What will we study?

In your previous mathematics courses, you have almost certainly studied some properties of geometric objects like circles, ellipses, triangles, rectangles, n-gons, parabolas, and spheres. Later in a differential calculus course, one generally studies curves and surfaces that are the graphs of functions, like a sinusoidal curve, a paraboloid, or a monkey saddle. In this course, we will formulate and study a type of space, called manifolds, which allow us to study the familiar shapes you know and also many new spaces. For example, circles and spheres are not graphs of functions, but they are manifolds. The key observation for careful study of spheres and circles is that they are locally graphs of functions, or locally parametrizable. That is, around any point $p = (a, b)$ in the circle $\mathbf{S}^1$, there is a continuous function $f_p : (-\epsilon, \epsilon) \to \mathbf{S}^1$ that describes the circle around the point p. We can arrange for the function f_p to be differentiable, and then use the menthods of differential calculus to study the circle.

Just as how some spaces were not studied in your calculus courses, we too will be unable to study all spaces in this course. Spaces like a figure eight, or the sphere with the z-axis passing through it are not locally like an open Euclidean ball. More paradoxical spaces are the set of rational numbers $\mathbf{Q}$ in real line $\mathbf{R}$ or the Cantor set in the unit interval $[0, 1]$. These are interesting spaces, but not manifolds. Why is $\mathbf{Q}$ not a manifold?

More examples of manifolds: $\mathbf{S}^2$, $\mathbf{RP}^2$, $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$, the 2-holed torus, Euclidean balls $\mathbf{B}^n(0; r)$.

Manifolds are what we will be study, but we must ask ourselves what properties of manifolds are important to us?

1. How far apart can two points on a manifold be from one another? On an honest sphere like $\mathbf{S}^2 = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1$, any pair of antipodal points (x, y, z) and $(-x, -y, -z)$ are maximally far apart. Thinking of a pair of antipodal points like $(1, 0, 0)$ and $(-1, 0, 0)$ in $\mathbf{R}^3$, the straight-line distance between them is clearly 2. But how far would a little bug have to walk to go from one point to the other if it has to stay on the sphere? The shortest distance a bug would have to walk is π units (if it walks along a great circle connecting the two points). So from wherever the little bug is on the sphere, it can reach any other point by walking at most π units.

   But what if instead of an honest sphere like $\mathbf{S}^2$, we ask the same question about a bumpy sphere, like the surface of the earth? On the earth, not all pairs of antipodal points are maximally far apart. For example, the equatorial radius of the earth is approximately 6378 km at the equator and 6356 km at the poles. Walking from the North pole to the South pole is then approximated by walking along half of an ellipse for roughly 20 002 km, while walking along the equator between a pair of antipodal points would be longer, as it is a distance of approximately $\pi \cdot 6378 \sim 20037$ kilometers.

   The answer to such a question depends a great deal on the very rigid nature of the shape in question.

2. Is a given manifold flat or curved? Spheres, tori, cylinders are all curved in our usual way of drawing them. However, the torus and the cylinder are flat in that they can be cut open and laid flat without changing the relative

distance between points. Yet any small disc in a sphere is truly curved, in that it cannot be squashed flat without altering the relative distance between points. This is the cause of much difficulty in map making!

3. What is the dimesion of a given manifold? That is, how many real parameters $x, y, z, \ldots$ are needed to describe the manifold locally about a point? The circle is 1-dimensional, since the angle of a point from $(1, 0)$ is sufficient to describe the circle. Spheres are 2-dimensional, since latitude and longitude can be used to describe positions on the sphere.

4. Can a given manifold be continuously squished or deformed to a point? Any disc can be deformed to its central point, yet the circle cannot be continuously deformed to a point; one would have to break the circle first.

5. How many "holes" does the manifold have?

6. What kind of closed loops are there on the manifold? Can you secure a loop of rope in the manifold, or will it slip off? Compare $\mathbf{S}^2$ and $\mathbf{T}^2$.

The first two properties are geometric properties of a manifold. These are certainly interesting questions and properties! They require a notion of how to measure distance on a manifold, which requires some effort to define and work with. This is studied in Riemannian geometry.

The other properties are topological. These are weaker properties in the sense that they don't require knowing how to measure distance on the manifold. For example, all circles and ellipses are topologically the same. And the famous example: a doughnut and a coffee mug are topologically equivalent. So it is the geometry of a coffee mug that enables it to hold coffee, not its topology!

Here are some examples of theorems we are going to prove during this class:

Fundamental Theorem of Algebra:
Let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ be a polynomial with complex coefficients, that is, $a_0, \ldots, a_{n-1} \in \mathbf{C}$. Then $P(X)$ has a zero in $\mathbf{C}$, that is, there exists at least one complex number $z \in \mathbf{C}$ such that $P(z) = 0$. That means of course that $P(X)$ has exaclty n zeroes in $\mathbf{C}$ (counted with multiplicities).

This has at first glance nothing to do with topology. But we can prove it using topology!

Brouwer Fixed Point Theorem: Every continuous map $f\colon D^n \to D^n$ has a fixed point, that is, there is an $x \in D^n$ such that $f(x) = x$. Here D^n is the n-dimensional unit disc

$$D^n = \{(x_1, \ldots, x_n) \in R^n : x_1^2 + \cdots + x_n^2 \leq 1\}$$

This may not look so exciting, but **how** can you show that a fixed point always exists?

Borsuk-Ulam theorem For any continuous function $f : \mathbf{S}^2 \to \mathbf{R}^2$ there is a point $p \in \mathbf{S}^2$ for which $f(p) = f(-p)$.

Pick a sphere about the earth in the earth's atmosphere, and consider the function which assigns to such a point its temperature and wind speed. Then at all times, there is some pair of antipodal points on the earth p and $-p$ which have the identical temperature and windspeed!

Hairy Ball Theorem Assume you have a ball with hairs attached to it. Then it is impossible to comb the hair continuously and have all the hairs lay flat. Some hair will always be sticking right up. A more mathematical formulation: Every smooth vector field on a sphere has a singular point. An even more general statement: The n-dimensional sphere S^n admits a smooth field of nonzero tangent vectors if and only if n is odd.

This just sounds like a fun fact. But wind speeds on the surface of the earth is an example of a vector field on a sphere. This means at all times there is some point on the earth with zero wind speed.

Topology of $\mathbf{R}^n$

A topology on a set X is a collection of subsets of X that are called "open subsets". These open sets are required to satisfy some nice properties, which we will discuss later. For $\mathbf{R}^n$, the open subsets are defined with the help of a metric.

Definition 1. The Euclidean metric on $\mathbf{R}^n$ is the function $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}. \tag{2}$$

This function $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is called a "metric" because it satisfies the following key properties.

1. For any $x, y \in \mathbf{R}^n$ we have $d(x, y) \geq 0$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. (Symmetry) For all $x, y \in \mathbf{R}^n$, $d(x, y) = d(y, x)$.
4. (Triangle inequality) For all $x, y, z \in \mathbf{R}^n$, $d(x, z) \leq d(x, y) + d(y, z)$.

Any function $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ satisfying the above properties is called a metric as well. The "sup" metric is another example

Definition 3. Define $d_{sup} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots |x_n - y_n|\}. \tag{4}$$

Proposition 5. d_{sup} is a metric.

Proof. It is easy to show that $d_{sup}(x, y)$ is always nonnegative, and that $d_{sup}(x, y)$ vanishes exactly when $x = y$. Symmetry is also straightforward, given that $|x - y| = |y - x|$. The triangle inequality requires a little work.

When $n = 1$, this is just the usual triangle inequality $|x-z| \leq |x-y|+|y-z|$, which is well known.[1] In general, we calculate as follows.

$$\begin{aligned} d(x, y) &= \max\{|x_i - y_i| \,|\, 1 \leq i \leq n\} & (6) \\ &= \max\{|x_i - z_i + z_i - y_i| \,|\, 1 \leq i \leq n\} & (7) \\ &\leq \max\{|x_i - z_i| + |z_i - y_i| \,|\, 1 \leq i \leq n\} & (8) \\ &\leq \max\{|x_i - z_i| \,|\, 1 \leq i \leq n\} + \{|z_i - y_i| \,|\, 1 \leq i \leq n\} & (9) \\ &= d(x, z) + d(z, y). & (10) \end{aligned}$$

[1] Try to prove it yourself if in doubt!

Note that inequality (8) follows from the usual triangle inequality. For inequality (9), note that the maximum on the left hand side must be achieved for some particular value of i, say it is $|x_j - z_j| + |z_j - y_j|$. But now, the two maxima on the right hand side are at least as big as $|x_j - z_j|$ and $|z_j - y_j|$ respectively; hence the sum of the maxima exceeds $|x_j - z_j| + |z_j - y_j|$. $\square$

Once we settle on a metric for $\mathbf{R}^n$, we can define various distinguished subsets of $\mathbf{R}^n$.

Definition 11. Let d be the Euclidean metric on $\mathbf{R}^n$.

1. The open ball centered at $x \in \mathbf{R}^n$ of radius $r \geq 0$ is
$$\mathbf{B}^n(x; r) = \{y \in \mathbf{R}^n \mid d(x, y) < r\}. \tag{12}$$
2. The closed disc centered at $x \in \mathbf{R}^n$ of radius $r \geq 0$ is
$$\mathbf{D}^n(x; r) = \{y \in \mathbf{R}^n \mid d(x, y) \leq r\}. \tag{13}$$
3. The sphere centered at $x \in \mathbf{R}^n$ of radius $r \geq 0$ is
$$\mathbf{S}^{n-1}(x; r) = \{y \in \mathbf{R}^n \mid d(x, y) = r\}. \tag{14}$$

The definitions of open ball, closed disc, and sphere, make sense if we replace the Euclidean metric with the sup metric. To indicate the difference, we will indicate the metric with a subscript, e.g., $B_{sup}(0; 1)$ is the unit open ball in the sup metric.

Definition 15. A set $U \subseteq \mathbf{R}^n$ is called open if for any $x \in U$ there exists an $\epsilon > 0$ for which $\mathbf{B}^n(x; \epsilon) \subseteq U$.

We can also define a set $U \subseteq \mathbf{R}^n$ to be sup-open if for any $x \in U$ there exists an $\epsilon > 0$ for which $\mathbf{B}^n_{sup}(x; \epsilon) \subseteq U$. What is the difference between a set being open and sup-open?

Exercise 16. Draw pictures of the unit open ball in $\mathbf{R}^2$ for the Euclidean and sup metrics.

Exercise 17. Show open balls are open sets in $\mathbf{R}^n$.

Proposition 18. The set of open subsets of $\mathbf{R}^n$ agrees with the set of sup-open subsets of $\mathbf{R}^n$.

To prove this proposition, we first relate the sup and Euclidean metrics.

Lemma 19. For any $x \in \mathbf{R}^n$ and any $r \geq 0$, there are inclusions
$$\mathbf{B}_{euc}(x; r) \subseteq \mathbf{B}_{sup}(x; r) \subseteq \mathbf{B}_{euc}(x; \sqrt{n}r).$$

Proof. It suffices to establish the inclusions when $x = 0$, since the ball $\mathbf{B}(x; r)$ is the translate of $\mathbf{B}(0; r)$ by x. That is, $\mathbf{B}(x; r) = x + \mathbf{B}(0; r) = \{x + y \mid y \in \mathbf{B}(0; r)\}$.

We establish the first inclusion by contraposition. So suppose $x \notin \mathbf{B}_{sup}(0; r)$. So as $d_{sup}(0, x) \geq r$ then there is some i for which $|x_i| \geq r$. Hence
$$d_{euc}(0, x) = \sqrt{x_1^2 + \cdots + x_i^2 + \cdots + x_n^2} \tag{20}$$
$$\geq \sqrt{r^2} = |r|, \tag{21}$$
so $x \notin \mathbf{B}_{euc}(0; r)$. Thus by contraposition, if $x \in \mathbf{B}_{euc}(0; r)$ then $x \in \mathbf{B}_{sup}(0; r)$.

For the second inclusion, suppose $x \in \mathbf{B}_{sup}(0; r)$. This implies $|x_i| \leq r$ for all $1 \leq i \leq n$. We compute

$$\sqrt{x_1^2 + \cdots + x_n^2} \leq \sqrt{r^2 + \cdots + r^2} = \sqrt{nr^2} = \sqrt{n}r. \tag{22}$$

Thus $x \in \mathbf{B}_{euc}(0; \sqrt{n}r)$ as desired. □

Proof of Proposition 18. Suppose $U \subseteq \mathbf{R}^n$ is open with respect to the Euclidean metric. We will show U is sup-open. If U is the empty set, then U is vaccuously sup-open as well. So suppose now U is non-empty. Let x be a member of U; we will construct a sup-open ball around x contained in U. Since U is open with respect to the Euclidean metric, there is an $\epsilon > 0$ for which $B(x; \epsilon) \subseteq U$. By Lemma 19

$$B_{sup}(x; \frac{1}{\sqrt{n}}\epsilon) \subseteq B_{euc}(x; \epsilon) \subseteq U. \tag{23}$$

Hence U is sup-open.

Now suppose $U \subseteq \mathbf{R}^n$ is sup-open. Again, we may suppose U is non-empty. If x is a member of U, then there is some ϵ for which $B_{sup}(x; \epsilon) \subseteq U$. Then $B_{euc}(x; \epsilon) \subseteq U$ by Lemma 19 as well. Hence U is open with respect to the Euclidean metric. □

The takeaway of these observations is that although the metrics d_{sup} and d_{euc} are very different geometrically[2], they nonetheless describe the same topological structure on $\mathbf{R}^n$. So the topology on $\mathbf{R}^n$ captures the qualitative notion of two points being close together, whereas a metric quantifies how close together two points are.

Not all metrics on $\mathbf{R}^n$ define the same topology on $\mathbf{R}^n$, however.

Exercise 24. Define $d : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases} \tag{25}$$

Show that d defines a metric on $\mathbf{R}^n$. Then show that the open sets defined by d are different from the Euclidean open sets.

Properties of open sets in $\mathbf{R}^n$

The collection of open sets in $\mathbf{R}^n$ satisfy some nice properties that are indispensible in our study of spaces.

Proposition 26.

1. The empty set $\varnothing$ and $\mathbf{R}^n$ are both open sets.
2. Let $U_1, U_2, \ldots, U_k$ be open sets in $\mathbf{R}^n$. Then their intersection $\bigcap_{i=1}^k U_i$ is also open.
3. Let $\{U_\alpha \,|\, \alpha \in \mathcal{A}\}$ denote a family of open subsets of $\mathbf{R}^n$.[3] Then their union $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is also open.

[2] Note that in $\mathbf{R}^2$ the Euclidean sphere is the unit circle, while sphere defined by the sup-metric is a square with side length 2, centered at the origin.

[3] Note that $\mathcal{A}$ can be any set.

Proof. The statement that $\varnothing$ is open is vaccuously true (there are no points in $\varnothing$ to find open balls around). It is evident that $\mathbf{R}^n$ itself is open as every open ball about any point is contained in $\mathbf{R}^n$.

Suppose $x \in \bigcap_{i=1}^k V_i$. Then for each i there is some ϵ_i for which $B(x;\epsilon_i) \subseteq V_i$. Let $\epsilon = \min\{\epsilon_i \,|\, 1 \leq i \leq k\}$. Then $B(x;\epsilon) \subseteq B(x;\epsilon_i) \subseteq V_i$ for all i; hence $B(x;\epsilon) \subseteq \bigcap_{i=1}^k V_i$. This establishes the second item.

Finally, let $x \in \bigcup U_\alpha$. Then there is some β for which $x \in U_\beta$. But then there is an ϵ for which $B(x;\epsilon) \subseteq U_\beta \subseteq \bigcup U_\alpha$. Thus $\bigcup U_\alpha$ is open. $\square$

Closed sets in $\mathbf{R}^n$. Closure, boundary, and interior.

Definition 27. A set $V \subseteq \mathbf{R}^n$ is closed if it is the complement of an open set. That is, V is closed if and only if $\mathbf{R}^n \setminus V$ is open.

Equivalently, one must check that for any point $x \notin V$ that there is an open ball $B(x;\epsilon)$ that does not intersect V, i.e., $V \cap B(x;\epsilon) = \varnothing$.

Definition 28. Let $A \subseteq \mathbf{R}^n$ be any subset.

1. The interior of A is the largest open subset contained in A. We write A^o for the interior of A.
2. The closure of A is the smallest closed subset that contains A. We write $\overline{A}$ for the closure of A.
3. The boundary of A is defined to be $\partial A = \overline{A} \setminus A^o$.

A slightly strange chore we must do is show that the interior and closure of an arbitrary subset of $\mathbf{R}^n$ actually exists. If A is a subset of $\mathbf{R}^n$, consider the set

$$A' = \{x \in A \,|\, \text{ there exists } \epsilon > 0 \text{ so that } B(x;\epsilon) \subseteq A\}. \tag{29}$$

I claim that A' is in fact the largest open subset contained in A. For suppose $U \subseteq A$ is open. If U is empty, then certainly $U \subseteq A'$. So suppose U is nonempty. Then about any point $x \in U \subseteq A$ there exists an open ball $B(x;\epsilon) \subseteq U \subseteq A$. Hence $x \in A'$. So A' contains all of the open sets U contained in A. It is straightforward to verify A' is open.

Alternatively, note that

$$A' = \bigcup_{\substack{U \subseteq A \\ U \text{ is open}}} U. \tag{30}$$

This clearly is an open set and is the largest open set contained in A. Thus $A' = A^o$.

Now that we have established interiors exist, we study closures. Define

$$A'' = \{x \in \mathbf{R}^n \,|\, \text{ for all } \epsilon > 0, B(x;\epsilon) \cap A \neq \varnothing\}. \tag{31}$$

We will check A'' is indeed the closure of A. Suppose V is any closed subset containing A. If x is not in V, then there is an open ball $B(x;\epsilon)$ which does not meet V, and so, does not meet A. Hence x is not in A''. Thus by contraposition, $A'' \subseteq V$. It is straightforward to verify A'' is itself closed.

Alternatively, note that

$$A'' = \bigcap_{\substack{A \subseteq V \\ V \text{ is closed}}} V. \tag{32}$$

Analogous to how arbitrary unions of open sets are open, it is true that arbitrary intersections of closed sets are closed.

Exercise 33. Show that the boundary of any subset of $\mathbf{R}^n$ is closed.

Exercise 34. Show that closed discs $D(x; r)$ are indeed closed subsets of $\mathbf{R}^n$.

Exercise 35. Identify the interior, boundary, and closure of the following subsets.

1. $\mathbf{B}^n(0; 1) \subseteq \mathbf{R}^n$.
2. $\mathbf{Q} \subseteq \mathbf{R}$
3. $\{\frac{1}{n} \,|\, n \in \mathbb{Z}, n > 0\}$
4. $S^n \subseteq \mathbf{R}^{n+1}$
5. $\mathbf{B}^n(0; 1) \cap \{x \in \mathbf{R}^n \,|\, x_n = 0\}$.