [0, 1]$ is open. So we can find $\epsilon > 0$ so that

$$(x - \epsilon, x + \epsilon) \cap [0, 1] \subseteq U_{\alpha_i}$$

So now, consider any $y \in ((x - \epsilon, x + \epsilon) \cap [0, 1])$. We observe

$$[0, y] \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$$

$$[0, y] \subseteq [0, x] \cup ((x - \epsilon, x + \epsilon) \cap [0, 1]))$$

so then $(x - \epsilon, x + \epsilon) \cap [0, 1]) \subseteq E$. which is open in $[0, 1]$. A similar argument will show $[0, 1] \setminus E$ is open.

(3) Connectedness of $[0,1]$ implies $E = [0,1]$. $1 \in E$, hence $[0,1]$ is covered by $U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$.

$\square$

Theorem 2.5.

$$X \subseteq \mathbb{R}^n$$

X is compact iff X is sequentially compact iff X is closed in $\mathbb{R}^n$ and bounded.

Proposition 2.6. $f : X \to Y$ is a homeomorphism, then X is compact iff Y is compact.

Proof. If X is compact, $f(X) = Y$ is compact.
Conversely if Y is compact, f^{-1} is continuous

$$X = f^{-1}(Y) \text{ is compact.}$$

$\square$

Theorem 2.7. Borsuk-Ulam
$f : S^1 \to \mathbb{R}$ continuous. There is some $p \in S^1$ such that $f(p) = f(-p)$.

Proof. Define $D : S^1 \to \mathbb{R}$ by $D(p) = f(p) - f(-p)$, if there is $p \in \S^1$ with $D(P) = 0 = f(p) - f(-p)$, then $f(p) = f(-p)$. Since S^1 is connnected and compact $D(S^1) = [a, b]$ where $a \leq b$.
I claim $0 \leq b$ and $a \leq 0$. Let's say $D(p) = f(p) - f(-p) \leq 0$ then $D(-p) = f(-p) - f(p) = -D(p) \geq 0$. So $0 \in D(S^1)$. $\square$

Lecture week 3 Difftop

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Definition 3.1. (Topological manifold)
A subspace $X \subseteq \mathbb{R}^n$ is a topological manifold if at any $x \in X$ there exists an open set $U \subseteq X$ and an open set $V \subseteq \mathbb{R}^k$ and a homeomorphism $\varphi : V \to U$.

Continuous maps can behave poorly. Instead of using continuity alone, we'll use differentiability.

Definition 3.2. $U \subseteq \mathbb{R}^n, f : U \to\subseteq \mathbb{R}^m$, we can write it as

$$f(x_1, \ldots, x_n) = f(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}) = \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{bmatrix} \in\subseteq \mathbb{R}^m$$

f is *smooth* if $\forall a \in U$, the partial derivatives of f exists at all $a \in U$.
$\frac{\partial f_i^n}{\partial x_{j_i} \cdots \partial x_{j_n}}(a)$ all exist.
$\frac{\partial f_i}{\partial x_j}(a) = \lim_{t \to 0} \frac{f(a+te_j) - f(a)}{t}$.

Ultimately we want to replace "homeomorphism" with "diffeomorphism" in definition of manifold.

Definition 3.3. Let $X \subseteq \mathbb{R}^n$ any subspace and consider a function

$$f : X \to \mathbb{R}^m$$

We say that f is smooth if $\forall x \in X$, there is an open set $U \subseteq X$ (rel. X), an open set $\tilde{U} \subseteq \mathbb{R}^n$ and a smooth function $\tilde{f} : \tilde{U} \to \mathbb{R}^m$.

$$U = \tilde{U} \cap X$$

so that $\forall a \in U$, $\tilde{f}(a) = f(a)$.

$$\begin{tikzcd}
\mathbb{R}^n \supseteq \tilde{U} \arrow[ddr, "\tilde{f}"] & \\
x \in U \arrow[u] \arrow[dr, "f_{|u}"] & \\
X \arrow[u, "\supseteq"] \arrow[r, "f"] & \mathbb{R}^m
\end{tikzcd}$$

$\tilde{f}$ smooth, then we say $f_{|u}$ smooth.

Example 3.1. $f : S^1 \to \mathbb{R}^1$, $f(x, y) = x$ when $x^2 + y^2 = 1$.
How can we see that f is smooth?
Observe that we can extend f to all of $\mathbb{R}^2$:

$$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x$$

$\tilde{f}$ is clearly smooth and $\widetilde{f_{|s^1}} = f$, we conclude f is smooth.

Example 3.2. $X = S^1 \cap \{(x, y)|y > 0\}$

$$L = \{(x, 1)|x \in\}$$

$Y = \mathbb{R}^2$. Define $f(x, y)$ to be the unique point common with L and $\{t(x, y)|t \in \mathbb{R}\}$.
f is smooth because we observe

$$f(x, y) =$$

Find t

$$t(x, y) \in \{(x, 1)|x \in \mathbb{R}\}$$

We need $ty = 1$, so $t = \frac{1}{y}$ if $y \neq 0$. Hence $f(x, y) = (\frac{x}{y}, 1)$. This description of f can clearly be extended to $\tilde{U} = \{(x, y) \in \mathbb{R}^2|y > 0\}$

$$\tilde{f} : \tilde{U} \to \mathbb{R}$$

$$(x, y) \mapsto (\frac{x}{y}, 1)$$

Clearly smooth, so hence f is also smooth.

Definition 3.4. If $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are subspaces then a function $f : X \to Y$ is smooth iff

$$X \xrightarrow{f} Y \xrightarrow{\subseteq} \mathbb{R}^m$$

is smooth.

Definition 3.5. $f : X \to Y$ is a diffeomorphism if f is smooth, f is bijective and f^{-1} is smooth.

Remark. If $f : X \to Y$ is a diffeomorphism, then $f : X \to Y$ is a homeomorphism.
BUT the converse need not be true. In fact there does not need exist any diffeomorphism at all. (not only f)

Example 3.3. There are uncountably many spaces X_α, and each is homeomorphic to $\mathbb{R}^4$.

$$f_\alpha : X_\alpha \to \mathbb{R}^4$$

but no two X_α and X_β are diffeomorphic.

Definition 3.6. $X \subseteq \mathbb{R}^n$ is a *smooth manifold* if $\forall x \in X$ there exists an open set $x \in U \subseteq X$ and an open set $V \in \mathbb{R}^k$, and a diffeomorphism

$$\varphi : V \to U$$

Example 3.4. S^1 is a smooth manifold.
Use the $exp : \mathbb{R} \to S^1, \theta \mapsto (\cos(\theta), \sin(\theta))$.
Use two "local parametrizations" of S^1

$$exp : (0, 2\pi) \xrightarrow{diffeo} S^1 \setminus \{(1, 0)\}$$

$$exp : (\pi, 3\pi) \xrightarrow{diffeo} S^1 \setminus \{(-1, 0)\}$$

all that is left is to prove

$$exp : (0, 2\pi) \xrightarrow{diffeo} S^1 \setminus \{(1, 0)\}$$

is a diffeo.

- $$exp(\theta) = (\cos(\theta), \sin(\theta))$$
  so exp is smooth.
- basic trig says exp is bijective.
- Show $exp^{-1} : S \setminus \{(1, 0)\} \to (0, 2\pi)$ is smooth.

$\arccos(x) : S^1 \cap \{(x, y)|y > 0\} \to (0, \pi)$, to see it is smooth,
Let's take

$$g : \mathbb{R}^2 \cap \{(x, y)|y > 0\} \to (0, \pi)$$
$$(x, y) \mapsto \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

defined on open subset of $\mathbb{R}^2$.

$$g = exp^{-1} \text{ on } S^1 \cap \{(x, y)|y > 0\}$$

for $S^1 \cap \{(x, y)|y < 0\}$, use

$$exp^{-1}((x, y)) = 2\pi - \arccos(x)$$

extends to

$$g(x, y) = 2\pi - \arccos\left(\frac{x}{\sqrt{x^2 - y^2}}\right)$$

Directional derivatives

Let $f : U$ open in $\mathbb{R}^n \to \mathbb{R}^m$ be smooth has *directional derivatives* $a \in U$ in the direction $v \in \mathbb{R}^n$ if

$$df_a(v) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}$$

exists.
$df_a(v)$ always exists when f is smooth
$df_a(v) : \mathbb{R}^n \to \mathbb{R}^m$ is in a fact linear map.

$$df_a(av + w) = cdf_a(v) + df_a(w)$$

df_a is represented by a matrix, use standard basis $\{e_1, \ldots, e_n\}$ and $\{e_1, \ldots, e_m\}$. Then

$$df_a = \begin{bmatrix} \frac{\partial f_1}{x_1}(a) & \cdots & \frac{\partial f_1}{x_n}(a) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{x_1}(a) & \cdots & \frac{\partial f_m}{x_n}(a) \end{bmatrix}$$

and

$$df_a(v) = \begin{bmatrix} \frac{\partial f_1}{x_1}(a) & \cdots & \frac{\partial f_1}{x_n}(a) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{x_1}(a) & \cdots & \frac{\partial f_m}{x_n}(a) \end{bmatrix} (v)$$

Theorem 3.1. (Chain rule)

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^p$$

U open in $\mathbb{R}^n$, V open in $\mathbb{R}^m$, f and g smooth. Then $g \circ f$ is smooth and

$$d(g \circ f)_a = (dg)_{f(a)} \circ (df)_a$$

(composition of linear functions or simply matrix multiplication).

Theorem 3.2. Let $U \subseteq \mathbb{R}^n$ open and $V \subseteq \mathbb{R}^m$ be open. Suppose $f : U \to V$ is a diffeo. Then $n = m$.
(no Peano space filling curve can give a diffeomorphism!)

Proof.

$$U \xrightarrow{f} V \xrightarrow{f^{-1}} U$$

$$f^{-1} \circ f = Id_u$$

So take the directional derivatives at $a \in U$.

$$\mathbb{R}^n \xrightarrow{df_a} \mathbb{R}^m \xrightarrow{df_a^{-1}} \mathbb{R}^m$$

$$d(Id)_a$$

$Id_u(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$

$$(dId_u)_a = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \ldots \\ 0 & \cdots & 1 \end{bmatrix}$$

Chain rule
$\left(df^{-1}\right)_{f(a)} \circ df_a = Id_{n\times n}$. Similarly

$$V \xrightarrow{f^{-1}} U \xrightarrow{f} V$$

shows

$$df_{f^{-1}(b)} \circ df_b^{-1} = Id_{m\times m}$$

i.e., $df_a \circ df_{f(a)}^{-1} = Id_{m\times n}$.
Since df_a is both $n \times n$ and $m \times m$ invertible matrix, must be that $n = m$. $\square$

Tangent spaces:

The vectors $v \in \mathbb{R}^n$ used in directional derivative, all directions we could move in at a point.

Definition 3.7. Let $U \subseteq \mathbb{R}^n$ be an open subset. We define the tangent space to U at $a \in U$ by

$$T_aU = \{(a, v) |\ v \in \mathbb{R}^n\}$$

Eig: at $(1, 1) \in \mathbb{R}^2$ a tangentt vector looks like $((1, 1), (v_1, v_2))$.
T_aU is a vector space

$$(a, v) + (a, w) = (a, v + w)$$

$$(a, (0, 0)) \text{ is the additive identity.}$$

Definition 3.8. Total tangent space

$$TU = \{(a, v)|a \in U, (a, v) \in T_aU\}$$

Extend to manifolds

Let $X \subseteq \mathbb{R}^n$ be a smooth manifold, $a \in X$. There exists open $a \in U \subseteq X$ and exists $V \subseteq \mathbb{R}^k$ open and $\varphi : V \to U$ diffeomorphism.

$$\varphi(y) = a$$

Define

$$T_aX = \{(a, d\varphi_y(v))|\ v \in \mathbb{R}^k\}$$

Note that $d\varphi_y(a) : \mathbb{R}^k \to \mathbb{R}^n$ is linear and its images k-dimensional.

4 23. jan

- Coordinate systems
- Tangent spaces
- Derivatives

Definition 4.1. Let $X \subseteq \mathbb{R}^n$ be a smooth manifold, we define T_xX for $x \in X$ by finding a local parametrization

$$\varphi : \tilde{u} \to u : x \in u \subseteq X, \tilde{U} \subseteq \mathbb{R}^n$$

$$\varphi(y) = x, \varphi \text{ diffeo}$$

Define $T_xX = \{(x, d\varphi_y(v)) | v \subseteq \mathbb{R}^k\}$. Related to image$(d\varphi_y : \mathbb{R}^k \to \mathbb{R}^n)$,
This is a linear subspace of $\mathbb{R}^n$, so it is a vector space. Hence T_xX is a vector space too.

$$(x, u) + (x, v) = (x, u + v)$$

Terminology
If $X \subseteq \mathbb{R}^n$ is a smooth manifold, $x \in X$ we know there exists $\varphi : \tilde{u} \to u$ with $x \in u, \tilde{u}$ open $\subseteq \mathbb{R}^k$.
We call φ a "local parametrization of X at $x \in X$". But this is equivalent ot giving a diffeo

$$\varphi^{-1} u \to \tilde{u} \subseteq \mathbb{R}^k$$

φ^{-1} is called a coordinate chart for X at $x \in X$.
We call thi generally a "local coordinate system at x".
Pick $1 \leq i \leq k$, and some $z \in \mathbb{Z}^k$, $z = (z_1, z_2, \ldots, z_k)$, and consider the lines

$$L_{i,z} = \{z + te_i | t \in \mathbb{R}\} \subseteq \mathbb{R}^k$$

Study the images $\varphi(L_{i,z} \cap \tilde{U}) \subseteq X$.
So φ^{-1} gives us coordinates to points on u. We often write

$$\varphi^{-1}(p) = (x_1(p), x_2(p), \ldots, x_k(p))$$

x_i is the i^{th} coordinate function of φ^{-1}. If we are really sloppy we will say

$$p = (x_1, x_2, \ldots, x_k)$$

Example 4.1. Polar coordinates
A system of local coordinates for $\mathbb{R}^2$ given by the diffeomorphism

$$\psi \, (-\pi, \pi) \times (0, \infty) \to \mathbb{R}^2 \setminus \{(x, 0) | x \geq 0\}$$

$$(\theta, r) \to (r\cos(\theta), r\sin(\theta))$$

Lemma 4.1. Let $X \subseteq \mathbb{R}^n$ smooth manifold. $\forall x \in X$ there always exists a local coordinate system at x,
$\varphi : \tilde{u} \to u$ with $u \subseteq X, \tilde{u}$ open $\subseteq \mathbb{R}^k$ satisfying $\varphi(0) = x$.

Proof. We know there exists *some* coordinate system at x. Call it $\mathbb{R}^k \supseteq \psi : \tilde{V} \to V \subseteq X$.
Write $y = \psi^{-1}(x) \in \tilde{V} \subseteq \mathbb{R}^k$. Translation is a diffeomorphism!
Consider

$$t_y : \mathbb{R}^k \to \mathbb{R}^k$$

$$v \mapsto v + y$$

is a diffeomorphism!
We take $\tilde{u} = t_y^{-1}(\tilde{V}) = \{z - y | z \in \tilde{V}\}$.
We take $u = V$ and finally

$$\varphi = \psi \circ t_{y_{|_{\tilde{u}}}} : \tilde{u} \to u$$

$$\tilde{u} \xrightarrow{t_y} \tilde{V} \xrightarrow{\psi} u = V$$

and this entire thing is a diffeo. $\square$

Lemma 4.2. Let $X \subseteq \mathbb{R}^n$ be a smooth manifold. For any $x \in X$ there is a coordinate system

$$\varphi : B^k(0; \varepsilon) \to u$$

at x.

Proof. Homework problem.
We know there is a coordinate system $\varphi : \tilde{u} \to u, 0 \in \tilde{u} \mapsto x \in u$.
$\tilde{u}$ is open, so $\exists \varepsilon > 0$ with $B^k(0; \varepsilon) \subseteq \tilde{u}$. Restrict φ to $B^k(0; \varepsilon)$.

$$\varphi|_{B^k(0;\varepsilon)} : B^k(0; \varepsilon) \to \varphi\left(B^k(0; \varepsilon)\right) \subseteq X.$$

$\square$

Lemma 4.3. Let $X \subseteq \mathbb{R}^n$ be a smooth manifold and $x \in X$, let

$$\varphi : \tilde{u} \to u \text{ and } \psi : \tilde{v} \to v$$

be coordinate systems at $x \in X$.
$\varphi^{-1}(u \cap v)$ is diffeomorphic to $\psi^{-1}(u \cap v)$ with diffeomorphism given by $\psi^{-1} \circ \varphi$ with restriction $\left(\varphi|_{\varphi^{-1}(u\cap v)}\right)$ or $\varphi^{-1} \circ \psi$ with restriction $\left(\psi|_{\psi^{-1}(u\cap v)}\right)$.

Proof. $\psi^{-1} \circ \varphi|_{\varphi^{-1}(u\cap v)}$ is inverse to $\varphi^{-1} \circ \psi|_{\psi^{-1}(u\cap v)}$.

$$\begin{aligned}&\left(\varphi^{-1} \circ \psi\right) \circ \left(\psi^{-1} \circ \varphi\right)(z)\\ &= \varphi^{-1}\left(\psi\left(\psi^{-1}\left(\varphi(z)\right)\right)\right)\\ &= \varphi^{-1}\left(\varphi(z)\right) = z\end{aligned}$$

To see $\varphi^{-1} \circ \psi$ is diffeo check they are smooth. $\square$

Proposition 4.4. Let $X \subseteq \mathbb{R}^n$ be a smooth manifold and $x \in X$. Then T_xX is well defined: independent of the choice of coordinate chart.
We can assume $\varphi(0) = \psi(0) = x$. We then have two definitions for T_xX:

$$\{(x, d\varphi_0(v)) | v \in \mathbb{R}^k\}$$

or

$$\{\left(x, d\psi_0(v) | v \in \mathbb{R}^k\right)\}$$

i.e., is

$$im\left(d\varphi_0 : \mathbb{R}^k \to \mathbb{R}^n\right)$$

equal to

$$im\left(d\psi_0 : \mathbb{R}^k \to \mathbb{R}^n\right)?$$

We can use the diffeo's $\psi^{-1} \circ \varphi$ and $\varphi^{-1} \circ \psi$ as follows. Let's say we have $d\varphi_0(v) \in im(d\varphi_0(v))$.

$$d\varphi_0(v) = d\left(\psi \circ \psi^{-1} \circ \varphi\right)_0 (v)$$

$$= d(\psi)_0 \cdot d\left(\psi^{-1} \circ \varphi\right)(v).$$

To finish, suppose

$$d\psi_0(w) = d\left(\varphi \circ \varphi^{-1}\psi\right)_0 (w)$$

$$= d\varphi_0 \left(d(\varphi^{-1} \circ \psi)_0(w)\right).$$

$$\in im\left(d\varphi_0\right)$$

Example 4.2. Tangent Spaces to S^1
Pick coordinates

$$exp : (0, 2\pi) :\to S^1 \setminus \{(1,0)\}$$

$$exp : (-\pi, \pi) \to S^1 \setminus \{(-1,0)\}$$

Consider $(x,y) \in S^1$. What is $T_{(x,y)}S^1$?

$$(dexp) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

if $exp(\theta) = (x,y)$, then

$$T_{(x,y)}S^1 = \{(x,y), (dexp_\theta)(v) | v \in \mathbb{R}\}$$

$$im(dexp_\theta) = Span_{\mathbb{R}}\{\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}\}$$

Example 4.3. S^4, $x = (1,0,0,0,0)$.
What is T_xS^4?

- Find local coord. system of S^4 at x

$$\varphi : B^4(0;1) \longrightarrow u \subseteq S^4$$

$$(x_1, x_2, x_3, x_4) \longmapsto \left(\sqrt{1 - x_1^2 - x_2^2 - x_3^2 - x_4^2}, x_1, x_2, x_3, x_4\right)$$

Observe that $x_1^2 + x_2^2 + x_3^2 + x_4^2 < 1$, so $\sqrt{1 - x_1^2 - x_2^2 - x_3^2 - x_4^2}$.
observe φ is the graph og

$$f : B^4(0;1) \to \mathbb{R}$$

$$(x_1, x_2, x_3, x_4) \mapsto \sqrt{1 - x_1^2 - x_2^2 - x_3^2 - x_4^2}.$$

Calculate $d\varphi_0$ as $\varphi(0,0,0,0) = x$.

$$d\varphi_{(0,0,0,0)}$$

$$= \begin{bmatrix} \frac{-x_1}{\sqrt{1-x_1^2-x_2^2-x_3^2-x_4^2}} \to 0 & \frac{-x_2}{\sqrt{1-x_1^2-x_2^2-x_3^2-x_4^2}} \to 0 & \frac{-x_3}{\sqrt{1-x_1^2-x_2^2-x_3^2-x_4^2}} \to 0 & \frac{-x_4}{\sqrt{1-x_1^2-x_2^2-x_3^2-x_4^2}} \to 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So

$$Im(d\varphi_0) = span\{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\}$$

$$T_xS^4 = \{x\}xim(d\varphi_0)$$

$$T_xS^n = \{(x,v)|v \in \mathbb{R}^{n+1} \text{ s.t. } v \cdot x = 0\}$$

Let $f : X \to Y$ be a smooth map where $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^n$ are smooth manifolds.
How to define the derivative of f?
What we want:

(1) $\forall x \in X$ we get a linear map

$$Df_x : T_xX \to T_{f(x)}Y.$$

(2) if $f : u \to v$, $u \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^n$ open, f smooth, then expect

$$Df_x : T_xu \to T_{f(x)}V$$

$$(x,v) \mapsto (f(x), df_x(v))$$

(3) D should satisfy the chain rule

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$T_xX \xrightarrow{Df_x} T_{f(x)}Y \xrightarrow{Dg_{f(x)}} T_{g(f(x))}Z$$

$$D(g \circ f)_x$$

$$D(g \circ f)_x = (Dg)_{f(x)} \circ Df_x$$

(4) if $\varphi : \tilde{u} \to u \subseteq X$ is a local coordinate system for X

$$D\varphi_x T_x\tilde{u} \to T_xX \subseteq T_x\mathbb{R}^n$$

Lecture Week 04 Differential Topology

5 28. jan

Derivative of a smooth map

$$f : X \to Y$$

$X \subseteq \mathbb{R}^n$ smooth manifold
$Y \subseteq \mathbb{R}^m$ smooth manifold
define $Df_x : T_xX \to T_{f(x)}Y$ by selecting cord. systems.
Diagram1
$f(u) \subseteq V$, $(\varphi(0) = x, \psi(0) = y)$ Define Df_x to be

$$Df_x = (D\psi)_0 \circ (Dh_0) \circ (D\varphi_0)^{-1}$$

Diagram 2
Main properties of Df.

(1) If $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ both open subsets,

$$f : u \to V \text{ smooth,}$$

$$\text{then } Df_x(x, v) = (x, df_x(v))$$

(2) Chain rule:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m, Z \subseteq \mathbb{R}^l$$

X, Y, Z all smooth manifolds. Check that

$$D(g \circ f)_x = D_{g_{f(x)}} \circ Df_x$$

We can take

$$x \in X, y = f(x), z = g(y)$$

We can find cord. systems
Diagram 3
Diagram 4
$D(g \circ f)_x = D\chi_0 \circ Dp_0 \circ (D\varphi_0)^{-1}$

$$= D\chi_0 \circ D(k \circ h)_0 \circ (D\varphi_0)^{-1}$$

$$= D\chi_0 \circ Dk_0 \circ Dh_0(D\varphi_0)^{-1}$$

$$= \left(D\chi_0 \circ Dk_0 \circ D\psi_0^{-1} \circ\right) \left(D\psi_0 \circ Dh_0(D\varphi_0)^{-1}\right)$$

$$Dg_y \circ Df_x.$$

Step three follows from the fact that the chain rule holds for open euclidean subsets.

Example 5.1.

$$f : S^1 \to S^1$$

$$f(x, y) = (x^2 - y^2, 2xy)$$

where $x^2 + y^2 = 1$. Have to check that

$$\|f(x, y)\| = 1$$

What is $Df_{(x,y)}$?

$$Df_{(x,y)} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$Df_{(x,y)} : T_{(x,y)}S^1(1 - dim) \to T_{(x,y)}S^1(1 - dim)$$

Why do we describe a 1-dimensional thing with a 2 by 2 matrix, so that is wrong.
We observe that $f : S^1 \to S^1$ can be extended to

$$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\tilde{f}(x, y) = (x^2 - y^2, 2xy)$$

$$D\tilde{f}_{(x,y)} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

is correct though.
How does this relate to $Df_{(x,y)}$? To find $df_{(x,y)}$, we need coordinate systems. E.g.

$$exp : (0, 2\pi) \to S^1 \setminus \{(1,0)\}$$

$$exp : (-\pi, \pi) \to S^1 \setminus \{(-1,0)\} = V$$

diagram 5

$$f(S^1 \setminus \{(-1,0)\}) = S^1.$$

Pick a good u. Try to take $f^{-1}(V)$. So what maps to $(-1,0)$?

$$f(x,y) = (x^2 + y^2, 2xy) = (-1,0)$$

with $x^2 + y^2 = 1$, if $x = 0, -y^2 = -1$, $y^2 = 1, y = \pm 1$.
So we take

$$u = S^1 \setminus \{(0,1),(0,-1)\}$$

Illustrasjon 1
using trig identities, we see that

$$\alpha = 2\theta + 2\pi \cdot n \text{ for some } n \in \mathbb{Z}$$

So h is the function

$$h(\theta) = \begin{cases} 2\theta \text{ if } \theta \in (-\pi/2, \pi/2) \\ 2\theta - 2\pi \text{ if } \theta \in (\pi/2, 3\pi/2) \end{cases}$$

Finally, to determine $Df_{(x,y)}$ where $(x,y) \in u$, we have that

$$Df_{(x,y)} = Dexp_{h(\theta)} \circ Dh_\theta \circ Dexp_\theta^{-1}$$

$$\varphi(\theta) = (x,y)$$

if $v \in T_{(x,y)}S^1$ and $(x,y) = (\cos\theta, \sin\theta)$,

$$v = \left((x,y), t \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right)$$

So $Df_{(x,y)}(v)$ is what?

$$= Dexp_{h(\theta)} \circ Dh_\theta \circ Dexp_\theta^{-1} \left((x,y), t \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right)$$

recall

$$Dexp_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

so $Df_{(x,y)}(v) = Dexp_{h(\theta)} \circ Dh_\theta(\theta, t)$

$$dh_\theta = \begin{cases} 2\theta \text{ if } \theta \in (-\pi/2, \pi/2) \\ 2\theta - 2\pi \text{ if } \theta \in (\pi/2, 3\pi/2) \end{cases}$$

$$\Rightarrow Dexp_{h(\theta)} \cdot (h(\theta), 2t)$$
$$= \left((\cos(2\theta), \sin(2\theta)), 2t \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\right)$$
$$D_{f(x,y)} : T_{(x,y)}S^1 \to T_{f(x,y)}S^1$$
$$t \cdot \begin{bmatrix} -\sin(\theta) \\ \cos()\theta) \end{bmatrix} \longmapsto 2t \cdot \begin{bmatrix} -\sin(2\theta) \\ \cos(2\theta) \end{bmatrix}$$

So if we pick a basis for

$$T_{(x,y)}S^1 = Span\{\left((x,y), \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\right)\}$$
$$T_{f(x,y)}S^1 = Span\{\left(f(x,y), \begin{bmatrix} -\sin 2\theta \\ \cos 2\theta \end{bmatrix}\right)\}$$
$$Df_{(x,y)} = [2]$$
$$X \subseteq \mathbb{R}^n$$

Remark. 1.
$$T_xX \subseteq T_x\mathbb{R}^n$$
$$\left(\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\right) \in T_{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}} S^1$$

2. Pick a good basis for T_xX. There is a standard basis for T_xX once we pick a coordinate system.
Diagram 6
$$\varphi^{-1}(p) = (x_1(p), x_2(p), \ldots, x_k(p))$$
if $\varphi(0) = x$
$$T_0\tilde{u} = \{(0,v) | v \in \mathbb{R}^k\} \simeq \mathbb{R}^k$$
Standard basis.
$$(0,e_1), (0,e_2), \ldots, (0,e_k) \in T_0\tilde{u}$$
So then taking $D\varphi_0$ gives a basis for T_xX.
$$D\varphi_0(0,e_i) = (x, d\varphi_0(e_i)) = \frac{\partial}{\partial x_i}|_{x\in X}$$
If we use coordinates
$$\varphi : \tilde{u} \to u \subseteq X$$
$$\psi : \tilde{v} \to V \subseteq Y$$
$$\varphi^{-1} = (x_1, \ldots, x_k)$$
$$\psi^{-1} = (y_1, \ldots, y_k)$$
then the matrix for Df_x with respect to $\{\frac{\partial}{\partial x_i}|_x\}$ and $\frac{\partial}{\partial y_i}|_{f(x)}$ is just dh_0.
Diagram 7
$$Df_x = \left(\frac{\partial}{\partial x_i}|_x\right) = \sum_{j=1}^{l} dh_0(i,j)\frac{\partial}{\partial y_i}|_{f(x)}$$

3.
$$X \xrightarrow{f} Y$$

at $x \in X$, find a smooth extension of f to $x \in u' \subseteq \mathbb{R}^n$ open
Diagram 8
Tegning 2
diagram 9
thus if $v \in T_x X$
$$Df_x(v) = D\tilde{f}_x(v) \in T_{f(x)}Y.$$

Caution! $D\tilde{f}_x$ depends on the choice of extension! We could pick another extension $\bar{f}$, and maybe $D\tilde{f}_x \neq D\bar{f}_x$. but

$$D\tilde{f}_x|_{T_xX} = D\bar{f}_x|_{TxX}$$

E.g. $\tilde{f}(x,y) = (x^2 - y^2, 2xy)$

$$f : S^1 \to S^1$$

$$D\tilde{f}_{(x,y)} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

we can only plug in

$$v \in T_{(x,y)}S^1 \Rightarrow \left((x,y), t \begin{bmatrix} -y \\ x \end{bmatrix} \right).$$

Then
$$Df_{(x,y)} = \left((x,y), t \begin{bmatrix} -y \\ x \end{bmatrix} \right) = \left(f(x,y), t \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \right)$$
$$\left(f(x,y), t \cdot 2 \begin{bmatrix} -2xy \\ x^2 - y^2 \end{bmatrix} \right)$$
$$= \left(f(x,y), 2t \begin{bmatrix} -f_y(x,y) \\ f_x(x,y) \end{bmatrix} \right)$$

This is indeed in $T_{f(x,y)}S^1$.

$$X = \{(x,0) \in \mathbb{R}^2 | x \in \mathbb{R}\}$$
$$f : X \to X$$
$$(x,0) \mapsto (x^2, 0)$$

What is $Df_{(x,0)}$? Extension of f:

$$\tilde{f}(x,y) = (x^2, 0)$$

defined on $\mathbb{R}^2$. OR.
$$\bar{f}(x,y) = (x^2, y)$$

only require $\bar{f}$ is smooth,

$$\bar{f}(x,0) = f(x,0) = (x^2, 0)$$

$$D\tilde{f} = \begin{bmatrix} 2x & 0 \\ 0 & 0 \end{bmatrix}, D\bar{f} = \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix}$$

$$Df_{(x,0)}\left(\begin{bmatrix} t \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2x & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 2xt \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 2xt \\ 0 \end{bmatrix}$$

$$Df_{(x,0)}\left((x,0)\begin{bmatrix} t \\ 0 \end{bmatrix}\right) = \left((x^2,0), \begin{bmatrix} 2xt \\ 0 \end{bmatrix}\right).$$

Definition 5.1. Let $X \subseteq \mathbb{R}^n$ be a smooth manifold. The local dimension of X at $x \in X$ is

$$dim_x X = dim_{\mathbb{R}} T_x X.$$

Proposition 5.1. *$dimX : X \to \mathbb{Z}$*
this function is locally constant.
i.e., ofr any $x \in X$, there exists $u \subseteq X$ open, $x \in u$, with $dim_y X = dim_x X$ for all $y \in u$, i.e., $dimX|_u$ is constant.

Consequence: if X is connected, there is $k \in \mathbb{Z}$ with $\forall x \in X$, $dim_x X = k$. When we say $dimX = k$, we implicitly assume X is connected.

Proof. We show $dimX : X \to \mathbb{Z}$ is continuous. Let $\{k\} \subseteq \mathbb{Z}$. This is an open set, so we need to show

$$\{x \in X | dim_x X = k\}$$

is an open set in X.
To see that this is open, we consider $x \in X$ with $dimT_x X = k$. Ther is a coordinate system

$$\varphi : \tilde{u} \to u \subseteq X$$

with $\tilde{u}$ open $\subseteq \mathbb{R}^k$ and $x \in u$.
That gives

$$(D\varphi)_{\varphi^{-1}(x)} : T_{\varphi^{-1}(x)}\mathbb{R}^k \xrightarrow{\simeq} T_x X$$

but for any other $y \in u$, we get

$$(D\varphi)_{\varphi^{-1}(y)} : T_{\varphi^{-1}(y)}\mathbb{R}^k \xrightarrow{\simeq} T_y X$$

therefore $dim_y X = k$ too.
hence

$$x \in u \subseteq \{x \in X | dim_x X = k\}$$

open, so $dimX^{-1}(\{k\})$ is also open. So oure dimension function is continuous. □

Example 5.2.

$$X = \{(x,y,0) | x^2 + y^2 = 1\} \cup S^2\left((10,10,10);1\right) \subseteq \mathbb{R}^3$$

6 30. jan

What is the local structure of a smooth map

$$f : X \to Y?$$

Idea: Use the derivative Df_x as a local approximation of f.

Definition 6.1. A smooth map $f : X \to Y$ is a local diffeomorphism at $x \in X$ if there exist $x \in u$ open $\subseteq X$ and $f(x) \in V$ open $subseteq Y$ so that $f|_u : u \xrightarrow{\simeq} V$.

Example 6.1.

$$can : \mathbb{R}^n \to \mathbb{R}^n$$
$$= id : x \mapsto x$$

obviously a global so local diffeo at all points.

Example 6.2.

$$exp : \mathbb{R} \to S^1$$
$$exp|_{((\theta-\varepsilon,\theta+\varepsilon))}(\theta - \varepsilon, \theta + \varepsilon) \xrightarrow{\simeq} I(\theta;\varepsilon)$$
$$\text{and } \varepsilon < \pi$$

A local diffeo at all $\theta \in \mathbb{R}$. Not injective, so not a global diffeomorphism.

Q: Can I use Df_x to see if f is a local diffeo at x?

Theorem 6.1. Inverse Function Theorem
(Calculus version): $u \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^m$ open and $f : u \to V$ smooth. Then f is a local diffeo at $x \in u$ iff df_x is an isomorphism, i.e. df_x is square matrix, $det(df_x) \neq 0$.

Theorem 6.2. Inverse Function Theorem
Let $f : X \to Y$ be a smooth map of smooth manifolds, hten f is a local diffeomorpshim at $x \in X$ iff $Df_x : T_xX \to T_{f(x)Y}$ is an isomorphism.

Proof. At $x \in X$, we can only find local coordinates
Diagram 1
if Df_x is an iso, then $D\psi \circ Dh_0 \circ D\varphi_0^{-1}$ where, $D\varphi_0^{-1}, D\psi$ are isomorphisms, so Dh_0 has to be iso too.
Diagram 2
diagram 3 □

Example 6.3.

$$Y = \{(x, y) \in \mathbb{R}^2 | y = x^2\}$$

Drawing of parabola

$$\mathbb{R} \xrightarrow{f} Y$$
$$f(x) = (x^2, x^4)$$

f is local diffeo at $x \neq 0$.
Diagram 4
h is how I think of f in coords.

$$\psi(h(x)) = f(x) = (x^2, x^4)$$

$$\big(h(x), h(x)^2\big) \Rightarrow h(x) = x^2$$

Pick better coordinates
Diagram 5

$$\psi(h(x)) = \big(h(x), h(x)^2\big) = f(\sqrt{x}) = \big(x, x^2\big)$$
$$h(x) = x$$

Identity map is obviously a diffeomorphism.

Reformulation of IFT
$f : X \to Y$ smooth, Df_x is an isomorphism, there are coordinate systems
Diagram 6
so that in cords., f is the identity map.
Diagram 7

Proof. The IFT gives us
Diagram 8 □

if $f : X \to Y$ is local diffeo at <u>all</u> x, it need not be a global diffeo.

Exercise 6.1. if $f : X \to Y$ is bijective, smooth, local diffeo, then f is a diffeomorphism.

"Nice maps $f : X \to Y$ where $dimX = dimY$"
"Nice maps $f : X \to Y$ with $dimX \leq dimY$"
$Df_x : T_xX \to T_{f(x)}Y$, $T_xX = ndim$, $T_{f(x)}Ymdim, n \leq m$. if Df_x is injective, what does it tell us about f locally?
If $f : X \to Y$, df_x is injective at $x \in X$, what can we say about f?

Definition 6.2. If $f : X \to Y$ is smooth, we say f is an immersion at $x \subseteq X$ is the derivative Df_x is injective.

Definition 6.3. Canonical immersion:

$$can : \mathbb{R}^k \to \mathbb{R}^{k+l}$$

$$(x_1, x_2, \ldots, x_k) \mapsto (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$$
$$dcan)_x =$$

Matrix 1
so can is immersion.

image of can is just the submanifold of $\mathbb{R}^{k+l}$ given by

$$\{x \in \mathbb{R}^{k+l} | x_{k+1} = x_{k+2} = \cdots = x_{k+l} = 0\}$$

Hopes: $f : X \to Y$ is an immersion, is the image a submanifold?
locally does this work? f immersion at x, is$f$ locally diffeo. to image around x?
we'll show: f immersion at x, then locally at x, f is just the canonical immersion (in some coords.)

Lemma 6.3. Say $u \subseteq \mathbb{R}^k$ open, $0 \in u$, and $h : u \to V \subseteq \mathbb{R}^{k+l}$ is an immersion at 0.
So dh_0 is injective.
Pick a linear diffeo. $P : \mathbb{R}^{k+l} \to \mathbb{R}^{k+l}$ so that
Diagram 9
and $dh'_0 =$ Matrix 1

Proof. $dh_0 : \mathbb{R}^k \to \mathbb{R}^{k+l}$ is injective. So $rank(dh_0) = k$. Do row operations on dh_0 to get:
Matrix 2
this corresponds to constructing $n+l\times n+l$ invertible matrix P so $p{\cdot}dh_0$ =Matrix 1.

$$V' = P(V)$$

$$h'0 = d(P \circ h)_0 = P \circ dh_0 = \text{ Matrix 1}$$

$\square$

Theorem 6.4. $f : X \to Y$ local immersion at x.
Diagram 10
$dh_0 =$ Matrix 1
Now we'll get new coords., so f is can in coords.

Definition 6.4. Define $\tilde{u} \in \mathbb{R}^k$

$$G : \tilde{u} \times \mathbb{R}^l \to \mathbb{R}^{k+l}$$

$$G(x_1, \dots, x_{k+l}) = h(x_1, \dots, x_k) + (0, \dots, 0, x_{k+1}, \dots, x_{k+l})$$

$$dG_0 = \begin{bmatrix} dh_0 & | & 0 \\ 0 & | & I_l \end{bmatrix} = \begin{bmatrix} I_k & | & 0 \\ 0 & | & I_l \end{bmatrix}$$

Hence G is a local diffeo. at 0. So we can pick neighborhoods at 0 so G is a diffeo.

$$G :\supseteq \mathbb{R}^k u' \times Z(\subseteq \mathbb{R}^l) \to v' \subseteq \tilde{v} \subseteq \mathbb{R}^{k+l}.$$

Observe that: $u' \subseteq \tilde{u}$.

$$u' \xrightarrow{can} u' \times z \xrightarrow{G} v'$$

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0) \mapsto (h(x_1, \dots x_k))$$

Observe we van take $u' \subseteq \tilde{u}$.
Diagram 11
f in coords. $\varphi : u' \to \hat{u}$ and $\psi \circ G : G : u' \times z \to \hat{V}$.

Corollary. $f : X \to Y$ an immersion at $x \in X \Rightarrow \exists u \subseteq X$ open, $\forall y \in u, Df_y$ injective.

Corollary. f is an immersion at $x \in X$, there is some $u \subseteq X$ where,

$$f|_u : u \to f(u)$$

is a diffeomorphism.

Remark. if $f : X \to Y$ is an immersion, we may not have $f : X \to f(X)$ a diffeo!

What if f is an injection <u>and</u> an immersion? Still $f : X \to f(X)$ not a diffeo. in general.
Illustrasjon 1

Definition 6.5. A map $f : X \to Y$ is called *proper* is for any compact $C \subseteq Y$, $f^{-1}(C)$ is also compact.

Definition 6.6. $f : X \to Y$ is an *embedding* if it is an immersion, injective and proper.

Consequence: if f is an embedding
$f(X) \subseteq Y$ is a manifold, and $f : X \to f(X)$ is a diffeomorphism.
f immersion at x
Diagram 12
Diagram 13

Proposition 6.5. if f is proper, injective, immersion then for any $u \subseteq X$ open, $f(u) \subseteq f(X)$ is open.

Proof. To show $f(u)$ open, we show $f(X) \setminus f(u)$ is closed.
Consider a sequence $y_i \in f(X) \setminus f(u)$ converging to: $\lim y_i = y \in f(X)$. We need to show $y \in f(X) \setminus f(u)$ or $y \notin f(u)$.

$$\{y_i | i \in \mathbb{N}\} \cup \{y\}$$

is compact.

$$f^{-1}\left(\{y_i | i \in N\} \cup \{y\}\right) \text{ compact too.}$$

f inj. so we can find $x_i \in X$

$$f(x_i) = y_i$$
$$f(x) = y$$

and $\{x_i | i \in \mathbb{N}\} \cup \{x\}$ is compact. So X_i has a convergent subsequence:

$$\lim x_{n_i} = z \in \{x_i | i \in \mathbb{N}\} \cup \{x\}$$

We check.

$$f(z) = \lim f(x_{n_i}) = \lim y_{n_i} = y = f(x)$$

$\Rightarrow z = x$ by injectivity. $x_i \notin u$ as $y_i \notin f(u)$. So $x_i \in X \setminus u$ is closed, so $z = x \imath X \setminus u$. Hence $y = f(x) \notin f(u)$. $\square$

Lecture week 5 DiffTop

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Local properties of smooth maps

$f : X \to Y$
$dimX \geq dimY$.

Definition 7.1. A map $f : X \to Y$ is a submersion at $x \in X$ if $Df_x : T_xX \to T_{f(x)}Y$ is surjective.

Example 7.1. Canonical submersion is a projection map

$$can : \mathbb{R}^{n+k} \to \mathbb{R}^n$$

$$(x_1, \ldots, x_m, x_{n+1}, \ldots, x_{n+k}) \mapsto (x_1, \ldots, x_n)$$

is a submersion at all points.

$$Dcan_x = \left[I_n | 0\right]$$

$n \times (n + k)$.

Proposition 7.1. Let $f : X \to Y$ be a submersion at $x \in X$, then there are coordinate systems about $x \in X$ and $f(x) \in Y$ so that the map f appears to be the canonical submersion in coordinates.

Proof. Find some local coordinates
Diagram 1
Assumption that Df_x is surjective implies Dh_0 is surjective.
We can find a matrix Q so that

$$dh_0 \cdot Q = \left[I_n|0\right]$$

with Q, we can modify our coordinates about $x \in X$. In the new coordinates, f appears to be $h \circ Q$, and so

$$d\left(h \circ Q\right)_0 = dh_0 \cdot Q = \left[I_n|0\right]$$

Assume we've done this, redefine our charts so
diagram 2

$$dh_0 = \left[I_n|0\right]$$

Now define a new map

$$G : \tilde{u} \to \tilde{v} \times \mathbb{R}^k$$

$$\tilde{u} \subseteq \mathbb{R}^{n+k}, \tilde{v} \subseteq \mathbb{R}^{n+k}$$

$$G(x) = (h(x), x_{n+1}, \ldots, x_{n+k})$$

$$dG_0 \begin{bmatrix} dh_0| & 0 \\ 0| & I_k \end{bmatrix} = \begin{bmatrix} I_n| & 0 \\ 0| & I_k \end{bmatrix} = I_{n+k}$$

Observe G is a local diffeo at 0 because $dG_0 = I_{n+k}$ by I.F.T. So we can pick $u' \subseteq \tilde{u}$ and $w \subseteq \tilde{v} \times \mathbb{R}^k$ where $G : u' \xrightarrow{\simeq} w$. Use coordinates $\varphi \circ G^{-1}$ about $x \in X$, then f is just canonical submersion in coordinates. □

Proposition 7.2. If $f : X \to Y$ is a submersion at $x \in X$, then f is a submersion in an open neighborhood around x.

Proof. Find good coordinate systems:
Diagram 3
for any point $u \in \mathcal{U}$,

$$Df_u = D\psi_{can(\varphi(u))} \circ Dcan_{\varphi(u)} \circ D\varphi^{-1}_{\varphi(u)}$$

isos plus just $[I_n|0]$ so surjective, $\Rightarrow Df_u$ is surjective. □

Proposition 7.3. If $f : X \to Y$ is smooth, and $f(x) = y$, and f is submersion at $x \in X$, then $f^{-1}(y)$ is locally euclidean.

Proof. Pick coords so f is the canonical submersion at $x \in X$.
diagram 4
consider $f^{-1}(y) \cap \mathcal{U}$ open relative to $f^{-1}(y)$. I claim this is diffeo to open subset of $\mathbb{R}^k$.

$$f^{-1}(y) \cap \mathcal{U} \simeq \varphi^{-1}(f^{-1}(y))$$

$$= can^{-1}(\psi^{-1}(y)) = can^{-1}(0) = \{x \in \tilde{\mathcal{U}} | x_1 = 0, x_2 = 0, \ldots, x_n = 0\}$$

first diffeo is diffeo via φ.
picture 1
to get coords for $can^{-1}(0)$, use another projection map.

$$\pi : \mathbb{R}^{n+k} \to \mathbb{R}^k$$

$$(x_1, \dots, x_{n+k}) \mapsto (x_{n+1}, \dots, x_{n+k})$$

if we look at

$$can^{-1}(0) = \{x \in \tilde{\mathcal{U}} | x_1 = 0, x_2 = 0, \dots, x_n = 0\}$$

$$can^{-1}(0) \xrightarrow{\pi} \mathbb{R}^k$$

we get $W = \pi(can^{-1}(0))$ and

$$\{x \in \tilde{\mathcal{U}} | x_1 = 0, x_2 = 0, \dots, x_n = 0\} \simeq W \subseteq \mathbb{R}^k \text{ open.}$$

$$can^{-1}(0) \xrightarrow{\pi, \simeq} W$$

is a diffeo. □

Example 7.2.

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

$$f(x) = x_1^2 + x_2^2 + \dots + x_{n+1}^2$$

$$f^{-1}(1) = \{x \in \mathbb{R}^{n+1} | \sum x_i^2 = 1\} = S^n.$$

Definition 7.2. Let $f : X \to Y$ be smooth. $y \in Y$ is a *regular value* of f if:

$$\forall x \in f^{-1}(y), Df_x \text{ is surjective}$$

i.e., f is a submersion at x.

Theorem 7.4. Let $f : X \to Y$ smooth, suppose X, Y connected, $dimX = n, dimY = k$. If $y \in Y$ is a regular value of f, $f^{-1}(y)$ is a manifold. If $f^{-1}(y) \neq \emptyset$, then $dimf^{-1}(y) = dimX - dimY = n - k$.

Proof.

$$\forall x \in f^{-1}(y)$$

as f is a submersion at x, we know $f^{-1}(y)$ is locally euclidean at x. So $f^{-1}(y)$ is a manifold. And $dimf^{-1}(y)$ is obtained from previous prop too. □

Example 7.3. S^1 is a smoth manifold.

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

$$f(x) = \sum_{i=1}^{n+1} x_i^2$$

$$df_x = [2x_1, 2x_2, \dots, 2x_{n+1}]$$

is surjective when?
$x \neq 0$
Hence $f^{-1}(y) \subseteq \mathbb{R}^{n+1}$ is a manifold as long as $0 \notin f^{-1}(y)$, i.e. for all $y \in \mathbb{R} \setminus \{0\}$.

$$S^n = f{-}1(1).$$

$$\emptyset = f^{-1}(-1)$$

Regular values of f are $\mathbb{R} \setminus \{0\}$.

Which manifolds can be obtained as the preimage of a regular value? Specifically

(1)
$$f : \mathbb{R}^n \to \mathbb{R}^k$$
When is a manifold $X \subseteq \mathbb{R}^n$ obtained as $X = f^{-1}(0)$ for 0 a regular value of f?
One way to think of this is
$$f = (f_1, \dots, f_k)$$
for $f_i : \mathbb{R}^n \to \mathbb{R}$ is a "smooth condition", then $f_i^{-1}(0) \subseteq \mathbb{R}^n$ are points with 1 less degree of freedom.
So $f^{-1}(0) = f_1^{-1}(0) \cap f_2^{-1}(0) \cap \cdots \cap f_k^{-1}(0)$.

(2) One modification, which submanifolds of a manifold X are obtained from $f : X \to \mathbb{R}^k$ $f^{-1}(0)$, 0 a regular value.
If we take $k = 0$. Consider $f : X \to \mathbb{R}^0$, $\mathbb{R}^0 = \{0\}, T_0\mathbb{R}^0 = \{0\}$.
$$Df_x : T_xX \to T_0\mathbb{R}^0$$
is surjective always, so $f^{-1}(0) = X$.

(3) "Partial answers"
GP Partial converse 1 not interesting.

(4)

Proposition 7.5. Let $Z \subseteq X$ be a submanifold, then there is a smooth function $f : X \to \mathbb{R}^k$ so that Z is locally $f^{-1}(0)$, i.e., for $x \in Z$, we can find $u \subseteq X$ open and $f : X \to \mathbb{R}^k$, so that $f^{-1}(0) \cap u = z \cap u$.

Proof. $Z \xrightarrow{i} X, x \mapsto x$ is an immersion.
We have
Diagram 5
Check that
$$f^{-1}(0) = Z \cap V$$
□

Example 7.4. $\{(0, 0, 1)\} \subseteq S^2$, there is no smooth function $f : S^2 \to \mathbb{R}^2$ so that $f^{-1}(0) = Z$, 0 a regular value.
Proof using degree theory.

Proposition 7.6. $f : X \to Y$ smooth, $Z = f^{-1}(y)$, y a regular value, then $T_xZ = ker(Df_x : T_xX \to T_yY)$.

Example 7.5. $S^n = f^{-1}(1)$
$$T_xS^n = ker(df_x = [2x_1, \dots, 2x_{n+1}]) = \{(x, v) | x \cdot v = 0\}$$

$$\mathbb{C} \simeq \mathbb{R}^2$$

$$x + iy \leftrightarrow (x, y)$$

We can think of S^2 as $\mathbb{C}$ (or $\mathbb{R}^2$) with "point at infinity".
Formula for $\varphi_N(x, y)$

$$\begin{aligned} l(t) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} tx \\ ty \\ 1-t \end{bmatrix} = \varphi_N(x, y) \end{aligned}$$

when t is such that $\in S^2$.
Need that

$$\begin{aligned} (tx)^2 + (ty)^2 + (1-t)^2 = 1 \\ t^2(x^2 + y^2) + 1 - 2t + t^2 = 1 \\ t\big(t(x^2 + y^2 + 1) - 2\big) = 0 \end{aligned}$$

need $t(x^2 + y^2 + 1) = 2$

$$t = \frac{2}{x^2 + y^2 - 1}$$

$$\varphi_N(x, y) = \frac{1}{x^2 + y^2 + 1} \cdot \begin{bmatrix} 2x \\ 2y \\ x^2 + y^2 - 1 \end{bmatrix}$$

$$\varphi_N^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \frac{1}{1-z} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\varphi_S, \varphi_S^{-1}$$

$$\varphi_S^{-1} \circ \varphi_N, \varphi_N^{-1} \circ \varphi_S$$

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Cauchy-Riemann says

$$P(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic implies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Example 8.1. Check this works for $P(z) = z^2 = (x^2 - y^2) + i(2xy)$

$$DP_{(x,y)} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

Outline of proof

Remark. $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n, so $a_n \neq 0$, there are at most n roots of P. If w_0 a root of P, can factor P.

$$P(z) = P_1(z)(z - w_0)$$

$$degP_n < n \text{ use I.H.}$$

(1) We define from $P(z) = \sum_{j=0}^{n} a_j z^j$ a smooth map

$$F : S^2 \to S^2$$

Diagram 1
if $a \in S^2$, $a \neq ((0,0,1) = N)$ then $a = \varphi_N(x,y) = \varphi_N(x + iy)$
define $F(a) = \varphi_N(P(x + iy))$.
for $N = (0,0,1) \in S^2$, define

$$F(N) = \begin{cases} N \text{ if } P \text{ is not constant} \\ a_0 \text{ if } P(z) = a_0 \end{cases}$$

(1) Construct $F : S^2 \to S^2$

(2) At any regular value

$$y \in S^2, \big(\forall x \in f^{-1}(y), DF_x \text{ is surjective.}\big)$$

we know $F^{-1}(y) \subseteq S^2$ is a manifold of dimension 0.
A 0-manifold is a *discrete subset of* S^2. i.e. $\{x\} \subseteq f^{-1}(y)$ is open for all $x \in f^{-1}(y)$.
$f^{-1}(y)$ is closed set, $\{y\} \subseteq S^2$ is closed, $f^{-1}(y) \subseteq S^2$ is thus compact. So $f^{-1}(y)$ is a finite set.

$$\{x\} | x \in f^{-1}(y)\}$$

open cover imp finite set.

(1) Construct $F : S^2 \to S^2$

(2) At any regular value $y \in S^2$, har $F^{-1}(y) = \{x_1, \ldots, x_N\}$ a finite set.

(3) What are critical points of F? i.e. DF_P not surjective? Possibly ∞, otherwise, it is the points where

$$P'(z) = 0 \leftrightarrow DF_{\varphi_N(z)} = 0$$

at most $n - 1$ points

$$z_1, \ldots, z_{n-1}$$

So regular values of F are at least

$$S^2 \setminus \{F(\infty), F(\varphi_N(z_1)), \ldots, F(\varphi_N(z_{n-1})\}$$

(1) Construct $F : S^2 \to S^2$

(2) At any regular value $y \in S^2$, har $F^{-1}(y) = \{x_1, \ldots, x_N\}$ a finite set.

(3) What are critical points of F?

$$S^2 \setminus \{F(\infty), F(\varphi_N(z_1)), \ldots, F(\varphi_N(z_{n-1})\}$$

(4) Define a function

$$C : S^2 \setminus \{F(\infty), F(\varphi_N(z_1)), \ldots, F(\varphi_N(z_{n-1})\} \to \mathbb{N}$$

$$C(y) = \#F^{-1}(y)$$

(5) Technical Theorem (Stack of Records Thm.)
Says C is continuous/locally constant.

(6)

$$S^2 \setminus \{F(\infty), F(\varphi_N(z_1)), \ldots, F(\varphi_N(z_{n-1})\}$$

is connected so C is globally constant

$$\forall y, C(y) = c \ \in \mathbb{N}$$

(7) Case (1) $C(y) = 0$
For almost all $y \in S^2$, $F^{-1}(y) = \emptyset$. So image of S^2 is not a regular value.

$$F(S^2) \subseteq \{\infty, F(\varphi_N(z_1)) \ldots F(\varphi_N(z_{n-1}))\}$$

So $F(S^2) = \{\varphi_N(a_0)\}$ is a constant function.

Cbse (2) Say $C(y) > 0$.
Note that $\varphi_N(0) = (0, 0, -1) = S$.
Either $S \in \{F(\varphi_N(z_1)) \ldots F(\varphi_N(z_{n-1}))\}$ in which case $S = F(\varphi_N(z_i)) \Rightarrow 0 = P(z_i)$.

Ccse (3) $C(y) > 0$ but now S is a regular value. So $F^{-1}(S)$ has $C(y) > 0$ elements. So there is some point $p \in S^2$, with $F(p) = S, p \neq N$ so

$$P(\varphi_N^{-1}(p)) = 0$$

Lemma 8.1. If X is discrete and compact, X is finite.

Proof. Cover X with

$$u = \{\{x\} | x \in X\}$$

only subcover is u itself. □

Theorem 8.2. Stack of records theorem
$f : X \to Y$ smooth,
$dim X = dim Y$
f is a submersion at all points. X is compact. Then for any $y \in Y$, $f^{-1}(y)$ is a finite set, $f^{-1}(y) = \{x_1, \ldots, x_N\}$ and there exists open sets

$$x_i \in u_i \subseteq X, 1 \leq i \leq N$$

$$y \in V \subseteq Y$$

Drawing

(1) $f|_{u_i} : u_i \to V$ is a diffeo

(2) $u_i \cap u_j = \emptyset$ if $i \neq j$

(3) $f^{-1}(V) = \cup_{i=1}^{N} u_i$

$$f^{-1}(y) = x_1, \ldots, x_N$$
$$\forall z \in V, f^{-1}(z) = \{f^{-1}|_{u_1}(z), f^{-1}|_{u_2}(z), \ldots, f^{-1}|_{u_N}(z)\}$$

Proof. $f^{-1}(y)$ is a 0-manifold, as y is a regular value for f, $dimX = dimY$.
Compactness of $X \Rightarrow f^{-1}(y)$ is compact, so it's finite, $f^{-1}(y) = \{x_1, \ldots, x_N\}$.
First off, $Df_{x_i} : T_{x_i}X \to T_yY$ is surjective $\Rightarrow Df_{x_i}$ is isomorphism

$$\Rightarrow f \text{ is a local diffeo at } x_i$$

So we find $x_i \in u_i \subseteq X$ open, $y \in v_i \subseteq Y$ open

$$f|_{u_i} : u_i \to v_i \text{ diffeo.}$$

Take $V = \cap_{i=1}^{N} v_i$. Redefine $u_i = f|_{u_i}^{-1}(V)$.
How to ensure $u_i \cap u_j = \emptyset$ when $i \neq j$?
There are open balls $B^n(x_i; \varepsilon)$, so that

$$B^n(x_i; \varepsilon) \cap B^n(x_j; \varepsilon) = \emptyset \text{ when } i \neq j$$

X is compact, $X \setminus \left(\cup_{i=1}^{N} u_i\right)$ closed, so compact.

$$f(X \setminus \left(\cup_{i=1}^{N} u_i\right)) \text{ is compact,}$$

$$Y \setminus f(X \setminus \left(\cup_{i=1}^{N} u_i\right)) \text{ open.}$$

Redefine $V = V \cap \left(Y \setminus f(X \setminus \left(\cup_{i=1}^{N} u_i\right))\right)$
Redefine $u_i = f^{-1}|_{u_i}(V)$. □

Lecture week 6

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Transversatility

Machine for making manifolds.
Take a smooth map $f : X \to Y$ ($f : \mathbb{R}^n \to \mathbb{R}^m$).
Find a regular value $y \in Y$. Then $f^{-1}(y) \subseteq X$ is a manifold.

Question:

If $p : \mathbb{R}^2 \to \mathbb{R}$ a polynomial $p(x, y)$ and $a \in \mathbb{R}$ a regular value of p. Then $p^{-1}(a) \subseteq \mathbb{R}^n$ is a manifold. Which one?

How to describe a 1-manifold?

Connected 1-manifold: diffeo to $\mathbb{R}$ or S^1 (these are the only possibilities)
In general a 1-manifold is a disjoint union of $\mathbb{R}$'s and S^1's.

Hornack's inequality

: If $p : \mathbb{R}^2 \to \mathbb{R}$ polynomial, $deg(p) = m, a \in R$ a regular value of p, then $p^{-1}(a) = \{(x, y) \in \mathbb{R}^2 | p(x, y) = a\}$ has at most

$$\frac{(m-1)(m-2)}{2} + 1$$

connected components.

Transversatility

Let $f : X \to Y$ be a smooth map and $Z \subseteq Y$ a smooth manifold, when is $f^{-1}(Z) \subseteq X$ a smooth manifold?
We first study one point at the time. Consider $x \in f^{-1}(Z)$. We'll see how we can find an open set about $x \in f^{-1}(Z)$ that's "euclidean".
Consider $x \in f^{-1}(Z)$, say $y \in f(x)$.
Find $g : V \to \mathbb{R}^2$ so that $g^{-1}(0) = V \cap Z$, 0 a regular value of g. Observe taht $f^{-1}(V) = u \subseteq X$ open so $u \cap f^{-1}(Z)$ open rel. $f^{-1}(Z)$. Now we can compose

$$g \circ f : u \to \mathbb{R}^2, f^{-1}(Z \cap V) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$$

Question, is 0 a regular value of $g \circ f$? If so $\Rightarrow u \cap f^{-1}(Z)$ is a manifold?
When is 0 a regular value of $g \circ f$? At the very least when is $D(g \circ f)x$ surjective? Which will give $f^{-1}(Z) \cap u$ locally euclidean at x.
So 0 is a regular value of g, so D_{gy} is surjective.
$D(g \circ f)_x$ surjective if Df_x is surjecive. This is however overkill.
Necessary and sufficient condition is that

$$Image(Df_x) + Ker(Dg_y) = T_y Y$$

$g : u \to \mathbb{R}^l, g^{-1}(0) = z \cap u$ submanifold and 0 regular value

$$\Rightarrow T_{yg}^{-1}(0) = Ker(Dg_y)$$

Then $f^{-1}(Z)$ will be a manifold at x if $Image(Df_x) + T_y Z = T_y Y$. (entirely dependent on f and Z).

Definition 9.1. We say a map $f : X \to Y$ is transverse to $Z \subseteq Y$ (a submanifold) if at all $f(x) = y$, $x \in f^{-1}(Z) \Rightarrow Image(Df_x) + T_y Z = T_y Y$.

Theorem 9.1. If $f : X \to Y$ is transverse to $Z \subseteq Y$, then $f^{-1}(Z)$ is a manifold.

Proof. Basically the argument before the definition. □

Remark. $Codim_Y Z = dimY - dimZ$

Corollary.
$$codim_x f^{-1}(Z) = codim_Y Z.$$

Proof. $dimZ = dimY - l, dimf^{-1}(Z) = dim((g \circ f)^{-1}(0)) = dimX - l$

$$\Rightarrow l = codim_Y Z, l = codim_X f^{-1}(Z)$$

□

Definition 9.2. Notation:

$$f \pitchfork Z = f \text{ is transverse to } Z.$$

Example 9.1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^n, df = \left[nx^{n-1}\right]$ is $f \pitchfork \{0\}$? We know $T_0\{0\} = \{0\}$ which is equivalent to $\pitchfork \Leftrightarrow df_0$ surjective. This only happens when $n = 1$. When $Z = \{y\}, \pitchfork \Leftrightarrow y$ a regular value.

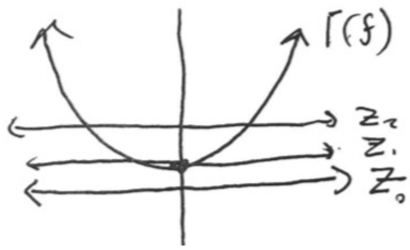
Example 9.2.

$$\Gamma(f)(x) = (x, x^n)$$

$$z = \{(x,0)|x \in \mathbb{R}\} = \mathbb{R} \times \{0\}$$

$$\Gamma(f)^{-1}(Z) \Leftrightarrow \{x \in \mathbb{R}|f(x) = 0\}$$

This is an intersection problem. $\Gamma(f)$ not transverse to Z.



$$\Gamma(f)^{-1}(Z) = \{x|x^n = 0\} \Leftrightarrow \{(x, x^n)|x \in \mathbb{R}\} \cap \{(x,0)|x \in \mathbb{R}\}$$

$$\Gamma(f) \cap z_0 = \emptyset$$

$$\Gamma(f) \cap z_1 \neq \emptyset$$

$$\Gamma(f) \cap z_2 \neq 0$$

10 13. feb.

- Homotopy and stability
- Don't use the entire week on the midterm

Definition 10.1. Let $f_0 : X \to Y$ and $f_1 : X \to Y$ be two smooth functions. We say $f_0 \sim f_1$, in words, f_0 *is homotopic to* f_1, if there exists smooth $F : X \times [0,1] \to Y$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. We'll use $f_t : X \to Y, f_t(x) = F(x,t)$. We'll think of F as a family of maps $f_t : X \to Y$ deforms f_0 into f_1.

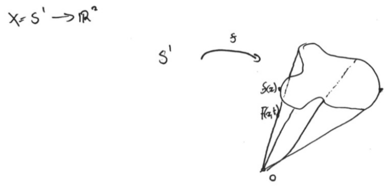
Example 10.1. $X = \mathbb{R}^0$ and $Y = \mathbb{R}^2$, $f : \mathbb{R}^0 \to \mathbb{R}^2$, picks at $x_0 = f(0), g : \mathbb{R}^0 \to \mathbb{R}^2, x_1 = g(0)$
$\Rightarrow f \sim g$, if there is $F : \{0\} \times [0,1] \to \mathbb{R}^2$.
$F(0,t) \in \mathbb{R}^2, F(0,0) = x_0$ and $F(0,1) = x_1$. Alternate name for this case: x_0 and x_1 are in the same path component of X if $x_0 \sim x_1$.

Exercise 10.1. X a manifold, X is connected, $iff \forall x_0, x_1 \in X, x_0 \sim x_1$.

In $\mathbb{R}^2$, any teo points $x_0, x_1 \in \mathbb{R}^2$ are connected by $F(t) = x_0 + t(x_1 - x_0), t \in \mathbb{R}, F(0) = x_0, F(1) = x_1$.

Proposition 10.1. Let $f : X \to \mathbb{R}^n$ be any smooth map. Then f is homotopic to $0 : X \to \mathbb{R}^n; x \mapsto 0$.

Proof. $F(x,t) = (1-t)f(x), F : X \times \mathbb{R} \to \mathbb{R}^n, F(x,0) = f(x), F(x,1) = 0.$ $\square$



Example 10.2. $f : S^1 \to \mathbb{R}^2 \setminus \{0\} : (x,y) \mapsto (x,y)$.
f is not homotopic to any constant map.

Definition 10.2. Two manifolds, X, Y are (smoothly) homotopy equivalent if we can find $f : X \to Y, g : Y \to X$ so that $g \circ f \sim id_X$ and $f \sim g = id_Y$. We say *g is the homotopy inverse of f.*

Definition 10.3. A manifold X is contractable if X is homotopy equivalent to a point ($\mathbb{R}^0$).

X is contractable if we have

$$\begin{tikzcd}
\mathbb{R}^0 \arrow[r, "f"] \arrow[rr, bend right, "0 \circ f = id_{\mathbb{R}^0}"'] & X \arrow[r, "0"] \arrow[rr, bend left, "f \circ 0 \sim id_X"] & \mathbb{R}^0 \arrow[r, "f"] & X
\end{tikzcd}$$

$f \circ 0(x) = f(0) = x_0 \in X$, we need a homotopy
$F : X \times [0,1] \to X$, $F(x,0) = x$ and $F(x,1) = x_0$

Proposition 10.2. $\mathbb{R}^n$ is contractable

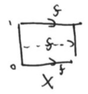
Proof. $id : \mathbb{R}^n \to \mathbb{R}^n$ is homotopic to $0 : \mathbb{R}^n \to \mathbb{R}^n$. “Straight line homotopy” $F(x,t) = (1-t)x$. $\square$

$$\mathbb{R}^n \sim \mathbb{R}^0, \forall n$$

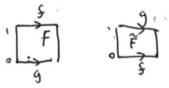
Notation 10.1. $C^\infty(X,Y)$ is the set of all smooth maps $f : X \to Y$.

Remark. Homotopy is an equivalence relation on $C^\infty(X,Y)$.

Proof. (Reflexive) $f \sim f$ (Easy) $F(x,t) = f(x)$

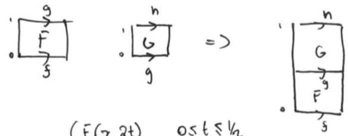


(Symmetric) $f \sim g \Rightarrow g \sim f$



$\tilde{F}(x,t) = F(x, 1-t)$

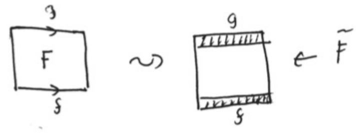
(Transitivity) $f \sim g$, $g \sim h \Rightarrow f \sim h$.



$$H(x,t) = \begin{cases} F(x, 2t) & 0 \le t \le 1/2 \\ G(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$H(x, 0) = f(x)$
$H(x, 1) = h(x)$, We need to show smoothness.

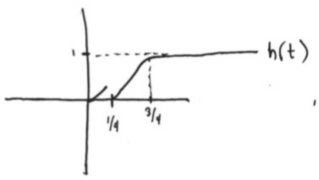
$\frac{\partial^k}{\partial t^k} H$, trick is: Make F, G be constant for $t \in [0, \varepsilon)$ and $t \in (1-\varepsilon, 1]$.



$\tilde{F}(x,t) = f(x)$ for $t \in [0,\varepsilon)$

$\tilde{F}(x,t) = g(x)$ for $t \in (1-\varepsilon, 1]$.

We need smooth step functions to do this:



$$h(t) = \begin{cases} 0 & t \le 1/4 \\ h(t) & 1/4 \le t < 3/4 \\ 1 & t \ge 3/4 \end{cases}$$

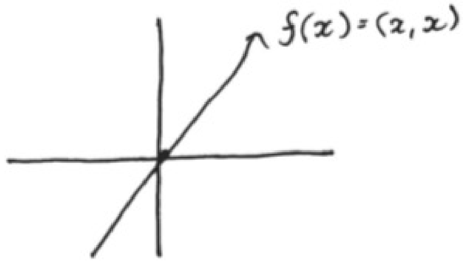
Not analytic!

$\sigma_{\frac{1}{4},\frac{3}{4}}$ be the step function:

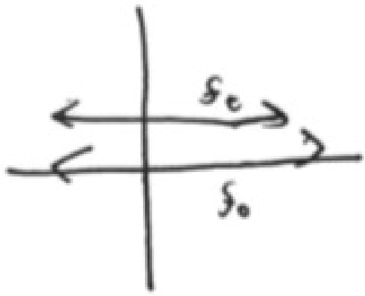
$\tilde{F}(x,t) = F(x, \sigma(t))$, $\tilde{G}(x,t) = G(x, \sigma(t))$

Definition 10.4. A class of maps $\mathcal{T} \subseteq C^\infty(X,Y)$ is *stable* if for any homotopy $F : X \times I \to Y$ with $f_0 \in \mathcal{T}$ there exists $\varepsilon > 0$ so $f_t \in \mathcal{T} \forall t < \varepsilon$.

Example 10.3. $\mathcal{T} = \{f : \mathbb{R}^1 \to \mathbb{R}^2 | f(\mathbb{R}^1) \cap \{(x,0) | x \in \mathbb{R}\} \neq \emptyset\}$ Stable family?



Example 10.4. $f(x) = (x,0)$



$f \in \mathcal{T}$ but $F(x,t) = (x,t)$ wich is smooth. Then if $t \neq 0$, $f_t(\mathbb{R}^1) = \{(x,t) |\, x \in \mathbb{R}\}$ is disjoint from x-axis.

Example 10.5. $\mathcal{T} = \{S^1 \to \mathbb{R}^n | f \,\overline{\pitchfork}\, \{(x,0)|x \in \mathbb{R}^2\}\}$ is stable.

Theorem 10.3. Stability theorem
Let X be compact, and a $Z \subseteq Y$ closed submanifold. The following classes of maps are stable:

(1) Immersion $X \to Y$

(2) Local diffeomorphisms $X \to Y$

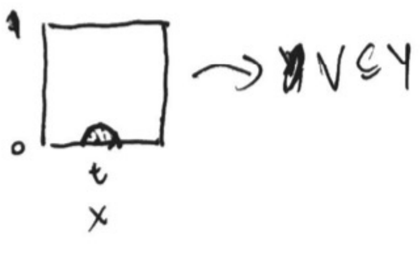
(3) Submersions $X \to Y$

(4) $f \,\overline{\pitchfork}\, Z$, $f : X \to Y$

(5) $f : X \to Y$ an embedding (this is amazing!)

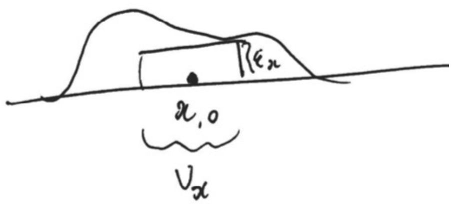
(6) $f : X \to Y$ diffeomorphisms

Proof. (1) $\Rightarrow$ (2) by def.
Consider $f : X \to Y$ an immersion and $F : X \times [0,1] \to Y$ a homotopy with $f_0 = f$. Must find $\varepsilon > 0$ so $\forall t < \varepsilon, f_t : X \to Y$ immersion.
The idea is $(Df_t)_x : T_xX \to Tf_t(x)Y$
We know that in some coords around $(x,0) \in X \times [0,1]$, (basically looks like $u \times [0,\varepsilon), x \in u \subseteq X$ open).



, get coords around $f(x) \in Y, V \subseteq Y$.).
So $(Df_t)_x$ is expressible as a matrix in this open set $(Df_0)_x$ is injective if f_0 is an immerison. So we can capture this as there is a $k \times k$-submatrix of $(Df_0)_x$ that has nonzero determinant.

E.g. $(Df_0)_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, take submatrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$.
So write $A(x,t)$ for this submatrix of $(Df_t)_x$. If $det(A(x,t)) \neq 0 \Rightarrow$ rank of $(Df_t)_x$ is k, and so Df_x is injective.
So $det(A(x,t)) : u \times [0, \varepsilon) \to \mathbb{R}$ is a smooth function. And at our particular point $(x, 0)$, $det(A(x, 0)) = b \neq 0$. Then $det(A(x, 0))^{-1}((b - \varepsilon, b + \varepsilon))$ which is open, at these points $(Df_t)_x$ is injective. For all x, we get open set $u_x \subseteq X \times [0,1]$ where $(Df_t)_x$ is injective. We then know $X \times \{0\} \subseteq \cup_{c \in X} u_x$. Since $X \times \{0\}$ is compact a finite subcover exists, $X \times \{0\} \subseteq \cup_{i=1}^n u_i$.
Modification: $V_x \times [0, \varepsilon_x) \subseteq u_x$, so $V_x \subseteq X$ open.



$X \times \{0\} \subseteq \cup_{x \in X} V_x \times [0, \varepsilon_x)$, so $X \times \{0\} \subseteq \cup_{i=1}^n V_i \times [0, \varepsilon_i)$. Conclusion $\varepsilon = min\{\varepsilon_i | 1 \leq i \leq n\}$ then for all $t < \varepsilon Df_t$ is injective. □

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Recall. $\mathcal{T} \subseteq C^\infty(X, Y)$, $\mathcal{T}$ stable if $\forall F : X \times [0,1] \to Y$ with $f_0 \in \mathcal{T} \Rightarrow \exists \varepsilon > 0 \forall t < \varepsilon, f_\varepsilon \in \mathcal{T}$.

$\{f : X \to Y | f$ a submersion $\}$ is stable when X compact.

Proof. $\forall a \in X, \exists u_a \subseteq X$ open neighborhood and $\varepsilon_a > 0$, s.t. $(a, 0) \in u_a \times [0, \varepsilon_a)$ that gives local coordinates, $f_0(a) \in V \subseteq Y$, s.t. $(Df_t)x$ at all $(x,t) \in u_a \times [0, \varepsilon_a)$ is represented as a matrix.
We can find an $n \times n$-submatrix of $(Df_t)x$, where $det(A(a, 0))$. Now A is smooth so $detA$ is smooth $\Rightarrow \exists u_a \times [0, \varepsilon_a)$ on which $det(A(x,t)) \neq 0$.

$$X \times \{0\} \subseteq \cup_{a \in X} u'_a \times [0, \varepsilon'_a) \subseteq_{compact} \cup_{i=1}^l u_i \times [0, \varepsilon_i),$$

and take $\varepsilon = min\{e_1, \ldots, e_l\}$. □

Lecture week 7

12 25. feb.

Theorem 12.1. Sard's theorem
Let $f : X \to Y$ be a smooth function, then the set of eigenvalues of f has measure 0 in Y. $f^{-1}(y)$ is a manifold for almost all $y \in Y$.

Morse functions

A function $f : \mathbb{R}^k \to \mathbb{R}$ is *Morse*, if at every critical point $p \in \mathbb{R}^k$.
($i.e. df_p = 0$) the hessian matrix of f at p is invertible

$$H(p) = \left[\frac{\partial^2}{\partial x_i \partial x_j}(p)\right]$$

We say p is a nondegenerate critical point when $H(p)$ is invertible.

Definition 12.1. Let X be a smooth manifold and let $f : X \to \mathbb{R}$ be smooth, we say f is *Morse* if at any critical point $p \in X$ we can find a coordinate system so that $h = f \circ \varphi : \tilde{u} \to \mathbb{R}$ is Morse, i.e., need Hessian

$$\left[\frac{\partial^2}{\partial x_i \partial x_j}(0)\right]$$

to be invertible.

Proposition 12.2. The critical points of a Morse function are *isolated* from one another. In other words, the set of critical points is a discrete subspace of X.

Proof. Say $p \in X$ is a critical point. Find coords around p:
coords
and look at $h = f \circ \varphi : \tilde{u} \to \mathbb{R}$. Define

$$g : \tilde{u} \to \mathbb{R}^k x \mapsto \begin{bmatrix} \frac{\partial h}{\partial x_1}(x) \\ \vdots \\ \frac{\partial h}{\partial x_k}(x) \end{bmatrix}$$

$$dg_x = \left[\frac{\partial^2}{\partial x_i \partial x_j}(0)\right]$$

is the Hessian of h. So we have

$$g(0) = 0 (0 \text{ is a critical value of } h)$$

dg_0 is an invertible linear map. Inverse function theorem $\Rightarrow$ g is a local diffoe at $0 \in \tilde{u}$. So in a negiborhood $0 \in \tilde{V} \subseteq \tilde{u}$ where $g : \tilde{V} \to \mathbb{R}^k$ diffeo to its image. Key: $g(x) = 0 iff x = 0, x \in \tilde{V}$. This means $p \in \varphi(\tilde{V})$ is the only critical value of f in $\varphi(\tilde{V})$, which is an open set, only critical point $p \in \varphi(\tilde{V})$. □

Proposition 12.3. On a compact space X, the critical points of a Morse function $f : X \to \mathbb{R}$ is a finite set.

Idea of Morse theory

$X \subseteq \mathbb{R}^n$ compact manifold. $f : X \to \mathbb{R}$ Morse function. (usually think of f as $f(x_1, \ldots, x_N) = x_N$). ($f$ is a height function).

$$T^2 \subseteq \mathbb{R}^3$$

$x \in X$ is critical when $T_x X | z$-axis.
critical points $\leftrightarrow$ local extrema of f on X.

from calculus, you can determine if f has a local min, max, saddle at a critical point by looking at Hessian of f at x.
For the torus:
Hessian at x_1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\leftrightarrow$ manifold is locally the graph of $x^2 + y^2$.
Hessian at x_2:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \leftrightarrow x^2 - y^2$$

Fact of Morse theory: If $s, t \in (a, c_1)$ or (c_1, c_2) or c_2, b then the manifolds $f^{-1}((-\infty, s))$ and $f^{-1}((-\infty, t))$ are homotopy equivalent.
When we cross c_1, the critical point has Hessian

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \leftrightarrow x^2 - y^2$$

Says we attach a saddle to $f^{-1}((-\infty, t))$, at next critical value c_3, we have to attach anothe saddle, at c_4 glue on a disc.

Lemma 12.4. Morse Lemma Let $f : X \to \mathbb{R}$ be a Morse function, $p \in X$ a critical point. In coords $\varphi : \tilde{u} \to X, 0 \mapsto p$ you have $h = f \circ \varphi$, Hessian $H(0) = \left[\frac{\partial^2 h}{\partial x_i \partial x_j}(0)\right] = \left[h_{ij}\right]$. Morse lemma says there is some coord. system

$$\psi : \tilde{V} \to X, 0 \mapsto p$$

so that $f \circ \psi(x) = f(p) + \sum h_{ij} x_i x_j$.

Compare $f : X \to \mathbb{R}$, $p \in X$ a regular point, Df_p surjective. Local submersion theorem says there are coords $\varphi : \tilde{u} \to X, o \mapsto p$ where $f \circ \varphi(x) = f(p) + x_k$.

Example 12.1.

$$S^2 \subseteq \mathbb{R}^3 \xrightarrow{h} \mathbb{R}$$

$$(x, y, z) \mapsto z$$

Is it Morse?

$$\varphi_N(x, y) = \frac{1}{1 + x^2 + y^2}\left(2x, 2y, x^2 + y^2 - 1\right)$$

$$\varphi_S(x, y) = \frac{1}{1 + x^2 + y^2}\left(2x, 2y, 1 - x^2 - y^2\right)$$

$$h_N = h \circ \varphi_N$$

$$h_N(x, y) = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}$$

$$h_s(x, y) = \frac{-(x^2 + y^2 - 1)}{1 + x^2 + y^2}$$

$$dh_N(x, y) = \frac{1}{(1 + x^2 + y^2)^2}[4x \quad 4y]$$

$$dh_s(x,y) = \frac{-1}{(1+x^2+y^2)}[4x \quad 4y]$$

Only critical points are

$$(0,0) in \varphi_N \leftrightarrow S \in S^2$$

$$(0,0) in \varphi_s \leftrightarrow N \in S^2$$

$$H_N(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$H_S(0,0) = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

Imagine $f : X \to \mathbb{R}$ with 2 critical points. X compact.

Theorem 12.5. Reeb's theorem If X is compact n-manifold with a Morse function $f : X \to \mathbb{R}$ that has exactly 2 critical points, then X is homeomorphic to S^n.

Exotic spheres: There are manifolds X that are homeomorphic to S^7 but are not diffeomorphic to S^7.

Lemma 12.6. $u \subseteq \mathbb{R}^k$ open, $f : u \to \mathbb{R}$ smooth. For almost every $a \in \mathbb{R}^k$

$$f_a(x) = f(x) + a_1x_1 + \cdots + a_kx_k$$

is Morse.

Proof. Define

$$g(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_k}(x) \end{bmatrix}, g : u \to \mathbb{R}^k$$

Recall dg_x is the Hessian of f at x.

$$(df_a)_x = g(x) + a.$$

$$g_a(x) = \begin{bmatrix} \frac{\partial f_a}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f_a}{\partial x_k}(a) \end{bmatrix}, (dg_a)_x = (dg)_x$$

A point $p \in u$ is critical point of f_a iff $(df_a)_p = 0 = g(p) + a$ iff $g(p) = -a$. Sard's theorem says there exists $-a \in \mathbb{R}^k$ that is a reg. value for g. So for any crit. point p of f_a, i.e., $g(p) = -a$, we have $(dg)_p$ invertible, Hessian of f_a is invertible. $\square$

Theorem 12.7. Say $X \subseteq \mathbb{R}^n$ is a manifold and $f : X \to \mathbb{R}$ smooth. Then for almost every $a \in \mathbb{R}^n$, $f_a(x) = f(x) + \sum_{i=1}^n a_ix_i$ is Morse.

Lemma 12.8. Let $X \subseteq \mathbb{R}^n$ be a k-manifold. Then at $p \in X$, there are coords $x_{i_1}, \ldots, x_{i_k}$ taken from the standard coords on $\mathbb{R}^n$, that give coords on X.

Proof. $T_xX = Span\{v_1, \ldots, v_k\}, v_i \in \mathbb{R}^n$. Look at $[v_1, \ldots, v_k]$ k-coords are lin.indep.. So can find k rows where the corresponding $k \times k$ submatrix is invertible. Say it is rows $i_1, i_2, \ldots, i_k$. Then look at $\pi : \mathbb{R}^n \to \mathbb{R}^k, x \mapsto (x_{i_1}, \ldots, x_{i_k})$.

$$D\pi|_x : T_xX \xrightarrow{\simeq} T_{\pi(x)}\mathbb{R}^k$$

$[v_1, \ldots, v_k] \mapsto k\times k$ submatrix that is invertible, in other words, $\{D\pi x_1, \ldots D\pi x_k\}$ is a a basis for $T_{\pi(x)}\mathbb{R}^k$. IFT says π is local diffeo.
On $u \subseteq X$ with coords $x_1, \ldots x_k$ look at functions

$$f_{(a,c)}(x) = f(x) + \sum_{i=1}^{k} a_i x_i + \sum_{i=k+1}^{N} c_i x_i$$

$f_{(0,c)}(x)$ on u, almost all $a \in \mathbb{R}^k$ give a Morse function $f_{(a,c)}(x)$ on u. $\square$

13 27. feb.

Remark. The 2-torus.

$$T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$$

But we think of T^2 as
Image of torus $\subseteq \mathbb{R}^3$
a surface of revolution.

What is the underlying structure of the torus shared by all of its representations?
For a manifold X, what kind of embeddings $X \to \mathbb{R}^n$ are possible?

Theorem 13.1. Whitney embedding theorem
If $X \subseteq \mathbb{R}^n$ a k-manifold, then there exists an embedding

$$X \xrightarrow{f} \mathbb{R}^{2k+1}.$$

Improvement:

Possible to embed X into $X \xrightarrow{f}{}^{2k}$.
$S^2 \subseteq \mathbb{R}^3$ Whitney says

$$S^2 \xrightarrow{f} \mathbb{R}^5$$

Klein bottle
Insert picture of Klein bottle with rep dia.
We'll show Whitney Embedding for X compact.

$$f : X \to \mathbb{R}^{2k+1} \text{ embedding.}$$

Properness is free.
Worry about: injectivity and immersion
Introduce the Tangent Bundle to a manifold X.

Definition 13.1. $X \subseteq \mathbb{R}^n$ a smooth manifold.

$$TX = \cup_{x\in X} T_xX \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

$$= \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n | (x, v) \in T_xX\}$$

TX is a smooth manifold.

Definition 13.2. if $f : X \to Y$ smooth, then we get smooth map

$$Df : TX \to TY$$

$$(x, v) \mapsto Df_x(x, v)$$

Notation 13.1.

$$Df_x(x, v) = (f(x), df_x(x, v))$$

find extension $\tilde{f}$ of f at x

$$\Big(f(x), d\tilde{f}_x(x, v)\Big) = Df_x(v) = (f(x), df_x(x, b))$$

Properties:

(1) (Functionality / Chain rule)

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$X \xrightarrow{g \circ f} Z$$

commute.

$$TX \xrightarrow{Df} TY \xrightarrow{Dg} TZ$$

$$TX \xrightarrow{D(g \circ f)} TZ$$

commute.

Proof.

$$\begin{aligned} D(g \circ f)(x, v)? &= D(g)\,(D(f)(x, v)) \\ &= (g(f(x)), d(g \circ f)_x(v)) \\ &= \big(g(f(x)), dg_{f(x)}(df_x(v))\big) \\ &= Dg\,(f(x), df_x(v)) \\ &= Dg \circ Df\,(x, v)\,. \end{aligned}$$

□

(2) (Functionality / Chain rule and Identity)

$$X \xrightarrow{id} X$$

$$TX \xrightarrow{Did} TX$$

$$(x, v) \mapsto (x, d(id)_x(v))$$

$$D(id_x) = id_{TX}$$

Consequence

if $X \xrightarrow{f} Y$ is a diffeo, then

$$TX \xrightarrow{Df} TY$$

is a diffeo.

$$X \xrightarrow{f} Y \xrightarrow{f^{-1}} X$$
$$X \xrightarrow{id_x} X$$

commute.

$$TX \xrightarrow{Df} TY \xrightarrow{Df^{-1}} TX$$
$$X \xrightarrow{id_{TX}} TX$$

commute.

(3) If $f : X \to Y$ is smooth, then

$$Df : TX \to TY \text{ is smooth.}$$

Need to find a smooth extension around $(x, v) \in TX$.
Since $f : X \to Y$ is smooth at x, there is $\tilde{u} \subseteq \mathbb{R}^n$ open,
Diagram
$T(\tilde{u} \cap X) = TX \cap (\tilde{u} \times \mathbb{R}^n)$.
What is $D\tilde{f}$?

$$\tilde{f}(x_1, \ldots, x_n) = \begin{bmatrix} \tilde{f}_1(x) \\ \vdots \tilde{f}_m(x) \end{bmatrix}$$

$$D\tilde{f}(x_1, \ldots, x_n, v_1, \ldots, v_n) = \begin{bmatrix} \tilde{f}_1(x) \\ \vdots \tilde{f}_m(x) \\ \sum \frac{\partial \tilde{f}_1}{\partial x_j}(x) \cdot v_j \\ \vdots \\ \sum \frac{\partial \tilde{f}_m}{\partial x_j}(x) \cdot v_j \end{bmatrix}$$

this is indeed smooth.

Lecture week 10

14 10. March

Came a little late to class. Get notes for the first 15 minutes.
Realize $\mathbb{R}P^2$ as a quotient of S^2. $S^2 \subseteq \mathbb{R}^3 \setminus \{0\}$ has an induced equivalence relation from $\sim$ on $\mathbb{R}^3 \setminus \{0\}$.

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \lambda \in \mathbb{R} \setminus \{0\}$$

but if $(x, y, z), (\lambda x, \lambda y, \lambda z) \in S^2, \lambda = \pm 1$.
diag 1
To see $\bar{\pi}$ is a homeo

diag 2
$(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$

$$\pi \circ f(x,y,z) = \pi \circ f(\lambda x, \lambda y, \lambda z)$$

$$u_0 = \{[x_0, x_1, x_2] | x_0 \neq 0\}$$

$$u_1 \subseteq S^2/\sim$$

$$[x_0, x_1, x_2], x_1 \neq 0$$

$$\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \in \mathbb{R}^2$$

Theorem 14.1. $\mathbb{R}P^n$ is a compact connected abstract smooth manifold

Proof. We have a continuous surjection $\pi : S^n \to \mathbb{R}P^n, (x_0, \ldots, x_n) \mapsto [x_0, x_1, \ldots, x_n]$. Since S^n is connected and compact, $\pi(S^n) = \mathbb{R}P^n$ is also connected and compact. □

Better pictures?
$H = \{(x,y,z) \in S^2 | z \geq 0\}$, $\sim$ on $H, (x,y,0) \sim (\lambda x, \lambda y, 0), \lambda = \pm 1$, $(x,y,z) \sim (x,y,z)$.
$\mathbb{R}P^2$ doesn't appear to have an embedding into $\mathbb{R}^3$. But it can be embedded in $\mathbb{R}^4$.
Goal: Embed a compact abstract smooth manifold into $\mathbb{R}^N$, for $N >> 0$.

Definition 14.1. $f : X \to \mathbb{R}$ smooth. *suppoert* of f is

$$supp(f) = \{x \in X | \bar{f}(x) \neq 0\}$$

Example 14.1. $f(x) = \begin{cases} e^{-lx} & x > 0 \\ 0 & \leq 0 \end{cases}$ $supp(f) = [0, \infty)$.

Definition 14.2. X an abstract manifold, $\{u_\alpha\}$ an open cover of X. Type 1: Partition of unity subordinate to $\{u_\alpha\}$ a collection of cuntions $p_\alpha : X \to \mathbb{R}$.

(1) $(\forall \alpha)(\forall x) 0 \leq p_\alpha(x) \leq 1$

(2) $\forall x \in X$ $\exists u$ open, $x \in u$ and only finitely many p_α are nonzero on u

(3) $(\forall x)$ $\sum_{\alpha \in A} p_\alpha(x) = 1$

(4) $supp(p_\alpha) \subseteq u_\alpha$

Type 2: Partition of unity with compact supports subordinate ot $\{u\alpha\}$.
$p_i : X \to \mathbb{R}, i \in \mathbb{N}$.
(1), (2), (3)

(4) $supp(p_i) \subseteq u_\alpha$ for some α and $supp(p_i)$ is compact.

Remark. P, O, U of either type always exist on an abstract manifold.

Theorem 14.2. Let X be a compact abstract smooth manifold, then there is an embedding $X \to \mathbb{R}^N$ for $N >> 0$.

Proof. let

$$\varphi_\alpha : \mathbb{R}^n \supseteq \tilde{u}_\alpha \to u_\alpha \subseteq X$$

be a collection of coordinate charts that cover X. By compactness, we only need finitely many of these to cover X.

$$X = \cup u_\alpha$$

So we have $\varphi_i : \tilde{u_i} \to u_i$ coordinates. Assume $\tilde{u_i} \subseteq \mathbb{R}^n$ open. There is a partition of unity subordinate to $\{u_1, u_2, \ldots, u_k\}$ we get $\rho_i : X \to \mathbb{R}, supp(\rho_i) \subseteq u_i$.

$$\varphi_i^{-1} : u_i \xrightarrow{\simeq} \tilde{u_i} \subseteq \mathbb{R}^n$$

locally embedding X into $\mathbb{R}^n$. Use P.O.U. to extend φ^{-1} to all of X:

$$g_i(x) = \begin{cases} \rho_i(x)\varphi^{-1}(x) \text{ if } x \in u_i \\ 0 \text{ if } x \neq u_i \end{cases}$$

$g_i : X \to \mathbb{R}^n$ is smooth.
Define $G : X \to \mathbb{R}^k \times \mathbb{R}^{n \cdot k}$

$$G(x) = (\rho_1(x), \ldots \rho(x), g_1(x), \ldots, g_k(x))$$

I claim G is an embedding.
G is smooth, since all component functions are smooth.
Injectivity: if $G(x) = G(y), x, y \in X$ then $\rho_i(x) = \rho_i(y) 1 \leq i \leq k$. So if $\rho_i(x) \neq 0$, then $\rho_i(y) \neq 0$ and $x, y \in supp(\rho_i) \subseteq u_i$. But $g_i(x) = g_i(y)$ so $\rho_i(x)\varphi_i^{-1}(x) = \rho_i(y)\varphi_i^{-1}(y)$. So $x, y \in u_i \xrightarrow{\simeq, \varphi_i^{-1}} \tilde{u_i}$ $\varphi_i^{-1}(x) = \varphi_i^{-1}(y)$

$$\Rightarrow x = y \text{ as } \varphi_i^{-1} \text{ is a bijection.}$$

□

G is an immersion. $x \in X$, there is some i with $\rho_i(x) \neq 0$. $x \in supp(\rho_i) \subseteq u_i$. There is an open set $x \in u \in supp(\rho_i)$ where $\forall y \in u$, $\rho_i(y) \neq 0$.

$$u \xrightarrow{G} \mathbb{R}^k \times \mathbb{R}^{n \cdot k} \xrightarrow{H_i} \mathbb{R}^n$$

$$(x_1, \ldots, x_k, w_1, \ldots, w_k) \mapsto \frac{w_i}{x_i}$$

Note $\rho_i(y) \neq 0 \forall y \in u$, so $H_i(G(y))$ well-defined. On u, $H_i \circ G(y) = \frac{g_i(y)}{\rho_i(y)}$

$$= \frac{\rho_i(y) \cdot \varphi_i^{-1}(y)}{\rho_i(y)}$$

$$= \varphi_i^{-1}(y).$$

Hence $D(H_i \circ G)_y = D(\varphi_i^{-1})_y$ is an isomorphism. Finally use chain rule

$$D(H_i)_{G(y)} \circ DG_y = (D\varphi_i^{-1})_y$$