Kompleks Analyse – Forelesning

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Definition 1.1. Let $D \subseteq \mathbb{C}$. A *path* or *curve* or *arc* in D is a continous map $\gamma : [a, b] \to D$, where $a, b \in \mathbb{R}, a \leq b$.

Remark. We say that γ is:

(a) *closed* provided $\gamma(a) = \gamma(b)$.

(b) *piecewise continously differentiable* (pwC^1 for short) provided there exist $a = t_0 < t_1 < \cdots < t_n = b$, s.t. $\gamma[t_{k-1}, t_k]$ is continously differentiable for $k = 1, \ldots, n$.

Definition 1.2. Let $a, b \in \mathbb{R}, a \leq b$.

(a) If $\gamma : [a, b] \to \mathbb{C}$ is a pwC^1 path, we define the *length* of γ as

$$L(\gamma) = \int_a^b |\gamma'(t)| dt \in \mathbb{R}_{\geq 0}.$$

(b) If $f : D \to \mathbb{C}$ is continous, D open subset of $\mathbb{C}$, and $\gamma : [a, b] \to D$ pwC^1 path, we define the *path integral* or *complex line integral* of f along γ as

$$\int_\gamma f(z) \cdot \gamma'(t) dz := \int_a^b f(\gamma(t)) dt \in \mathbb{C}.$$

$$= \left(\int_\gamma u dx - v dy \right) + i \left(\int_\gamma v dx + u dy \right), \text{ where u=Re f, v=Im f}$$

"pseudo calculations"

$$L(\gamma) = \int |\gamma'(t) dt = \int |\frac{dz}{dt}| dt = \int |dz|$$

$$f(z) dz = f(z) \frac{dz}{dt} dt = \int f(\gamma(t)) \cdot \gamma'(t) dt$$

Example 1.1. 1. $\gamma : [0, 2\pi] \to \mathbb{C} : \quad t \mapsto z_0 + r \cdot e^{it}$, $z_0 \in \mathbb{C}, r \in \mathbb{R}_{>0}$.

$$L(\gamma) = \int_0^{2\pi} |r \cdot i \cdot e^{it}| dt = \int_0^{2\pi} r dt = 2\pi r.$$

Let $f : \mathbb{C} \setminus \{z_0\} \to \mathbb{C} : z \mapsto \frac{1}{z - z_0}$. Then

$$\int_\gamma f dz = \int_0^{2\pi} \frac{1}{r \cdot e^{it}} r \cdot e^{it} dt = \int_0^{2\pi} i dt = 2\pi i$$

2. If $\gamma : [a,b] \to \mathbb{C}$ is a C^1 path along the real axis with $c = \gamma(a), d = \gamma(b)$, and if $f : \mathbb{R} \to \mathbb{R}$ is continous, then:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(d)) \cdot \gamma'(dt) = \int_{\gamma(a)}^{\gamma(b)} f(s)dz = \int_c^d f(s)ds$$

Remark. $D \subseteq \mathbb{C}, \gamma : [a,b] \to \mathbb{C}$ pwC^1 path. If $\psi : [c,d] \to [a,b]$ is increasing and continously differentiable with $\psi(c) = a, \psi(d) = b$ and if $f : D \to \mathbb{C}$ is continous, then

$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz, \text{ where } \tilde{\gamma} = \gamma \circ \psi.$$

Proof.

$$\int_{\tilde{\gamma}} f(z)dz = \int_c^d f(\gamma(\psi(t))) \cdot (\gamma\circ)'(t)dt = \int_c^d f(\psi(t)) \cdot \gamma'(\psi(t)) \cdot \psi'(t)dt$$

substitution $s = \psi(t)$

$$= \int_a^b f(\gamma(s)) \cdot \gamma'(s)ds = \int_\gamma f(z)dz$$

□

Remark. If $\psi : [c,d] \to [a,b]$ is C^1, decreasing with $\psi(c) = b, \psi(d) = a$, then

$$\int_{\tilde{\gamma}} f(z)dz = -\int_\gamma f(z)dz.$$

Remark. If $\gamma : [a,b] \to \mathbb{C}$ is pwC^1, then it can be reparametrized into a C^1 path.

Proof. Let $a = t_0 < t_1 < \cdots < t_n = b$, s.t. $\gamma[t_{k-1}, t_k]$ is C^1 for $k = 1, \ldots, n$ and set $\tilde{\gamma} = \gamma \circ \psi$. Pick $\psi : [a,b] \to [a,b]$ s.t.

- ψ is C^∞
- ψ is strictly increasing
- $\psi(t_k) = t_k$ for $k = 0, \ldots, n$
- $\psi'(t_k) = 0$ for $k = 0, \ldots, n$

On $[t_{k-1}, t_k]$: $\left(\tilde{\gamma}_{|[t_{k-1},t_k]}\right)'(t_k) = \left(\gamma_{|[t_{k-1},t_k]}\right)'(t_k)\psi'(t_k) = 0$. Analogously $\left(\tilde{\gamma}_{|[t_{k-1},t_k]}\right)'(t_{k-1}) = 0$. Hence the parts $(\tilde{\gamma}_{|[t_{k-1},t_k]}$ can be "pieced together" into a path $\tilde{\gamma}$, which is C^1 on $[a,b]$. □

Example 1.2. $\gamma : [-1,1] \to \mathbb{C}$

$$t \mapsto \begin{cases} t - if, & \text{if } t < 0 \\ t + if, & \text{if } t \geq 0 \end{cases}$$

γ not differentiable in 0.

$$\psi : [-1,1] \to [-1,1], \quad s \mapsto s^3$$

$$\tilde{\gamma}$$

Example 1.3. $r > 0, \gamma_1 : [0, 2\pi] \to \mathbb{C} : t \mapsto re^{it}$

$$\gamma_2 : [0, 2\pi] \to \mathbb{C} \quad : \quad t \mapsto r$$

$\gamma_1(0) = r = \gamma_2(0), \gamma_1(2\pi) = r = \gamma_2(2\pi)$, but

$$\int_{\gamma_1} \frac{1}{z} dz = 2\pi i \neq 0 = \int_{\gamma_2} \frac{1}{z} dz.$$

Hence:
Value of path integral does in general not only depend on start and end point.

Definition 1.3. Let D open subset of $\mathbb{C}$, $f : D \to \mathbb{C}$ (continous). A function $F : D \to \mathbb{C}$ is called a *complex antiderivative* of f, provided F is holomorphic on D and $F'(z) = f(z) \forall z \in D$.

Lemma 1.1. Let D open subset of $\mathbb{C}$, $f : D \to \mathbb{C}$ (continous), $F : D \to \mathbb{C}$ complex antiderivative of f. Assume $\gamma[a, b] \to D$ is of class C^1. Then

$$\int_\gamma f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, $\int_\gamma f dz$ only depends on start and end point of γ, and not on the path itself. If γ is closed, then $\int_\gamma f(z) dz = 0$.

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Lemma 2.1. Let D open in $\mathbb{C}$, $f : D \to \mathbb{C}$ (continous), $F : D \to \mathbb{C}$ complex antiderivative of f. Assume that $\gamma : [a, b] \to D$ of class C^1. Then

$$\int_\gamma f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof.

$$\int_\gamma f(z) dz = \int_a^b f(\gamma(t)) \cdot \gamma'(f) dt = \int_a^b F'(\gamma(t)) \cdot \gamma'(t) dt$$

$$F = U + iV$$

$$U = ReF, V = ImF$$

$$= \int_a^b \left[\frac{\partial u}{\partial x}(\gamma_1(t), \gamma_2(t)) + i \frac{\partial V}{\partial x}(\gamma_1(t), \gamma_2(t)) \right] \cdot [\gamma_1'(t) + i\gamma_2'(t)] \, dt$$

$$= \int_a^b \left[\frac{\partial u}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1' - \frac{\partial V}{\partial x}(\gamma_1(t), \gamma_2(t)) \cdot \gamma_2'(t) \right] + i \left[\frac{\partial V}{\partial x}(\gamma_1(t), \gamma_2(t)) \cdot \gamma_1'(t) \frac{\partial U}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_2'(t) \right] dt$$

$$= \int_a^b \left(\left[\frac{\partial U}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + \frac{\partial U}{\partial y}(\gamma_1(t), \gamma_2(t))\gamma_2'(t) \right] + i \left[\frac{\partial V}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + \frac{\partial V}{\partial y}(\gamma_1(t), \gamma_2(t))\gamma_2'(t) \right] \right)$$

$$= \int_a^b \left[(U \circ \gamma)'(t) + i (V \circ \gamma)'(t) \right] dt = [U \circ \gamma]_a^b + i[V \circ \gamma]_a^b$$
$$= U(\gamma(b)) - U(\gamma(a)) + iV(\gamma(b)) - iV(\gamma(a)) = F(\gamma(b)) - F(\gamma(a))$$

□

Corollary. D open in $\mathbb{C}$, $F : D \to \mathbb{C}$, $\gamma : [a, b] \to D$ of class C^1. Then $\forall f \in [a, b]$:
$$(F \circ \gamma)'(t) = F'(\gamma(t)) \cdot \gamma'(t)$$

Corollary. The holomorphic function $f : \mathbb{C} \setminus 0 \to \mathbb{C} : z \mapsto \frac{1}{z}$ has no complex antiderivative.

Example 2.1. If $n \in \mathbb{Z}$ eith $n \neq 1$, then $F(z) = \frac{(z-z_0)^{n+1}}{n+1}$ defines an antiderivative of $f(z) = (z - z_0)^n$ on $\mathbb{C}$ if $n \geq 0$, on $\mathbb{C} \setminus \{0\}$ if $n \leq -z$.
So if $\gamma[0, 2\pi] \to \mathbb{C}$, $t \mapsto z_+ r \cdot e^{it}$, then we have for $u \in \mathbb{Z}$:

$$\int_\gamma (z - z_0)^n dz = \begin{cases} 2\pi i, & \text{if } n = -1 \\ 0, & \text{if } n \neq -1 \end{cases}$$

Convention

If $\gamma : [0, 2\pi] \to \mathbb{C}$: $t \mapsto z_0 + r \cdot e^{it}$, then we will denote

$$\int_\gamma ...dz \text{ as } \int_{|z-z_0|=r} ...dz$$

Proposition 2.2. Let $D := \mathbb{C} \setminus \mathbb{R}_{\leq 0} = \{z = r \cdot e^\varphi : r \in \mathbb{R}_{>0}, -\pi\varphi < \pi$ open contained in $\mathbb{C}$.
Then

(a)
$$\log : D \to \mathbb{C}$$
$$r \in \mathbb{R}_{>0}, \varphi \in (-\pi, \pi)$$
$$\log(r \cdot e^\varphi) := \log(r) + i \cdot \varphi$$
is well-defined and holomorphic on D. Furthermore $\log'(z) = \frac{1}{z}$ for $z \in D$. This is called the complex logarithm.

(b) Let $z_1, z_2 \in D$ and $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ be of class C^1 with $\gamma(a) = z_1, \gamma(b) = z_2$. Then $\exists k \in \mathbb{Z}$, s.t.
$$\int_\gamma \frac{1}{z} dz = \log(z_2) - \log(z_1) + 2\pi i \cdot k \tag{1}$$
Intuitively speaking the k measures how many times the path winds around zero. (Because of the way the unit circle jumps from $-\pi$ to π).

(c) Conversely, given $k \in \mathbb{Z}$, there exists $z_1, z_2 \in D$ and a C^1 path $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$, s.t. (1) is satisfied.

Proof. exercise. hint for b: compute exp(LHS) and exp(RHS). □

Definition 2.1. Let U open in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be of class C^1. Then f is said to be *conformal* at $z_0 \in U$, provided it preserves angles (and orientation thereof) between curves at z_0. More precisely, if $\gamma_1 : [a_1, b_1] \to U, \gamma_2[a_2, b_2] \to \mathbb{C}$ are of class C^1 and if furthermore $t_1 \in [a_1, b_1], t_2 \in [a_2, b_2]$ satisfy $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1) \neq 0, \gamma_2'(t_2) \neq 0$. Then:

- $$(f \circ \gamma_1)'(t_1) \neq 0, (f \circ \gamma_2)'(t_2) \neq 0$$
- if $\gamma_2'(t_2) = r \cdot e^{i\theta} \cdot \gamma_1'(t_1)$ for $r \in \mathbb{R}_{>0}, \theta \in \mathbb{R}$, then $\exists s \in \mathbb{R}_{>0} : (f \circ \gamma_2)'(t_2) = s \cdot e^{i\theta} \cdot (f \circ \gamma_1)'(t_1)$.

f is called *conformal* if it is conformal at every point in U.

Example 2.2. $f : \mathbb{C} \to \mathbb{C}$: $z \mapsto \bar{z}$
If $\gamma_1, \gamma_2, t_1, t_2, r, \theta$ are as in the definition, then:

$$(f \circ \gamma_2)'(t) = (Re(\gamma_2) - Im(\gamma_2))'(t)$$

$$= (Re(\gamma_2))'(t_2) - i\,(Im(\gamma_2))'(t_2) = (Re(\gamma_2))'(t_2) + i\,(Im(\gamma_2))'(t_2)$$

$$= \gamma_2'\bar{(}t) = r \cdot e^{i\theta} \bar{\cdot} \gamma_1'(t_1) = r \cdot e^{-i\theta} \cdot \gamma_1'\bar{(}t_1)$$

$$r \cdot e^{i\theta} \cdot (f \circ \gamma_1)'(t_1)$$

f preserves angles, but reverses the orientation $\Rightarrow f$ nor conformal.

Proposition 2.3. Let U open subset in $\mathbb{C}$, let $f : U \to \mathbb{C}$ be of class C^1. Then TFAE:

(a) f is conformal

(b) f is holomorphic and $f'(z) \neq 0 \forall z \in U$

Proof. *b)* $\Rightarrow$ *a)* Let $z_0 \in U$. Let $\gamma_1, \gamma_2, t_1, t_2, r, \theta$ be as in the definition. Then, by "Corollary of proof" , we have

$$(f \circ \gamma_j)'(t_j) = f'(\gamma_j(t_j)) \cdot \gamma_j'(t_j) = f'(z_0)\,\gamma_j'(t_j) \neq 0$$

Furthermore,

$$(f \circ \gamma_2)'(t_2) = f'(z_0) \cdot \gamma_2'(t_2) = f'(z_0) \cdot r \cdot e^{i\theta} \cdot \gamma_1'(t_1)$$

$$= r \cdot e^{i\theta} \cdot (f \circ \gamma_1)'(t_1)$$

as desired. □

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Proposition 2.4. U open in $\mathbb{C}$, $f : U \to \mathbb{C}$ of class C^1. Then TFAE:

(a) f is conformal

(b) f is holomorphic and $f'(z) \neq 0 \forall z \in U$

Proof. $(b) \Rightarrow (a)$ Last lecture
$(a) \Rightarrow (b)$ Let $z_0 \in U$, $f = u + iv$ as usual.
For $\omega \in \mathbb{R}$, consider $\gamma_\omega : [-\varepsilon, \varepsilon] \to U$, $t \mapsto z_0 + te^{i\omega}$ ($\varepsilon > 0$ small enough, independent from ω)
$\gamma_\omega(0) = z_0$ and $\gamma'_\omega(0) = e^{i\omega} = \cos\omega + i\sin\omega$

$$(f \circ \omega)'(0) = \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}$$

If $\omega \in \mathbb{R}$, then

$$\gamma\omega'(0) = e^{i\omega} = e^{i\omega} \cdot 1 = e^{i\omega} \cdot \gamma'_0(0)$$

Conformality $\Rightarrow \exists s_\omega \in \mathbb{R}_{>0}$:

$$0 \neq (f \circ \gamma_\omega)'(0) = s_\omega \cdot e^{i\omega} \cdot (f \circ \gamma_0)'(0)$$

$$s_\omega \cdot e^{i\omega} \cdot (u_x(z_0) + i \cdot v_x(z_0))$$

In particular $(u_x(z_0))^2 + (v_x(z_0))^2 > 0$. We get

$$\begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} = \begin{bmatrix} u_x(z_0) & -v_x(z_0) \\ v_x(z_0) & u_x(z_0) \end{bmatrix} \cdot \begin{bmatrix} s_\omega \cos\omega \\ s_\omega \sin\omega \end{bmatrix}$$

$$det \begin{bmatrix} u_x(z_0) & -v_x(z_0) \\ v_x(z_0) & u_x(z_0) \end{bmatrix} > 0,$$

so the matrix is invertible. Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = M := \begin{bmatrix} u_x(z_0) & -v_x(z_0) \\ v_x(z_0) & u_x(z_0) \end{bmatrix}^{-1} \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix}$$

Then, $\forall \omega \in \mathbb{R}$:

$$M \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} = s_\omega \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}$$

Noting that $s_0 = 1$, we get:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = M \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow a = 1, c = 0$$

For $\omega = \frac{\pi}{2}$, we get

$$s_{\frac{\pi}{2}} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = M \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow b = 0, M = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$$

For $\omega = \frac{\pi}{4}$, we get

$$s_{\frac{\pi}{2}} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = M \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow s_{\frac{\pi}{4}} = 1, d = 1, M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} = \begin{bmatrix} u_x(z_0) & -v_x(z_0) \\ v_x(z_0) & u_x(z_0) \end{bmatrix}.$$

Cauchy Riemann
$\Rightarrow f$ complex differentiable in z_0. Furthermore,

$$f'(z_0) = u_x(z_0) + iv_x(z_0) \neq 0$$

$\square$

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$
$$\mathcal{S}^{\in} = \{x_1, x_2, x_3 \in \mathbb{R} : x_1^2 + x_2^2 + x_3^2 = 1\}$$
$$\mathbb{C} \simeq \{(x, y, 0) \in \mathbb{R}^3\}$$
$$\phi : \mathcal{S}^2 \to \hat{\mathbb{C}} : \phi(x_1, x_2, x_3) = \begin{cases} \infty, \text{ if } (x_1, x_2, x_3) = (0,0,1) \\ \frac{1}{1-x_3}(x_1, x_2, 0) \text{ otherwise} \end{cases}$$

ϕ is bijective with $\phi^{-1} : \hat{\mathbb{C}} \to \mathcal{S}^2$ given by

$$\infty \mapsto (0,0,1), x + iy \mapsto \frac{1}{x^2 + y^2 + 1}(2x, 2y, x^2 + y^2 - 1)$$

This defines a topology on $\hat{\mathbb{C}}$. $\hat{\mathbb{C}}$ equipped with this topology, is called the *extended complex plane* or *the Riemann sphere.*

Remark. The topology on $\mathbb{C}$ induced by $\mathbb{C} \subseteq \hat{\mathbb{C}}$ is the same as the classic topology on $\mathbb{C}$.

$$\frac{az+b}{cz+d}$$

If $(c, d) = (0, 0)$, we get division by 0, so assume $(c, d) \neq (0, 0)$. If

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0,$$

then $(a, b) = \gamma \cdot (c, d)$ for some $\gamma \in \mathbb{C}$, so $\frac{az+b}{cz+d} = \frac{\gamma cz + \gamma d}{cz+d} \equiv \gamma$.

Definition 2.2. Given $a, b, c, d \in \mathbb{C}$ with

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0,$$

the uniquely determined continuous map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$f(z) = \frac{az+b}{cz+d} \forall z \in \mathbb{C} \text{ with } cz + d \neq 0$$

is called a *linear fractional transformation* or *Mobius transformation.*
We denote this map as $T_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$.

Remark.

$$T_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} : z \mapsto \begin{cases} \frac{a}{b}z + \frac{b}{d}, \text{ if } c = 0, z \neq \infty \\ \infty, \text{ if } c = 0, z = \infty \\ \frac{az+b}{cz+d}, \text{ if } c \neq 0, z \neq \infty, z \neq \frac{-d}{c} \\ \frac{a}{z}, \text{ if } c \neq 0, z = \infty \\ \infty, \text{ if } c \neq 0, z = -\frac{d}{c} \end{cases}$$

Notation

$$GL_2(\mathbb{C}) = \{M \in \mathbb{C}^{2\times 2} : detM \neq 0\}$$

Lemma 2.5. Let $\mathcal{T} := \{T_M : M \in GL_2(\mathbb{C})\}$ be the set of Mobius transformations. Then $(\mathcal{T}, \circ)$ is a group and

$$GL_2(\mathbb{C}) \to \mathcal{T} : M \mapsto T_M \text{ is a group homomorphism.}$$

In particular, T_M is bijective for all $M \in GL_2(\mathbb{C})$ and $(T_M)^{-1} = T_{M^{-1}}$. And furthermore $T_M \circ T_N = T_{M\cdot N} \forall M, N \in GL_2(\mathbb{C})$.

Remark. Not isomorphism. if $\alpha \neq 0$, $M \in GL_2(\mathbb{C})$, then $T_M = T_{\alpha M}$·¡¿

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(week 3 friday)

Recall. Given $A \in GL_2(\mathbb{C})$, the continuous map

$$T_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} : z \mapsto \frac{az+b}{az+d}, \forall z \in \mathbb{C} \text{ with } cz+d \neq 0$$

is called a *Mobius transformation.* If $M, N \in GL_2(\mathbb{C})$:

- T_M is bijective, $T_M = T_{M^{-1}}$.
- $T_M \cdot T_N = T_{MN}$

Remark. Every Mobius transformation has a finite composition of Mobius transforms of the following type:

- translation
$$\mathbb{C} \ni z \mapsto z + b$$
$$b \in \mathbb{C}, \infty \mapsto \infty$$
- rotation
$$\mathbb{C} \ni z \mapsto e^{i\theta} \cdot z$$
$$\theta \in \mathbb{R}, \infty \mapsto \infty$$
- Holomorphic transformation
$$\mathbb{C} \ni z \mapsto r \cdot z, r \in \mathbb{R}_{>0}$$
$$\infty \mapsto \infty$$
- inversion
$$\mathbb{C} \setminus \{0\} \ni z \mapsto \frac{1}{z}$$
$$0 \mapsto \infty$$
$$\infty \mapsto 0$$

Consider $T_M : \hat{\mathbb{C}} \to \mathbb{C}$ for $M \in GL_2(\mathbb{C})$.
Invertible $\Rightarrow M$ finite product of matrices of the following form:

- $$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \alpha \in \mathbb{C} \setminus 0$$
- $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
- $$\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \beta \ \in \mathbb{C}$$

Definition 3.1. Let $\mathcal{C}_{\mathcal{L}} = \{\{z \in \mathbb{C} : |z - z_0| = r\} : r \in \mathbb{R}_{>0}, z_0 \in \mathbb{C}\}$

$$\cup \{\{\infty\} \cup \{ax + b : x \in \mathbb{R}\} : b \in \mathbb{C}, a \in \mathbb{C} \setminus \{0\}\}.$$

We include $\{\infty\}$ so that $\mathcal{C}_{\mathcal{L}}$ is compact, as it must be because Riemann sphere is compact.

Definition 3.2. A circle on $\mathcal{S}^2$ is a set C of the form $C = \mathcal{S}^2 \cap H$, where H is an affine hyperplane in $\mathbb{R}^3$ and C har more than one point. The set of those cirlces i denoted as $\mathcal{C}_{\mathcal{S}^2}$.

Remark. $\varphi : \mathcal{S}^2 \to \hat{\mathbb{C}}$ induces a bijection $\mathcal{CL} \xleftarrow{|:|} \mathcal{C}_{\mathcal{S}^2}$.
Note. Given $\alpha, \beta, \gamma \in \hat{\mathbb{C}}$ pairwise distinct, $\exists! A \in \mathcal{CL}$ such that $\alpha, \beta, \gamma \in A$.

Definition 3.3. (kind of a remark as well?)
Let $z_2, z_3, z_4 \in \hat{C}$ be pairwise distinct. Then there exists a a uniquely determined Mobius transformation

$$D(\cdot, z_2, z_3, z_4) : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ mapping } 1, 0, \infty \text{ respectively.}$$

Given $z_1 \in \mathbb{C}$ we call $D(z_1, z_2, z_3, z_4)$ the *cross ratio of* z_1, z_2, z_3, z_4.

Proof. WTS: Existence,

Case (1)

$$z_2 = \infty \to \text{ set } D(\cdot, z_2, z_3, z_4) := T_{\begin{bmatrix} 1 & -z_3 \\ 1 & -z_4 \end{bmatrix}}$$

, which is well-defined.

Cbse (2)

$$z_3 = \infty \to \text{ set } D(\cdot, z_2, z_3, z_4) := T_{\begin{bmatrix} 0 & z_2 - z_4 \\ 1 & -z_4 \end{bmatrix}}$$

, which is also well defined.

Ccse (3)

$$z_4 = 0 \to \text{ set } D(\cdot,_2, z_3, z_3) := T_{\begin{bmatrix} 1 & \cdot z_3 \\ 0 & z_2 - z_3 \end{bmatrix}},$$

also well-defined.

Cdse (4)

$$z_2, z_3, z_4 \in \mathbb{C} \rightarrow \text{ set } D(\cdot, z_2, z_3, z_4) := T_{\begin{bmatrix} 1 & -z_3 \\ \frac{z_2-z_3}{z_2-z_4} & -z_4\frac{z_2-z_3}{z_2-z_4} \end{bmatrix}}$$

is also well-defined.

We have shown existence.
WTS: Uniqueness
Let $S : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a mobius transf. mapping z_2, z_3, z_4 to $1, 0, \infty$ respectively. Then

$$T := D(\cdot, z_2, z_3, z_4) \circ S^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

is a Mobius transf. with $T(1) = 1, T(0) = 0$ and $T(\infty) = \infty$.
To prove, $T = id_{\hat{\mathbb{C}}}$
Let $T = T_M$, where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C})$. Since $T(\infty) = \infty$, we get $c = 0$. Since $ad - bc \neq 0$, we get $a, d \neq 0$, $\rightarrow$ WLOG $a = 1$, so $M = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$ with $d \in \mathbb{C} \setminus \{0\}$.

$$0 = T(0) = \frac{1 \cdot 0 + b}{d} = \frac{b}{d} \Rightarrow b = 0,$$
$$1 = T(1) = \frac{1 \cdot 1 + 0}{d} = \frac{1}{d} \Rightarrow d = 1$$
$$\Rightarrow M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Lemma 3.1. If $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ pairwise distinct, $z_1 \in \hat{\mathbb{C}}$, $T : \hat{\mathbb{C}} \to \mathbb{C}$ Mobius transformation, then $D(z_1, z_2, z_3, z_4) = D(T(z_1), T(z_2), T(z_3), T(z_4))$.

Proof. $D(\cdot, z_2, z_3, z_4) \circ T^{-1}$ maps $T(z_2), T(z_3), T(z_4)$ to $1, 0, \infty$ respectively. Hence, by uniqueness:

$$D(\cdot, z_2, z_3, z_4) \circ T^{-1} = D(\cdot, T(z_2), T(z_3), T(z_4))$$

Now evaluate $T(z_1)_i$ which is trivial. □

Proposition 3.2. Let $z_1, z_3, z_4 \in \hat{\mathbb{C}}$ be pairwise distinct and $z_1 \in \hat{\mathbb{C}}$. Then $D(z_1, z_2, z_3, z_4) \in \mathbb{R} \cup \{\infty\}$

$$\Leftrightarrow \exists A \in \mathcal{CL} : z_1, z_2, z_3, z_4 \in A$$

Theorem 3.3. If $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Mobius transf. and $A \in \mathcal{CL}$, then $T(A) \in \mathcal{CL}$.

Proof. (assuming proposition)
Let $A \in \mathcal{CL}$ and let $z_2, z_3, z_4 \in A$ be pairwise distinct. Since T bijective, $T(z_2), T(z_3), T(z_4)$ pairwise distinct $\Rightarrow \exists! B \in \mathcal{CL}$ s.t. $T(z_2), T(z_3), T(z_4)$ are in B. To prove $T(A) \supseteq B$: If $z \in A$, then $D(z, z_2, z_3, z_4) \in \mathbb{R} \cup \{\infty\}$ by prop. so $D(T(z), T(z_2), T(z_3), T(z_4)) \in \mathbb{R} \cup \{\infty\}$ by lemma. So by proposition and uniqueness of B, we get $T(z) \in B \Rightarrow T(A) \subseteq B$. If $w \in B$, then $\exists z \in \hat{\mathbb{C}}$ with $T(z) = w$. To show $z \in A$, $D(z, z_2, z_3, z_4) = D(w, T(z_2), T(z_3), T(z_4)) \in \mathbb{R} \cup \{\infty\}$ since $w \in B$. $\Rightarrow \exists C \in \mathcal{CL}$ s.t. $z, z_2, z_3, z_4 \in C$. But $z_2, z_3, z_4 \in A$ are pairwise distinct. By uniqueness $C = A \Rightarrow z \in A$. □

Forelesning Week 04 Complex Analysis

Recall. $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ pairwise distinct $\to D|\cdot, z_2, z_3, z_4| : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ Mobius transf. with $z_2 \mapsto, z_3 \mapsto 0, z_4 \mapsto \infty$.

Proposition 3.4. $z_2, z_3, z_4 \in \hat{\mathbb{C}}$ pairwise distinct, $z_1 \in \hat{\mathbb{C}}$. then:

$$D|\cdot, z_2, z_3, z_4| \subseteq \mathbb{R} \cup \{\infty\} \Leftrightarrow \exists A \in \mathcal{CL} : z_1, z_2, z_3, z_4 \in A$$

Proof. It suffices to show that a Mobius transf. maps $\mathbb{R} \cup \{\infty\}$ onto an element of $\mathcal{CL}$. To this end, consider a Mobius transf.

$$T := T_M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{C})$$

Case (1) $c = 0 \Rightarrow a, d \neq 0$

$$T(\mathbb{R} \cup \{\infty\}) = \{\infty\} \cup \{\frac{a}{d} \cdot x + \frac{b}{d} : x \in \mathbb{R}\} \in \mathcal{CL}$$

Cbse (2) $c \neq 0, \frac{-d}{c} \in \mathbb{R}$
$T_M = T_{\frac{1}{c} \cdot M}$, so WLOG $c = 1$, so $d \in \mathbb{R}$.
$T(\infty) = \frac{a}{c} = a, T(-d) = T(-\frac{d}{c}) = \infty$
If $x \in \mathbb{R} \setminus \{-d\}$, then

$$T(x) = \frac{ax + b}{x + d} = \frac{ax + ad + b - ad}{x + d} = a - \frac{ad - b}{x + d}$$

$$T(\mathbb{R} \cup \{\infty\}) = \{a\} \cup \{\infty\} \cup \{a - \frac{ad - b}{x + d} : x \in \mathbb{R}, x \neq -d\}$$

$\{a\} \cup \{\infty\} \cup \{a + (b - ad) \cdot t : t \in \mathbb{R}, t \neq 0\} = \{\infty\} \cup \{a + (b - ad) \cdot t : t \in \mathbb{R}\} \in \mathcal{CL}, b - ad \neq 0$

Ccse (3) $c \neq 0, \frac{-d}{c} \notin \mathbb{R}$.
WLOG $c = 1$, i.e. $d \notin \mathbb{R}$. Note $T(\infty) = a$. For $x \in \mathbb{R}$:

$$T(x) = \frac{ax + b}{x + d} = a + \frac{b - ad}{x + d}$$

$$t = x + Re(d) : \text{ bijection } \mathbb{R} \to \mathbb{R}$$

$$T(\mathbb{R}) = \{a + \frac{b - ad}{x + d} : x \in \mathbb{R}\} = \{a + \frac{b - ad}{t + i \cdot Im(d)} : t \in \mathbb{R}\}, Im(d) \neq 0 \Leftarrow d \notin \mathbb{R}$$

$$T(\mathbb{R} \cup \{\infty\}) = \{a\} \cup \{a + \frac{b - ad}{t + i \cdot Im(d)} : t \in \mathbb{R}\}$$

$$= a + \left(\{0\} \cup \{\frac{b - ad}{t + i \cdot Im(d)} : t \in \mathbb{R}\}\right)$$

$$= a + \frac{b - ad}{Im(d)} \cdot \left(\{0\} \cup \{\frac{1}{t/Im(d) + i} : t \in \mathbb{R}\}\right)$$

$$y = \frac{t}{Im(d)} \text{ definess bijection } \mathbb{R} \to \mathbb{R}$$

$$= a + \frac{b - ad}{Im(d)} \cdot \left(\{0\} \cup \{\frac{1}{y + c} : y \in \mathbb{R}\}\right)$$

$a \in \mathbb{C}, \frac{b - ad}{Im(d)} \in \mathbb{C} \setminus \{0\}$.

Note. If $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, then $f : \mathbb{C} \to \mathbb{C} : z \mapsto c_1 + c_2 \cdot z$ maps circles to circles:

$$f(\{z \in \mathbb{C} : |z - z_0| = r\}) = \{w \in \mathbb{C} : |w - (c_1 + c_2 \cdot z_0)| = |c_2| \cdot r\}$$

So, by Note, it is enough to show that $\{0\} \cup \{\frac{1}{y+i} : y \in \mathbb{R}\}$ is a circle. claim:

$$\{0\} \cup \{\frac{1}{y+i} : y \in \mathbb{R}\} = \{z \in \mathbb{C} : |z - (-\frac{i}{2})| = \frac{1}{2}\}$$

($\subseteq$) $0 \in RHS.$, so consider $\frac{1}{y+y}, y \in \mathbb{R}$. Have

$$|\frac{1}{y+y} - (-\frac{i}{2})| = |\frac{y}{y^2+1} + i\left(\frac{1}{2} - \frac{1}{y^2+1}\right) = \sqrt{\left(\frac{y}{y^2+1}\right)^2 + \left(\frac{1}{2} - \frac{1}{y^2+1}\right)^2} = \frac{1}{2}$$

($\supseteq$) Let $z \in \mathbb{C}$ with $|z - (-\frac{i}{2})| = \frac{1}{2}$. If $z = 0$, then $z \in LHS$, so assume $z \neq 0$.

<u>To prove:</u> $Im(\frac{1}{z}) = 1$.

Let $\theta \in \mathbb{R}$, s.t. $z = -\frac{i}{2} + \frac{1}{2}e^{i\theta}$. We compute:

$$Im\left(\frac{1}{z}\right) = Im\left(\frac{2}{e^{i\theta} - i}\right) = Im\left(\frac{2}{\cos\theta + i(\sin\theta - 1)}\right) = 1.$$

□

Remark. "Trick" in the proof: $t = \frac{1}{x+d}$ defines a bijection from $\mathbb{R} \setminus \{0\}$ and $\mathbb{R} \setminus \{-d\}$.

Notation: If $Q = [a, b] + [c, d] \subseteq \mathbb{R}^2$ is a box, its boundary is the image of the piecewise C^1 curve

$$[0, 2(b-a) + 2(d-c)] \to \mathbb{R}^2 : s \mapsto \begin{cases} (a+s, c), & \text{if } 0 \leq s \leq b-a \\ \ldots, \ldots \\ \ldots, \ldots \end{cases}$$

By a slight abuse of notation, we denote this curve as ∂Q.

- counter-clockwise
- can be re-parametrized

Theorem 3.5. Cauchy Integral Theorem
Let $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic. Further let $Q = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ be a box ahd $\psi : Q \to D$ continuously diferentiable. Then:

$$\int_{\psi(\partial Q)} f(z)dz = 0.$$

Example 3.1. <u>triangles</u>
$\psi : [0, 1] \times [0, 1] \to \mathbb{C} : \psi(t, s) = t(1 + s \cdot i)$

$$\int_{\psi(\partial Q)} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

Example 3.2. discs
$z_0 \in \mathbb{C}, r \in \mathbb{R}_{>0}$

$$\psi : [0, r] \times [0, 2\pi] \to \mathbb{C} : (t, s) \mapsto z_0 + t \cdot e^{is}$$

$$\int_{\psi(\partial Q)} f(z)dz = \int_{|z-z_0|=r} f(z)dz$$

Lecture week 5

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Corollary. If $D \subseteq \mathbb{C}$ open, is simply connected (path-connected, every closed C^0-path is C^0-contractible in D), then the complex line integral of holomorphic functions on D only depends on start- and end point.

Proof.

$$0 = \int_\gamma f(z)dz - \int_{\tilde{\gamma}} f(z)dz$$

□

Proposition 4.1. Let $D \subseteq \mathbb{C}$ open, be simply connected. Then every holomorphic function $f : D \to \mathbb{C}$ admits a complex antiderivative.

Proof. Exercise. Hint:

- fix $z_0 \in D$.
- for $z \in D$ pick arbitrary C^1-curve γ_z from z_0 to z
- define $F : D \to \mathbb{C} : z \mapsto \int_\gamma f(z)dz$
- $$\frac{F(z+h) - F(z)}{h} = \frac{\int_{\gamma_{z+h}} f(z)dz - \int_{\gamma_z} f(z)dz}{h} = \frac{1}{h} \int_{\text{straight line segment from z to z+h}} f(z)dz$$

□

Theorem 4.2. Cauchy Integral Formula
Let $D \subseteq \mathbb{C}$ open, $z_0 \in D$ and assume $\overline{D_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\} \subseteq D$. If $f : D \to \mathbb{C}$ is holomorphic, then we have $\forall z \in D_r(z_0)$:

$$f(z) = \frac{1}{2\pi i} = \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw$$

In particular, the values of f inside the disc are uniquely determined by the values in the boundary of the disc.

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Theorem 5.1. Cauchy Integral Formula
Let $D \subseteq \mathbb{C}^n, z_0 \in D, r \in \mathbb{R}_{>0}, \overline{D_r(z_0)} \subseteq D$.
If $f : D \to \mathbb{C}$ is holomorphic, then, $\forall z \in D_r(z_0)$

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw.$$

If γ is a closed C^1 curve homotopic to $|w - z_0| = r$ in $D \setminus \{z\}$, then $f(z) = \frac{1}{2\pi\varepsilon} \int_\gamma \frac{f(w)}{w-z} dw$.

Proof. Fix a path $z \in D_r(z_0)$. Then $w \mapsto \frac{f(w)}{w-z}$ is holomorphic in $D \setminus \{z\}$. So by invariance of complex line integral under homotopies, we have for $0 < \varepsilon << 1 : \overline{D_\varepsilon(z)} \subseteq D_r(z_0)$ and

$$\frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z} dw$$

$$= \frac{1}{2\pi i} \int_{|w-z|=\varepsilon} \frac{f(w) - f(z)}{w-z} dw + \frac{1}{2\pi i} \int_{|w-z|=\varepsilon} \frac{f(z)}{w-z} dw$$

Let's look at just $\frac{1}{2\pi i} \int_{|w-z|=\varepsilon} \frac{f(z)}{w-z} dw$

$$= \frac{1}{2\pi i} f(z) \cdot \int_{|w-z|=\varepsilon} \frac{1}{w-z} dw$$

$$= \frac{1}{2\pi i} f(z) \cdot \int_0^{2\pi} \frac{1}{\varepsilon \cdot e^{i\theta}} \cdot \varepsilon \cdot i \cdot e^{i\theta} d\theta$$
$$= f(z)$$

Define

$$g : D \to \mathbb{C} : w \mapsto \begin{cases} \frac{f(w)-f(z)}{w-z} \text{ if } w \neq z \\ f'(z) \text{ if } w = z \end{cases}$$

g is continuous $\Rightarrow \exists M \in \mathbb{R}_{>0} : |g| \leq M$ on $\overline{D_r(z_0)}$.

$$\Rightarrow \left| f(z) - \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw \right|$$

$$\left| \frac{1}{2\pi i} \int_{|w-z|=\varepsilon} g(w) dw \right| \leq \frac{1}{2\pi i} (2\pi\varepsilon) \cdot max_{w:|w-z|=\varepsilon} |g(w)|$$
$$\leq \varepsilon \cdot M \to 0$$

as $\varepsilon \to 0$. $\square$

$$D = \mathbb{C} \setminus \{0\}, f : D \to \mathbb{C} : w \mapsto \frac{1}{w}$$

Given $z \in D_1(0) \cap D = D_1(0) \setminus \{0\}$, do we have

$$\frac{1}{z} = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)}{w-z} dw??$$

$$|\frac{1}{z}| \xrightarrow{z \to 0} \infty,$$

but

$$\left| \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)}{w-z} dw \right| \leq \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{1}{e^{i\theta} \cdot (e^{i\theta} - z)} \cdot i \cdot e^{i\theta} d\theta \right|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|e^{i\theta} - z|} d\theta \leq (0 < z < 1) \frac{1}{2\pi} \cdot \int_0^{2\pi} \frac{1}{|e^{i\theta}| - |z|} d\theta$$

$$= \frac{1}{2\pi} \frac{1}{1 - |z|} d\theta$$

$$= \frac{1}{1 - |z|} \xrightarrow{z \to 0} 1$$

Lemma 5.2. Let $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic, $\overline{D_r(z_0)} \subseteq D$.
Then $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r \cdot e^{i\theta}) d\theta$.

Proof.

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r \cdot e^{i\theta}} \cdot r \cdot i \cdot e^{i\theta} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r \cdot e^{i\theta}) d\theta$$

□

Proposition 5.3. Maximum Principle, 1st version

$D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic. Further, let $z_0 \in D$ be a local maximum of $|f|$. Then f is constant in a neighborhood of z_0.

Proof. $\exists \varepsilon > 0$: the restriction of $|f|$ to $\overline{D_\varepsilon |z_0|}$ has a global maximum in z_0. Hence, $\forall 0 < r \leq \varepsilon$:

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r \cdot e^{i\theta}) d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + r \cdot e^{i\theta})| d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| d\theta = |f(z_0)|$$

$$\Rightarrow 0 = \int_0^{2\pi} \left(|f(z_0)| - |f(z_0 + r \cdot e^{i\theta})| \right) d\theta$$

So, by continuity:

$$|f(z_0)| = |f(z_0 + r \cdot e^{i\theta})| \forall \theta \in [, 0, 2\pi].$$

$$\Rightarrow f \text{ constant on } \overline{D_\varepsilon(z_0)}.$$

Exercise Week 3 $\Rightarrow$ f constant on $D_\varepsilon(z_0)$, since the latter set is open, connected.

□

Corollary. $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic. If $|f|$ has a local minimum in z_0, then $f(z_0) = 0$ or f is constant in a neighborhood of z_0.

Proof. If $f(z_0) \neq 0$, then $\frac{1}{f}$ is holomorphic in a neighborhood of z_0. $\square$

Proposition 5.4. Maximum Principle, 2nd version

$D \subseteq \mathbb{C}$ open, D bounded, $f : D \to \mathbb{C}$ is continuous on $\overline{D}$, holomorphic on D. Then $\exists p \in \partial D$ s.t.

$$|f(p)| = max_{z \in \overline{D}} |f(z)|.$$

If furhtermore f has no zero on D, then $\exists q \in \partial D$, s.t. $|f(q)| = min_{z \in \overline{D}} |f(z)|$.

Proof. $\overline{D}$ is compact, $|f$ is continuous $\Rightarrow$ $M = max_{z \in \overline{D}} |f(z)| \in \mathbb{R}_{\geq 0}$ well-defined. Let $K := |f|^{-1}(\{M\}) \subseteq \overline{D}$ closed. So K is compact and non-empty. If $K \cap \partial D \neq \emptyset$, we are done, so assume FTSOAC that $K \cap \partial D = \emptyset$. Then $K \subseteq D$.
Let $a \in \partial_{\mathbb{C}} K \subseteq K \subseteq D$.
$a \in K \subseteq D$, so $|f|$ has a local maximum at $a \in D$.
$\Rightarrow$ constant in a neighborhood in $\mathbb{C}$ of a $\Rightarrow$ an open (in $\mathbb{C}$) neighborhood of a is contained in K
$a \notin \partial_{\mathbb{C}} K$. Contradiction. $\square$

Forelesninger Week 6 Complex Analysis

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$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 \text{ if } x = 0 \\ exp(-\frac{1}{x^2}) \text{ if } x \neq 0 \end{cases}$$

f is of class C^∞ and $f^n(0) = 0 \forall n \in \mathbb{Z}_{n \geq 0}$.
$\sum \frac{f^{(n)}}{n!} x^n \equiv 0 \to f$ is not locally given by a power series.

Taylor expansion of holomorphic functions

Let $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic, $r \in \mathbb{R}_{>0}, z_0 \in D$, and assume $D_r(z_0) \subseteq D$. Then

(a) On $D_r(z_0)$ the function f is given as a power series centered at z_0, whose radius of convergence is $\geq r$. In particular, f is infinitely many times complex differentiable and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} (z - z_0)^n, \forall z \in D_r(z_0)$$

(b) If $0 < s < r$, then $\forall n \in \mathbb{Z}_{n \geq 0}$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

Proof. Let $z \in D_r(z_0)$. Pick an $s \in (0,r)$ with $z - z - 0| < s$ i.e. $z \in D_s(z_0)$. Since $D_s(z_0) \subseteq D_r$ we can apply the Cauchy Integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=s} dw = \frac{1}{2\pi i} \int_{|w-z_0|=s} \frac{f(w)}{w-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}} dw$$

$$\left| \frac{z-z_0}{w-z_0} \right| \leq 1???$$

This is a geometric series so

$$= \frac{1}{2\pi i} \int_{|w-z_0|=s} \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n dw$$

$$(*) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|w-z_0|=s} \frac{f(w)}{w-z_0} \left(\frac{z-z_0}{w-z_0} \right)^n dw$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{|w-z_0|=s} \frac{f(w)}{(w-z_0)^{n+1}} dw \right] (z-z_0)^n$$

The term inside the brackets is constant in $\mathbb{C}$, independent from z, and from $s \in (0,r)$ by homotopy invariance. This is a power series around z_0, convergence for $x \in D_r(z_0)$ implicit from calculation. By uniqueness of power series it follows that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|w-z_0|=s} \frac{f(w)}{(w-z_0)^{m+1}} dw$$

Now we have to verify $(*)$: this follows straight away from the dominated convergence theorem:

$$|f_n| = \left| \frac{f(re^{i\varphi} + z_0)}{(re^{i\varphi} + z_0) - z_0} \cdot \left(\frac{z - z_0}{re^{i\varphi} + z_0 - z_0} \right)^n \cdot ire^{i\varphi} \right| = \left| \frac{f(re^{i\varphi} + z_0) \cdot (z-z_0)^n \cdot r}{re^{i\varphi} \cdot r^n (e^{i\varphi})^n} \right|$$

$$= \left| \frac{z-z_0}{r} \right|^n \left| f(re^{i\varphi} + z_0) \right| \leq \left| f(re^{i\varphi} + z_0) \right| \leq max_{0 \leq \varphi \leq 2\pi} \left| f(re^{i\varphi} + z_0) \right| < \infty$$

□

Assuming f has a power sereies $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, we set by termwise differentiation that $a_n = \frac{f^{(n)}(z_0)}{n!}$

Example 6.1.

$$f : \mathbb{C} \setminus \{-i, i\} \to \mathbb{C}, Z \mapsto \frac{1}{z^2+1}$$

f is given as a power series centered in $D_{\sqrt{2}}(1)$.

Remark. $D \subseteq \mathbb{C}$, $f : D \to \mathbb{C}$ holomorphic, $D_r(z_0) \subseteq D$ for some $z_0 \in D$, $r \in \{R_{>0}$. Then $\forall z \in D_r(z_0)$:

$$\forall n \in \mathbb{Z}_{\geq 0} : f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z)^{n+1}} dw$$

"Generalized Cauchy Integral formula" – observe that z now don't have to be the center of the disk z_0, but can take any value in the interior $D_r(z_0)$.

Proof. (Sketch)
The proof is by homotopy invariance. $s > 0 : D_s(z_0) \subseteq D_r(z_0)$. Now we apply our previous theorem

$$f^{(n)}(z) = \frac{n!}{22\pi i} \int_{|w-z|=s} \frac{f(w)}{(w-z)^{n+1}} dw$$

Now let γ_1, γ_2 be freely $\mathbb{C}^2$-homotopic in $D \setminus \{z\}$

$$= \frac{n!}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z)^{n+1}} dw$$

□

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Corollary. Let $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ a function. TFAE:

(i) f is complex differentiable (holom.)

(ii) f is infinitely many times comp. diff.

(iii) f admits a complex antiderivative locally at every point in D

(iv) f can be written as a complex power series loccally at every point in D

(v) f is (real) totally differentiable and satisfies Cauchy-Riemann

(vi) f i of class c^∞ and satisfies Cauchy-Riemann

Theorem 7.1. Liouville's theorem
Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant.

Proof. $\exists M \in \mathbb{R}_{>0} : |f(z)| < M \quad \forall z \in \mathbb{C}$. Furthermore $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad \forall z \in \mathbb{C}$ and $\forall r > 0 \quad \forall n \in \mathbb{Z}_{\geq 1}$

$$\left| \frac{f^{(n)}(0)}{n!} \right| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right|$$

$$\leq \frac{1}{2\pi} \int_{|z|=r} \left| \frac{f(z)}{z^{n+1}} \right| dz$$

$$\nleq \frac{1}{2\pi} \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{M}{r^n} \xrightarrow{r\to\infty} 0$$

$$\Rightarrow f(z)|_{D_r} = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} z^n \xrightarrow{r\to\infty} f(0)$$

□

Lemma 7.2. Let $\emptyset \neq D \subseteq \mathbb{C}$ open, be connected and let $f : D \to \mathbb{C}$ be holomorphic. TFAE:

(i) $f \equiv 0$ on D

(ii) $f|_A \equiv 0$ for some set $A \subseteq D$, which has an accumulation point in D.

(iii) $\exists z_0 \in D : f^{(n)}(z_0) = 0 \forall n \in z_{\geq 0}$.

Proof. ($(i) \Rightarrow (ii)$) take $A = D$
($(ii) \Rightarrow (iii)$) Let $z_0 \in D$ be an accumulation point of A. Assume FTSOAC $\exists n \in \mathbb{Z}_{\geq 0} : f^{(n)}(z_0) \neq 0$. Let n be chosen to be minimal with this property $\Rightarrow \exists u \subseteq D$ open with $z_0 \in u$, such that $\forall z \in u$:

$$f(z) = \sum_{k=n}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \Rightarrow g : u \to \mathbb{C} : z \mapsto \sum_{k=0}^{\infty} \frac{f^{(n+k)}(z_0)}{k!}(z - z_0)^k$$

is holomorphic on u and $\forall z \in u \setminus \{z_0\} g(z) = \frac{f(z)}{(z-z_0)^n}$
$\Rightarrow$ the function $h(z_0) : D \to \mathbb{C}, z \mapsto \begin{cases} f(z)/(z - z_0)^n \text{ if } z \neq z_0 \\ f^{(n)}/n! \text{ if } z = z_0 \end{cases}$.
Let $(a_m)_{m \in \mathbb{Z}_{\geq 0}}$ be a sequence in $A \setminus \{z_0\}$ converging to z_0. We get

$$0 \neq \frac{f^{(n)}(z_0)}{n!} = h(z_0) = \lim_{m \to \infty} h(a_m) = \lim_{m \to \infty} \frac{f(a_m)}{(a_m - z_0)^n} = 0$$

Contradiction
($(iii) \Rightarrow (i)$) $S := \{.z \in D : f^{(n)}(z) = 0 \quad \forall n \in \mathbb{Z}_{\geq 0}\}$. Have to show that $S = D$. D connected $\Rightarrow$ enough to show $a) S \neq \emptyset$ $b) S \subseteq D$ closed , $c) S \subseteq D$ open.

a) $z_0 \in S$

b) $f^{(n)}$ is continous $\forall n \in \mathbb{Z}_{\geq 0}$. Let $z \in D$ be the point(?) of a sequence $(s_m)_{m \in \mathbb{Z}_{\geq 0}} \in S$. Then $\forall n$, we have

$$f^{(n)}(z) = f^{(n)}\left(\lim_{m \to \infty} s_m\right) (= cont) \quad lim_{m \to \infty} f^{(n)}(s_m) = \lim_{m \to \infty} 0 = 0$$

Hence the limit is in S so $\overline{S} = S$, that is S is closed in D.

c) Let $w \in S \Rightarrow \exists$ open neighborhood u of w in D such that

$$\forall z \in u : f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(w)}{n!}(z - w)^n = 0$$

$$\Rightarrow f \equiv 0 \text{ on } U \Rightarrow u \subseteq S$$

□

Example 7.1. (a) $f : \mathbb{C} \to \mathbb{C}$ holomorphic, $f(\frac{1}{n}) = 0 \forall n \in \mathbb{Z}_{>0}, A = \{\frac{1}{n} : n \in \mathbb{Z}_{>0}\} \Rightarrow f \equiv 0$

(b) $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ holomorphic, $g(\frac{1}{n}) = 0 \forall n \in \mathbb{Z}_{>0}$ can choose $g(z) = \sin\left(\frac{\pi}{2}\right) \not\equiv 0$ but $A = \{\frac{1}{n} : n \in \mathbb{Z}_{Z>0}\}$ has accumulation point – the problem is that it is in the boundary.

Theorem 7.3. Identity theorem
Let $D \subseteq \mathbb{C}$ open, be connected. If two holom. functions $f, g : D \to \mathbb{C}$ coincide on a set $A \subseteq D$ which has an accumulation point in D, then they coincide on all of D.

Proof. Apply the previous lemma to $f - g$. □

Maximum principle Version 3

$D \subseteq \mathbb{C}$ open, connected, $f : D \to \mathbb{C}$ holomorphic. If $|f|$ has a local maximum in D.

Corollary. $D \subseteq \mathbb{C}$ open, connected, $f : D \to \mathbb{C}$ holomorphic and non-constant. If $z_0 \in D$ and $f(z_0) = 0$, then there exists an open neighborhood $u \subseteq D$ of z_0 such that z_0 is the only zero of $f|_u$.

Proof. Do it yourself. $\square$

Theorem 7.4. Open mapping theorem
Let $D \subseteq \mathbb{C}$ open, be connected, $f : D \to \mathbb{C}$ holomorphic and non-constant. Then $f(D)$ is open in $\mathbb{C}$.

Proof. Let $w_0 \in f(D)$, i.e. $z_0 \in D : f(z_0) = w_0$. Since $f - w_0$ is holomorphic and non-constant ont the connected open set D, we find $\varepsilon > 0$ such that $\overline{D_\varepsilon}(z_0) \subseteq D$ and z_0 is the only zero of $f - w_0$ on $\overline{D_\varepsilon}(z_0)$. Let $\Delta := min_{z:|z-z_0|=\varepsilon}|f(z) - w_0| > 0$. WTS: $D_{\Delta/2}(w_0) \subseteq f(D)$. Let $w \in D_{\Delta/2}(w_0)$ and consider $f - w$. For $z \in D$ with $|z - z_0| = \varepsilon$ we get by triangle inequality

$$||f(z) - |w|| \geq |f(z) - w_0| - |w_0 - w| \geq \Delta - |w - w_0| > \Delta - \frac{\Delta}{2} = \frac{\Delta}{2}$$

But

$$|f(z_0) - w| = |w_0 - w| < \frac{\Delta}{2} \Rightarrow \left| f|_{\overline{D_\varepsilon(z_0)}} - w \right|$$

does not attain its minimum on the boundary of $D_\varepsilon(z_0)$, which by the maximum principle (2nd version) implies that $f|_{\overline{D_\varepsilon(z_0)}} - w$ must have a zero $\tilde{z} \in D_\varepsilon(z_0)$. $\square$

Lecture Week 7 Complex Analysis

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Definition 8.1. A Laurent SEries centered at $z_0 \in \mathbb{C}$ is a formal expression of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n := \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} + \sum_{n=0}^{\infty} A_n(z - z_0)^n$$

$$f^-(z) = \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}, \quad f^+ = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

Spør Markus
The series is said to converge for some $z \in \mathbb{C}$ provided both $f^+(z)$ and $f^-(z)$ both converge.

Remark. $\lim_{N\to\infty} \sum_{n=-N}^{N} a_n(z - z_0)^n$ exists does not imply $f(z)$ converges.

Because look at $a_n = \begin{cases} 0 \text{ if } n = 0 \\ (-1)^n \text{ if } n > 0 \\ (-1)^{n+1} \text{ if } n < 0 \end{cases}$, then

$$f^+(z) = -z + z^2 - z^3 \quad , f^-(z) = \frac{1}{2} - \frac{1}{z^2} + \frac{1}{z^3} - \dots$$

With $z = 1$ this celarly doesn't converge for $f^+(z)$ and same for $f^-(z)$, but look at the limit: $\sum_{n=-N}^{N} a_n \cdot 1^n = 0 \forall N$, so $\lim_{N\to\infty} \sum_{n=-N}^{N} A_n \cdot 1^n = 0$.

Remark. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be a Laurent series. Set

$$r := \lim_{n\to\infty} \sup \sqrt[n]{|a_{-n}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

$$R := \lim_{n\to\infty} \sup \frac{1}{\sqrt[n]{|a_n|}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\ \}$$

$$U := \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

then

(a) $f(z)$ converges absolutely if $z \in u$

(b) $f(z)$ diverges if $|z - z_0| < r$ pr $> R$

(c) The partial sums of f^+ and f^- converge uniformly on compact subsets of u. u is called the *annulus of convergence* of f.

(d) f defines a holomorphic function on u, which can be differnetiated term by term. Teh resulting Laurent series has the same annulus of convergence.

Proof. If $g_1(w) = \sum_{n=0}^{\infty} a_n w^n, g_2(w) = \sum_{n=1} = a_{-n}w^n$, then radius of convergence of g_1 is R, the radius of convergence of g_2 is $\frac{1}{r}$ (root test, i.e. Cauchy-Hadamard Formula). The claims follows by applying the corresponding results for power series. □

Laurent Series expansion of holomorphic functions

Let $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic. Furthermore let $r, R \in \mathbb{R}_{\geq 0} \cup \{+\infty\}, r < R$, such that

$$u :)\{z \in \mathbb{C} : r < |z - z_0| < R\} \subseteq D.$$

Then

(a) On u, the function f is given by $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \forall z \in u$.

(b) The coefficients are uniquely determined: they satisfy $\forall n \in \mathbb{Z} : a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}}$ for some (and hence any by homotopy) ρ with $r < \rho < R$.

Proof sketch:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw$$

By figure we get
Sett inn figur

$$\int_{\gamma} \frac{f(w)}{w - z} dw = \int_{|w-z_0|=r_2} \frac{f(w)}{w - z} - \int_{|w-z_0|=r_1} \frac{f(w)}{w - z}$$

Let the two integrals be denoted I and II respecitvely.
Now we use the power series trick as

$$\left|\frac{z-z_0}{w-z_0}\right| < 1 \text{ and } \left|\frac{w-z_0}{z-z_0}\right| < 1$$

For I:

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}} = \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n$$

For II:

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}} = \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n$$

The rest of the proof is analogous to the way we proved this for power series. This proves existence, now we prove uniqueness. Assume that $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is one such representation of f on u. Then for $n \in \mathbb{Z}, \rho \in (r, R)$:

$$\frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} \int_{|z-z_0|=\rho} a_k \frac{(z-z_0)^k}{(z-z_0)^{n+1}} dz$$

$$\frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z-z_=)^{k-n-1} dz$$

Recall.

$$\int_{|w|=1} w^n dw = \begin{cases} 0 \text{ if } n \neq -1 \\ 2\pi i \text{ if } n = -1 \end{cases}$$

By recall we have

$$\int_{|z-z_0|=\rho} (z-z_0)^{k-n-1} dz = \begin{cases} 2\pi i \text{ if } k-n-1 = -10 \text{ otherwise} \end{cases}$$

Hence

$$\frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=\rho} (z-z_0)^{k-n-1} dz = \frac{1}{2\pi i} a_n \cdot 2\pi i = a_n$$

Remark. (Notation as above)
If $D_R(z_0) \subseteq D$, then the Laurent expansion of f around z_0 and the Taylor expansion of f around z_0 coincide.

Example 8.1. $f : \mathbb{C} \setminus \{-1, 1, -i, i\} \to \mathbb{C} : z \mapsto \frac{1}{1-z^n}, z_0 = 1$

$$u_1 = \{z \in \mathbb{C} : 0 < |z-1| < \sqrt{2}\}$$

$$u_2 = \{z \in \mathbb{C} : \sqrt{2} < |z-1| < 2\}$$

$$u_3 = \{z \ \in \mathbb{C} : 2 < |z.1|\}$$

Hence f has three Laurent expansions as we need to partition them to avoid the singularities.

Definition 8.2. $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic, $z \in \mathbb{C} \setminus D$.

1. If $\exists \varepsilon > 0 : D_\varepsilon \setminus \{z_0\} \subseteq D$, then z_0 is calle an *isolated singulairty.*
2. If z_0 is an isolated singularity and f extends holomorphically to $D \cup \{z_0\}$, then z_0 is called a *removable singularity.*

Example 8.2. $f, y : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}, z \xrightarrow{f} \frac{1}{z}, z \xrightarrow{g} \frac{e^z - 1}{z}$
0 and 1 are isolated singularities of f and g. 1 is a removable singularity of f and g. 0 is not a removable singularity of f.

$$0 \neq g(z) = \frac{1}{z}\left(-1 + \sum_{k=0}^{\infty} \frac{z^k}{k!}\right) = \frac{1}{z}\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)!} z^k \Rightarrow 0 \text{ is a removable singularity of } g.$$

Definition 8.3. $f : D \to \mathbb{C}$ holomorphic, $z_0 \in \mathbb{C}$ ($z_0 \in D$ or $z_0 \in \mathbb{C} \setminus D$ isolated singularity of f). So for $0 < \varepsilon << 1$ (small enough), we have $D_\varepsilon(z_0) \setminus \{z_0\} \subseteq D$, i.e. on this set f admits a uniquely determined expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, z \in D_\varepsilon(z_0) \setminus \{z_0\}$$

which is independent from $0 < \varepsilon << 1$. If $f \not\equiv 0$ on $D_\varepsilon(z_0) \setminus \{z_0\}$, then we call

$$ord_{z_0} f := \begin{cases} min\{n \in \mathbb{Z} : a_n \neq 0, \text{ if this set is bounded from below} \\ -\infty, \text{ otherwise} \end{cases}$$

the *order* of f in z_0. Write $m = ord_{z_0} f \in \mathbb{Z} \cup \{-\infty\}$.

(a) If $m > 0$ then z_0 is called a zero of order m (of f)

(b) If $m < 0$, then z_0 is called a pole of order $-m, m \neq -\infty$ of f.

(c) If $m = -\infty$, then z_0 is calledc an essential singularity of f

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Lemma 9.1. Let $f : D \to \mathbb{C}$ holom., $z_0 \in \mathbb{C}$ s.t. $z_0 \in \mathbb{C}$ or $z_0 \in \mathbb{C} \setminus D$ isolated sing. of f. Assume $f \not\equiv 0$ on $D_\varepsilon(z_0) \setminus \{z_0\}$ for $0 < \varepsilon << 1$ and assume $ord_{z_0} f \neq -\infty$. Then $ord_{z_0} f$ is the uniquely determined integer $m \in \mathbb{Z}$, s.t. $\exists g : D \cup \{z_0\} \to \mathbb{C}$ holom. with $g(z_0) \neq 0$ and $\forall z \in D \setminus \{z_0\} f(z) = (z - z_0)^m \cdot g(z)$.

Corollary. (a) $f, f_1, f_2 : D \to \mathbb{C}$ holom., z_0 isolated sing. of f, f_1, f_2. Assume $f, f_1, f_2 \not\equiv 0$ on $D_\varepsilon(z_0) \setminus \{z_0\}$ for $0 < \varepsilon << 1$ and $ord_{z_0} f$, and $ord_{z_0} f_1$, $ord_{z_0} f_2 \neq -\infty$. Then

$$ord_{z_0}(f_1 \cdot f_2) = ord_{z_0} f_1 + ord_{z_0} f_2 \text{ and } ord_{z_0}(\frac{1}{f}) = -ord_{z_0} f$$

(b) $f : D \to \mathbb{C}$ holom., z_0 isolated sing. of f, $f \not\equiv 0$ on $D_\varepsilon(z_0) \setminus \{z_0\}$ for $0 < \varepsilon << 1$. Then z_0 is a *removable* singularity $\Leftrightarrow ord_{z_0} f \geq 0$

Theorem 9.2. Riemann Removable Singularity Theorem $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holom., $z_0 \in \mathbb{C} \setminus D$ is of singularity of f. If $\exists$ neigbourhood u of z_0 in $\mathbb{C}$, s.t. $u \setminus \{z_0\} \subseteq D$ and f is bounded on $u \setminus \{z_0\}$, then z_0 is a removable singularity of f.

Proof. Let $M \in \mathbb{R}_{>0}$ and $\varepsilon > 0$ s.t.: $D_{\varepsilon_0}(z_0) \setminus \{z_0\} \subseteq D$ and $|f| < M$ on $D_{\varepsilon_0}(z_0) \setminus \{z_0\}$. Then $\forall z \in D_{\varepsilon_0}(z_0) \setminus \{z_0\}$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \cdot (z - z_0)^n, \text{ where } \forall n \in \mathbb{Z}$$

$$\forall 0 < \varepsilon < \varepsilon_0 : a_n = \frac{1}{2\pi i} \int_{|w-z_0|=\varepsilon} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

$$|a_n| \leq \frac{1}{2\pi} \cdot 2\pi\varepsilon \frac{M}{\varepsilon^{n+1}} = M \cdot \varepsilon^{-n} \to 0(\varepsilon \to 0(n < 0))$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \forall z \in D_\varepsilon(z_0) \setminus \{z_0\}$$

$$\Rightarrow f \text{ extends holomorphically to } z_0 \text{ via } f(z_0) = a_0.$$

□

Behaviour near isolated singularity

$D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holom., $z_0 \in \mathbb{C} \setminus D$ isolated singularity, i.e. $D_{\varepsilon_0}(z_0) \setminus \{z_0\} \subseteq D$ for some $\varepsilon > 0$.

- if $f \equiv 0$ on $D_{\varepsilon_0}(z_0) \setminus \{z_0\}$, then z_0 is a removable sing.
- if $f \not\equiv 0$ on $D_{\varepsilon_0}(z_0) \setminus \{z_0\}$ and $ord_{z_0} f \geq 0$, then z_0 is a removabl sing.
- if $f \not\equiv 0$ on $D_{\varepsilon_0}(z_0) \setminus \{z_0\}$ and $ord_{z_0} f < 0$, but $\neq -\infty$, then

$$|f(z)| \to \infty \text{ as } z \to z_0$$

- (Casarati-Weierstrass)

  if $f \not\equiv 0$ on $D_{\varepsilon_0}(z_0) \setminus \{z_0\}$ and $ord_{z_0} f = -\infty$, then $\forall u \subseteq \mathbb{C}$ open with $z_0 \in u$, $u \setminus \{z_0\} \subseteq D$, we have: $f(u \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proof. Assume FTSOAC $\exists u \subseteq \mathbb{C}$ open, with $z_0 \in u$ and $u \setminus \{z_0\} \subseteq D$, s.t.

$$f(u \setminus \{z_0\}) \cap D_\delta(w_0) = \emptyset \text{ for some } \delta > 0, w_0 \in \mathbb{C}, i.e.$$

$$|f(z) - w_0| \geq \delta \forall z \in u \setminus \{z_0\}. \Rightarrow g : u \setminus \{z_0\} \to \mathbb{C}, z \mapsto \frac{1}{f(z) - w_0}$$

is well-defined, holom. bounded, so by RRST, g extends holomorphically to

$$g : u \to \mathbb{C}. \text{ On } u \setminus \{z_0\} : f(z) - w_0 = \frac{1}{g(z)}$$

$$\Rightarrow \mathbb{Z} \ni -ord_{z_0} g = ord_{z_0} \frac{1}{g} = ord_{z_0}(f - w_0) = ord_{z_0} f = -\infty \Rightarrow -\infty \in \mathbb{Z}$$

Contradiction. Above we use $g : u \to \mathbb{C}$ holomorphic, g nowhere 0 on $u \setminus \{z_0\}$. □

Definition 9.1. Let $z_0 \in \mathbb{C}$ and let γ be a closed pwC^1 curve in $\mathbb{C} \setminus \{z_0\}$. Then

$$ind_{z_0}\gamma := \frac{1}{2\pi i} \cdot \int_\gamma \frac{1}{z - z_0} dz$$

is an integer. It is called the *index* or *winding number of* γ around z_0.
see exercises week 4

Remark. Let $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ be a closed pwC^1 curve and assume that
We choose $arg \in (-\pi, \pi]$

(a) $T = \{t \in [a, b] : \gamma(t) \in \mathbb{R}_{<0}$ is a finite set.

(b) $a, b \notin T$.

(c) $T = T^+ \cup (dot)T^-$, where

- $T^+ = \{t \in T : \lim_{s \nearrow t} arg\gamma(s) = \pi, \lim_{s \searrow t} arg\gamma(s) = -\pi\}$
- $T^- = \{t \in T : \lim_{s \nearrow t} arg\gamma(s) = -\pi, \lim_{s \searrow t} arg\gamma(s) = \pi\}$

Then $ind_0\gamma = cardT^+ - cardT^-$.
Lecture week 9

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Theorem 11.1. Mantel's Theorem

Lemma 11.2. $D \subseteq \mathbb{C}$ open, $f : D \to \mathbb{C}$ holomorphic, $\overline{D_{4r}(z_0)} \subseteq D$, $|f| \leq 1$ on ..

11.1 After Break

$(g_k)_k$ that converges pointwise at every a_j. Let $z_0 \in \Omega$ and $r > 0$ s.t. $\overline{D_{4r}(z_0)} \subseteq \Omega$. Let $0 < \varepsilon << 1$ be given. Then $\overline{D_r(z_0)}$ can be covered by a finite collection of open discs $\Delta_l, l = 1, \ldots, N$ of radius $< \frac{\varepsilon \cdot r}{4}$. Since $\{a_j : j \in \mathbb{Z}_{\geq 0}\}$ is dense in Ω, we can pick some $w_l \in \Delta_l \cap \{a_j : j \in \mathbb{Z}_{\geq 0}\}$. Then, given $z \in \overline{D_r(z_0)}$, we find $l \in \{1, \ldots, N\}$, s.t. $z \in \Delta_l$.

$$|g_k(z)| - g_m(z)| \leq |g_k(z) - g_k(w_l)| + |g_k(w_l) - g_m(w_l)| + |g_m(w_l) - g_m(z)|$$

$$\leq |z - w_l| \cdot \frac{2}{2r} + |g_k(w_l) - g_m(w_l)| + |z - w_l| \cdot \frac{2}{2r}$$

This second ineequality comes from the lemma.

$$= |g_k(w_l) - g_m(w_l)| + 2\varepsilon < 3\varepsilon \text{ when } k, m > N_\varepsilon,$$

where $N_\varepsilon \in \mathbb{Z}_{>0}$ can be chosen independently from z and w_l. Hence $(g_k)_k$ is uniformly Cauchy on $\overline{D_r(z_0)}$, i.e. it converges uniformly on $\overline{D_r(z_0)}$. Define $g : \Omega \to \mathbb{C} : G(z_0) = \lim_{k\to\infty} g_k(z_0)$. g is well-defined and holomorphic. Let $\emptyset \neq K \subseteq \Omega$ be compact. Then we find $p_1, \ldots, p_M \in \mathbb{R}_{>0}$, s.t. $\overline{D_{4r_j}(p_j)} \subseteq \Omega$ and $K \subseteq \cup_{j=1}^M D_{r_j}(p_j)$. Since $(g_k)_k$ converges uniformly to g on $\overline{D_{r_j}(p_j)} \forall j = 1, \ldots, M$ it also converges uniformly to g on K (finite union).