Verdier Duality for stable ∞-categories

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May 27, 2023

Contents

Abstract

In 1965, Jean-Louis Verdier introduced Verdier duality for locally compact topological spaces, thus generalizing the classical theory of Poincaré duality for manifolds. Verdier Duality is a cohomological duality that allows exchanging cohomology for cohomology with compact support. More precisely it states that the derived functor of the compactly supported direct image functor has a right adjoint in the derived category of sheaves. By using sheaf cohomology one can derive the classical Poincaré duality as a special case. In his book "Higher Algebra" Jacob Lurie extends the theory to the ∞-categorical setting by showing there is an equivalence between sheaves and cosheaves valued in stable ∞-categories. This thesis follows this proof closely, expanding and adding details where necesarry. To introduce the relevant background on sheaves and $\mathcal{K}$-sheaves valued in stable ∞-categories we introduce and utilize Kan extensions, a ubiquitous concept in category theory, as well as giving an expository account of the basic theory of ∞-categories.

TODO: Skriv norsk abstract.

Acknowledgements

Introduction

The main goal of this thesis is to understand and reproduce Jacob Lurie's version of Verdier duality for sheaves valued in stable ∞-categories. The first four chapters in this thesis are dedicated to building the framework and gathering the tools we need to prove the theorem.

In chapter **??**, we introduce simplicial sets and ∞-categories. There are many excellent introductions to the world of ∞-categories and beyond. We mostly follow [**kerodon**] and [**Rezk**].

In chapter **??**, we continue our introduction to ∞-categories and introduce the machinery required to define ∞-categorical versions of limits and colimits. We also introduce the notion of a cofinal functor and present a result that greatly simplifies the calculation of limits and colimits of certain functors. Finally, we define stable ∞-categories and introduce some examples and valuable results.

In chapter **??**, we first recall the definitions of Kan extensions in ordinary category theory and present some selected results and examples. Kan extensions are a ubiquitous concept in category theory, and their properties will be central in proving Verdier duality.

In chapter **??**, we introduce sheaves on locally compact Hausdorff spaces and their compact counterpart $\mathcal{K}$-sheaves. The main goal of this chapter is to prove Theorem **??** which says that for an ∞-category $\mathcal{C}$ and a locally compact Hausdorff space X there is a categorical equivalence between the ∞-category $\mathrm{Shv}(X;\mathcal{C})$ of $\mathcal{C}$-valued sheaves on X and the ∞-category $\mathrm{Shv}_{\mathcal{K}}(X;\mathcal{C})$ of $\mathcal{C}$-valued $\mathcal{K}$-sheaves. We start the chapter by recalling the definition of sheaves valued in 1-categories and give a short recollection of the history of sheaf theory.

In chapter **??**, we prove Theorem **??**: Verdier duality. The theorem says that for a stable ∞-category and a locally compact Hausdorff space X there is an equivalence between the ∞-category $\mathrm{Shv}(X;\mathcal{C})$ of $\mathcal{C}$-valued sheaves on X and the ∞-category $\mathrm{Shv}(X;\mathcal{C}^{op})^{op}$ of $\mathcal{C}$-valued cosheaves on X. Using Theorem **??**, we restate the theorem as an equivalence between ∞-categories of $\mathcal{K}$-sheaves instead, and while this is a crucial part of the proof, it also clarifies the connection to the classical Verdier duality. We start the chapter by recalling classical Verdier duality and its relation to Poincaré duality.

Chapter 1

∞-categories

What Lurie [**HTT**] calls ∞-categories were originally called restricted Kan complexes by Boardman and Vogt [**BoardmanVogt**], but without the intent of using them for ∞-categories. The first development of such a theory was done by Joyal in [**Joyal**], who called them quasicategories. As most of this thesis follows Lurie's works very closely, we will follow his convention and use the name ∞-categories.[1] While [**HTT**] gives a good introduction to ∞-categories extending on the work of Joyal, his web-project [**kerodon**] reworks a lot of the foundations, and we take a lot of inspiration from this presentation.

1.1 Simplicial sets

Originally, simplicial sets were used to rephrase the homotopy theory of spaces in combinatorial terms. There are many good introductions to simplicial sets, depending on what you want to use them for, but Friedman's [**friedman2021elementary**] was enlightening for the author of this thesis. For algebraic topologists, Peter May's [**MAY**] is a good introduction to semi-simplicial topology.

Definition 1.1.1. Usually denoted by Δ, the simplex category or the simplicial category is the category with linearly ordered sets $[n] = \{0, 1, 2, \cdots, n\}$ as its objects and order-preserving maps between them as its morphisms. That is, for a map $\varphi\colon [m] \to [n]$ we have that $0 \leq \varphi(i) \leq \varphi(j) \leq n$ for each $0 \leq i \leq j \leq m$.

We denote by δ^i the elementary face operator $[n-1] \to [n]$ and by σ^i the elementary degeneracy operator $[n+1] \to [n]$ given by

$$\delta^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}, \quad \sigma^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

[1] It should be noted that in other sources "∞-categories" might refer to other models than the one we use.

Remark 1.1.2. All morphisms in Δ are finite compositions of such morphisms.

Definition 1.1.3. We define the category sSet$:=$ Fun$(\Delta^{op}, \text{Set})$ (also denoted Set$_\Delta$ by Lurie) of simplicial sets as Set-valued presheaves on Δ, i.e. functors $\Delta^{op} \to \text{Set}$.

Let $X \in$ sSet. We will denote by X_n the set $X([n])$ of n-simplices (also called n-cells) of X. We define the standard n-simplex as $\Delta^n := y([n])$ where y is the Yoneda embedding, meaning Δ^n is the presheaf $\text{Hom}_\Delta(-, [n])$. By the Yoneda lemma $\text{Hom}_{\text{Fun}(\Delta^{op},\text{Set})}(\Delta^n, X) \simeq X_n$, so we can identify each simplex $x \in X_n$ with a map $x\colon \Delta^n \to X$. This application of the Yoneda lemma is a crucial part of the theory of simplicial sets and we will more often than not consider n-simplices of a simplicial set X as maps of simplicial sets instead. Observe, moreover, that composition with the elementary face operator gives us a map $\Delta^{n-1} \to \Delta^n$.

Definition 1.1.4. For a simplicial set X, we define the face and degeneracy maps

$$d_i := X(\delta^i) : X_n \to X_{n-1}, \quad s_i := X(\sigma^i) : X_n \to X_{n+1}$$

where both maps are given by composition with δ^i and σ^i respectively.

Example 1.1.5. The standard 0-simplex $\Delta^0 := \text{Hom}(-, [0])$ is a terminal object in sSet, meaning it maps any $[m] \in \Delta$ to a singleton. This is usually just referred to as the point and denoted $*$.

Example 1.1.6 ([**kerodon**]). Let $X \in$ sSet and suppose we have subsets $T_n \subseteq X_n$ for every $n \geq 0$ such that $d_i(T_n) \subseteq T_{n-1}$ and $s_i(T_n) \subseteq T_{n+1}$. Then the collection $\{T_n\}_{n\geq 0}$ is a simplicial set we will call a simplicial subset $T \subseteq X$.

Definition 1.1.7. We define the boundary $\partial\Delta^n$ of Δ^n as the simplicial set

$$(\partial\Delta^n)_m = (\partial\Delta^n)([m]) := \{\alpha \in \text{Hom}_\Delta([m], [n]) |\ [n] \not\subseteq \text{im}(\alpha)\}.$$

Observe that $\partial\Delta^0 = \emptyset$ because every map $[m] \to [0]$ is surjective.

Definition 1.1.8. For $0 \leq i \leq n$, we define the horn Λ^n_i as the simplicial set

$$(\Lambda^n_i)_m = (\Lambda^n_i)([m]) := \{\alpha \in \text{Hom}_\Delta([m], [n]) |\ \delta^i[n] \not\subseteq \text{im}(\alpha)\}.$$

Observe that the horn is inside the boundary. We usually refer to Λ^n_i as the ith horn in Δ^n and we will call the horns such that $0 < i < n$ the inner horns.

Example 1.1.9. We define the topological n-simplex:

$$\Delta^n_{\text{Top}} := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} |\ \sum_i x_i = 1, x_i \geq 0\},$$

and for a topological space X we define its singular complex $\text{Sing}(X)$ as the simplicial set with cells

$$[n] \mapsto \text{Hom}_{\text{Top}}(\Delta^n_{\text{Top}}, X).$$

Example 1.1.10. We define the nerve $\mathrm{N}(\mathcal{C})$ of a 1-category $\mathcal{C}$ by

$$\mathrm{N}(\mathcal{C}) := \mathrm{Hom}_{\mathsf{Cat}}([-], \mathcal{C})$$

where we view the sets $[n]$ as categories (posets with a map i to j whenever $i \leq j$). Observe that for any order-preserving morphism $\alpha : [m] \to [n]$ we get a map

$$\mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C}) \xrightarrow{-\circ\alpha} \mathrm{Hom}_{\mathsf{Cat}}([m], \mathcal{C})$$

and it is clear that the nerve is a simplicial set with $\mathrm{N}(\mathcal{C})_n = \mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C})$.
Observe furthermore that for a functor $F : \mathcal{C} \to \mathcal{D}$ we get a simplicial map $\mathrm{N}(F) : \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ by sending n-cells $\varphi : [n] \to \mathcal{C}$ in $\mathrm{N}(\mathcal{C})_n$ to n-cells $F(\varphi) \colon [n] \to \mathcal{D}$ in $\mathrm{N}(\mathcal{D})_n$, so the construction is functorial. It should also be clear that the set of objects of $\mathcal{C}$ is identified with the 0-cells $\mathrm{N}(\mathcal{C})_0$ and the morphisms with the 1-cells $\mathrm{N}(\mathcal{C})_1$. Additionally, the 2-cells $\mathrm{N}(\mathcal{C})_2$ is in bijection with the set of composable pairs of morphisms in $\mathcal{C}$ and likewise the n-cells are the strings of n composable morphisms. We will talk more about composition of morphisms in the next section.

Definition 1.1.11. We define the connected components $\pi_0 X$ of a simplicial set X as the colimit of X.

This is equivalently the quotient of X_0 by the equivalence relation on X_0 generated by the relation $x \sim y$ if and only if there exists an edge $f \in X_1$ such that $f_0 = x$ and $f_1 = y$.

1.2 ∞-categories

Before we give a precise definition, we will take a closer look at the nerve construction. Clearly, we want the nerve of a 1-category to give us an ∞-category, and most of this thesis will revolve around nerves of certain poset-categories of topological spaces. Nerves of categories are not just any ordinary simplicial sets but simplicial sets with some more structure inherited from the underlying 1-category. For instance, 1-categories have composition of morphisms. Take for example

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

in some ordinary 1-category $\mathcal{C}$. This diagram gives us a morphism $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ of simplicial sets, but in $\mathcal{C}$ the maps f and g can be composed to a morphism $h : X \to Z$ which in turn gives a unique way to extend the simplicial map $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ to a map $\Delta^2 \to \mathrm{N}(\mathcal{C})$. If we instead look at the outer horns Λ^2_0 and Λ^2_2, we will not necessarily have a way to extend morphisms to Δ^2 in general. For example the diagram

$$\begin{tikzcd} X \arrow[dr, "\mathrm{id}_X"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

gives a map $\Lambda^2_2 \to \mathrm{N}(\mathcal{C})$, but extending this to a morphism $\Delta^2 \to \mathrm{N}(\mathcal{C})$ would amount to finding a right inverse to g, which of course is not something we can always do in general, unless $\mathcal{C}$ was a groupoid. This property of extending a morphism from a horn to the standard n-simplex is sometimes also called filling the horn, and we will see that it is a defining property for ∞-categories. In fact, the existence of horn fillings completely classifies those simplicial sets which are nerves of categories:

Proposition 1.2.1 ([**HTT**]). *Let $X \in$ sSet. Then the following conditions are equivalent:*

1. *There exists a small category $\mathcal{C}$ with an isomorphism $X \simeq \mathrm{N}(\mathcal{C})$.*
2. *Every inner horn $\Lambda^n_i \to X$ of X can be filled in a unique way. Or, in other words, for any solid diagram as below, there is a unique dotted arrow making it commute:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

□

Simplicial sets which admits extensions for all horn inclusions are called Kan complexes:

Definition 1.2.2. A simplicial set X is a Kan complex if it satisfies the following condition: For $0 \leq i \leq n$, any map $\sigma_0 \colon \Lambda^n_i \to X$ can be extended to a map $\sigma \colon \Delta^n \to X$.

Example 1.2.3. The singular complex $\mathrm{Sing}(X)$ of a topological space is a Kan complex.

Proposition 1.2.4. *Nerves of groupoids are Kan complexes.*

Proof. The idea of the proof is that all morphisms are invertible, so all horns can be filled. □

As we saw in the example of a map $\Lambda^2_2 \to \mathrm{N}(\mathcal{C})$ above, whenever there's non-invertible morphisms around some outer horns will be impossible to fill. This motivates the definition of an ∞-category. The following definition is due to Boardman and Vogt [**BoardmanVogt**] who defined weak Kan complexes as simplicial sets satisfying what they called the restricted Kan condition: [2]:

Definition 1.2.5 (Boardman and Vogt [**BoardmanVogt**])**.** A simplicial set X is an ∞-category if it satisfies the following condition: For $0 < i < n$, any map $\sigma_0 \colon \Lambda^n_i \to X$ can be extended to a map $\sigma \colon \Delta^n \to X$.

This means that any Kan complex is an ∞-category, and in particular, so is $\mathrm{Sing}(X)$ for a topological space X. Additionally, observe that the nerve $\mathrm{N}(\mathcal{C})$ of an ordinary category $\mathcal{C}$ is an ∞-category. Because the nerve functor is fully faithful (see example **??**), many authors choose to omit its notation altogether. In many ways, ∞-categories behave similarly to ordinary categories, and we will often

[2] Maybe more commonly known as the weak Kan condition.

write about them almost as if they were ordinary categories instead. For example, we will use the terminology of ordinary category theory and refer to the vertices and edges of our simplicial sets as objects and morphisms in our ∞-categories. There are however some obvious differences between ordinary categories and ∞-categories which need adressing before adopting complete 1-categorical language. For example, and perhaps most crucially, we have higher-level maps given by simplices of dimension $n \geq 2$. While we have seen that nerves of categories admit unique horn extensions, this condition is dropped for general ∞-categories, and hence composition of morphisms in an ∞-category is not necessarily unique but rather unique up to homotopy. Before we can make this precise, we must define what we mean by homotopy.

1.2.1 Homotopy

Definition 1.2.6 ([**kerodon**]). Let $\mathcal{C}$ be an ∞-category and $f, g \colon X \to Y$ morphisms in $\mathcal{C}$. We define a homotopy between f and g as a 2-simplex $\sigma \in \mathcal{C}$ with boundary specified by $d_0(\sigma) = \mathrm{id}_Y, d_1(\sigma) = g$ and $d_2(\sigma) = f$ as illustrated in the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "\mathrm{id}_Y"] \\ & Y. & \end{tikzcd}$$

We say f and g are homotopic if such a homotopy σ exists.

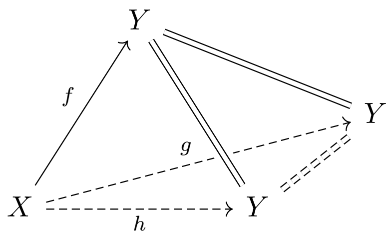
Example 1.2.7. For a 1-category $\mathcal{C}$ two morphisms $f, g \colon X \to Y$ in $\mathcal{C}$ are homotopic in $\mathrm{N}(\mathcal{C})$ if and only if $f = g$.

Proposition 1.2.8 ([**kerodon**]). *Let $\mathcal{C}$ be an ∞-category and X, Y objects of $\mathcal{C}$. Then homotopy is an equivalence relation on the collection of morphisms from X to Y in $\mathcal{C}$.*

Proof. Let $f \colon X \to Y$ be a map in $\mathcal{C}$.
First observe that reflexivity follows from considering the degenerate 2-simplex $s_1(f)$:

$$\begin{tikzcd} & Y \arrow[dr, equal] & \\ X \arrow[ur, "f"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

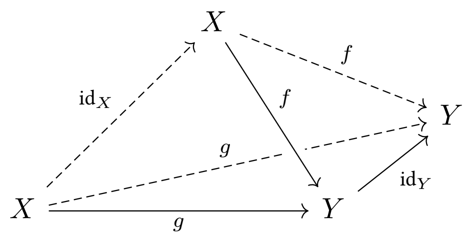
which is a homotopy from f to f. Let us now consider three maps $f, g, h \colon X \to Y$ and let σ_2 be a homotopy between f and h, σ_3 a homotopy between f and g, and let σ_0 witness the degenerate 0-face. Then we can picture the map $\tau_0 \colon \Lambda^3_1 \to \mathcal{C}$ induced by the tuple $(\sigma_0, -, \sigma_2, \sigma_3)$ with the following diagram:



where the dashed lines represent the boundary of the "missing" face. Now, since $\mathcal{C}$ is an ∞-category we can fill in this face and extend τ_0 to a 3-cell $\tau\colon \Delta^3 \to \mathcal{C}$. Observe now that the face $d_1(\tau)$ gives a homotopy between g and h, so we have shown transitivity.
Finally setting $f = h$ in the above diagram shows homotopy is symmetric since f is always homotopic to itself. $\square$

Proposition 1.2.9 (**[kerodon]**). *Two maps $f, g\colon X \to Y$ in an ∞-category $\mathcal{C}$ are homotopic if and only if they are homotopic as morphisms in $\mathcal{C}^{op}$.*

Proof. One must show that demanding the existence of a 2-cell $\sigma \in \mathcal{C}_2$ such that $d_0(\sigma) = \mathrm{id}_Y, d_1(\sigma) = g$ and $d_2(f)$ is equivalent to demanding the existence of a 2-cell $\tau \in \mathcal{C}_2$ such that $d_0(\tau) = f, d_1(\tau) = g$ and $d_2(\tau) = \mathrm{id}_X$.
Let f be homotopic to g. By the symmetry of the homotopy relation there exists a homotopy σ from g to f. Then the tuple $(\sigma, s_1(g), -, s_0(g))$ yields a map $\rho_0\colon \Lambda^3_2 \to \mathcal{C}$ as follows:



where once again the dashed lines represent the boundary of the "missing" face from Λ^3_2. We fill in to get a 3-cell $\rho\colon \Delta^3 \to \mathcal{C}$ with $d_2(\rho) = \tau$ as "demanded" above. The other direction is very similar and like most of these proofs it comes down to drawing the correct horn. $\square$

Now that we know what it means for morphisms of ∞-categories to be homotopic, we can define a composition of morphisms.

Definition 1.2.10 (**[kerodon]**). Let $\mathcal{C}$ be an ∞-category with morphisms

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

We define h to be a composition of f and g if there exists some 2-simplex $\sigma \in \mathcal{C}$ such that $d_0(\sigma) = g, d_1(\sigma) = h$ and $d_2(\sigma) = f$. We say σ witnesses h as a composition of f and g and we will use the usual notation $h = g \circ f$.

Observe that we have only defined composition up to homotopy. We make this precise in the following proposition:

Proposition 1.2.11 ([**kerodon**]). *Let $\mathcal{C}$ be an ∞-category with morphisms f and g as follows:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Then there exists a composition h of f and g and any other morphism $X \to Z$ is a composition of f and g if and only if it is homotopic to h. □

Furthermore, compositions respect homotopy in the following sense:

Proposition 1.2.12 ([**kerodon**]). *Let $\mathcal{C}$ be an ∞-category with homotopic morphisms $f \sim f' \colon X \to Y$ and $g \sim g' \colon Y \to Z$. Let $h = g \circ f$ and $h' = g' \circ f'$. Then h is homotopic to h'.*

Remark 1.2.13. The nerve construction preserves compositions in the sense that for a 1-category $\mathcal{C}$ with morphisms f, g as above, there is a unique morphism $h \colon X \to Z$ in $\mathrm{N}(\mathcal{C})$ which is given by $f \circ g$ in $\mathcal{C}$.

One can show that the nerve construction $\mathrm{Cat} \xrightarrow{\mathrm{N}} \mathrm{sSet}$ admits a left adjoint h and moreover that the counit of this adjunction is an isomorphism which in turn means that the nerve is fully faithful. This can be shown directly and the interested reader can see for example [**Rezk**] or [**kerodon**] for proofs. We now construct h directly, but only on ∞-categories, but we will delay the proof of the adjunction to chapter **??** to illustrate the usefulness of Kan extensions.

TODO: Look at Rune's comment.

Analogously to the construction of the fundamental groupoid $\pi_{\leq 1}(X)$ of a topological space X, we can construct the homotopy category $\mathrm{h}\mathcal{C}$ of an ∞-category $\mathcal{C}$.

Preben: Writing about the fundamental gruopoid could be smart.

Definition 1.2.14. Let $\mathcal{C}$ be an ∞-category. We denote by $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ homotopy classes of morphisms $X \to Y \in \mathcal{C}$ and for a morphism $f \colon X \to Y$ in $\mathcal{C}$, we denote by $[f]$ its equivalence class in $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$.

Proposition 1.2.15 ([**kerodon**]). *We have a unique composition of morphisms*

$$\circ \colon \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Z)$$

such that $[h] = [f] \circ [g]$ for any $h = f \circ g \in \mathcal{C}$. This composition law is both

1. *associative in the sense that any triple $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{C}$ yields an equivalence*

$$([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f]) \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(W, Z).$$

2. *unital in the sense that for any $X \in \mathcal{C}$ the homotopy class $[\mathrm{id}_X]$ of the identity on X is a two-sided identity with respect to the composition law. In other words, for every $W \xrightarrow{f} X$ and every $X \xrightarrow{g} Y$ in $\mathcal{C}$, we have $[id_X] \circ [f] = [f]$ and $[g] \circ [id_X] = [g]$.*

TODO: Place the following in the correct place: This means that $([h] \circ [g]) \circ [f] = [w]$ and $[h] \circ ([g] \circ [f]) = [h] \circ [u]$, so it remains to show that $[w] = [h] \circ [u]$.

Proof. The existence of the composition law follows directly from the previous two propositions. To prove 1. we pick compositions $u = g \circ f$, $v = h \circ g$ and $w = v \circ f$. Choosing 2-cells $\sigma_0, \sigma_2, \sigma_3$ witnessing the compositions $v = g \circ h, u = g \circ f$ and $w = v \circ f$, respectively, yields a map $\Lambda_1^3 \to \mathcal{C}$ as depicted in the following diagram:

$$\begin{tikzcd}
 & X \arrow[rr, "v"] \arrow[d, "g"] & & Z \\
W \arrow[ur, "f"] \arrow[r, dashed, "u"'] \arrow[urrr, dashed, "w"] & Y \arrow[urr, dashed, "h"'] & &
\end{tikzcd}$$

where the dashed lines represent the "missing" 2-cell. Since $\mathcal{C}$ is an ∞-category we can extend this map to a 3-cell $\Delta^3 \to \mathcal{C}$ essentially "filling" in the missing 2-cell witnessing the desired composition $w = h \circ u$.

To prove 2. pick $X \in \mathcal{C}$ and maps $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ and observe that the degenerate 2-cells with boundaries as in the following diagrams:

$$\begin{tikzcd}
 & X \arrow[dr, "g"] & \\
X \arrow[ur, equal, "\mathrm{id}_X"] \arrow[rr, "g"'] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dr, equal, "\mathrm{id}_X"] & \\
W \arrow[ur, "f"] \arrow[rr, "f"'] & & X
\end{tikzcd}$$

witness the compositions $g \circ \mathrm{id}_X = g$ and $\mathrm{id}_X \circ f = f$. $\square$

We can now define the homotopy category $\mathrm{h}\mathcal{C}$ of an ∞-category $\mathcal{C}$.

Definition 1.2.16. Let $\mathcal{C}$ be an ∞-category. Then we define $\mathrm{h}\mathcal{C}$ to be the 1-category with objects of $\mathcal{C}$ as its objects and homotopy classes of morphisms as defined in **??** as its morphisms. The previous proposition provides identity morphisms $[\mathrm{id}_X]$ for any object $X \in \mathcal{C}$ and a composition law satisfying the axioms for being a 1-category.

Example 1.2.17.

1. $\mathrm{h}\Delta^n = [n] \simeq \{0 < 1 < \cdots < n\}$ and in general $\mathrm{hN}(\mathcal{C}) \cong \mathcal{C}$.
2. For a topological space X one can identify $\mathrm{hSing}(X)$ with $\pi_{\leq 1}(X)$.

1.2.2 Isomorphims

Definition 1.2.18. Let $\mathcal{C}$ be an ∞-category and let $X \xrightarrow{f} Y$ in $\mathcal{C}$. We say f is an isomorphism if $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$. Isomorphims are also often called equivalences.

Example 1.2.19. Let $\mathcal{C}$ be a 1-category. A morphism in $\mathcal{C}$ is an isomorphism if and only if it is an isomorphism in $\mathrm{N}(\mathcal{C})$.

Definition 1.2.20. An ∞-groupoid is an ∞-category such that $\mathrm{h}\mathcal{C}$ is a groupoid, or in other words an ∞-category where every morphism is an isomorphism.

Example 1.2.21. Every Kan complex K is an ∞-groupoid because every horn can be filled and filling the horns $\Lambda^2_0 \to K$ and $\Lambda^2_1 \to K$ yields inverses for any morphisms in K.

As one should maybe expect, this works the other way around as well; ∞-groupoids are Kan complexes. Thankfully, this is true, but it is a non-trivial and technical theorem which is the main focus of [**Joyal**]. For a proof, see [**Joyal**] or [**Rezk**]. Inspired by [**Groth**] we can write the following commutative diagram of fully faithful functors:

$$\begin{tikzcd}
\mathrm{Grpd} \arrow[r] \arrow[d, "\mathrm{N}"'] & \mathrm{Cat} \arrow[d, "\mathrm{N}"] \arrow[dr, "\mathrm{N}"] & \\
\mathrm{Kan} = \mathrm{Grpd}_\infty \arrow[r] & \mathrm{Cat}_\infty \arrow[r] & \mathrm{sSet}
\end{tikzcd}$$

Definition 1.2.22.

TODO: The subcategory of a ∞-category

Definition 1.2.23. The core of a 1-category is the subcategory consisting of only the isomorphisms.

Definition 1.2.24. The core of an ∞-category $\mathcal{C}$ is the ∞-groupoid $\mathcal{C}^{\simeq}$ (also written $\mathcal{C}^{\mathrm{core}}$ by some authors) obtained as the subcategory of $\mathcal{C}$ corresponding to the core of $\mathrm{h}\mathcal{C}$.

We will say two objects in an ∞-category are isomorphic whenever there exists an isomorphism between them. Furthermore being isomorphic is an equivalence relation on the objects of an ∞-category which means we can sensibly speak of isomorphism classes.

Preben: Rezk mentions $\pi_0(\mathcal{C}^{\mathrm{core}})$. Do we care?

In addition to having a notion of isomorphism or equivalence of objects in an ∞-category we would like a notion of natural isomorphism or equivalence of ∞-categories, but first we define the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors between ∞-categories.

1.2.3 Mapping spaces

For ordinary 1-categories $\mathcal{C}$ and $\mathcal{D}$ we can create the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with functors its objects and natural transformations as its morphisms. We want to create an ∞-categorical analogue:

Definition 1.2.25. Let $X, Y \in \mathrm{sSet}$. We define $\mathrm{Fun}(X, Y)$ by $\mathrm{Fun}(X, Y)_n := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times X, Y)$.

If σ is some map $[m] \to [n]$ in Δ, the induced map

$$\sigma^* : \mathrm{Fun}(X, Y)_n \to \mathrm{Fun}(X, Y)_m$$

is defined by

$$(X \times \Delta^n \xrightarrow{f} Y) \mapsto (X \times \Delta^m \xrightarrow{\mathrm{id}_X \times \sigma} X \times \Delta^n \xrightarrow{f} Y).$$

In particular, this means that $\mathrm{Fun}(X, Y)_0$ is precisely the set of maps between the simplicial sets X and Y. Observe that Fun defines a functor $\mathrm{sSet}^{op} \times \mathrm{sSet} \to \mathrm{sSet}$ and for each n it is clear that we have a bijection between $\mathrm{Hom}(\Delta^n \times X, Y)$ and $\mathrm{Hom}(\Delta^n, \mathrm{Fun}(X, Y))$. Furthermore, we can extend the bijection to any simplicial set:

Proposition 1.2.26 (**[Rezk]**). *Let $X, Y, Z \in \mathrm{sSet}$, then there is a bijeciton*

$$\mathrm{Hom}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Fun}(Y, Z)). \qquad \square$$

This proposition yields a natural isomorphism of simplicial sets $\mathrm{Fun}(X \times Y, Z) \cong \mathrm{Fun}(X, \mathrm{Fun}(Y, Z))$.

It can be shown that applying the same construction to ∞-categories $\mathcal{C}$ and $\mathcal{D}$ yields a new ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with maps of simplicial sets as objects (0-cells) and natural transformations as morphisms (1-cells)

TODO: mention that these are maps $\mathcal{C} \times \Delta^1 \to \mathcal{D}$

. Proving this is indeed an ∞-category uses machinery due to Joyal that I will not introduce in this text. See for example [**Rezk**] or the proof of [**HTT**] on p.94. This allows us to define an equivalence between ∞-categories:

Definition 1.2.27. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $f, g \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ functors between them. We will say that a natural transformation φ between f and g is a natural isomorphism or a natural equivalence if it is an equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 1.2.28. Let $f \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be a functor of ∞-categories. Then f is a categorical equivalence if there exists a functor $g \in \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ and natural equivalences between gf and $\mathrm{id}_{\mathcal{C}}$ and between $\mathrm{id}_{\mathcal{D}}$ and fg.

Proposition 1.2.29 ([Rezk]). *Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary 1-categories. Then* $\mathrm{N}(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \simeq \mathrm{Fun}(\mathrm{N}(\mathcal{C}),\mathrm{N}(\mathcal{D}))$.

Proof. We will show they are the same on the level of n-cells for all n. First use $\Delta^n = \mathrm{N}([n])$ and that the nerve preserves finite products to observe the following:

$$\begin{aligned} \mathrm{Fun}(\mathrm{N}(\mathcal{C}),\mathrm{N}(\mathcal{D}))_n :&= \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times \mathrm{N}(\mathcal{C}),\mathrm{N}(\mathcal{D})) && (1.1)\\ &= \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{N}([n]) \times \mathrm{N}(\mathcal{C}),\mathrm{N}(D)) && (1.2)\\ &= \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{N}([n] \times \mathcal{C}),\mathrm{N}(D)) && (1.3)\end{aligned}$$

Finally, we use fully-faithfulness of the nerve to get

$$\begin{aligned} \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{N}([n]\times\mathcal{C}),\mathrm{N}(\mathcal{D})) &\cong \mathrm{Hom}_{\mathrm{Cat}}([n]\times\mathcal{C},\mathcal{D}) && (1.4)\\ &\cong \mathrm{Hom}_{\mathrm{Cat}}([n],\mathrm{Fun}(\mathcal{C},\mathcal{D})) && (1.5)\\ &= (\mathrm{N}(\mathrm{Fun}(\mathcal{C},\mathcal{D})))_n. && (1.6)\end{aligned}$$

□

Definition 1.2.30. Let $\mathcal{C}$ be an ∞-category with objects X and Y. Then we define the mapping space $\mathrm{Map}_{\mathcal{C}}(X,Y)$ as the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1,\mathcal{C}) \arrow[d] \\ \{(X,Y)\} \arrow[r] & \mathcal{C}\times\mathcal{C} \end{tikzcd}$$

which, in fact, is an actual Kan complex[3].

For a proof that this is always a Kan complex, see [**Rezk**]. It is worth mentioning that we only define the mapping space up to homotopy so there are other useful ways to model it that we are not mentioning here.

Definition 1.2.31. We say a functor $f \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$ of ∞-categories is essentially surjective if the functor $\mathrm{h}f \in \mathrm{Fun}(\mathrm{h}\mathcal{C},\mathrm{h}\mathcal{D})$ is essentially surjective.

Definition 1.2.32. We say a functor $f \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$ of ∞-categories is fully faithful if the functor $\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{D}}(f(X),f(Y))$ is an equivalence for all $X,Y \in \mathcal{C}$.

Charles Rezk names the following the "fundamental theorem" of ∞-categories (and he names the corresponding result for 1-categories the "fundamental theorem of category theory" though he mentions Yoneda's lemma might be more deserving of that name).

[3]It is named mapping space because of the convention among many authors to call Kan complexes and ∞-groupoids spaces.

Theorem 1.2.33 ([**Rezk**]). *A functor $f: \mathcal{C} \to \mathcal{D}$ of ∞-categories is a categorical equivalence if and only if it is fully faithful and essentially surjective.* □

We can also define a weaker notion of equivalence between simplicial sets.

Definition 1.2.34. We say a map $f: X \to Y$ of simplicial sets is a weak homotopy equivalence if for any space Z, the induced functor

$$\mathrm{Fun}(f, Z): \mathrm{Fun}(Y, Z) \to \mathrm{Fun}(X, Z)$$

is a categorical equivalence.

An immmediate observation is that any categorical equivalence between ∞-categories X and Y is also a weak homotopy equivalence, but the converse is not necessarily true:

Example 1.2.35. $\Delta^0 \hookrightarrow \Delta^1$ is a weak homotopy equivalence but not a categorical equivalence.

1.2.4 Adjunctions

We give an ∞-categorical definition of adjoint functors.

Definition 1.2.36 ([**kerodon**]). Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. We say that a pair of natural transformations $\eta: \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\varepsilon: F \circ G \to \mathrm{id}_{\mathcal{D}}$ are compatible up to homotopy if the following conditions are satisfied:

1. The identity isomorphism $\mathrm{id}_F: F \to F$ is a composition of natural transformations:

$$F = F \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{\mathrm{id}_F \circ \eta} F \circ G \circ F \xrightarrow{\varepsilon \circ \mathrm{id}_F} \mathrm{id}_{\mathcal{D}} \circ F = F.$$

2. The identity isomorphism $\mathrm{id}_G: G \to G$ is a composition of natural transformations:

$$G = \mathrm{id}_{\mathcal{D}} \circ G \xrightarrow{\eta \circ \mathrm{id}_G} G \circ F \circ G \xrightarrow{\mathrm{id}_G \circ \varepsilon} G \circ \mathrm{id}_{\mathcal{D}} = G.$$

Definition 1.2.37. We say that a natural transformation $\eta: \mathrm{id}_{\mathcal{C}} \to G \circ F$ is the unit of an adjunction if there exists a natural transformation $\varepsilon: F \circ G \to \mathrm{id}_{\mathcal{D}}$ which is compatible with η up to homotopy. We say that a natural transformation $\varepsilon: F \circ G \to \mathrm{id}_{\mathcal{D}}$ is the counit of an adjunction if there exists a natural transformation $\eta: \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is compatible with ε up to homotopy.

Definition 1.2.38. Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. We say that F is a left adjoint to G, and G a right adjoint to F, if there exists a natural transformation $\eta: \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is the unit of an adjunction between F and G.

Example 1.2.39. An equivalence between ∞-categories is both a left and a right adjoint.

Example 1.2.40. A composition of two left/right adjoints is again a left/right adjoint.

It can be shown that this notion of adjunctions is compatible with the 1-categorical notion of adjunctions in the sense that a natural transformation $\eta\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ is the unit of a 1-categorical adjunction if and only if its nerve is the unit of an ∞-categorical adjunction. For $\mathcal{C}$ and $\mathcal{D}$ 1-categories one can equivalently define an adjunction as a pair of functors $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ such that there is a bijection $\mathrm{Hom}_{\mathcal{D}}(L(-), -) \simeq \mathrm{Hom}_{\mathcal{C}}(-, R(-))$. Similarly, it can be shown that a pair of functors between ∞-categories is an adjunction if and only if we have a bijection of mapping spaces:

Proposition 1.2.41 ([**kerodon**])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural transformation. Then η is the unit of an adjunction between F and G if and only if, for any pair of objects $X \in \mathcal{C}, Y \in \mathcal{D}$, the following composition is an isomorphism of ∞-groupoids:*

$$\mathrm{Map}_{\mathcal{D}}(F(X), Y) \xrightarrow{G} \mathrm{Map}_{\mathcal{C}}((G \circ F)(X), G(Y)) \xrightarrow{\circ[\eta_X]} \mathrm{Map}_{\mathcal{C}}(X, G(Y)).$$

Chapter 2

Limits and Colimits

This chapter aims to give ∞-categorical versions of limits and colimits. To accomplish this, some machinery is required, and we define most of the important notions necessary as well as stating without proofs the central results of the theory.

2.1 Joins and slices

In this section we will introduce the join and slice constructions. We will start with a recollection of what these constructions are in the case of ordinary 1-categories before defining the appropriate ∞-categorical notions. For most people, at least for me, the slice construction is very familiar while the join maybe not so much. These two constructions will ultimately give us a way to talk about the right notions of limits and colimits in the world of ∞-categories.

Definition 2.1.1. Let $\mathcal{C}$ be a 1-category and $C \in \mathcal{C}$. The slice category, or over-category, $\mathcal{C}_{/C}$ is the category with arrows $C' \to C$ in $\mathcal{C}$ as objects and commutative triangles in $\mathcal{C}$ as its morphisms. The coslice category, or under-category, $\mathcal{C}_{C/}$ is the category with arrows $C \to C'$ in $\mathcal{C}$ as objects and commutative triangles in $\mathcal{C}$ as its morphisms.

Remark 2.1.2. We have pullbacks

$$\begin{tikzcd}
\mathcal{C}_{C/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],\mathcal{C}) \arrow[d, shift right, "\mathrm{ev}_0"'] \arrow[d, shift left, "\mathrm{ev}_1"] & \mathcal{C}_{/C} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\{C\} \arrow[r] & \mathcal{C} & \{C\} \arrow[l]
\end{tikzcd}$$

where $\mathrm{ev}_0 \colon \mathrm{Fun}([1],\mathcal{C}) \to \mathrm{Fun}(\{0\},\mathcal{C}) \simeq \mathcal{C}$ is evaluation at 0 and $\mathrm{ev}_1 \colon \mathrm{Fun}([1],\mathcal{C}) \to \mathrm{Fun}(\{1\},\mathcal{C}) \simeq \mathcal{C}$ is evaluation at 1.

The above remark says that we can identify the slice and coslice categories with fibers of the evaluation functors ev_0 and ev_1 and we will use this idea to define the notion of slicing over (and under) diagrams.

Definition 2.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be 1-categories. For each $C \in \mathcal{C}$, we let $\underline{C}\colon \mathcal{D} \to \mathcal{C}$ denote the constant functor sending each $D \in \mathcal{D}$ to C and each morphism to id_C. For each functor $F\colon \mathcal{D} \to \mathcal{C}$ we denote by $\mathcal{C}_{/F}$ the fiber product

$$\begin{tikzcd} \mathcal{C}_{/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{D},\mathcal{C})_{/F} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathrm{Fun}(\mathcal{D},\mathcal{C}) \end{tikzcd}$$

where the bottom arrow is given by $C \mapsto \underline{C}$. Dually, we denote by $\mathcal{C}_{F/}$ the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D},\mathcal{C})_{F/}$. Here $\mathrm{Fun}(\mathcal{D},\mathcal{C})_{/F}$ and $\mathrm{Fun}(\mathcal{D},\mathcal{C})_{F/}$ are simply the slice and coslice categories of definition **??**.

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be 1-categories. We define the join $\mathcal{C} \star \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ as the category with $\mathrm{ob}\mathcal{C} \coprod \mathrm{ob}\mathcal{D}$ as its objects and for objects X, Y morphisms given by:

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Y) := \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \text{if } X, Y \in \mathcal{C}, \\ \mathrm{Hom}_{\mathcal{D}}(X,Y) & \text{if } X, Y \in \mathcal{D}, \\ \{*\} & \text{if } X \in \mathcal{C}, Y \in \mathcal{D}, \\ \emptyset & \text{if } X \in \mathcal{D}, Y \in \mathcal{C}, \end{cases}$$

with composition defined such that $\mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D} \hookleftarrow \mathcal{D}$ are functors.

Remark 2.1.5. These inclusions are isomorphisms to full subcategories of the join. It is usual to abuse notation a bit and identify $\mathcal{C}$ and $\mathcal{D}$ with these subcategories.

Remark 2.1.6. The functor $- \star \mathcal{D}\colon \mathrm{Cat} \to \mathrm{Cat}_{/\mathcal{D}}$ is left adjoint to the slice functor that takes a functor $f\colon \mathcal{D} \to \mathcal{C}$ to $\mathcal{C}_{/f}$. Dually, the functor $\mathcal{C} \star -\colon \mathrm{Cat} \to \mathrm{Cat}_{\mathcal{C}/}$ is left adjoint to the slice functor that takes a functor $g\colon \mathcal{C} \to \mathcal{D}$ to $\mathcal{D}_{f/}$.

Example 2.1.7. Maybe the most important examples of joins, at least in this text, are the left and right cone of a category. Letting $[0]$ denote the category with one object and one morphism, we denote by $\mathcal{C}^{\triangleleft} := [0] \star \mathcal{C}$ the left cone of a 1-category $\mathcal{C}$ and $\mathcal{C}^{\triangleright} := \mathcal{C} \star [0]$ the right cone of $\mathcal{C}$. In practice, the right cone of $\mathcal{C}$ is the category obtained by adjoining an additional object X_0 to $\mathcal{C}$ and for every $C \in \mathcal{C}^{\triangleright}$ a unique morphism $C \to X_0$ so that X_0 becomes terminal in $\mathcal{C}^{\triangleright}$. Dually, the left cone is obtained by adjoining an additional object which becomes initial in $\mathcal{C}^{\triangleleft}$.

The usefulness of cones materializes when considering limits and colimits. Lurie denotes the category of functors extending F to the cones by $\mathrm{Fun}_F(\mathcal{C}^{\triangleright}, \mathcal{D}) := \{G \in \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D}) |\, G|_{\mathcal{C}} = F\}$ and $\mathrm{Fun}_F(\mathcal{C}^{\triangleleft}, \mathcal{D}) := \{G \in \mathrm{Fun}(\mathcal{C}^{\triangleleft}, \mathcal{D}) |\, G|_{\mathcal{C}} = F\}$ and colimits and limits of F can be identified with initial and terminal objects in $\mathrm{Fun}_F(\mathcal{C}^{\triangleright}, \mathcal{D})$ and $\mathrm{Fun}_F(\mathcal{C}^{\triangleleft}, \mathcal{D})$ respectively.

We now define the join of two simplicial sets.

Definition 2.1.8. Let $X, Y \in$ sSet and let Δ^+ be the full subcategory of ordered sets obtained by adding the empty set $[-1] := \emptyset$ to Δ. We define the join $X \star Y$ on n-cells:

$$(X \star Y)_n := \coprod_{[n]=[n_1]\sqcup[n_2]} X_{n_1} \times Y_{n_2},$$

where $[n_1], [n_2] \in \Delta_+$ and $\sqcup : \Delta_+ \times \Delta_+ \to \Delta_+$ is the ordered disjoint union. That is, $[p] \sqcup [q] = [p+1+q]$. We allow $[n_1]$ and $[n_2]$ to be -1 with $[-1] = \emptyset$ and $X_{-1} = * = Y_{-1}$.

Remark 2.1.9. This is the left Kan extension of the product $X \times Y$ along the ordered disjoint union.

TODO: See Rune's comment.

Example 2.1.10. We denote by $X^{\triangleleft}$ the left cone $\Delta^0 \star X$ and by $X^{\triangleright}$ the right cone $X \star \Delta^0$.

One can show that the nerve of 1-categories commutes with joins in the sense that $\mathrm{N}(\mathcal{C} \star \mathcal{D}) = \mathrm{N}(\mathcal{C}) \star \mathrm{N}(\mathcal{D})$, which in particular means that the nerve commutes with the cone constructions:

$$\mathrm{N}(\mathcal{C}^{\triangleright}) \cong \mathrm{N}(\mathcal{C})^{\triangleright} \quad \text{and} \quad \mathrm{N}(\mathcal{C}^{\triangleleft}) \cong \mathrm{N}(\mathcal{C})^{\triangleleft}.$$

Furthermore, it can be shown that the join of ∞-categories is again an ∞-category. See for example [**kerodon**] or [**Rezk**]. To define limits and colimits for ∞-categories we will need ∞-categorical versions of the slice functors introduced above and we do this by finding right adjoints to the functors $X \star - \colon \mathrm{sSet} \to \mathrm{sSet}_{X/}$ and $- \star X \colon \mathrm{sSet} \to \mathrm{sSet}_{X/}$.

Proposition 2.1.11. *For $X \in$ sSet, the functors $X \star -$ and $- \star X$ preserve colimits.*

Proof. Writing out definition **??**, we get

$$(X \star Y)_n = X_n \coprod (X_{n-1} \times Y_0) \coprod \cdots \coprod (X_0 \times Y_{n-1}) \coprod Y_n,$$

which means that cell-wise we have a functor to $\mathrm{Set}_{X_n/}$ and here colimits and products commute. □

Consequently, we can find right adjoints going from $\mathrm{sSet}_{X/}$ to sSet which we call slice functors. For a map $f \colon X \to Y$ and a simplicial set K, we have bijections

$$\mathrm{Hom}\left(K, Y_{f/}\right) \cong \mathrm{Hom}_{X/}(X \star K, Y)$$

and

$$\mathrm{Hom}\left(K, Y_{/f}\right) \cong \mathrm{Hom}_{X/}(K \star X, Y).$$

These universal properties will serve as our definitions, but we can define slices by what they do on n-cells by considering the special case $K = \Delta^n$:

$$(Y_{f/})_n \cong \mathrm{Hom}_{\mathrm{sSet}_{X/}}(X \star \Delta^n, X) \quad \text{and} \quad (Y_{/f})_n \cong \mathrm{Hom}_{\mathrm{sSet}_{X/}}(\Delta^n \star X, Y).$$

Important examples arise when considering the map $y\colon \Delta^0 \to Y$ giving descriptions of the slices $Y_{y/}$ and $Y_{/y}$:

$$(Y_{y/})_n = \{\sigma\colon \Delta^{n+1} \to Y | \sigma(0) = y\},$$
$$(Y_{/y})_n = \{\sigma\colon \Delta^{n+1} \to Y | \sigma(n+1) = y\}.$$

As one might expect, the nerve of 1-categories commutes with taking slices as it does for joins, i.e. for a functor $F : \mathcal{C} \to \mathcal{D}$ between 1-categories, we have

$$\mathrm{N}(\mathcal{D}_{F/}) \cong \mathrm{N}(\mathcal{D})_{\mathrm{N}(F)/} \quad \text{and} \quad \mathrm{N}(\mathcal{D}_{/F}) \cong \mathrm{N}(\mathcal{D})_{/\mathrm{N}(F)}.$$

Furthermore, as one might expect — or at least desire — the slices of an ∞-category are again ∞-categories; see for example [**Rezk**] for a proof.

2.2 Initial and terminal objects

We now have what we need to define initial and terminal objects of ∞-categories.

Definition 2.2.1. Let $\mathcal{C}$ be an ∞-category. An object $c \in \mathcal{C}$ is initial if for all $n \geq 1$, every $f\colon \partial\Delta^n \to \mathcal{C}$ such that $f|_{\{0\}} = c$ can be extended to a map $f'\colon \Delta^n \to \mathcal{C}$. Dually, a terminal object is an object c' such that for all $n \geq 1$, every $f\colon \partial\Delta^n \to \mathcal{C}$ such that $f|_{\{n\}} = c'$ can be extended to a map $f'\colon \Delta^n \to \mathcal{C}$. Equivalently initial and terminal objects are objects such that we can always solve the respective extension problems:

$$\begin{tikzcd} \{0\} \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, "c"] & \partial\Delta^n \arrow[r] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad \begin{tikzcd} \{n\} \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, "c'"] & \partial\Delta^n \arrow[r] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

Remark 2.2.2. It can be shown that initial and terminal objects are invariant under isomorphisms.

In his notes, Rezk shows that the slice categories give alternate definitions of initial and terminal objects:

Proposition 2.2.3 ([**Rezk**][1])**.** *Let $\mathcal{C}$ be an ∞-category and X an object in $\mathcal{C}$. Then*

1. *X is initial if and only if the projection $\mathcal{C}_{X/} \to \mathcal{C}$ is a categorical equivalence.*
2. *X is terminal if and only if the projection $\mathcal{C}_{/X} \to \mathcal{C}$ is a categorical equivalence.* $\square$

In ordinary category theory we define initial and terminal objects by contractibility of Hom-sets and after defining the mapping space between two ∞-categories one would hope for a ∞-categorical analogue of this classification.

[1]Rezk shows that the maps are trivial fibrations rather than categorical equivalences but instead of introducing fibrations we remedy this by referring to [**Rezk**] which states that isofibrations that are also categorical equivalences are trivial fibrations and claim the projections in question are isofibrations. This is essentially [**Rezk**].

Definition 2.2.4. We say an ∞-category is contractible if it is categorically equivalent to Δ^0.

This definition leads us to the desired result:

Proposition 2.2.5 ([**Rezk**]). *Let $\mathcal{C}$ be an ∞-category and X an object in $\mathcal{C}$. Then*

1. *X is initial if and only if $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is contractible for any object $Y \in \mathcal{C}$.*
2. *X is terminal if and only if $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is contractible for any object $Y \in \mathcal{C}$.* □

While initial and terminal objects in 1-categories are unique up to unique isomorphism, initial and terminal objects in ∞-categories are unique up to equivalence in the sense that the full subcategories spanned by the initial and terminal objects are either empty or contractible. Furthermore, any object isomorphic to an initial (terminal) object is itself initial (terminal).

2.3 Limits and colimits

Now we can finally define limits and colimits.

Definition 2.3.1. Let $\mathcal{C}$ be an ∞-category and $f\colon K \to \mathcal{C}$ a map of simplicial sets. A colimit of f is an initial object in $\mathcal{C}_{f/}$, and a limit of f is a terminal object in $\mathcal{C}_{/f}$.

This means that a colimit of $f\colon K \to \mathcal{C}$ is a map $\bar{f}$ such that we can always solve the following extension problem

$$\begin{tikzcd}
\{0\} \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, "\bar{f}"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{f/} \\
& \Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Recalling that we defined the slice of a simplicial set as right adjoint to the join, we know that maps $\{0\} \to \mathcal{C}_{f/}$ are in bijection with maps $K \star \Delta^0 = K^{\triangleright} \to \mathcal{C}$. In summary, the adjunction tells us that a colimit of f is a map $\bar{f}\colon K^{\triangleright} \to \mathcal{C}$ extending f such that we can always solve the following extension problem

$$\begin{tikzcd}
K \star \{0\} \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, "\bar{f}"] & K \star \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\
& K \star \Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Dually, a limit of f is a map $\bar{f}\colon K^{\triangleleft} \to \mathcal{C}$ such that we can always solve the following extension problem

$$\begin{tikzcd}
\{n\} \star K \arrow[r, hook] \arrow[rr, bend left, "\bar{f}"] \arrow[dr] & \partial\Delta^n \star K \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^n \star K \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

Lurie [**HTT**] refers to the object colim $f \in \mathcal{C}_{f/}$ as a colimit of f and the map $\bar{f}\colon K^{\triangleright} \to \mathcal{C}$ as a colimit diagram. We will probably be lazy and not care about this distinction. It follows immediately that limits and colimits are unique in the same sense that initial and terminal objects are: the full subcategory spanned by the (co)limits is either empty or contractible. When saying something is unique in ∞-category theory it is usually this notion we mean; it is unique up to a contractible space of choices.

Remark 2.3.2. As the slice and join constructions commute with the nerve construction, so do initial and terminal objects and hence also limits and colimits.

2.4 Cofinality

A major part of the proofs in this thesis will revolve around calculating certain (co)limits and in particular certain Kan extensions. Following Lurie we will very often do these calculations by using cofinality of certain maps to replace our diagrams with simpler diagrams without changing the (co)limit. We start this section by defining the notion of cofinal maps between ∞-category and state some results that will turn out to be very important in proving our main theorems.

Definition 2.4.1. A map $f\colon X \to Y$ of simplicial sets is a right fibration if the following extension problem can be solved for any $0 < i \leq n$, and a left fibration if it can be solved for any $0 \leq i < n$:

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists"'] & Y
\end{tikzcd}$$

Definition 2.4.2. We say a simplicial set is weakly contractible if it is weakly homotopy equivalent to Δ^0.

Definition 2.4.3. [**kerodon**] Let $p\colon S \to Y \in$ sSet. We say p is colimit-cofinal/limit-cofinal [2] if, for any right/left fibration $X \to Y$, precomposition with p induces a homotopy equivalence

$$\mathrm{Fun}_{/Y}(Y, X) \xrightarrow{\simeq} \mathrm{Fun}_{/Y}(S, X).$$

Proposition 2.4.4 ([**HTT**]). *Let $f\colon X \to Y$ be a map of simplicial sets. Then the following conditions are equivalent:*

[2] The naming conventions here are highly confusing and different author's have adopted similar but different conventions. Lurie himself use left cofinal in [**HA**] for colimit-cofinal but changed it to right cofinal in [**kerodon**]. We take the approach of for instance

1. *The map f is colimit-cofinal.*
2. *For any ∞-category $\mathcal{C}$ and functor $p\colon Y \to \mathcal{C}$ the induced map $\mathcal{C}_{f/} \to \mathcal{C}_{(p\circ f)/}$ is a categorical equivalence.*
3. *For any ∞-category $\mathcal{C}$ and colimit $\overline{p}\colon Y^{\triangleright} \to \mathcal{C}$, the induced map $\overline{(p\circ f)}\colon X^{\triangleright} \to \mathcal{C}$ is a colimit.* □

Corollary 2.4.4.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then the following conditions are equivalent:*

1. *F is colimit-cofinal,*
2. *for any ∞-category $\mathcal{E}$ and functor $G\colon \mathcal{D} \to \mathcal{E}$, the colimit $\operatorname{colim}_{\mathcal{D}} G$ exists if and only if the colimit $\operatorname{colim}_{\mathcal{C}} GF$ exists, and when they exist they are equivalent in $\mathcal{E}$;*

and the following conditions are equivalent:

1. *F is limit-cofinal,*
2. *for any ∞-category $\mathcal{E}$ and functor $G\colon \mathcal{D} \to \mathcal{E}$, the limit $\lim_{\mathcal{D}} G$ exists if and only if the limit $\lim_{\mathcal{C}} GF$ exists, and when they exist they are equivalent in $\mathcal{E}$.* □

This corollary makes it clear that cofinal maps are very useful for calculating (co)limits, but the definition is not the easiest to work with when determining whether a map is cofinal or not. The following theorem due to Joyal remedies this by giving a very convenient way of checking cofinality.

Theorem 2.4.5. *[**kerodon**] Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map of simplicial sets, where $\mathcal{D}$ is an ∞-category. Then the following conditions are equivalent:*

1. *The functor f is colimit-cofinal,*
2. *for every $D \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible;*

and the following conditions are equivalent:

1. *The functor f is limit-cofinal,*
2. *for every $D \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible.* □

This means that checking cofinality of functors reduces to checking cofinality of certain simplicial sets. In the next part of this thesis we will mostly consider functors from certain poset categories and this means that we can use the theory of filtered ∞-categories to simplify the process of checking contractibility. While much can be said about filtered ∞-categories we only state the definition and the results we need and refer the reader to [**kerodon**] and [**HTT**] for more details and proofs.

Definition 2.4.6 ([**kerodon**]). We say that a non-empty 1-category $\mathcal{C}$ is filtered if it satisfies the following conditions:

1. For any pair of objects X and Y in $\mathcal{C}$, there exists an object $Z \in \mathcal{C}$ and a pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$.

2. For any pair of objects X and Y in $\mathcal{C}$ and parallel pair of morphisms $X \overset{f_1}{\underset{f_0}{\rightrightarrows}} Y$, there exists a morphism $g\colon Y \to Z$ such that $g \circ f_0 = g \circ f_1$.

Definition 2.4.7. We say that an ∞-category $\mathcal{C}$ is filtered if, for any finite $K \in$ sSet any $f\colon K \to \mathcal{C}$ can be extended to $K^{\triangleright} \to \mathcal{C}$. Dually, we say it is cofiltered if $\mathcal{C}^{op}$ is filtered (we can extend to $K^{\triangleleft} \to \mathcal{C}$).

Proposition 2.4.8 ([**kerodon**])**.** *Filtered ∞-categories are weakly contractible.* □

Proposition 2.4.9 ([**kerodon**])**.** *A 1-category $\mathcal{C}$ is filtered if and only if its nerve $\mathrm{N}(\mathcal{C})$ is filtered.* □

Remark 2.4.10. In the case of partially ordered sets there are no parallel arrows, so a partially ordered set is filtered if and only if it is directed, that is; every finite subset has an upper bound.

2.5 Stable ∞-categories

Quoting Lurie [**HA**], "The theory of stable ∞-categories can be regarded as an axiomatization of the essential features of stable homotopy theory: most notably, that fiber sequences and cofiber sequences are the same." For the purpose of this thesis (proving Verdier Duality), we really don't need much more of the theory than the fact that pullbacks squares are always also pushout squares, but nevertheless we think it useful to state the basic theory and examples of the categories we care about.

Definition 2.5.1 ([**HA**])**.** A zero-object, usually denoted 0, in an ∞-category is an object that is both initial and terminal. A pointed ∞-category is an ∞-category containing a zero-object.

Remark 2.5.2. An object $0 \in \mathcal{C}$ is a zero-object if $\mathrm{Map}_{\mathcal{C}}(X, 0)$ and $\mathrm{Map}_{\mathcal{C}}(0, X)$ are contractible for every object $X \in \mathcal{C}$, and such an object is determined up to equivalence (since we saw that initial and terminal objects are unique up to a contractible space).

Remark 2.5.3 ([**HA**]). Let $\mathcal{C}$ be a pointed ∞-category. For any $X, Y \in \mathcal{C}$, the natural morphism

$$\mathrm{Map}_{\mathcal{C}}(X, 0) \times \mathrm{Map}_{\mathcal{C}}(0, Y) \xrightarrow{0} \mathrm{Map}_{\mathcal{C}}(X, Y)$$

has contractible domain, which means that we have a well-defined zero-morphism $X \to Y$ in $\mathrm{h}\mathcal{C}$.

Definition 2.5.4 ([**HA**])**.** Let $\mathcal{C}$ be a pointed ∞-category. We define a triangle in $\mathcal{C}$ to be a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$, depicted as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{2.1}$$

where 0 is a zero-object of $\mathcal{C}$.

Definition 2.5.5. We say that a triangle is a fiber sequence if it is a pullback square and a cofiber sequence if it is a pushout square.

Definition 2.5.6 ([**HA**]). For a morphism $g\colon Y \to Z$ in a pointed ∞-category $\mathcal{C}$, we say that a fiber of g is a fiber sequence as depicted in **??**. For a morphism $f\colon X \to Y$ in a pointed ∞-category $\mathcal{C}$, we say that a cofiber of f is a cofiber sequence as depicted in **??**. We will abuse terminology by writing $X = \mathrm{fib}(g)$ and $Z = \mathrm{cofib}(f)$.

Remark 2.5.7 ([**HA**]). The functor $\mathrm{cofib}\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ can be identified with a left adjoint to the left Kan extension [3] functor $\mathcal{C} \simeq \mathrm{Fun}(\{1\}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ which associates a zero morphism $0 \to X$ to each object $X \in \mathcal{C}$. This means that cofib preserves all colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$.

We now define stable ∞-categories:

Definition 2.5.8 ([**HA**]). We say that an ∞-category is stable if it satisfies the following conditions:

1. It is pointed.
2. Any morphism in $\mathcal{C}$ has a fiber and a cofiber.
3. A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Proposition 2.5.9. *Stable ∞-categories have finite limits and colimits.*

Proof. Let $\mathcal{C}$ be a stable ∞-category. By [**HTT**] it is enough to show that $\mathcal{C}$ has pushouts and pullbacks. We will only consider pushouts, but the argument is the same. Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd} \mathrm{fib}(g) \arrow[r, "f"] \arrow[d] & X \arrow[r, "h"] \arrow[d, "g"] & Z \arrow[d] \\ 0 \arrow[r] & Y \arrow[r] & \mathrm{cofib}(hf) \end{tikzcd}$$

By stability of $\mathcal{C}$, both the left and outer squares are both pushouts and pullbacks. Lemma [**HTT**] says that if the left square is a pushout, then the right square is also a pushout if and only the outer square is a pushout, so we are done. ☐

In fact, an even stronger statement can be made:

Proposition 2.5.10 ([**HTT**]). *Let $\mathcal{C}$ be a pointed ∞-category. Then $\mathcal{C}$ is stable if and only if the following conditions are satisfied:*

1. *$\mathcal{C}$ admits finite limits and colimits.*
2. *A square is a pushout if and only if it is a pullback.* ☐

The fact that pushouts are also pullbacks in stable ∞-categories will be helpful later when proving Verdier Duality.

We see that, like for additive 1-categories, stability is a property of ∞-categories rather than additional data [**HA**][Remark 1.1.1.14.]. While additive categories are often presented as categories equipped

[3] We define Kan extensions in chapter **??**.

with an abelian group structure on Hom-sets, this structure is determined by the underlying category. A similar structure can be found in stable ∞-categories: they are weakly enriched over what is known as the ∞-category Sp of spectra. We will not go into the details of this comment, but we will at least define Sp as it is maybe the most important example of a stable ∞-category.

Example 2.5.11 ([**HA**]). We say that a spectrum consists of an infinite sequence $\{X_i\}_{i\geq 0}$ of pointed topological spaces, together with homeomorphisms $X_i \simeq \Omega X_{i+1}$, where Ω is the loop space functor $\Omega(X) := \mathrm{fib}(0 \to X)$. The collection of such spectra can be organized into a stable ∞-category Sp and this category is in some sense the universal example of a stable ∞-category.

Example 2.5.12 ([**HA**]). For an abelian category $\mathcal{A}$ one can construct a stable ∞-category $\mathcal{D}(\mathcal{A})$ such that its homotopy category $h\mathcal{D}(\mathcal{A})$ can be identified with the classical notion of the derived category of $\mathcal{A}$ from homological algebra.

Theorem 2.5.13 ([**HA**]). *Let $\mathcal{C}$ be a stable ∞-category. Then $h\mathcal{C}$ is a triangulated category.* □

Chapter 3

Kan Extensions

We will start with a detour into the world of ordinary 1-categories. In the classic [**MacLane**], Saunders MacLane writes, "The notion of Kan Extensions subsumes all the other fundamental concepts of category theory". In her introduction to category theory [**CatContext**], Emily Riehl devotes a whole chapter to the slogan "All concepts are Kan extensions". Ubiquitous in the toolbox of any category theorist, Kan extensions are central to almost everything we do in this thesis. As we have seen, a lot of ∞-categorical concepts can be thought of as if we are working with ordinary 1-categories, and we will therefore start by defining Kan extensions in ordinary categories.

3.1 Kan extensions in 1-categories

Let $\mathcal{C}$ be an ordinary 1-category and $i\colon I \to J$ a functor of 1-categories. Composing with i now yields a "pullback" functor $i^*\colon \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ and whenever $\mathcal{C}$ is a category with nice properties, like having small limits and colimits, we can find left and right adjoints to i^* which will become what we call left and right Kan extensions along i. A nice special case of this construction, which hints at the slogan "all concepts are Kan extensions," arises when considering $J = *$, the category with one object. Now, the pullback i^* is just the diagonal functor, and it is well known that colimits and limits can be described as left and right adjoints to the diagonal functor, so Kan extensions are, in some sense, generalizations of the notions of limits and colimits. In this thesis's final part, we will mostly consider the opposite special case in which i is not the projection to a point but rather the inclusion of a full subcategory.

Definition 3.1.1 ([**CatContext**])**.** Given functors F and K as in the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, dashed, "K_!F"'] & \end{tikzcd}$$

a left Kan extension of F along K is a functor $K_!F\colon \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta\colon F \Rightarrow K_!F \circ K$ such that for any other pair $G\colon \mathcal{D} \to \mathcal{E}, \gamma\colon F \Rightarrow G \circ K$, γ factors uniquely through η as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\swArrow\gamma"] & \mathcal{E} \\ \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\swArrow\eta"] & \mathcal{E} \\ \mathcal{D} \arrow[ur, bend left, "K_!F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\seArrow\exists!"] & \end{tikzcd}$$

Dually, a right Kan extension of F along K is a functor $K_*F\colon \mathcal{D} \to \mathcal{E}$ with a natural transformation $\epsilon\colon K_*F \circ K \Rightarrow F$ such that any functor $G\colon \mathcal{D} \to E$ and any natural transformation $\delta\colon G \Rightarrow F$, δ factors uniquely through ϵ.

The following result justifies the choice to denote left and right Kan extensions by lower shriek and star:

Proposition 3.1.2 ([**CatContext**]). *Let K be a functor $\mathcal{C} \to \mathcal{D}$ and $\mathcal{E}$ some category. If the left and right Kan extensions of any functor F along K exists, these define left and right adjoints to the pre-composition functor $K^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and by uniqueness of adjoints, any left or right adjoint to the pre-composition functor yields left or right Kan extensions.*

While the description of Kan extensions as certain adjoints is useful, there is usually another description available to us that turns out to be even more useful. Whenever $\mathcal{C}$ and $\mathcal{D}$ are respectively small and locally small and $\mathcal{E}$ has certain limits and colimits, potential Kan extensions of functors $\mathcal{C} \to \mathcal{E}$ along $K\colon \mathcal{C} \to \mathcal{D}$ exist and are what we call pointwise Kan extensions. For a functor $K\colon \mathcal{C} \to \mathcal{D}$, [**MacLane**] denotes by $d \downarrow K$ the category $\mathcal{C} \times_{\mathcal{C}} \mathcal{D}_{d/}$ and we will choose to denote it by $K_{d/}$. Likewise the category $K \downarrow d = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d}$ will be denoted $K_{/d}$. These categories come with projection functors Π_d and Π^d, respectively, which send the objects $(c, d \to Kc)$ and $(c, Kc \to d)$ to the object $c \in \mathcal{C}$. The following theorem gives a formula for calculating certain left and right Kan extensions as colimits and limits.

Theorem 3.1.3 ([**CatContext**]). *Let $\mathcal{D} \xleftarrow{K} \mathcal{C} \xrightarrow{F} \mathcal{E}$ be functors. If the following colimit exists for every object $d \in D$, then it defines the left Kan extension $K_!F$:*

$$K_!F(d) := \mathrm{colim}(K_{/d} \xrightarrow{\Pi^d} \mathcal{C} \xrightarrow{F} \mathcal{E})$$

and the unit transformation $\eta\colon F \to K_!F \circ K$ can be extracted from the colimit cone. Dually, if the following limit exists for every object $d \in D$, then it defines the right Kan extension K_*F:

$$K_*F(d) := \lim(K_{d/} \xrightarrow{\Pi_d} \mathcal{C} \xrightarrow{F} \mathcal{E})$$

and the counit transformation $\epsilon\colon K_*F \circ K \to F$ can be extracted from the limit cone.

Proof. Proofs can be found in [**CatContext**] and [**MacLane**] □

When $\mathcal{D}$ and $\mathcal{E}$ are locally small, we call Kan extensions that can be calculated by the colimit and limit formulae above pointwise Kan extensions. In [**CatContext**], Riehl says the consensus among category theorists is that the important Kan extensions are the pointwise Kan extensions and quotes [**Kelly**]: "Our present choice of nomenclature is based on our failure to find a single instance where a weak Kan extension plays any mathematical role whatsoever." This thesis is no different; we will only care about pointwise Kan extensions from here on out. We will see that there are analogous limit formulae for Kan extensions in ∞-categories which are central to most of the proofs in this thesis. Before we extend the theory of Kan extensions from ordinary categories to the world of ∞-categories, we will consider some important examples, but first observe that Theorem **??** gives the following immediate consequence:

Corollary 3.1.3.1. *If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then for a functor $K\colon \mathcal{C} \to \mathcal{D}$ we have:*

1. *If $\mathcal{E}$ is cocomplete, left Kan extensions of functors $\mathcal{C} \to \mathcal{E}$ along K exist and are given by the colimit formula of Theorem* **??**.
2. *If $\mathcal{E}$ is complete, right Kan extensions of functors $\mathcal{C} \to \mathcal{E}$ along K exist and are given by the limit formula of Theorem* **??**. □

Example 3.1.4. Let i_n be the functor given by restricting along the inclusion $i\colon \Delta_{\leq n} \hookrightarrow \Delta$. Since Set is both cocomplete and complete, we have both left and right Kan extensions:

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, "i_n^*" description] & \mathrm{sSet}_{\leq n} \arrow[l, bend right, "(i_n)_!"'] \arrow[l, bend left, "(i_n)_*"] \end{tikzcd}$$

where $\mathrm{sSet}_{\leq n} := \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathrm{Set})$.

Lemma 3.1.5 (Kan extension along fully faithful functors)**.** *Let $K\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then, up to natural isomorphism, any pointwise Kan extension along K defines an actual on the nose extension.*

Proof. Observe that for $c \in \mathcal{C}$, $\mathcal{C}_{/c} \simeq K_{/K(c)}$ is an equivalence of categories because K is fully faithful, so we can calculate the left Kan extension $K_!F$ on objects by the colimit formula:

$$K_!F(K(c)) = \mathrm{colim}(K_{/K(c)} \simeq \mathcal{C}_{/c} \xrightarrow{\Pi} \mathcal{C} \xrightarrow{F} \mathcal{E}).$$

Since the identity on c is terminal in $\mathcal{C}_{/c}$ the colimit reduces to evaluation at the terminal object $K(c) \xrightarrow{id} K(c)$ in $K_{/Kc}$, so $\eta_c\colon F(c) \cong K_!F(K(c))$ is an isomorphism. The proof for pointwise right Kan extensions is completely dual. □

Example 3.1.6. Let $p\colon J \to \mathcal{C}$ be a functor and i the inclusion $J \hookrightarrow J^{\triangleleft}$. The right Kan extension of p along i has to be equal to $p(x)$ for any $x \in J$, so the only interesting part is the Kan extension at the cone point where the formula just gives a limit over J. This means that we have $p_*(x) = \lim_J p$ and $p_!(x) = \mathrm{colim}_J\, p$.

Example 3.1.7 (Yoneda extension). Let $\mathcal{C}$ be small, $\mathcal{E}$ locally small and cocomplete. By the corollary above, any functor $F\colon \mathcal{C} \to \mathcal{E}$ admits a left Kan extension $y_!F$ along the Yoneda embedding y and since y is fully faithful the unit $F \cong y_!F \circ y$ is an isomorphism. In fact, $y_!F$ has a right adjoint R, defined on objects by

$$R(e)\colon = \mathrm{Hom}_{\mathcal{E}}(F(-), e)\colon \mathcal{C}^{op} \to \mathrm{Set}.$$

The full proof showing that this is in fact right adjoint to $y_!F$ can be found in [**CatContext**] from which we have taken this example.

The process of left Kan extending along the Yoneda embedding is called Yoneda extension in [**CatsSheaves**] and it provides lots of interesting examples of Kan extensions. We will look at a couple examples in the special case $\mathcal{C} = \Delta$.

Example 3.1.8. Let $\Delta_{\mathrm{Top}}\colon \Delta \to \mathrm{Top}$ be the functor known as the standard topological n-simplex:

$$[n] \mapsto \Delta^n_{\mathrm{Top}} := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | \sum_i x_i = 1, x_i \geq 0\}.$$

By the discussion of Yoneda extension above we have an adjoint pair $(y_!\Delta_{\mathrm{Top}}, R)$ where the right adjoint is given by

$$R(e) = \mathrm{Hom}_{\mathrm{Top}}(\Delta_{\mathrm{Top}}(-), e)$$

and this is what is known as the total singular complex functor of example **??**, also written Sing.

Example 3.1.9 (The Nerve construction). Let F be the embedding $\Delta \hookrightarrow \mathrm{Cat}$. Yoneda extension yields an adjoint pair $(y_!F, R)$ where the right adjoint R is given by

$$R(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}(F(-), \mathcal{C}).$$

Recall from Example **??** that this is the definition of the nerve of $\mathcal{C}$. The left adjoint $y_!F$ is the homotopy category h of a simplicial set. It can be shown that the counit $\mathrm{h}(\mathrm{N}(X)) \to X$ is an isomorphism which implies that the nerve construction is a fully faithful functor.

Yoneda extension produces even more examples of adjunctions. In [**CatContext**], Riehl fixes a topological space X and constructs the inclusion $\mathrm{Open}(X) \to \mathrm{Top}_{/X}$ by sending open subsets $U \subseteq X$ to the inclusion map $U \hookrightarrow X$. Yoneda extension now yields an adjunction

$$\begin{tikzcd} \mathrm{Top}_{/X} \arrow[r, shift right, "\perp"'] & \mathrm{Fun}(\mathcal{U}(X)^{op}, \mathrm{Set}) \arrow[l, shift right] \end{tikzcd}$$

and as Riehl writes, all adjunctions restrict to an equivalence of subcategories which in this case yields the equivalence between the category $\mathrm{Shv}(X)$ of sheaves on X and the category $\mathrm{Et}(X)$ of étale spaces on X.

3.2 Kan extensions for ∞-categories

We now want to generalize the concept of Kan extensions to the ∞-categorical setting. As the theory of Kan extending along inclusions is simpler, we start there, and give a more general definition later.

We will however see that the proof of Verdier duality mostly uses the simpler notion of extending along the inclusion of full subcategories.

Definition 3.2.1. Let $\mathcal{A}$ be an ∞-category with a full subcategory $\mathcal{A}^0$ and $p\colon K \to \mathcal{A}$ a diagram. Following [**HTT**] we write $\mathcal{A}^0_{/p}$ for the fiber product $\mathcal{A}_{/p} \times_{\mathcal{A}} \mathcal{A}^0$. If $A \in \mathcal{A}$, then $\mathcal{A}^0_{/A}$ is the full subcategory of $\mathcal{A}_{/A}$ spanned by the morphisms $A' \to A$ where $A' \in \mathcal{A}^0$.
Analogously $\mathcal{A}^0_{p/}$ denotes $\mathcal{A}_{p/} \times_{\mathcal{A}} \mathcal{A}^0$ and $\mathcal{A}^0_{A/}$ is the full subcategory spanned by morphisms $A \to A'$.

Definition 3.2.2 ([**kerodon**])**.** For a functor $F\colon \mathcal{A} \to \mathcal{C}$ between ∞-categories where $\mathcal{A}$ has a full subcategory $\mathcal{A}^0$, we say F is left Kan extended from $\mathcal{A}^0$ if

$$(\mathcal{A}^0_{/A})^{\triangleright} \hookrightarrow (\mathcal{A}_{/A})^{\triangleright} \xrightarrow{c} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$ for every object $A \in \mathcal{A}$. Here c is the slice contraction morphism of [**kerodon**], i.e. $c|_{\mathcal{A}_{/A}}$ is the projection and $c|_{\Delta^0} = A$. Recalling the adjoint relationship between joins and slices, this is the counit of the adjunction.
Right Kan extensions are opposite to left Kan extensions, i.e. F is right Kan extended from $\mathcal{A}^0$ if

$$(\mathcal{A}^0_{A/})^{\triangleleft} \hookrightarrow (\mathcal{A}_{A/})^{\triangleleft} \xrightarrow{c'} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

is a limit diagram, where c' is the coslice contraction morphism.

Example 3.2.3. The notion of Kan extensions in ∞-categories is compatible with the notion of Kan extensions in 1-categories in the sense that it respects the nerve construction. In other words, a functor of 1-categories $F : \mathcal{C} \to \mathcal{D}$ is left Kan extended from a full subcategory $\mathcal{C}^0$ if and only if $\mathrm{N}(F)\colon \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ is left Kan extended from $\mathrm{N}(\mathcal{C}^0)$.

A very useful result about Kan extensions of full subcategories is their transitive property:

Proposition 3.2.4 (Transitivity of Kan extensions [**HTT**])**.** *For a functor $F\colon \mathcal{A} \to \mathcal{C}$ of ∞-categories where $\mathcal{A}^0 \subseteq \mathcal{A}^1$ are full subcategories of $\mathcal{A}$, if $F|_{\mathcal{A}^1}$ is left Kan extended from $\mathcal{A}^0$, then F is left Kan extended from $\mathcal{A}^1$ if and only if it is left Kan extended from $\mathcal{A}^0$.* □

In the previous section, we defined Kan extensions as functors together with natural transformations satisfying certain universal properties. While there is a similar notion of a universal property for the ∞-categorical Kan extensions, there are some details we have suppressed. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let F_0 be the restriction to some full subcategory C^0 of $\mathcal{C}$ and i be the inclusion $\mathcal{C}^0 \hookrightarrow \mathcal{C}$. Then, if F is left Kan extended from F_0 there certainly is a natural transformation $F_0 \to F \circ i$ "belonging" to this Kan extension as we defined for 1-categories. However, we have chosen to omit it because it is an isomorphism in this case where we Kan extend along the inclusion of a full subcategory. For Kan extensions along general functors, however, this natural transformation is no longer necessarily an isomorphism, so it has to be part of the specified data. Following Lurie's Kerodon, we define the left Kan extension along a general functor of ∞-categories:

Definition 3.2.5 ([**kerodon**])**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Suppose we are given a simplicial set K together with functors $\delta\colon K \to \mathcal{C}$ and $F_0\colon K \to \mathcal{D}$ and a natural transformation

$\beta\colon F_0 \to F \circ \delta$ as illustrated in the following diagram:

$$\begin{tikzcd} K \arrow[r, "F_0"] \arrow[d, "\delta"'] & \mathcal{D} \arrow[dl, phantom, "\Swarrow\beta"] \\ \mathcal{C} \arrow[ur, "F"'] & \end{tikzcd}$$

We say that β exhibits F as a left Kan extension of F_0 along δ if, for any $X \in \mathcal{C}$, we can calculate $F(X)$ as a colimit of the diagram

$$K_{/X} = K \times_{\mathcal{C}} \mathcal{C}_{/X} \to K \xrightarrow{F_0} \mathcal{D}.$$

Remark 3.2.6. Here we have intentionally given a slightly imprecise definition, because the actual definition is unnecesarrily technical for our purposes. Lurie calls this an informal way of stating definition [**kerodon**], and we refer the reader to [**kerodon**] for all the details.

Remark 3.2.7 ([**kerodon**]). This characterizes $F(X)$ up to isomorphism and, in fact; for functors δ and F_0 as in the definition, a left Kan extension is uniquely determined up to isomorphism as an object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Remark 3.2.8. This definition is compatible with the definition for Kan extensions along inclusions; just use the identity natural transformation for β.

As promised, we have universal properties also for ∞-categorical Kan extensions:

Proposition 3.2.9 ([kerodon]). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\delta\colon K \to \mathcal{C}$ and $F_0\colon K \to D$ be functors of simplicial sets and let $\beta\colon F_0 \to F \circ \delta$ be natural transformation exhibiting F as a left Kan extension of F_0 along δ. Then for any $G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the following composition is an equivalence of Kan complexes:*

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{D})}(F \circ \delta, G \circ \delta) \xrightarrow{\circ[\beta]} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{D})}(F_0, G \circ \delta).$$

The description of Kan extensions as adjoints to the restriction functor is now a direct consequence of Proposition **??**:

Corollary 3.2.9.1 ([kerodon]). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\delta\colon K \to \mathcal{C}$ be a functor of simplicial sets. Suppose that any functor $F_0 : K \to \mathcal{D}$ has a left Kan extension along δ. Then the restriction functor*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\circ\delta} \mathrm{Fun}(K, \mathcal{D})$$

has a left adjoint carrying any functor $F_0 : K \to \mathcal{D}$ to a left Kan extension of F_0 along δ.

The theory of right Kan extensions is dual, so we can of course realize right Kan extensions as right adjoints to restriction.

Finally, we note that Lemma **??** also holds for ∞-categories:

Proposition 3.2.10 ([**kerodon**]). *Let $G\colon \mathcal{C}^0 \to \mathcal{C}$, $F_0\colon \mathcal{C}^0 \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories where G is fully faithful. Then:*

- *If $\alpha\colon F \circ G \to F_0$ is a natural transformation exhibiting F as a right Kan extension of F_0 along G, then α is an isomorphism in the ∞-category* $\mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$.
- *If $\beta\colon F_0 \to F \circ G$ is a natural transformation exhibiting F as a left Kan extension of F_0 along G, then β is an isomorphism in the ∞-category* $\mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$.

Chapter 4

Sheaves and K-sheaves

4.1 A little history

The origins of sheaf theory is quite a remarkable story, and we quote Armand Borel [**Borel**]:

> "The Second World War broke out in 1939 and J. Leray [then Professor at the Sorbonne and an officer in the French army] was made prisoner by the Germans in 1940. He spent the next five years in captivity in an officers' camp, Oflag XVIIA in Austria [not far from Salzburg]. With the help of some colleagues, he founded a university there, of which he became the Director ("recteur"). His major mathematical interests had been so far in analysis, on a variety of problems which, though theoretical, had their origins in, and potential applications to, technical problems in mechanics or fluid dynamics. Algebraic topology had been only a minor interest, geared to applications to analysis. Leray feared that if his competence as a "mechanic" ("mécanicien," his word) were known to the German authorities in the camp, he might be compelled to work for the German war machine, so he converted his minor interest to his major one, in fact to his essentially unique one, presented himself as a pure mathematician, and devoted himself mainly to algebraic topology."

The result of this short and involuntary detour into the field of algebraic topology was as Haynes Miller [**H.Miller**] puts it "a spectacular flowering of highly original ideas, ideas which have, through the usual metamorphism of history, shaped the course of mathematics in the sixty years since then." While used to using topological invariants to prove the existence of solutions of certain partial differential equations, he motivated his study of algebraic topology in captivity by seeking methods that applied to a wide class of topological spaces, all in the hope of proving more directly the same kind of theorems he had proved before the war in an indirect manner. In 1942 he announced the first part of his work, and it was approved by Heinz Hopf and published in 1945 with the subtitle "Un cours de topologie algébrique professé en captivité" – "A course in algebraic topology taught in captivity". As with many new discoveries, it took some time before his work was fully appreciated. Part of the reason was that his 1945 papers [**Leray45a2**],[**Leray45b2**] and [**Leray45c2**] according to Armand Borel [**BorelAMS**] "did not seem to go drastically beyond those of mainstream algebraic topology (even though a closer examination would have revealed a novel approach and more general assumptions for a number of familiar results), so [**Leray45a2**],[**Leray45b2**] and [**Leray45c2**] did not create such a big impression."

But Leray was not done. As Gray writes in [**Gray**], "Leray 1946 must be one of the first instances of the French saying that they are going to take what they have done for spaces and generalize it to mappings." While well-known that a continuous map induces a homomorphism in (co)homology, Leray's insight was going to break entirely new ground: in a conversation with A. Weil [**BorelAMS**] he spoke of a homology with "variable coefficients"[1]. It was thus the two subsequent papers by Leray [**Leray46a**] and [**Leray46b**] that first introduced the notions of sheaves, sheaf cohomology and spectral sequences. The impact of these three notions on algebraic topology and homological algebra is hard to understate.

Inspired by [**Alexander**], the fundamental idea was to equip a module with a support function taking its values in the closed subsets of a topological space subject to some properties. In [**Leray46a**], Leray defined a sheaf on a topological space X by associating a module $\mathcal{F}(K)$ over a ring R to each closed subset K of X and a morphism $\mathcal{F}(K) \to \mathcal{F}(K')$ of modules to each inclusion $K' \subseteq K$. A first basic example is the p-th cohomology sheaf assigning $H^p(K; R)$ to K. His definition was not quite what we use today, but this was nevertheless the first time the word "faisceau" was used for anything resembling today's notion. For a more detailed exposition of the original definition and how it evolved in the following years, [**Gray**] gives a very detailed rendition of the origins of sheaf theory.

According to [**Gray**], in the winter of 1947 to 1948, Leray gave a course in algebraic topology, which he wrote up for the proceedings of a 1947 colloquium in Paris and published as [**Leray49**]. The presentation was now much clearer than in the earlier papers, and notions such as subsheaves, quotient sheaves and direct image sheaves were first described. Perhaps more importantly, he describes the notion of a so-called complex and how to construct a sheaf from one. He then defines certain special complexes and calls them "couvertures", and then he uses certain "fine couvertures" to define a notion of a relative cohomology ring on a topological space X. Fine couvertures inspired many related, and perhaps more familiar, notions such as Cartan's "homotopically fine" in [**Cartan_50_4**], Godement's "flasque" (flabby) and "soft" sheaves in [**Godement58**] and Grothendieck's "injective" in [**Tohoku**]. These notions were then used to construct special resolutions of sheaves consisting of sheaves that have trivial (co)homology. Isomorphism theorems and duality theorems (such as Verdier and Poincaré duality) were then proved by reducing to show a certain resolution were for instance, flasque or injective.

It was Cartan that in [**Cartan_50_4**] completely reformulated the theory using open subspaces [**H.Miller**], and the switch to open subsets allowed people like Cartan and Serre to introduce sheaves in several complex variables, in algebraic geometry over $\mathbb{C}$ and over any algebraically closed field [**BorelAMS**]. Here a "faisceau" was now defined to be what we today would call an espace étalé. It was the book [**Godement58**] by functional analyst Godement that "standardized the terminology once and for all" [**Gray**]. The "faisceau" in [**Cartan_50_4**] became espace étalé, a presheaf was defined simply as a contravariant functor on the category of open subsets of a topological space, and a sheaf or "faisceau" became a special kind of presheaf.

[1] We know from a footnote in [**Leray45c2**] that he already thought of this idea while in captivity.

Leray's work was always concerned with sheaf cohomology and spectral sequences for compactly supported cohomology of locally compact topological spaces, and it is fitting that we in this chapter also only concern ourselves with sheaves on locally compact topological spaces. If one appreciates that 1-categories such as Set are also ∞-categories with the properties required in Theorem **??**, the main theorem in this chapter says that the modern notion of a sheaf on a locally compact space using the open subsets is equivalent to a slight modification of Leray's definition using the compact subsets.

4.2 Sheaves on topological spaces

For the theory of sheaves valued in ∞-categories we will closely follow Lurie [**HTT**]. Let $\mathcal{U}(X)$ denote the partial order of open subsets of a topological space X. A presheaf on X is just a contravariant functor with source $\mathcal{U}(X)$. For a 1-category $\mathcal{C}$ admitting all limits we say that a presheaf $\mathcal{F}$ is a sheaf if the diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer for any open cover $\{U_i\}$ of any $U \in \mathcal{U}(X)$. Grothendieck wanted a way of defining a sheaf not just on a topological space, but on "any" category, and he did this by endowing categories with what is now called a Grothendieck topology, and then using essentially the same definition as above. A Grothendieck topology is a category T eqiupped with a set of maps called coverings. We follow Lurie's [**HTT**] and skip right to defining Grothendieck topologies on ∞-categories while noting that the definition is perfectly analogous for 1-categories.[2]

Definition 4.2.1 ([**HTT**])**.** Let $\mathcal{C}$ be an ∞-category. We define a sieve on $\mathcal{C}$ to be a full subcategory $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ where $Y \in C^{(0)}$, X must also belong to $C^{(0)}$. Furthermore, for an object $X \in \mathcal{C}$, we define a sieve on X to be a sieve on $\mathcal{C}_{/X}$ in the above sense.

Definition 4.2.2 ([**HTT**])**.** We define a Grothendieck topology on an ∞-category $\mathcal{C}$ as a collection of sieves called covering sieves on every object $X \in \mathcal{C}$. These collections are required to satisfy the following conditions:

1. For an object $X \in \mathcal{C}$, the sieve $\mathcal{C}_{/X} \subseteq \mathcal{C}_{/X}$ is a covering sieve.
2. For a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and covering sieve $\mathcal{C}^{(0)}_{/X}$ on Y, $f^*\mathcal{C}^{(0)}_{/X}$ is a covering sieve on X.
3. For an object $X \in \mathcal{C}$, a covering sieve $\mathcal{C}^{(0)}_{/X}$ on X, and an arbitrary $\mathcal{C}^{(1)}_{/X}$ sieve on X, if the pullback $f^*\mathcal{C}^{(1)}_{/X}$ is a covering sieve on Y for any $f\colon X \to Y$, then $\mathcal{C}^{(1)}_{/X}$ is a covering sieve on X.

For a topological space X we can equip the poset $\mathcal{U}(X)$ of opens with a Grothendieck topology in which the covering sieves on U are those sieves $U_\alpha \subseteq U$ such that $U = \bigcup_\alpha U_\alpha$.

[2] In fact, the ∞-categorical definition is equivalent whenever we are considering nerves of 1-categories and in general a Grothendieck topology on an ∞-category is the same as a Grothendieck topology on its homotopy category.

Definition 4.2.3 ([HTT]). Let $X \in \mathrm{Top}$ and $\mathcal{C}$ an ∞-category. We define a $\mathcal{C}$-valued sheaf on X to be a presheaf $\mathcal{F}\colon \mathcal{U}(X)^{op} \to \mathcal{C}$ such that for every $U \in \mathcal{U}(X)$ and every covering sieve $\mathcal{W} \subseteq \mathcal{U}(X)_{/U}$, the diagram

$$\mathrm{N}(\mathcal{W})^{\triangleright} \hookrightarrow \mathrm{N}(\mathcal{U}(X)_{/U})^{\triangleright} \to \mathrm{N}(\mathcal{U}(X)) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

is a colimit.

Remark 4.2.4. We will often utilize the fact that this is equivalent to the following limit diagram:

$$\mathrm{N}((\mathcal{W})^{op})^{\triangleleft} \hookrightarrow \mathrm{N}((\mathcal{U}(X)_{/U})^{op})^{\triangleleft} \to \mathrm{N}(\mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

Equivalently one can define sheaves as the presheaves $\mathcal{F}\colon \mathcal{U}(X)^{op} \to \mathcal{C}$ such that for any open cover $\{U_\alpha\}$ of an open set $U \in \mathcal{U}(X)$, the map

$$\mathcal{F}(U) \to \lim_{V} \mathcal{F}(V)$$

is an equivalence in $\mathcal{C}$, where the limit is taken over all open subsets $V \subseteq U$ contained in some U_α.

TODO: Mention the "normal definitions using covers and Cech nerves and briefly discuss relation to classical definition of sheaves in a 1-category.

We write $\mathrm{Presh}(X, \mathcal{C})$ for the ∞-category $\mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ of $\mathcal{C}$-valued presheaves on X and $\mathrm{Shv}(X; \mathcal{C})$ for the full subcategory of $\mathrm{Presh}(X; \mathcal{C})$ spanned by the $\mathcal{C}$-valued sheaves on X.

4.3 Sheaves on locally compact spaces

In this section we will show that for locally compact Hausdorff spaces there is an equivalence of ∞-categories between $\mathrm{Shv}(X; \mathcal{C})$ and $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ where the latter denote so-called $\mathcal{K}$-sheaves and $\mathcal{C}$ is a presentable ∞-category with left exact filtered colimits. These are sheaves defined on the collection of compact subsets instead of the opens. From here on out we generally omit the notation for the nerve construction and assume it is clear from context that we use the nerve construction almost everywhere.

Definition 4.3.1. For a locally compact Hausdorff space X, we write $\mathcal{K}(X)$ for its collection of compact subsets.

Definition 4.3.2. If $K, K' \subseteq X$, we write $K \Subset K'$ if there exists an open subset $U \subseteq X$ between K and K', i.e. $K \subseteq U \subseteq K'$.

Definition 4.3.3. If $K \subseteq X$ is compact, we write $\mathcal{K}_{K\Subset}(X)$ for the set $\{K' \in \mathcal{K}(X) | K \Subset K'\}$ which gives a poset category $\mathcal{K}(X)$.

Definition 4.3.4. A presheaf $\mathcal{F} : \mathcal{K}(X)^{op} \to \mathcal{C}$ is a $\mathcal{K}$-sheaf if it satisfies the following:

1. $\mathcal{F}(\emptyset)$ is terminal.

2. For every pair $K, K' \in \mathcal{K}(X)$, the diagram

$$\begin{tikzcd} \mathcal{F}(K \cup K') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(K) \arrow[d] \\ \mathcal{F}(K') \arrow[r] & \mathcal{F}(K \cap K') \end{tikzcd}$$

is a pullback in $\mathcal{C}$.

3. For each $K \in \mathcal{K}(X)$, $\mathcal{F}(K)$ is a colimit of $\mathcal{F}|\mathcal{K}_{K\Subset}(X)^{op}$.

Definition 4.3.5. We denote the full subcategory of Presh$(\mathcal{K}(X); \mathcal{C})$ spanned by the $\mathcal{K}$-sheaves by Shv$_{\mathcal{K}}(X; \mathcal{C})$.

Remark 4.3.6. We can use finite coverings to define a Grothendieck topology on $\mathcal{K}(X)$ such that condition 1. and 2. are equivalent to the sheaf condition, so in particular a $\mathcal{K}$-sheaf is also a member of Shv$(\mathcal{K}(X); \mathcal{C})$.

Lemma 4.3.7 ([HTT]). *Let X be locally compact and Hausdorff, and let $\mathcal{C}$ be an ∞-category with small limits and colimits and left exact filtered colimits. Let $\mathcal{W}$ be an open cover of X and denote by $\mathcal{K}_{\mathcal{W}}(X)$ the compact subsets of X that are contained in some element of $\mathcal{W}$. Then any $\mathcal{K}$-sheaf $\mathcal{F} \in$ Shv$_{\mathcal{K}}(X; \mathcal{C})$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$.*

Proof. We begin by saying that an open covering $\mathcal{W}$ of X is good if a $\mathcal{K}$-sheaf $\mathcal{F}$ is right Kan extended from the restriction to $\mathcal{K}_{\mathcal{W}}(X)^{op}$. Furthermore, we observe that a covering $\mathcal{W}$ is good if the open sets in K given by $\{K \cap W \mid W \in \mathcal{W}\}$ form a good covering for every $K \in \mathcal{K}(X)$. This means that proving any covering of X is good reduces to showing that any covering of a compact topological space X is good, and hence we can assume the covering has a finite subcovering.
We will use induction on $n \geq 0$. We want to show that if $\mathcal{W}$ is a collection of open subsets of X such that there exists $W_1, \cdots, W_n \in \mathcal{W}$ with $\bigcup_{1 \leq i \leq n} W_i = X$, then $\mathcal{W}$ is a good covering of X. For $n = 0$, we must only show $\mathcal{F}(\emptyset)$ is terminal, but this is given by the definition of $\mathcal{F}$ being a $\mathcal{K}$-sheaf.
By transitivity of Kan extensions (Proposition **??**), if $\mathcal{W} \subseteq \mathcal{W}'$ are two coverings of X such that for every $W' \in \mathcal{W}'$ the covering $\{W \cap W' \mid W \in \mathcal{W}\}$ is a good covering of W', then $\mathcal{W}'$ is a good covering of X if and only if $\mathcal{W}$ also is.
This means that for $n > 0$ it suffices to show that $\mathcal{W}' = \mathcal{W} \cup \{V\}$ is a good covering of X, where $V = \bigcup_{2 \leq i \leq n} W_i$. Observe now that $\mathcal{W}'$ contains W_1 and V which together cover X and using transitivity of Kan extensions once more further reduces the proof to $n = 2$ and showing that $\mathcal{W} = \{W_1, W_2\}$ is a good covering.
By definition of right Kan extensions along inclusions we must show that $\mathcal{F}(K)$ is the limit of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$ for any compact set $K \in \mathcal{K}(X)$. For a compact set $K \in \mathcal{K}(X)$ we define

$$P = \{(K_1, K_2) \in \mathcal{K}(X) \times \mathcal{K}(X) \mid K_1 \subseteq W_1, K_2 \subseteq W_2 \text{ and } K_1 \cup K_2 = K\}$$

and observe that this set is filtered as a poset ordered by inclusion in the sense of **??**, so it is furthermore weakly contractible by Proposition **??**. Denoting an element $\alpha = (K_1, K_2)$ in P, we define $\mathcal{K}_\alpha$ as the set of compacts belonging to either K_1 or K_2 and note that a set $K' \subseteq K$ belongs to $\mathcal{K}_{\mathcal{W}}(X)$ if and only if it belongs to some $\mathcal{K}_\alpha$. Observe also that the inclusion $(K_1 \hookleftarrow K_1 \cap K_2 \hookrightarrow K_2) \hookrightarrow \mathcal{K}_\alpha$ is limit-cofinal by Theorem **??** and hence, by $\mathcal{F}_{\mathcal{K}}$ being a $\mathcal{K}$-sheaf, condition 2. implies $\mathcal{F}(K)$ is a limit of the diagram $\mathcal{F}|_{\mathcal{K}_\alpha^{op}}$ for every $\alpha \in P$. Now, we observe that we can use the theory of cocartesian fibrations to calculate the limit from $K_{\mathcal{W}}(X)^{op}$ as a limit of limits.

Let $p\colon E \to P$ be a cocartesian fibration associated to the functor $P \to \mathrm{Cat}_\infty$ which sends $\alpha \in P$ to the ∞-category $\mathcal{K}_\alpha$. Then it is known that for a functor $\varphi\colon E \to \mathcal{C}$ we can calculate the colimit of φ over E as the colimit of $p_!\varphi$ over P where $p_!\varphi$ is the left Kan extension of φ along p. Moreover, for a cocartesian fibration, we know the map $E_\alpha \to E_{/\alpha}$ is colimit-cofinal. To see how we obtain this map, we identify the fiber E_α of p with $\mathcal{K}_\alpha$ and consider the following diagram:

$$\begin{tikzcd}
K_\alpha \arrow[rr] \arrow[dr, dashed, "\exists!"] \arrow[ddd] & & E \arrow[rr, "\varphi"] \arrow[dr] \arrow[ddd, "p"'] & & \mathcal{C} \\
& E_{/\alpha} \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{K}_{\mathcal{W}}(X) \arrow[ur] & \\
& P_{/\alpha} \arrow[dr] & & & \\
\alpha \arrow[ur] \arrow[rr, hook] & & P \arrow[uuurr, bend right, "p_!\varphi"'] & &
\end{tikzcd}$$

The map $g\colon K_\alpha \to E_{/\alpha}$ is now the obvious map between the two pullback squares. By our definition of left Kan extensions, we calculate $p_!\varphi$ as the following colimit:

$$(p_!\varphi)(\alpha) = \mathrm{colim}\left(E_{/\alpha} \to E \xrightarrow{\varphi} \mathcal{C}\right)$$

and by colimit-cofinality of g we get

$$\mathrm{colim}_{\alpha\in P}\, p_!\varphi \simeq \mathrm{colim}_{\alpha\in P}\, \mathrm{colim}_{\mathcal{K}_\alpha} \mathcal{F} = \mathrm{colim}_{\alpha\in P}\, \mathcal{F}(K),$$

but we have already argued that P is weakly contractible, so the map $P \to \Delta^1$ is colimit-cofinal and we conclude that this colimit is also evaluated as $\mathcal{F}(K)$.

□

Theorem 4.3.8 (**[HTT]**). *Let X be locally compact and Hausdorff and $\mathcal{C}$ an ∞-category with small limits and colimits and left exact filtered colimits. Let $\mathcal{F}\colon (\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \to \mathcal{C}$. The following conditions are equivalent:*

1. *The presheaf $\mathcal{F}_{\mathcal{K}} := \mathcal{F}|_{\mathcal{K}(X)^{op}}$ is a $\mathcal{K}$-sheaf, and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$.*
2. *The presheaf $\mathcal{F}_{\mathcal{U}} := \mathcal{F}|_{\mathcal{U}(X)^{op}}$ is a sheaf, and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$.*

We will split the theorem into a few lemmas for readability and keep the notation in the theorem statement through the rest of the section.

Lemma 4.3.9. *If $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf, then $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$.*

Proof. By definition we want to show that

$$(\mathcal{U}(X)^{op}_{/K})^{\triangleright} \hookrightarrow ((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} (\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$. The assumption that $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf means that for each $K \in \mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}(K)$ is a colimit of $\mathcal{F}|\mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}$. We will "transfer" this colimit to the colimit we want by colimit-cofinal maps

$$(\mathcal{U}(X)^{op})_{/K} \xrightarrow{p} ((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K}) \xleftarrow{p'} \mathcal{K}_{K\Subset}(X)^{op}.$$

Recall that by Theorem **??** checking cofinality reduces to checking weak contractibility of certain simplicial sets. For p we must check that the pullback $\mathcal{U}(X)^{op}_{/K} \times_{(\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{/K}} ((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})_{K'/}$ is weakly contractible for every $K' \in (\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K}$. This is the simplicial set obtained by taking the nerve of the partially ordered set $\{U \in \mathcal{U}(X) \mid K \subseteq U \subseteq K'\}$. By **??** filtered ∞-categories are weakly contractible, and our partially ordered set is filtered as it is nonempty and stable under finite unions, and taking nerves preserve the property of being filtered by **??**. The pullback we must check for p' is given by $\{K'' \in \mathcal{K}(X) \mid K \Subset K'' \subseteq K'\}$ and is weakly contractible by exactly the same argument, and hence p and p' are colimit-cofinal maps. By colimit-cofinality of p and p', the diagram

$$(\mathcal{U}(X)^{op})^{\triangleright}_{/K} \hookrightarrow ((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} (\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram if and only if

$$((\mathcal{K}(X)^{op}_{\Subset K}))^{\triangleright} \hookrightarrow ((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} (\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram, which it is by the assumption that $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf. □

Lemma 4.3.10. *If $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf, and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$, then $\mathcal{F}_{\mathcal{U}}$ is a sheaf.*

Proof. By definition **??** we must show that for every $U \in \mathcal{U}(X)$ and every covering sieve $\mathcal{W}$ covering U,

$$\mathcal{W}^{\triangleright} \hookrightarrow \mathcal{U}(X)^{\triangleright}_{/U} \to \mathcal{U}(X) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

is a colimit diagram, or equivalently that

$$\mathcal{W}^{op,\triangleleft} \hookrightarrow \mathcal{U}(X)^{op,\triangleleft}_{/U} \to \mathcal{U}(X)^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a limit diagram. Let $\mathcal{K}_{\mathcal{W}}(X)$ be the set $\{K \in \mathcal{K}(X) \mid (\exists W \in \mathcal{W}) \text{ with } K \subseteq W\}$. We will once again use cofinality by observing that **??** implies limit-cofinality of the inclusion

$$\mathcal{W} \subseteq \mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X)$$

if and only if, for any $Y \in \mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X)$ the simplicial set $\mathcal{W} \times_{\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X)} \mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X)_{Y/}$ is weakly contractible. This fiber product is equivalent to the nerve of the partially ordered set $\{W' \in \mathcal{W} \mid [\exists W \in$

$\mathcal{W}][K \subseteq W \subseteq Y]\}$ and this is filtered, and hence, the fiber product is weakly contractible by **??** and **??**.[3] Therefore it is enough to show the limit starting from $\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X)^{op}$. By example **??** this is equivalent to showing $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X))^{op,\triangleleft}}$ is right Kan extended from $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X))^{op}}$. By the assumption that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ and the observation that

$$\mathcal{F}(U) = \lim_{K \in \mathcal{K}(X)^{op}_{U/}} \mathcal{F}(K) = \lim_{K \in \mathcal{K}_{\mathcal{W}}(X)^{op}_{U/}} \mathcal{F}(K)$$

we see that $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}})^{op}}$ is a right Kan extension of $\mathcal{F}|_{(K_{\mathcal{W}}(X))^{op}}$. Hence, it suffices to prove that $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. Outside of U this is clear from the fact that $F|_{(\mathcal{W}\cup\mathcal{K}^{op}_{\mathcal{W}})}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. This means we only need to prove $\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. Observe that by assumption

$$\mathcal{F}(U) = \lim_{K \in \mathcal{K}(X)^{op}_{/U}} \mathcal{F}(K) = \lim_{K \in (\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \mathcal{F}(K)$$

so $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}}$ is right Kan extended from $\mathcal{K}(X)^{op}_{/U}$. Lemma **??** tells us that $\mathcal{F}|_{(\mathcal{K}(X)_{/U})^{op}}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. We have $\mathcal{K}_{\mathcal{W}}(X)^{op} \subseteq \mathrm{N}(\mathcal{K}(X)_{/U})^{op} \subseteq (\mathcal{K}(X)_{/U} \cup \{U\})^{op}$, with Kan extensions as in proposition **??**, so we get that $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}}$ is right Kan extended from $(\mathcal{K}_{\mathcal{W}}(X))^{op}$. To summarize, we have the following square of inclusions

$$\begin{tikzcd}
\mathcal{K}_{\mathcal{W}}(X)^{op} \arrow[r, hook, "i"] \arrow[d, hook, "a"] & (\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op} \arrow[d, hook, "b"] \\
\mathcal{K}(X)^{op}_{/U} \arrow[r, hook, "j"] & (\mathcal{K}(X)_{/U} \cup \{U\})^{op}
\end{tikzcd}$$

where $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \simeq b_*(\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}\cup\{U\})^{op}})$ and $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \simeq (j \circ a)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$. We want to show $\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}} \simeq i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$. Since b is fully faithful (it is the inclusion of a full subcategory), we know $b^* b_* \simeq \mathrm{id}$, so we get

$$\begin{aligned}
\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}} &\simeq b^* b_*(\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}) && (4.1)\\
&\simeq b^*(j \circ a)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (4.2)\\
&\simeq b^*(b \circ i)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (4.3)\\
&\simeq b^* b_* i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (4.4)\\
&\simeq i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (4.5)
\end{aligned}$$

□

Lemma 4.3.11. *If $\mathcal{F}_{\mathcal{U}}$ is a sheaf, and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$, then $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf.*

---

[3]

TODO: Check this calculation is correct with Tallak or Rune.

Proof. By definition we need to show three things: Firstly, observe that $\mathcal{F}_{\mathcal{K}}(\emptyset) = \mathcal{F}_U(\emptyset)$ and since $\mathcal{F}_U$ is a sheaf $\mathcal{F}_{\mathcal{K}}(\emptyset)$ is terminal. Secondly, we need the following diagram to be a pullback in $\mathcal{C}$ for any $K, K' \in \mathcal{K}(X)$.

$$\begin{tikzcd}
\mathcal{F}(K \cup K') \arrow[r] \arrow[d] & \mathcal{F}(K) \arrow[d] \\
\mathcal{F}(K') \arrow[r] & \mathcal{F}(K \cap K')
\end{tikzcd} \tag{4.6}$$

We will do this by using that $\mathcal{F}_U$ is a sheaf. Let $P = \{(U, U')|K \subseteq U, K' \subseteq U'\}$ and $\sigma\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ denote diagram **??**. Now $\mathcal{F}$ induces a map $\sigma_P\colon \mathrm{N}(P^{op})^{\triangleright} \to \mathcal{C}^{\Delta^1 \times \Delta^1}$ taking each (U, U') to

$$\begin{tikzcd}
\mathcal{F}(U \cup U') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F}(U) \arrow[d] \\
\mathcal{F}(U') \arrow[r] & \mathcal{F}(U \cap U')
\end{tikzcd}$$

and the cone point is sent to σ. This is a pullback by the fact that $\mathcal{F}_U$ is a sheaf. Evaluating σ_P in each of the four vertices of $\Delta^1 \times \Delta^1$ we get four maps $\mathrm{N}(P^{op})^{\triangleright} \to \mathcal{C}$. We now check that evaluating in the final vertex yields a colimit diagram. By assumption $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_U$ which by definition means that the following is a colimit diagram:

$$(\mathcal{U}(X)_{/(K\cap K')})^{op,\triangleright} \hookrightarrow ((\mathcal{U}(X) \cup \mathcal{K}(X))_{/(K\cap K')})^{op,\triangleright} \xrightarrow{c} (\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

Observe that for every $U'' \in \mathcal{U}(X)_{/(K\cap K')}$, the set $P_{U''} = \{(U, U') \in P \mid U \cap U' \subseteq U''\}$ is nonempty and stable under finite intersections, which implies it is filtered and hence its nerve is contractible. By **??** this implies $P^{op} \to (\mathcal{U}(X)_{/(K\cap K')})^{op}$ is colimit-cofinal and we get a colimit diagram

$$P^{op,\triangleright} \to (\mathcal{U}(X)_{/(K\cap K')})^{op,\triangleright} \hookrightarrow (\mathcal{U}(X) \cup \mathcal{K}(X))^{op,\triangleright}_{/(K\cap K')} \xrightarrow{c} (\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}.$$

We can show that evaluating the three other vertices also yields colimit diagrams by similar arguments. Since σ_P yields a colimit diagram when evaluated in each of the four vertices of $\Delta^1 \times \Delta^1$, we conclude that σ_P is itself a colimit diagram by [**HTT**]. Observe now that σ_P is a filtered colimit in $\mathcal{C}$ and hence it is left exact. This concludes the argument that **??** is a pullback. Finally, we need to show that for each $K \in \mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}$ is a colimit of $\mathcal{F}_{\mathcal{K}}|_{\mathcal{K}_{K\Subset}(X)^{op}}$. We do this by showing

$$\mathcal{K}_{K\Subset}(X)^{op,\triangleright} \to (\mathcal{K}(X) \cup \mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. We use Proposition **??** to show that $\mathcal{F}|_{(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}}$ and $\mathcal{F}|_{(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}\cup\{K\}}$ are left Kan extensions of $\mathcal{F}|_{\mathcal{U}(X)^{op}}$ which again implies $\mathcal{F}|_{(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}\cup\{K\}}$ is a left Kan extension of $\mathcal{F}|_{(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}}$. Now observe that

$$(\mathcal{K}(X) \cup \mathcal{U}(X))^{op,\triangleright}_{/K} = (\mathcal{K}_{K\Subset}(X) \cup \mathcal{U}(X)_{/K})^{op,\triangleright},$$

so in particular

$$(\mathcal{K}(X) \cup \mathcal{U}(X))^{op,\triangleright}_{/K} \to (\mathcal{K}(X) \cup \mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram, and the statement is reduced to showing that $\mathcal{K}_{K\Subset}(X) \subseteq (\mathcal{K}(X) \cup \mathcal{U}(X))^{op}_{/K}$ is colimit-cofinal. Let $Y \in (\mathcal{K}(X) \cup \mathcal{U}(X))_{/K}$ and let R be the partially ordered set $\{K' \in \mathcal{K}(X) \mid K \Subset K' \subseteq Y\}$. Since R is nonempty and stable under intersections, R^{op} is filtered and hence $\mathrm{N}(R)$ is weakly contractible. By **??** the inclusion is colimit-cofinal and we have shown that $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf. □

Lemma 4.3.12. *If $\mathcal{F}_{\mathcal{U}}$ is a sheaf, then $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$.*

Proof. We will show $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ in a similar manner to how we showed $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$ in the start of the proof, but we will consider the partial order on $\mathcal{U}(X)$ given by writing $V \Subset U$ whenever $V \in \mathcal{U}(X)$ and its closure $\overline{V}$ is compact and contained in U. Writing $\mathcal{U}(X)_{U/}$ for the set $\{V \in \mathcal{U}(X) \mid V \Subset U\}$, we need to show that

$$\mathcal{K}(X)^{op,\triangleleft}_{U/} \hookrightarrow (\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{U/} \xrightarrow{c} (\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. As earlier we do this by finding cofinal inclusions

$$\mathcal{K}(X)^{op}_{U/} \xrightarrow{f} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{U/} \xleftarrow{f'} \mathcal{K}(X)^{op}_{/U}.$$

By Theorem **??** f and f' are colimit-cofinal inclusions if for any $Y \in (\mathcal{U}(X) \cup \mathcal{K}(X))_{U/}$ the partially ordered sets

$$\{V \in \mathcal{U}(X) | Y \subseteq V \Subset U\}$$

and

$$\{K \in \mathcal{K}(X) | Y \subseteq K \subseteq U\}$$

have weakly contractible nerves, which they have by the usual argument; they are nonempty and stable under unions, hence filtered. Since $\mathcal{U}(X)_{U/}$ is a sieve covering U and $\mathcal{F}_{\mathcal{U}}$ is a sheaf,

$$\mathcal{U}(X)^{op}_{U/} \to \mathcal{U}(X)^{op,\triangleleft}_{U/} \to \mathcal{C}$$

is a colimit diagram and this completes the proof that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$. □

Corollary 4.3.12.1. *Let X be a locally compact Hausdorff space and $\mathcal{C}$ an ∞-category with left exact filtered colimits and small limits, then $\mathrm{Shv}(X;\mathcal{C}) \simeq \mathrm{Shv}_{\mathcal{K}}(X;\mathcal{C})$ is an equivalence of ∞-categories.*

Proof. Let $\mathrm{Shv}_{\mathcal{K}\mathcal{U}}(X;C)$ be the full subcategory of $\mathrm{Fun}((\mathcal{K}(X) \cup \mathcal{U}(X))^{op}, \mathcal{C})$ spanned by those presheaves satisfying the equivalent conditions of theorem **??**. We get restrictions

$$\mathrm{Shv}(X;\mathcal{C}) \leftarrow \mathrm{Shv}_{\mathcal{K}\mathcal{U}}(X;C) \to \mathrm{Shv}_{\mathcal{K}}(X;C),$$

and these are equivalences of ∞-categories because inclusions of full subcategories are fully faithful and Kan extensions along fully faithful functors give isomorphisms by Proposition **??**. □

Chapter 5

Verdier Duality

The goal of this chapter is to prove the following theorem:

Theorem 5.0.1 (Verdier Duality [**HA**]). *Let X be a locally compact Hausdorff space and $\mathcal{C}$ be a stable ∞-category with small limits and colimits. Then we have an equivalence of ∞-categories*

$$\mathbb{D}\colon \mathrm{Shv}(X;\mathcal{C})^{op} \simeq \mathrm{Shv}(X;\mathcal{C}^{op}).$$

5.1 Classical Verdier Duality

Classical Verdier Duality can be seen as a generalization of Poincaré duality, replacing the pairing on cohomology with a pairing in the derived category. It was introduced in 1965 by Jean Louis Verdier [**Verdier95**]. Grothendieck had, a few years earlier, introduced a Poincaré duality in étale cohomology for schemes in algebraic geometry and Verdier duality serves as an analog of this duality for the theory of locally compact topological spaces. Verdier duality applies to morphisms of locally compact spaces and reduces to classical Poincaré duality when considering the projection of a manifold to the point. Recall that for a map $X \to Y$ between topological spaces and a sheaf $\mathcal{F}$ on X we can define a so-called direct image sheaf $f_*\mathcal{F}$ on Y by considering preimages of f:

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)) \text{ for any } U \in \mathcal{U}(Y)$$

Furthermore, viewing f_* as a functor $\mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(Y;\mathcal{C})$, there is a right adjoint f^* taking a sheaf $\mathcal{G}$ on Y to the so-called inverse image sheaf on X. Explicitly, this sheaf is given by the formula

$$f^*\mathcal{G}(U) = \mathrm{colim}_{f(V)\subseteq U}\, \mathcal{G}(U),$$

where the colimit is taken over all open neighborhoods of $f(V)$. Observe that considering the case when Y is just the point, the projection map f induces a functor $f_*\colon \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(*;\mathcal{C}) \simeq \mathcal{C}$; the global sections functor $\Gamma(X; -)$. As sheaf cohomology is defined as the right derived functor of the global sections functor, we can write

$$R^i f_* (\mathcal{F}) \cong R^i\Gamma(X; -)(\mathcal{F}) = H^i(X; \mathcal{F})$$

and this motivates the perspective of seeing the derived direct image as a kind of relative version of cohomology. Next, one constructs a so-called proper (or exceptional) direct image functor $f_!$ by considering the compactly supported subfunctor of f_*:

$$\Gamma(U; f_!\mathcal{F}) := \{s \in \Gamma(U; f_*\mathcal{F}) | \ s \text{ has compact support}\}.$$

If we once again considers the special case where $Y = *$, one recovers compactly supported cohomology:

$$R^i f_! (\mathcal{F}) \cong H^i_c(X; \mathcal{F})$$

The natural follow-up question to seeing this construction is what about proper inverse image? In general $Rf_!$ does not need to admit a right adjoint, but for locally compact Hausdorff spaces it exists and we denote it $f^!$. Together with the tensor product and the internal hom, the four functors f_*, f^*, $f_!$ and $f^!$ produces a six functor formalism. It is this final adjunction we call Verdier Duality:

Theorem 5.1.1. *Let $f : X \to Y$ be a continuous map of locally compact Hausdorff spaces such that $f_!$ has finite cohomological dimension*[1]*. Then the derived functor of the proper direct image has a right adjoint $f^!$ in the derived category of sheaves:*

$$R\mathrm{Hom}_{D(\mathrm{Shv}(Y;\mathcal{C}))}(Rf_!\mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}_{D(\mathrm{Shv}(X;\mathcal{C}))}(\mathcal{F}, f^!\mathcal{G}),^{2}$$

where $\mathcal{F}$ and $\mathcal{G}$ are sheaves of for instance A*-modules.*

Again, taking f to be the projection $X \to *$ yields fruitful special cases. We consider the simpler case when X is a compact and orientable n-dimensional manifold and consider the constant sheaf k_X on X for some field k. Verdier Duality now gives an equivalence

$$\mathrm{Hom}(Rf_!(k_X), k) \cong \mathrm{Hom}(k_X, f^!k).$$

Using the fact that we can find an injective resolution $I^\bullet$ of k_X and some homological algebra, one can show that maps $\mathrm{Hom}(Rf_!(k_X), k)$ are further equivalent to the zeroth cohomology group

$$H^0(\mathrm{Hom}^\bullet(\Gamma_c(X; I^\bullet), k)) = H^0_c(X; k_X)^\vee,$$

where $\vee$ denotes the dual vector space. Additionally, it can be shown that $f^!k = k_X[n]$ which is something called the dualizing complex for a manifold, which means that studying $\mathrm{Hom}(k_X, f^!k)$ reduces to studying $\mathrm{Hom}(k_X, k_X[n])$ which is isomorphic to the n-th cohomology of $\mathrm{Hom}^\bullet(x_k, x_k)$, so we have an isomorphism

$$H^0_c(X; k_X)^\vee \cong H^n(X; k_X).$$

[1]This means that there is some bound d for which all cohomology groups $H^r_c(f^{-1}(y); \mathrm{A})$ above d vanish for all $y \in Y$. This holds for instance if all the fibres $f^{-1}(y)$ are at most d-dimensional CW-complexes.

[2]Here we are really meaning complexes of sheaves.

Now, shifting the complexes i degrees and repeating the above arguments recovers classical Poincaré duality:

$$H^i_c(X; k_X)^\vee \cong H^{n-i}(X; k_X).$$

For a more detailed and precise account of the above arguments, see Iversen's book [**Iversen**] which serves as a good introduction to sheaves and sheaf cohomology as a whole and for a slightly more amateurish account, see [**BACH**].

5.2 Verdier Duality in stable ∞-categories

Let k be a field and Ch_k the category of chain complexes of k-vector spaces. The nerve construction $\mathrm{N}(\mathrm{Ch}_k)$ is now euqivalent to the derived ∞-category of the category of k-vector spaces. Vector space duality gives a limit preserving functor $\mathrm{N}(\mathrm{Ch}_k^{op}) \to \mathrm{N}(\mathrm{Ch}_k)$ which induces a functor

$$\mathrm{Shv}(X; \mathrm{N}(\mathrm{Ch}_k)^{op}) \to \mathrm{Shv}(X; \mathrm{N}(\mathrm{Ch}_k))$$

for any locally compact Hausdorff space. Composing with the equivalence of Theorem **??** yields a contravariant functor from $\mathrm{Shv}(X; \mathrm{N}(\mathrm{Ch}_k))$ to itself:

$$\mathbb{D}' \colon \mathrm{Shv}(X; \mathrm{N}(\mathrm{Ch}_k))^{op} \to \mathrm{Shv}(X; \mathrm{N}(\mathrm{Ch}_k))$$

and it is this functor that is usually called Verdier Duality. As we have composed the equivalence $\mathbb{D}$ with vector space duality, $\mathbb{D}'$ is not necessarily an equivalence of ∞-categories unless certain finiteness conditions are imposed.

We will be using the theory of $\mathcal{K}$-sheaves set up in the previous chapter to prove the theorem. By corollary **??** we can rewrite theorem **??** in terms of $\mathcal{K}$-sheaves instead:

Theorem 5.2.1. *Let X be a locally compact Hausdorff space and $\mathcal{C}$ be a stable ∞-category with small limits and colimits. Then we have an equivalence of ∞-categories:*

$$\mathbb{D}_{\mathcal{K}} \colon \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}).$$

Definition 5.2.2 ([**HA**])**.** Let X be a locally compact Hausdorff space and $\mathcal{C}$ a pointed ∞-category with small limits and colimits. For a sheaf $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ and K compact we denote by $\Gamma_K(X; \mathcal{F})$ the fiber product $\mathcal{F}(X) \times_{\mathcal{F}(X-K)} 0$. For U open, we denote by $\Gamma_c(U; \mathcal{F})$ the filtered colimit $\mathrm{colim}_{K \in \mathcal{K}(X)_{/U}} \Gamma_K(X; \mathcal{F})$ where K ranges over all compact subsets of U. The construction $U \mapsto \Gamma_c(U; \mathcal{F})$ determines a functor

$$\Gamma_c(-; \mathcal{F}) \colon \mathrm{N}(\mathcal{U}(X)) \to \mathcal{C}.$$

Remark 5.2.3. Observe that for K a compact subset of an open subset U, $\mathcal{F}$ is a sheaf means that we have pullbacks:

$$\begin{tikzcd}
\mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma(X; \mathcal{F}) \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner"] & \Gamma(U; \mathcal{F}) \arrow[d, "g"] \\
0 \arrow[r] & \Gamma(X - K; \mathcal{F}) \arrow[r] & \Gamma(U - K; \mathcal{F})
\end{tikzcd}$$

As the composition of pullbacks is again a pullback we get $\Gamma_K(X;\mathcal{F}) = \mathrm{fib}(f) = \mathrm{fib}(g) = \Gamma_K(U;\mathcal{F})$.

Definition 5.2.4 ([**HA**])**.** Let X be a locally compact Hausdorff space. We define a partially ordered set M as follows:

1. The objects of M are pairs (i,S) where $0 \leq i \leq 2$ and $S \subseteq X$ such that $i = 0$ implies S is compact and $i = 2$ implies $X - S$ is compact.
2. We have $(i,S) \leq (j,T)$ if either $i \leq j$ and $S \subseteq T$, or $i = 0$ and $j = 2$.

Remark 5.2.5 ([**HA**]). Observe that projecting $(i,S) \to i$ gives a map $\varphi : M \to [2]$ of partially ordered sets. For $0 \leq i \leq 2$ denote the fiber $\varphi^{-1}\{i\}$ by M_i. Also, observe that $M_0 \simeq \mathcal{K}(X)$, $M_2 \simeq \mathcal{K}(X)^{op}$ and M_1 is isomorphic to the powerset poset of X.

Definition 5.2.6. Let M' denote the partially ordered sets of pairs (i,S), where $0 \leq i \leq 2$ and $S \subseteq X$ such that $i = 0$ implies S is compact and $i = 2$ implies $X - S$ is either open or compact. Let $(i,S) \leq (j,T)$ if $i \leq j$ and $S \subseteq T$ or if $i = 0$ and $j = 2$. For $0 \leq i \leq 2$, let M'_i denote the subset $\{(j,S) \in M' | j = i\} \subseteq M'$.

We will prove Theorem **??** as a simple corollary of the following proposition.

Proposition 5.2.7 ([**HA**])**.** *Let X be a locally compact Hausdorff space, $\mathcal{C}$ be a stable ∞-category with small limits and colimits and M be as in* **??***. Let $F\colon M \to \mathcal{C}$ be a functor. Then the following conditions are equivalent:*

1. *The restriction $F|^{op}_{M_0}$ determines a $\mathcal{K}$-sheaf $\mathcal{K}(X)^{op} \to \mathcal{C}^{op}$, the restriction $F|_{M_1}$ is zero, and F is left Kan extended from $M_0 \cup M_1$.*
2. *The restriction $F|_{M_2}$ determines a $\mathcal{K}$-sheaf $\mathcal{K}(X)^{op} \to \mathcal{C}$, the restriction $F|_{M_1}$ is zero, and F is right Kan extended from $M_1 \cup M_2$.*

The proof of this is a bit long, so we will split the theorem into a few lemmas keeping the notation of the theorem statement.

Lemma 5.2.8. *It is enough to show that condition* 2. *implies condition* 1..

Proof. Observe that the map $(i,S) \mapsto (2-i, X-S)$ is an order-reversing bijection $M \to M$ which is moreover self-inverse. Now, it is enough to observe that we can safely swap $\mathcal{C}$ and $\mathcal{C}^{op}$.

Preben: Is this good enough? Should probably say why we can safely swap.

□

Lemma 5.2.9. *Let $\mathcal{D}$ denote the full subcategory of* $\mathrm{Fun}(M', \mathcal{C})$ *spanned by those functors F that satisfy the following conditions:*

1. *$F|_{M_2}$ is a $\mathcal{K}$-sheaf on X.*

2. *$F|_{M_2'}$ is a right Kan extension of $F|_{M_2}$.*
3. *$F|_{M_1'}$ is zero.*
4. *$F|_{M'}$ is a right Kan extension of $F|_{M_1' \cup M_2'}$.*

Then any $F' \in \mathcal{D}$ can be restricted to a sheaf $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ and is given by the fiber $\Gamma_K(X; \mathcal{F})$ when restricted to M_0.

Proof. Observe that we have a bijection between $\mathcal{U}(X)^{op}$ and the partially ordered set of closed subsets of X by sending $\mathcal{U}(X) \ni U \mapsto (X - U)$ and we have a natural inclusion $\mathcal{U}(X)^{op} \hookrightarrow M_2'$. By Theorem **??** we can restrict $\mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ to $\mathrm{Shv}(X; \mathcal{C})$, so we can also restrict M_2' and even $\mathcal{D}$ to $\mathrm{Shv}(X; \mathcal{C})$. Let $\mathcal{F}$ be the sheaf obtained by restricting F'. Define $\varphi \colon \mathrm{N}(M_0) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{N}(M'))$ by sending an object $(0, K) \in M_0$ to the diagram

$$\begin{tikzcd} (0,K) \arrow[r] \arrow[d] & (1,K) \arrow[d] \\ (2,\emptyset) \arrow[r] & (2,K). \end{tikzcd}$$

We can regard $\varphi(0, K)$ as a map $i : \Lambda^2_2 \to (M_1' \cup M_2')_{(0,K)/}$:

$$\begin{tikzcd} & a \arrow[d] & {} \arrow[r] & {} & (0,K) \to (1,K) \arrow[d] \\ b \arrow[r] & c & & (0,K) \to (2,\emptyset) \arrow[r] & (0,K) \to (2,K) \end{tikzcd}$$

Here we have abused notation to write the fiber product $M'_{(0,K)/} \times_{M'} M_1' \cup M_2'$ as $(M_1' \cup M_2')_{(0,K)/}$. By **??** i is limit-cofinal if and only if for every $(m, A) \in (M_1' \cup M_2')_{(0,K)/}$ the fiber product

$$\begin{tikzcd} \mathrm{PB} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \big((M_1' \cup M_2')_{(0,K)/}\big)_{/(m,A)} \arrow[d, hook, "j"] \\ \Lambda^2_2 \arrow[r, hook, "i"] & (M_1' \cup M_2')_{(0,K)/} \end{tikzcd}$$

is weakly contractible. As we have partially ordered sets j is just the inclusion given by sending objects $((0, K) \leq (r, B) \leq (m, A))$ in $\big((M_1' \cup M_2')_{(0,K)/}\big)_{/(m,A)}$ to objects $((0, K) \leq (r, B))$ in $(M_1' \cup M_2')_{(0,K)/}$. Since $i(a) = (2, \emptyset), i(b) = (1, K)$ and $i(c) = (2, K)$ and the pullback of a mono is mono, PB has to be a subcategory of

$$\begin{tikzcd} & (b,(1,K)) \arrow[d] \\ (a,(2,\emptyset)) \arrow[r] & (c,(2,K)). \end{tikzcd}$$

Observe that such a subcategory fails to be contractible only if (m, A) is chosen such that the pullback is either empty or consists of two disjoint objects. If $r = 1$ we know $(1, K) \leq (1, B)$ and have no arrows from $(2, \emptyset)$ or $(2, K)$ to $(1, B)$. If $r = 2$ we must have $(2, \emptyset) \leq (2, K) \leq (2, B)$, so the pullback is always weakly contractible. By condition 4., $F'|_{M'}$ is right Kan extended from $F'|_{M_1' \cup M_2'}$, which by definition ?? means

$$(M_1' \cup M_2')^{\triangleleft}_{(0,K)/} \hookrightarrow M'^{\triangleleft}_{(0,K)/} \to M' \xrightarrow{F'} \mathcal{C}$$

is a limit diagram. In other words, we have

$$\lim_{(M_1' \cup M_2')_{(0,K)/}} F' = F'(0, K)$$

and by the limi-cofinality of i we get

$$F'(0, K) = \lim_{(M_1' \cup M_2')_{(0,K)/}} F' = \lim_{\Lambda_2^2}(F' \circ i) = \lim F'((2, \emptyset) \to (2, K) \leftarrow (1, K))$$

which means condition 4. is equivalent to requiring that F' composed with $\varphi(0, K)$ yields another pullback diagram

$$\begin{tikzcd} F'(0,K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F'(1,K) \arrow[d] \\ F'(2,\emptyset) \arrow[r] & F'(2,K) \end{tikzcd}$$

Observe now that by condition 3. $F'(1, K) = 0$ and hence

$$F'(0, K) = F(0, K) \simeq \mathrm{fib}(F'(2, \emptyset) \to F'(2, K)).$$

Recall that we defined $\mathcal{F}$ as the restriction of F' to $\mathrm{Shv}(X; \mathcal{C})$ by identifying open sets U with their complements. This means that $F'(2, \emptyset) = \mathcal{F}(X)$ and $F'(2, K) = \mathcal{F}(X - K)$ which in turn means that

$$F'(0, K) = F(0, K) \simeq \mathrm{fib}(F'(2, \emptyset) \to F'(2, K)) \simeq \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(X - K)) = \Gamma_K(X; \mathcal{F})$$

which completes the proof that $F'|_{M_0}(K)$ is given by $\Gamma_K(X; \mathcal{F})$. □

Corollary 5.2.9.1. *In particular, $F \colon M \to \mathcal{C}$ satisfying condition 2. in ?? is given by $\Gamma_K(X; \mathcal{F})$ when restricted to M_0.*

Proof. Observe first that M is a full subcategory of M' and then that $\mathrm{Fun}(M \to \mathcal{C})$ is a full subcategory of $\mathcal{D}$. By the fact that the inclusion of full subcategories is fully faithful and Kan extending along fully faithful functors give actual on the nose extensions by Proposition **??**, we can extend F to $\mathcal{D}$. $\square$

Lemma 5.2.10. *Let $F|_{M_0} = \mathcal{G}$. Then $\mathcal{G}^{op}\colon M_0^{op} \to \mathcal{C}^{op}$ is a $\mathcal{K}$-sheaf valued in $\mathcal{C}^{op}$.*

Proof. Here $\mathcal{G}^{op}$ does the same on objects as $\mathcal{G}$ but we think of it as a functor $M_0^{op} \to \mathcal{C}^{op}$. We must show that it satisfies the following three properties:

1. $\mathcal{G}(\emptyset)$ is a zero object, because $\mathcal{G}(\emptyset) = \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(X - \emptyset)) = 0$.
2. For any compact subsets K and K' of X we must show the following diagram is a pullback:

$$\begin{tikzcd}
\mathcal{G}(K \cap K') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}(K') \arrow[d] \\
\mathcal{G}(K) \arrow[r] & \mathcal{G}(K \cup K')
\end{tikzcd}$$

   Observe that the diagram can be identified with the fiber of the map

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathcal{F}(X) \arrow[dd] & & \mathcal{F}(X - (K \cap K')) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathcal{F}(X - K') \arrow[dd] \\
 & {} \arrow[rr] & & {} & \\
\mathcal{F}(X) \arrow[r] & \mathcal{F}(X) & & \mathcal{F}(X - K) \arrow[r] & \mathcal{F}(X - (K \cup K'))
\end{tikzcd}$$

   As $\mathcal{F}$ is a sheaf, this is a map between pullbacks, so our diagram is also a pullback.
3. For any compact subset K of X we must show that the map $\theta\colon \mathcal{G}(K) \to \lim_{K \Subset K'} \mathcal{G}(K')$ is an equivalence in $\mathcal{C}$. Observe now that θ gives us a map between two fiber sequences

$$\begin{tikzcd}
\mathcal{G}(K) \arrow[r, "\theta"] \arrow[d] & \lim_{K \Subset K'} \mathcal{G}(K') \arrow[d] \\
\mathcal{F}(X) \arrow[r, "\theta'"] \arrow[d] & \lim_{K \Subset K'} \mathcal{F}(X) \arrow[d] \\
\mathcal{F}(X - K) \arrow[r, "\theta''"] & \lim_{K \Subset K'} \mathcal{F}(X - K')
\end{tikzcd}$$

   Since the partially ordered set $\{K' \in \mathcal{K}(X) | K \Subset K'\}$ is filtered it is weakly contractible and hence θ' is an equivalence. Since $\mathcal{F}$ is a sheaf and the set $\{X - K' | K \Subset K'\}$ is a covering sieve on $X - K'$, θ'' is also an equivalence. As we have shown that θ' and θ'' are equivalences, θ must also be an equivalence, and we have shown that $\mathcal{G}^{op}$ determines a $\mathcal{K}$-sheaf $(\mathcal{K}(X)^{op} \to \mathcal{C}^{op})$.

$\square$

To complete the proof of Proposition **??**, we must show the following lemma:

Lemma 5.2.11. *F is left Kan extended from $F|_{M_0 \cup M_1}$.*

Proof. F is left Kan extended from $F|_{M_0 \cup M_1}$. Let $M'' = \{(i,S) \in M_0 \cup M_1 | (i,S) \in \mathcal{K}(X)\}$. We can observe that $F|_{M_0 \cup M_1}$ is left Kan extended from $F|_{M''}$ (F is zero on M_1). By Proposition **??** it is enough to show that F is a left Kan extension of $F|_{M''}$, and this is enough to check at every $(2,S) \in M_2$. We will instead show that $F'|_{M'' \cup M_2'}$ is a left Kan extension of $F|_{M''}$ and for this we define

$$B := \{(2, X - U) \subseteq M_2' \mid U \in \mathcal{U}(X) | \overline{U} \in \mathcal{K}(X)\}.$$

By Proposition **??** it is enough to show that

(a) $F'|_{M'' \cup M_2'}$ is a left Kan extension of $F'|_{M'' \cup B}$.
First observe that M'' and M_2' are disjoint so it is enough to check that for every $(2, X - K) \in M_2' - B$, the following

$$(M'' \cup B)^{\triangleright}_{/(2,X-K)} \hookrightarrow (M'' \cup M_2')^{\triangleright}_{/(2,X-K)} \to (M'' \cup M_2') \to \mathcal{C}$$

is a colimit diagram. According to Theorem **??** we can restrict the colimit from $(M'' \cup B)_{/(2,X-K)}$ to $B_{/(2,X-K)}$ if we can show that the pullback

$$\begin{tikzcd}
PB \arrow[r] \arrow[d] & \left((M'' \cup B)_{/(2,X-K)}\right)_{/(2,X-U)} \arrow[d] \\
\left(B_{/(2,X-K)}\right)_{/(2,X-U)} \arrow[r] & (M'' \cup B)_{/(2,X-K)}
\end{tikzcd}$$

is weakly contractible. As this is just the partially ordered set $\{(2, X-U) \in B | \ (i,S) \leq (2, X - U) \leq (2, X-K)\}$ it is weakly contractible by the usual argument (it is nonempty and stable under finite unions hence filtered). This means that it is enough to show that $F'|_{M_2'}$ is left Kan extended from B. Assumption 2. says that $F|_{M_2}$ determines a $\mathcal{K}$-sheaf, $F|_{M_1} = 0$ and that F is a right Kan extension from $M_1 \cup M_2$. Identifying $M_2 = \{(2,S) | (X - S) \in \mathcal{K}(X)\}$ with $\mathcal{K}(X)^{op}$ we see that we are in the situation of Theorem **??**. As $M_2' = \{(2,S) | (X-S) | \in \mathcal{U}(X) \cup \mathcal{K}(X)\}$ we can identify it with $(\mathcal{U}(X) \cup \mathcal{K}(X))^{op}$ and by Theorem **??** we get that $F'|_{M_2'}$ is a left Kan extension of $F'|_{\mathcal{U}(X)^{op}}$. By observing that for a $K \in \mathcal{K}(X)$ the collection of open neighborhoods of K with compact closure is colimit-cofinal in the collection of all open neighborhoods of K in X we get that $F'|_{M_2'}$ is furthermore left Kan extended from B, which was what we wanted to show.

$$\begin{tikzcd} B \arrow[r, hook, "i"] & \mathcal{U}(X)^{op} \arrow[rr] \arrow[dr, hook] & & \mathcal{C} \\ & & M_2' \arrow[ur] & \end{tikzcd}$$

Here we have used that we calculate Kan extensions as colimits, so i being colimit-cofinal over some fixed K means restricting the colimit from $\mathcal{U}(X)^{op}$ back to B is an equivalence.

(b) $F'|_{M''\cup B}$ is a left Kan extension of $F|_{M''}$ Fix $U \in \mathcal{U}(X)$ such that $\overline{U} \in \mathcal{K}(X)$. By **??** we want to show that $F'(2, X-U)$ is a colimit of the diagram $F'|_{M''/(2,X-U)}$. For $K \in \mathcal{K}(X)$ denote by M''_K the subset of M'' consisting of pairs (i, S) such that $(0, K) \leq (i, S) \leq (2, X-U)$. Now, observe that $M''_{/(2,X-U)}$ is a filtered colimit of $\{M''_K\}_{K\in\mathcal{K}(X)_{U/}}$. By **[HTT]** we can identify $\mathrm{colim}(F'|_{M''})_{/(2,X-U)}$ with the filtered colimit of the diagram $\{\mathrm{colim}(F'|_{M''_K})\}_K$. This means that we are reduced to showing that for every $K \in \mathcal{K}(X)_{U/}$, F' exhibits $F'(2, X-U)$ as a colimit of $F'|_{M''_K}$. By Theorem **??** the diagram

$$\begin{tikzcd} (0, K-U) \arrow[r] \arrow[d] & (1, K-U) \\ (0, K) & \end{tikzcd}$$

is limit-cofinal in $\mathrm{N}(M''_K)$ and hence it is enough to show that

$$\begin{tikzcd} F'(0, K-U) \arrow[r] \arrow[d] & F'(1, K-U) \arrow[d] \\ F'(0, K) \arrow[r] & F'(2, X-U) \end{tikzcd}$$

is a pushout in $\mathcal{C}$. We will show this by considering the larger diagram

$$\begin{tikzcd}
F'(0, K-U) \arrow[r] \arrow[d] & F'(1, K-U) = 0 \arrow[d] & \\
F'(0, K) \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(1, K) = 0 \arrow[d] \\
F'(2, \emptyset) \arrow[r] & F(2, K-U) \arrow[r] \arrow[d] & F(2, K) \arrow[d] \\
 & F(2, X-U) \arrow[r] & F(2, X)
\end{tikzcd}$$

where we already know that the middle rectangle is a pullback (we have shown $F'(0, K)$ to be the fiber of the map $F'(2, \emptyset) \to F(2, K)$), so the middle left square is also a pullback. As we have shown $F'(0, K-U) = \mathrm{fib}(F(2, \emptyset) \to F(2, K-U))$ the left vertical rectangle is also a pullback, so the upper left must be as well. Since $\mathcal{C}$ is stable it is also a pushout. As $F(1, K) = 0$ and $F(2, X) = \mathcal{F}(\emptyset) = 0$ we have an equivalence $F(1, K) \to F(2, K) \to F(2, X)$ which means that if we can show the composite square

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & F(1, K) = 0 \arrow[d] \\
F(2, X-U) \arrow[r] & F(2, X) = \mathcal{F}(\emptyset) = 0
\end{tikzcd}$$

is a pullback, we have shown the desired equivalence $Z \to F(2, X-U)$ using the fact that pullback along an equivalence is again an equivalence.
To complete the proof it is therefore enough to show that the lower right square is a pullback. Replacing F by $\mathcal{F}$ we get

$$\begin{tikzcd}
\mathcal{F}((X-K) \cup U) \arrow[r] \arrow[d] & \mathcal{F}(X-K) \arrow[d] \\
\mathcal{F}(U) \arrow[r] & \mathcal{F}(\emptyset)
\end{tikzcd}$$

which is a pullback because $\mathcal{F}$ is a sheaf (U and $X-K$ are disjoint).

$\square$

We can now prove Verdier Duality (Theorem **??**):

Proof. Let $\mathcal{E}(\mathcal{C}) \subseteq \mathrm{Fun}(M)$ be the full subcategory spanned by those functors satisfying the conditions of Proposition **??** and observe that the inclusions $M_0 \hookrightarrow M \hookleftarrow M_2$ give restrictions

$$\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}) \xleftarrow{\theta} \mathcal{E}(\mathcal{C})^{op} \xrightarrow{\theta'} \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op}.$$

Because we Kan extend along inclusions of full subcategories which are fully faithful these are equivalences of ∞-categories by Proposition **??**. This proves Theorem **??** and by Corollary **??** we have shown Theorem **??**. □

Proposition 5.2.12 (**[HA]**). *Let X be a locally compact Hausdorff space and $\mathcal{C}$ a pointed and stable ∞-category with small limits and colimits. Then the equivalence of ∞-categories*

$$\mathbb{D} \colon \mathrm{Shv}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}(X; \mathcal{C}^{op}).$$

given in Theorem **??** *is given by $\mathbb{D}(\mathcal{F})(U) = \Gamma_c(U; \mathcal{F})$.*

Proof. It follows from the proof of Theorem **??** that the equivalence

$$\theta \colon \mathrm{Shv}(X; \mathcal{C}^{op})^{op} \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op}$$

is given by the formula $\theta(\mathcal{F})(U) = \mathrm{colim}_{K \subseteq U} \mathcal{F}(K)$.

TODO: Make sure this colimit is taken in the correct category. It should maybe just be a limit in $\mathcal{C}$?

Preben: Should I write out theta more explicitly in the proof of **??**. Not a lot of time.

Let $\psi \colon \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ be the equivalence of Corollary **??** and ψ' the equivalence $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}) \to \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op}$ of Theorem **??**. Composing, we get a string of equivalences

$$\mathbb{D}^{op} \colon \mathrm{Shv}(X; \mathcal{C}) \xrightarrow{\psi} \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}) \xrightarrow{\psi'} \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op} \xrightarrow{\theta} \mathrm{Shv}(X; \mathcal{C}^{op})^{op}.$$

Let $\mathcal{D}$ be as in the proof of Lemma **??**. By Theorem **??** the restriction $\mathcal{D} \to \mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ is a categorical equivalence onto $\mathrm{Shv}(X; \mathcal{C})$, since we Kan extend along the fully faithful inclusion of $\mathrm{Shv}(X; \mathcal{C})$ (it is a full subcategory of $\mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$.)
In the other direction we restrict $\mathcal{D} \to \mathrm{Fun}(M_0, \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{K}(X), \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{K}(X)^{op}, \mathcal{C}^{op})^{op}$ and $\psi' \circ \psi$ is given by the composition $\mathrm{Shv}(X; C) \to \mathcal{D} \to \mathrm{Fun}(\mathcal{K}(X)^{op}, \mathcal{C}^{op})^{op}$ and as we saw in the proof of Lemma **??**, restriction from $\mathcal{D}$ to functors from M_0 is given by $\Gamma_K(X; \mathcal{F})$. This means that $\psi' \circ \psi \colon \mathcal{F} \mapsto (K \mapsto \Gamma_K(X; \mathcal{F}))$ so by Remark **??** we have

$$(\theta \circ \psi' \circ \psi)(\mathcal{F})(U) = \mathrm{colim}_{K \subseteq U}(\Gamma_K(X; F)) = \mathrm{colim}_{K \subseteq U}(\Gamma_K(U; F)) = \Gamma_c(U; \mathcal{F}).$$

□

Remark 5.2.13. This is the infinity-categorical generalization of the classical fact that conjugation by Verdier duality exchanges cohomology and cohomology with compact support. The construction $\Gamma_c(U; \mathcal{F})$ really is analogous to the construction

$$\Gamma_c(X; \mathcal{F}) := \{s \in \Gamma(X; \mathcal{F}) \mid s \text{ has compact support}\}$$

from classical sheaf theory in 1-categories. Also, recall that in the introductio of this chapter we defined direct image with proper support as the functor

$$\Gamma(U; f_! \mathcal{F}) := \{s \in \Gamma(U; f_* \mathcal{F}) \mid s \text{ has compact support}\}.$$

and said Verdier duality is the existence of a right adjoint $f^!$.

In [**Volpe**], Marco Volpe shows that one can extend the classical six functor formalism for sheaves on locally compact Hausdorff spaces to sheaves with values in any closed symmetric monoidal ∞-category which is stable and bicomplete, and Lurie's Verdier duality (Theorem **??**) is central in proving the ∞-categorical adjunction.

Remark 5.2.14. As we saw in the introduction to this chapter, classical Verdier duality can be used to give a proof of Poincaré duality and in [**HA**] Jacob Lurie uses his version of Verdier duality to prove a version of what he calls non-abelian Poincaré duality.

Bibliography