

title

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Abstract

In 1965 Jean-Louis Verdier introduced Verdier duality for locally compact topological spaces, thus generalizing the classical theory of Poincaré duality for manifolds. Verdier Duality is a cohomological duality allowing for exchanging cohomology for cohomology with compact support. More precisely it states that the derived functor of the compactly supported direct image functor has a right adjoint in the derived category of sheaves. By using sheaf cohomology one can derive the classical Poincaré duality as a special case. In his book “Higher Algebra” Jacob Lurie extends the theory to the ∞ -categorical setting by showing there is an equivalence between sheaves and cosheaves valued in ∞ -categories. This thesis follows this proof closely, expanding and adding details where necessary. To introduce the relevant background on sheaves and \mathcal{K} -sheaves valued in stable ∞ -categories we introduce and utilize Kan extensions, an ubiquitous concept in category theory.

Chapter 1

∞ -categories

TODO: Speak about (∞, n) -categories.

1.1 Simplicial sets

Definition 1.1.1. Usually denoted by Δ , the simplex category or the simplicial category is the category with linearly ordered sets $[n] = \{0, 1, 2, \dots, n\}$ as its objects and order-preserving maps between them as its morphisms. That is, for a map $\varphi : [m] \rightarrow [n]$ we have that $0 \leq \varphi(i) \leq \varphi(j) \leq n$ for each $0 \leq i \leq j \leq m$.

We denote by δ^i the elementary face operator $[n-1] \rightarrow [n]$ and by σ^i the elementary degeneracy operator $[n+1] \rightarrow [n]$ given by

$$\delta^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}, \quad \sigma^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

Remark 1.1.2. All morphisms in Δ are compositions of such morphisms.

Definition 1.1.3. We define the category \mathbf{sSet} of simplicial sets as \mathbf{Set} -valued presheaves on Δ .

Let $X \in \mathbf{sSet} := \mathbf{Fun}(\Delta^{op}, \mathbf{Set})$. We will denote by X_n the set $X([n])$ of n -simplices in X . We define the standard n -simplex as $\Delta^n := y([n])$ where y is the Yoneda embedding, meaning Δ^n is the presheaf $\mathbf{Hom}_{\Delta}(-, [n])$. By the Yoneda lemma $\mathbf{Hom}_{\mathbf{Fun}(\Delta^{op}, \mathbf{Set})}(\Delta^n, X) \simeq X_n$, so we can identify each simplex $x \in X_n$ with a map $x : \Delta^n \rightarrow X$. This application of the Yoneda lemma is a crucial part of the theory of simplicial sets and we will more often than not consider n -simplices of a simplicial set X as maps of simplicial sets instead. Observe, moreover that composition with the elementary face operator gives us a map $\Delta^{n-1} \rightarrow \Delta^n$.

Definition 1.1.4. For a simplicial set X , we define the face and degeneracy maps

$$d_i := X(\delta^i) : X_n \rightarrow X_{n-1}, \quad s_i := X(\sigma^i) : X_n \rightarrow X_{n+1}$$

where both maps are given by composition with δ^i and σ^i respectively.

Example 1.1.5. The standard 0-simplex $\Delta^0 := \text{Hom}(-, [0])$ is a terminal object in \mathbf{sSet} , meaning it maps any $[m] \in \Delta$ to a singleton.

Example 1.1.6 ([Lur22, Remark 000P]). Let $X \in \mathbf{sSet}$ and suppose we have subsets $T_n \subseteq X_n$ for every $n \geq 0$ such that $d_i(T_n) \subseteq T_{n-1}$ and $s_i(T_n) \subseteq T_{n+1}$. Then the collection $\{T_n\}_{n \geq 0}$ is a simplicial set we will call a simplicial subset $T \subseteq X$.

Definition 1.1.7. We define the boundary $\partial\Delta^n$ of Δ^n as the simplicial set

$$(\partial\Delta^n)_m = (\partial\Delta^n)([m]) := \{\alpha \in \text{Hom}_\Delta([m], [n]) \mid [n] \not\subseteq \text{im}(\alpha)\}.$$

Observe that $\partial\Delta^0 = \emptyset$ because every map $[m] \rightarrow [0]$ is surjective.

Definition 1.1.8. For $0 \leq i \leq n$, we define the horn Λ_i^n as the simplicial set

$$(\Lambda_i^n)_m = (\Lambda_i^n)([m]) := \{\alpha \in \text{Hom}_\Delta([m], [n]) \mid \delta^i[n] \not\subseteq \text{im}(\alpha)\}.$$

Observe that the horn is inside the boundary. We usually refer to Λ_i^n as the i th horn in Δ^n and we will call the horns such that $0 < i < n$ the inner horns.

Example 1.1.9.

TODO: Write about $\text{Sing}(X)$ for a topological space X .

Example 1.1.10. We define the nerve $\mathbf{N}(\mathcal{C})$ of a 1-category \mathcal{C} by

$$\mathbf{N}(\mathcal{C}) := \text{Hom}_{\text{Cat}}([-], \mathcal{C})$$

where we view the sets $[n]$ as categories (posets with a map i to j whenever $i \leq j$). Observe that for any order-preserving morphism $\alpha : [m] \rightarrow [n]$ we get a map

$$\text{Hom}_{\text{Cat}}([n], \mathcal{C}) \xrightarrow{- \circ \alpha} \text{Hom}_{\text{Cat}}([m], \mathcal{C})$$

and it is clear that the nerve is a simplicial set with $\mathbf{N}(\mathcal{C})_n = \text{Hom}_{\text{Cat}}([n], \mathcal{C})$.

1.2 ∞ -categories

What Joyal [Joy02] named quasi-categories and Lurie [Lur09] calls ∞ -categories were originally called weak Kan complexes by Boardman and Vogt [BV73]. As most of this thesis follows Lurie's works very closely, we will follow his convention and use the name ∞ -categories. Before we give a

precise definition, we will take a closer look at the nerve construction. Clearly, we want the nerve of a 1-category to give us an ∞ -category and most of this thesis will revolve around nerves of certain poset-categories of topological spaces. Nerves of categories are not just any ordinary simplicial sets, but simplicial sets with some more structure inherited from the underlying 1-category. For instance, 1-categories have composition of morphisms. Take for example

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \\ & Z & \end{array}$$

in some ordinary 1-category \mathcal{C} . This diagram gives us a morphism $\Lambda_1^2 \rightarrow \mathbf{N}(\mathcal{C})$ of simplicial sets, but in \mathcal{C} f and g can be composed to a morphism $h : X \rightarrow Z$ which in turn gives a unique way to extend the simplicial map $\Lambda_1^2 \rightarrow \mathbf{N}(\mathcal{C})$ to a map $\Delta^2 \rightarrow \mathbf{N}(\mathcal{C})$. If we instead look at the outer horns Λ_0^2 and Λ_2^2 we will not necessarily have a way to extend morphisms to Δ^2 in general. For example the diagram

$$\begin{array}{ccc} X & & Y \\ \searrow \text{id}_X & & \swarrow g \\ & X & \end{array}$$

gives a map $\Lambda_2^2 \rightarrow \mathbf{N}(\mathcal{C})$, but extending this to a morphism $\Delta^2 \rightarrow \mathbf{N}(\mathcal{C})$ would amount to finding a right-inverse to g , which of course is not something we can always do in general, unless \mathcal{C} was a groupoid. This property of extending a morphism from a horn to the standard n -simplex is sometimes also called filling the horn, and we will see that it is a defining property for ∞ -categories. In fact, the existence of horn fillings completely classifies the simplicial sets which are nerves of categories:

Proposition 1.2.1 ([Lur09, Proposition 1.1.2.2]). *Let $X \in \mathbf{sSet}$. Then the following conditions are equivalent:*

1. *There exists a small category \mathcal{C} with an isomorphism $X \simeq \mathbf{N}(\mathcal{C})$.*
2. *Every inner horn $\Lambda_i^n \rightarrow X$ of X can be filled in a unique way. Or, in other words, for any solid diagram as below, there is a unique dotted arrow making it commute:*

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \text{!} & \\ \Delta^n & & \end{array}$$

Simplicial sets which admits extensions for all horn are called Kan complexes:

Definition 1.2.2. A simplicial set X is a Kan complex if it satisfies the following condition: For $0 \leq i \leq n$, any map $\sigma_0 : \Lambda_i^n \rightarrow X$ can be extended to a map $\sigma : \Delta^n \rightarrow X$.

As we saw in the example of a map $\Lambda_2^2 \rightarrow \mathbf{N}(\mathcal{C})$ above, whenever there's non-invertible morphisms around some outer horns will be impossible to fill. This motivates the definition of an ∞ -category. The following definition is due to Boardman and Vogt [BV73] who defined weak Kan complexes as simplicial sets satisfying what they called the restricted Kan condition:¹

Definition 1.2.3 (Boardman and Vogt [BV73]). A simplicial set X is an ∞ -category if it satisfies the following condition: For $0 < i < n$, any map $\sigma_0 : \Lambda_i^n \rightarrow X$ can be extended to a map $\sigma : \Delta^n \rightarrow X$.

This means that any Kan complex is an ∞ -category, and in particular so is $\mathrm{Sing}(X)$ for a topological space X . Additionally, observe that the nerve $\mathbf{N}(\mathcal{C})$ of an ordinary category \mathcal{C} is an ∞ -category. Because the nerve functor is fully faithful, many authors choose to omit its notation altogether.

TODO: Write fully-faithfulness of the nerve as a proposition or something.

Preben: Consider if we should omit it as well. Currently we have been using it, but it is fucking annoying to remember writing \mathbf{N} all the time.

In many ways ∞ -categories behave similarly to ordinary categories and we will often write about them almost as if they were ordinary categories instead. For example, we will use the terminology of ordinary category theory and refer to the vertices and edges of our simplicial sets as objects and morphism in our ∞ -categories. There are however some obvious differences between ordinary categories and ∞ -categories which needs addressing before adopting complete 1-categorical language. For example, and perhaps most crucially, we have higher-level maps given by simplices of dimension $n \geq 2$.

TODO: Write about Map.

While we have seen that nerves of categories admit unique horn extensions, this condition is dropped for general ∞ -categories and hence, composition of morphisms in an ∞ -category are not necessarily unique, but rather unique up to homotopy. Before we can make this precise, we must define what we mean by homotopy.

1.2.1 Homotopy

Definition 1.2.4 ([Lur22, Definition 003V]). Let \mathcal{C} be an ∞ -category and $f, g \in \mathrm{Hom}_{\mathcal{C}}(C, D)$. We define a homotopy between f and g as a 2-simplex $\sigma \in \mathcal{C}$ with boundary specified by $d_0(\sigma) = \mathrm{id}_D$, $d_1(\sigma) = g$ and $d_2(\sigma) = f$ and we say f and g are homotopic if such a homotopy exists.

Example 1.2.5. For a 1-category \mathcal{C} two morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ are homotopic in $\mathbf{N}(\mathcal{C})$ if and only if $f = g$.

¹Maybe more commonly known as the weak Kan extension condition. Expand on this with a historical account.

Proposition 1.2.6 ([Lur22, Proposition 003Z]). *Let \mathcal{C} be an ∞ -category and C, D objects of \mathcal{C} . Then homotopy is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(C, D)$.*

Proof.

TODO: Prove. Both Rune and Lurie have nice proofs.

□

Proposition 1.2.7. *$f, g \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ are homotopic if and only if they are homotopic as morphisms in \mathcal{C}^{op} .*

Proof.

TODO: Lurie.

□

TODO: Consider adding Corollary [Lur22, Corollary 00V0]

Now that we know what it means for morphisms of ∞ -categories to be homotopic, we can define a composition of morphisms.

Definition 1.2.8 ([Lur22, Definition 0042]). *Let \mathcal{C} be an ∞ -category with morphisms*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow h & \swarrow g \\ & Z & \end{array}$$

We define h to be a composition of f and g if there exists some 2-simplex $\sigma \in \mathcal{C}$ such that $d_0(\sigma) = g$, $d_1(\sigma) = h$ and $d_2(\sigma) = f$. We say σ witnesses h as a composition of f and g and we will use the usual notation $h = g \circ f$.

Observe that we have only defined composition up to homotopy. We make this precise in the following proposition:

Proposition 1.2.9 ([Lur22, Proposition 0043]). *Let \mathcal{C} be an ∞ -category with morphisms f and g as follows:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow h & \swarrow g \\ & Z & \end{array}$$

Then there exists a composition h of f and g and any other morphism $X \rightarrow Z$ is a composition of f and g if and only if it is homotopic to h .

Preben: The proof seems kind of technical, so might not be worth doing. Might do it if there's time or Rune thinks I should. Should consider adding [Lur22, Exercise 000Z] as this exercise is used in a lot of the proofs in this subsection.

Furthermore, compositions respect homotopy in the following sense:

Proposition 1.2.10 ([Lur22, Proposition 0048]). *Let \mathcal{C} be an ∞ -category with homotopic morphisms $f \sim f' : X \rightarrow Y$ and $g \sim g' : Y \rightarrow Z$. Let $h = f \circ g$ and $h' = f' \circ g'$. Then h is homotopic to h' .*

Remark 1.2.11. The nerve construction preserves compositions in the sense that for a 1-category \mathcal{C} with morphisms f, g as above, there is a unique morphism $h : N(X) \rightarrow N(Z)$ in $N(\mathcal{C})$ which is given by $f \circ g$ in \mathcal{C} .

In a later chapter on Kan extensions we will see that the nerve construction $\Delta \xrightarrow{N} \mathcal{C}$ admits a left adjoint h and moreover that the counit of this adjunction is an isomorphism which in turn means that the nerve is fully faithful. We now construct this left adjoint directly, but delay the proof of the adjunction. Analogously to the construction of the fundamental groupoid $\pi_{\leq 1}(X)$ of a topological space X , we can construct the homotopy category $h\mathcal{C}$ of an ∞ -category \mathcal{C} .

Preben: Writing about the fundamental groupoid could be smart.

Definition 1.2.12. Let \mathcal{C} be an ∞ -category. We denote by $\text{Hom}_{h\mathcal{C}}(X, Y)$ homotopy classes of morphisms $X \rightarrow Y \in \mathcal{C}$ and for a morphism $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, we denote by $[f]$ its equivalence class in $\text{Hom}_{h\mathcal{C}}(X, Y)$.

Proposition 1.2.13. *We have a unique composition of morphisms*

$$\circ : \text{Hom}_{h\mathcal{C}}(Y, Z) \times \text{Hom}_{h\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{h\mathcal{C}}(X, Z)$$

such that $[h] = [f] \circ [g]$ for any $h = f \circ g \in \mathcal{C}$.

Proof. This follows directly from the previous two propositions. □

1.3 Stable ∞ -categories

Definition 1.3.1. A zero-object, usually denoted 0 , in an ∞ -category is an object that is both initial and terminal. A pointed ∞ -category is an ∞ -category containing a zero-object.

Remark 1.3.2. An object $0 \in \mathcal{C}$ is a zero-object if $\text{Map}_{\mathcal{C}}(X, 0)$ and $\text{Map}_{\mathcal{C}}(0, X)$ are contractible for every object $X \in \mathcal{C}$ and such an object is determined up to equivalence.

TODO: Define Map and contractibility.

Chapter 2

Limits and Colimits

2.1 Limits and Colimits

2.2 Cofinality

Definition 2.2.1. [Lur09, Definition 4.1.1.1] Let $p : S \rightarrow T \in \mathbf{sSet}$. We say p is *cofinal* if, for any right fibration $X \rightarrow T$, the induced map of simplicial sets

$$\mathrm{Map}_T(T, X) \rightarrow \mathrm{Map}_T(S, X)$$

is a homotopy equivalence.

We will probably want to write more about cofinality later, but for now we state a result that will be extremely helpful later.

Theorem 2.2.2. [Lur22, Theorem 02NY] Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a map of simplicial sets, where \mathcal{D} is an ∞ -category. Then the following conditions are equivalent:

1. The functor f is left cofinal.
2. For every $D \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible.

and the following conditions are equivalent:

1. The functor f is right cofinal.
2. For every $D \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible.

2.3 Kan Extensions

We will start with a detour into the world of ordinary 1-categories. In the classic [Mac71] Saunders MacLane says “The notion of Kan Extensions subsumes all the other fundamental concepts of category theory”. In the fantastic introduction to category theory [Rie17] Emily Riehl devotes a whole chapter to the slogan “All concepts are Kan extensions”.

Preben: This intro made me cringe.

Ubiquitous in the toolbox of any category theorist, Kan extensions are central for most everything we do in this thesis. As we have seen, a lot of ∞ -categorical concepts can be thought of as if we are working with ordinary 1-categories, and we will therefore start by defining Kan extensions in ordinary categories.

Definition 2.3.1 ([Rie17, Definition 6.1.1.]). Given functors F and K as in the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ K \downarrow & \nearrow K_! F & \\ \mathcal{D} & & \end{array}$$

a left Kan extension of F along K is a functor $K_! F : \mathcal{D} \rightarrow \mathcal{E}$ together with a natural transformation $\eta : F \Rightarrow K_! F \circ K$ such that for any other pair $G : \mathcal{D} \rightarrow \mathcal{E}, \gamma : F \rightarrow G \circ K$, γ factors uniquely through η as in the following diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ K \downarrow & \nearrow \gamma & \nearrow G \\ \mathcal{D} & & \end{array} = \begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ K \downarrow & \nearrow \eta & \nearrow K_! F \\ \mathcal{D} & \nearrow \exists! & \nearrow G \end{array}$$

Dually, a right Kan extension of F along K is a functor $K_* F : \mathcal{D} \rightarrow \mathcal{E}$ with a natural transformation $\epsilon : K_* F \circ K \Rightarrow F$ such that any functor $G : \mathcal{D} \rightarrow \mathcal{E}$ and any natural transformation $\delta : G \Rightarrow F$, δ factors uniquely through ϵ .

Preben: Diagram is exactly the same but all 2-cells go the other way.

The following result justifies the choice to denote left and right Kan extensions by lower shriek and star:

Proposition 2.3.2 ([Rie17, Proposition 6.1.5.]). *Let K be a functor $\mathcal{C} \rightarrow \mathcal{D}$ and \mathcal{E} some category. If the left and right Kan extensions of any functor F along K exists, these define left and right*

adjoints to the pre-composition functor $K^* : \text{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{E})$ and by uniqueness of adjoints, any left or right adjoint to the pre-composition functor yields left or right Kan extensions.

While the description of Kan extensions as certain adjoints are useful, there's usually another description available to us that turns out to be even more useful. Whenever \mathcal{C} and \mathcal{D} are respectively small and locally small potential Kan extensions along $K : \mathcal{C} \rightarrow \mathcal{D}$ are what we call pointwise Kan extensions. More precisely, the existence of left or right Kan extensions along such functors coincides with the existence of certain colimits or limits, respectively. For a functor $K : \mathcal{C} \rightarrow \mathcal{D}$, [Mac71] denotes by $d \downarrow K$ the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ and we will choose to denote it by $K_{d/}$. Likewise the category $K \downarrow d = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d}$ will be denoted $K_{/d}$. These categories come with projection functors Π_d and Π^d , respectively, which send the objects $d \rightarrow Kc$ and $Kc \rightarrow d$ to the object $c \in \mathcal{C}$. The following theorem gives a formula for calculating certain left and right Kan extensions as colimits and limits.

Theorem 2.3.3 ([Rie17, Theorem 6.2.1.]). *Let $\mathcal{D} \xleftarrow{K} \mathcal{C} \xrightarrow{F} \mathcal{E}$ be functors. If it exists for every object $d \in \mathcal{D}$, the following colimits defines the left Kan extension $K_!F$:*

$$K_!F(d) := \text{colim}(K_{/d} \xrightarrow{\Pi_d} \mathcal{C} \xrightarrow{F} \mathcal{E})$$

and the unit transformation $\eta : F \rightarrow K_!F \circ K$ can be extracted from the colimit cone. Dually, if the following limits exists for every object $d \in \mathcal{D}$, they define the right Kan extension K_*F :

$$K_*F(d) := \text{lim}(K_{d/} \xrightarrow{\Pi^d} \mathcal{C} \xrightarrow{F} \mathcal{E})$$

and the counit transformation $\epsilon : K_*F \circ K \rightarrow F$ can be extracted from the limit cone.

Preben: Is it clear what it means to extract the 2-cells from the cones?

Proof. Proofs can be found in [Rie17, Theorem 6.2.1.] and [Mac71, Theorem X.3.1.] □

When \mathcal{D} and \mathcal{E} are locally small we call Kan extensions that can be calculated by colimit and limit formulae above for pointwise Kan extensions. In [Rie17], Riehl says the consensus among category theorists is that the important Kan extensions are the pointwise Kan extensions and quotes [Kel05, §4]: “Our present choice of nomenclature is based on our failure to find a single instance where a weak Kan extension plays any mathematical role whatsoever.” This thesis is no different, and we will only care about pointwise Kan extensions from here on out. We will see that there are analogous limit formulae for Kan extensions in ∞ -categories which are central to most of the proofs in this thesis. Before we extend the theory Kan extensions from ordinary categories to the world of ∞ -categories, we will consider some important examples, but first observe that Theorem 2.3.3 gives the following immediate consequence:

Corollary 2.3.3.1. *If \mathcal{C} is small and \mathcal{D} is locally small, then for a functor $K : \mathcal{C} \rightarrow \mathcal{D}$ we have:*

1. If \mathcal{E} is cocomplete, left Kan extensions exists and are given by the colimit formula of Theorem 2.3.3.
2. If \mathcal{E} is complete, right Kan extensions exists and are given by the limit formula of Theorem 2.3.3.

Example 2.3.4. n -truncation is given by restricting along the inclusion $i : \Delta_{\leq n} \hookrightarrow \Delta$, and since \mathbf{Set} is both cocomplete and complete, we have both left and right Kan extensions to n -truncation.

$$\begin{array}{ccc}
 & (i_n)! & \\
 \swarrow & \perp & \searrow \\
 \mathbf{sSet} & \xrightarrow{i_n^*} & \mathbf{sSet}_{\leq n} \\
 \nwarrow & \perp & \nearrow \\
 & (i_n)_* &
 \end{array}$$

where $\mathbf{sSet}_{\leq n} := \mathbf{Fun}(\Delta_{\leq n}^{op}, \mathbf{Set})$.

Example 2.3.5 (Kan extension along fully faithful functors). Let $K : \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful functor. Then, up to natural isomorphism, any pointwise Kan extension along K define actual on the nose extensions.

Proof. Observe that $\mathcal{C}_{/c} \simeq K_{/K(c)}$ is an equivalence of categories because K is fully faithful, so we can calculate the left Kan extension $K_!F$ on objects by the colimit formula:

$$K_!F(K(c)) = \operatorname{colim}(K_{/K(c)} \simeq \mathcal{C}_{/c} \xrightarrow{\Pi} \mathcal{C} \xrightarrow{F} \mathcal{E}).$$

Since the identity on c is terminal in $\mathcal{C}_{/c}$ the colimit reduces to evaluation at the terminal object $K(c) \xrightarrow{id} K(c)$ in $K_{/K(c)}$, so $\eta_c : F(c) \cong K_!F(K(c))$ is an isomorphism. The proof for pointwise right Kan extensions is completely dual. \square

Example 2.3.6 (Yoneda extension). Let \mathcal{C} be small, \mathcal{E} locally small and cocomplete. By the corollary above, any functor $F : \mathcal{C} \rightarrow \mathcal{E}$ admits a left Kan extension $y_!F$ along the Yoneda embedding y and by fully faithfulness of y the unit $F \cong y_!F \circ K$ is an identity natural transformation. In fact, $y_!F$ has a right adjoint R , defined on objects by

$$R(e) := \operatorname{Hom}_{\mathcal{E}}(F(-), e) : \mathcal{C}^{op} \rightarrow \mathbf{Set}.$$

The full proof showing that this is in fact right adjoint to $y_!F$ can be found in [Rie17, Remark 6.5.9.] from which we have taken this example.

The process of left Kan extending along the Yoneda embedding is called Yoneda extension in [KS06, pp.62-64] and it provides lots of interesting examples of Kan extensions. We will look at a couple examples in the special case $\mathcal{C} = \Delta$.

Example 2.3.7. Sing

Example 2.3.8 (The Nerve construction). This is awesome.

2.3.1 Kan extensions for ∞ -categories

Preben: THIS IS CURRENTLY JUST ALONG INCLUSIONS

Definition 2.3.9.

TODO: This is chaos. Fix all slice notations in separate subsection of infcat chapter.

Let \mathcal{A} be an ∞ -category with a full subcategory \mathcal{A}^0 and $p : K \rightarrow \mathcal{A}$ a diagram. Following [Lur09, Notation 4.3.2.1] we write $\mathcal{A}_{/p}^0$ for the fiber product $\mathcal{A}_{/p} \times_{/\mathcal{A}} \mathcal{A}^0$. If $A \in \mathcal{A}$, $\mathcal{A}_{/A}^0$ is just the full subcategory of $\mathcal{A}_{/A}$ spanned by the morphisms $A' \rightarrow A$ where $A' \in \mathcal{A}^0$.

Analogously $\mathcal{A}_{p/}^0$ denotes $\mathcal{A}_{p/} \times_{\mathcal{A}} \mathcal{A}^0$ and $\mathcal{A}_{A/}^0$ is the full subcategory spanned by morphisms $A \rightarrow A'$.

Definition 2.3.10 ([Lur22, Definition 02YQ]). For a functor $F : \mathcal{A} \rightarrow \mathcal{C}$ between ∞ -categories where \mathcal{A} has a full subcategory \mathcal{A}^0 , we say F is left Kan extended from \mathcal{A}^0 if

$$(\mathcal{A}_{/A}^0)^\triangleright \hookrightarrow (\mathcal{A}_{/A})^\triangleright \xrightarrow{c} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

is a colimit diagram in \mathcal{C} for every object $A \in \mathcal{A}$. Here c is the slice contraction morphism of [Lur22, Tag 0188], i.e. $c|_{\mathcal{A}_{/A}}$ is the projection and $c|_{\Delta^0} = A$. Recalling the adjoint relationship between joins and slices, this is just the counit of the adjunction.

Right Kan extensions are opposite to left Kan extensions, i.e. F is right Kan extended from \mathcal{A}^0 if the following is a limit diagram

$$(\mathcal{A}_{A/}^0)^\triangleleft \hookrightarrow (\mathcal{A}_{A/})^\triangleleft \xrightarrow{c'} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

where c' is the coslice contraction morphism.

Later on we will need the following result about Kan extensions of full subcategories.

Proposition 2.3.11 ([Lur09, Proposition 4.3.2.8]). For a functor $F : \mathcal{A} \rightarrow \mathcal{C}$ of ∞ -categories where $\mathcal{A}^0 \subseteq \mathcal{A}^1$ are full subcategories of \mathcal{A} , if $F|_{\mathcal{A}^1}$ is left Kan extended from \mathcal{A}^0 then F is left Kan extended from \mathcal{A}^1 if and only if it is left Kan extended from \mathcal{A}^0 .

Remark 2.3.12. Original result is for a categorical fibration $p : \mathcal{C} \rightarrow \mathcal{C}'$ and p -left Kan extensions. Okay to just do this??

Chapter 3

Sheaves and K-sheaves

We follow Lurie [Lur09] closely. We write $\mathcal{U}(X)$ for the partial order of open subsets of a topological space X . Let's define sheaves and K-sheaves.

Definition 3.0.1 ([Lur09, Definition 7.3.3.1]). Let $X \in \mathbf{Top}$ and \mathcal{C} an ∞ -category. We define a \mathcal{C} -valued sheaf on X to be a presheaf $\mathcal{F} : \mathcal{U}(X)^{op} \rightarrow \mathcal{C}$ such that, for every $U \subseteq X$ and every covering sieve $\mathcal{W} \subseteq \mathcal{U}(X)_{/U}$,

$$\mathbf{N}(\mathcal{W})^\triangleright \hookrightarrow \mathbf{N}(\mathcal{U}(X)_{/U})^\triangleright \rightarrow \mathbf{N}(\mathcal{U}(X)) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

is a colimit diagram.

Preben: Is this equivalent to the following limit diagram?

$$\mathbf{N}((\mathcal{W})^{op})^\triangleleft \hookrightarrow \mathbf{N}((\mathcal{U}(X)_{/U})^{op})^\triangleleft \rightarrow \mathbf{N}(\mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

Preben: This is almost exactly definition of left Kan extension! Is it true that a presheaf is a sheaf if (and only if??) it is a left Kan extension of every possible covering sieve on X ?

We will write $\mathbf{Presh}(X, \mathcal{C})$ for the ∞ -category $\mathbf{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ of \mathcal{C} -valued presheaves on X . Moreover, we will write $\mathbf{Shv}(X; \mathcal{C})$ for the full subcategory of $\mathbf{Presh}(X; \mathcal{C})$ spanned by the \mathcal{C} -valued sheaves on X . Whenever we write $\mathbf{Shv}(X)$ without specifying the target category \mathcal{C} , we will always mean sheaves valued in spaces, i.e. $\mathbf{Shv}(X; \mathcal{S})$.

3.0.1 Sheaves on locally compact spaces

In this section we will show that for locally compact Hausdorff spaces there is an equivalence of ∞ -categories between $\mathrm{Shv}(X; \mathcal{C})$ and $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ where the latter denote so-called \mathcal{K} -sheaves and \mathcal{C} is a presentable ∞ -category with left exact filtered colimits. These are sheaves defined on the collection of compact subsets instead of the opens. Classically it is known that sheaves of sets on such spaces are determined by compact subsets as well as the opens.

TODO: Expand on this with references.

Definition 3.0.2. For a locally compact Hausdorff space X , we write $\mathcal{K}(X)$ for its collection of compact subsets.

Definition 3.0.3. If $K, K' \subseteq X$, we write $K \subseteq K'$ if there exists an open subset $U \subseteq X$ between K and K' , i.e. $K \subseteq U \subseteq K'$.

Definition 3.0.4. If $K \subseteq X$ is compact, we write $\mathcal{K}_{K \subseteq}(X)$ for the set $\{K' \in \mathcal{K}(X) \mid K \subseteq K'\}$.

Definition 3.0.5. A presheaf $\mathcal{F} : \mathbf{N}(\mathcal{K}(X))^{op} \rightarrow \mathcal{C}$ is a \mathcal{K} -sheaf if it satisfies the following:

1. $\mathcal{F}(\emptyset)$ is terminal.
2. For every pair $K, K' \in \mathcal{K}(X)$, the diagram

$$\begin{array}{ccc} \mathcal{F}(K \cup K') & \longrightarrow & \mathcal{F}(K) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(K') & \longrightarrow & \mathcal{F}(K \cap K') \end{array}$$

is a pullback in \mathcal{C} .

3. For each $K \in \mathcal{K}(X)$, $\mathcal{F}(K)$ is a colimit of $\mathcal{F}|\mathbf{N}(\mathcal{K}_{K \subseteq}(X))^{op}$.

Definition 3.0.6. We denote the full subcategory of $\mathrm{Presh}(\mathbf{N}(\mathcal{K}(X)); \mathcal{C})$ by $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$.

Lemma 3.0.7 ([Lur09, Lemma 7.3.4.8]). *Let X be locally compact and Hausdorff and \mathcal{C} be a presentable ∞ -category with left exact filtered colimits. Let \mathcal{W} be an open cover of X and denote by $\mathcal{K}_{\mathcal{W}}(X)$ the compact subsets of X that are contained in some element of \mathcal{W} . Any \mathcal{K} -sheaf $\mathcal{F} \in \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ is a right Kan extension of $\mathcal{F}|\mathbf{N}(\mathcal{K}_{\mathcal{W}}(X))^{op}$.*

Theorem 3.0.8 ([Lur09, Theorem 7.3.4.9]). *Let X be locally compact and Hausdorff and \mathcal{C} a presentable¹ ∞ -category with left exact filtered colimits. Let $\mathcal{F} : \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \rightarrow \mathcal{C}$. TFAE:*

1. *The presheaf $\mathcal{F}_{\mathcal{K}} := \mathcal{F}|\mathbf{N}(\mathcal{K}(X))^{op}$ is a \mathcal{K} -sheaf, and \mathcal{F} is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$.*
2. *The presheaf $\mathcal{F}_{\mathcal{U}} := \mathcal{F}|\mathbf{N}(\mathcal{U}(X))^{op}$ is a sheaf, and \mathcal{F} is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$.*

¹Don't really need presentable here, but need existence of small limits and colimits.

Preben: I'm pretty sure we must regard the union $\mathcal{U}(X) \cup \mathcal{K}(X)$ as a poset contained in the powerset of X here for everything to make sense

Proof.

Preben: We regard $\mathcal{K}(X)$ as partially ordered by the \subseteq relation. (I think)

We start by assuming the first condition and want to show that F is left Kan extended from $\mathbf{N}(\mathcal{U}(X))^{op}$. By definition we want to show that

$$\mathbf{N}(\mathcal{U}(X)_{/K}^{op})^{\triangleright} \hookrightarrow \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/K}^{op})^{\triangleright} \xrightarrow{c} \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram in \mathcal{C} . The assumption that $\mathcal{F}_{\mathcal{K}}$ is a \mathcal{K} -sheaf means that For each $K \in \mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}(K)$ is a colimit of $\mathcal{F}|\mathbf{N}(\mathcal{K}_{K \in \cdot}(X))^{op}$. We will "transfer" this colimit to the colimit we want by cofinal maps

$$\mathbf{N}((\mathcal{U}(X)_{/K}^{op}) \xrightarrow{p} \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/K}^{op}) \xleftarrow{p'} \mathbf{N}(\mathcal{K}_{K \in \cdot}(X))^{op}.$$

Recall that by 2.2.2 checking cofinality reduces to checking weak contractibility of certain simplicial sets. For p we must check $\mathbf{N}((\mathcal{U}(X))_{/K}^{op}) \times_{\mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/K}^{op})} \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/K'}^{op})_{K'}$ is weakly contractible for every $K' \in \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/K}^{op})$. This is the simplicial set obtained by taking the nerve of the partially ordered set $\{U \in \mathcal{U}(X) \mid K \subseteq U \subseteq K'\}$. By [Lur09, Lemma 5.3.1.20] filtered ∞ -categories are weakly contractible, and our partially ordered set is filtered as it is nonempty and stable under finite union and taking nerves preserve the property of being filtered. The simplicial set $\mathbf{N}(\{K'' \in \mathcal{K}(X) \mid K \subseteq K'' \subseteq K'\})$ is weakly contractible by the exact same argument, and hence p and p' are cofinal maps. By cofinality of p and p' , the diagram

$$\mathbf{N}((\mathcal{U}(X))_{/K}^{op})^{\triangleright} \hookrightarrow \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/K}^{op})^{\triangleright} \xrightarrow{c} \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram if and only if

$$\mathbf{N}((\mathcal{K}(X)_{\in K})^{op})^{\triangleright} \hookrightarrow \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/K}^{op})^{\triangleright} \xrightarrow{c} \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram, which it is by the assumption that $\mathcal{F}_{\mathcal{K}}$ is a \mathcal{K} -sheaf. We now show $\mathcal{F}_{\mathcal{U}}$ is a sheaf. By definition 3.0.1 we must show that for every $U \in \mathcal{U}(X)$ and every covering sieve \mathcal{W} covering U ,

$$\mathbf{N}(\mathcal{W})^{\triangleright} \hookrightarrow \mathbf{N}(\mathcal{U}(X)_{/U})^{\triangleright} \rightarrow \mathbf{N}(\mathcal{U}(X)) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

is a colimit diagram. We will once again use cofinality by observing that 2.2.2 implies cofinality of the inclusion

$$\mathbf{N}(\mathcal{W}) \subseteq \mathbf{N}(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))$$

where $\mathcal{K}_{\mathcal{W}}(X)$ is the set $\{K \in \mathcal{K}(X) \mid (\exists W \in \mathcal{W})(K \subseteq W)\}$.

Preben: Might be good to write out the details of why this is contractible.

Observe that $\text{RKan}(\mathcal{F}|(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))^{op}) = \mathcal{F}|(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op}$ together with the cofinal inclusion above implies the limit.

Preben: I have not found a nice proof of this implication, but I think the Kan extension is a stronger statement.

By the assumption that \mathcal{F} is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ and the observation that

$$\mathcal{F}(U) = \lim_{K \in \mathcal{K}(X)_{/U}^{op}} \mathcal{F}(K) = \lim_{K \in \mathcal{K}_{\mathcal{W}}(X)_{/U}^{op}} \mathcal{F}(K) = K'_* \mathcal{F}(U)$$

we see that $\mathcal{F}|(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}})^{op}$ is a right Kan extension of $\mathcal{F}|(\mathcal{K}_{\mathcal{W}}(X))^{op}$. Hence, it suffices to prove that $\mathcal{F}|(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. Outside of U this is clear from the fact that $\mathcal{F}|(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}})^{op}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. This means we only need to prove $\mathcal{F}|(\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. Observe that by assumption

$$\mathcal{F}(U) = \lim_{K \in \mathcal{K}(X)_{/U}^{op}} \mathcal{F}(K) = \lim_{K \in (\mathcal{K}(X)_{/U} \cup \{U\})^{op}} \mathcal{F}(K)$$

so $\mathcal{F}|(\mathcal{K}(X)_{/U} \cup \{U\})^{op}$ is right Kan extended from $\mathcal{K}(X)_{/U}^{op}$. Lemma 3.0.7 tells us that $\mathcal{F}|(\mathcal{K}(X)_{/U})^{op}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. We have $\mathbf{N}(\mathcal{K}_{\mathcal{W}}(X))^{op} \subseteq \mathbf{N}(\mathcal{K}(X)_{/U})^{op} \subseteq \mathbf{N}(\mathcal{K}(X)_{/U} \cup \{U\})^{op}$, with Kan extensions as in proposition 2.3.11, so we get that $\mathcal{F}|_{\mathbf{N}(\mathcal{K}(X)_{/U} \cup \{U\})^{op}}$ is right Kan extended from $\mathbf{N}(\mathcal{K}_{\mathcal{W}}(X))^{op}$. To summarize, we have the following square of inclusions

$$\begin{array}{ccc} \mathcal{K}_{\mathcal{W}}(X)^{op} & \xhookrightarrow{i} & (\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op} \\ \downarrow a & & \downarrow b \\ \mathcal{K}(X)_{/U}^{op} & \xhookrightarrow{j} & (\mathcal{K}(X)_{/U} \cup \{U\})^{op} \end{array}$$

where $\mathcal{F}|(\mathcal{K}(X)_{/U} \cup \{U\})^{op} \simeq b_*(\mathcal{F}|(\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op})$ and $\mathcal{F}|(\mathcal{K}(X)_{/U})^{op} \simeq (j \circ a)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$. We want to show $\mathcal{F}|(\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op} \simeq i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$.

TODO: Notation i_* comes from it being right adjoint to i^* . Write about adjoint description of Kan extensions in Kan chapter.

Preben: This is probably obvious or easier to show than what I am about to do, so please let me know.

We use the commutativity of the square and the adjoint relationship between Kan extensions

and “pullbacks” and get:

$$\mathcal{C}^{(\mathcal{K}_W(X) \cup \{U\})^{op}}(b^*(-), \mathcal{F}|_{(\mathcal{K}_W(X) \cup \{U\})^{op}}) \simeq \mathcal{C}^{(\mathcal{K}(X)/U \cup \{U\})^{op}}(-, \mathcal{F}) \quad (3.1)$$

$$\simeq \mathcal{C}^{(\mathcal{K}(X)/U \cup \{U\})^{op}}(-, a_*(\mathcal{F}|_{\mathcal{K}_W(X)^{op}})) \quad (3.2)$$

$$\simeq \mathcal{C}^{\mathcal{K}_W(X)^{op}}(a^*(-), \mathcal{F}|_{\mathcal{K}_W(X)^{op}}) \quad (3.3)$$

$$\simeq \mathcal{C}^{\mathcal{K}_W(X)^{op}}((b \circ i)^*(-), \mathcal{F}|_{\mathcal{K}_W(X)^{op}}) \quad (3.4)$$

$$\simeq \mathcal{C}^{\mathcal{K}_W(X)^{op}}(i^* \circ b^*(-), \mathcal{F}|_{\mathcal{K}_W(X)^{op}}) \quad (3.5)$$

$$\simeq \mathcal{C}^{(\mathcal{K}_W(X) \cup \{U\})^{op}}(b^*(-), i_*(\mathcal{F}|_{\mathcal{K}_W(X)^{op}})) \quad (3.6)$$

and Yoneda’s lemma tells us that

$$\mathcal{C}^{(\mathcal{K}_W(X) \cup \{U\})^{op}}(b^*(-), \mathcal{F}|_{(\mathcal{K}_W(X) \cup \{U\})^{op}}) \simeq \mathcal{C}^{(\mathcal{K}_W(X) \cup \{U\})^{op}}(b^*(-), i_*(\mathcal{F}|_{\mathcal{K}_W(X)^{op}}))$$

$$\Rightarrow \mathcal{F}|_{(\mathcal{K}_W(X) \cup \{U\})^{op}} \simeq i_*(\mathcal{F}|_{\mathcal{K}_W(X)^{op}}).$$

In conclusion we have shown that \mathcal{F}_U is a sheaf, so 1. implies 2..

Preben: This calculation could maybe be a lemma in the Kan chapter instead.

For the other direction we assume 2. and want to show \mathcal{F}_K is a \mathcal{K} -sheaf. By definition we need to show three things: Firstly, observe that $\mathcal{F}_K(\emptyset) = \mathcal{F}_U(\emptyset)$ and since \mathcal{F}_U is a sheaf $\mathcal{F}_K(\emptyset)$ is terminal. Secondly, we need the following diagram to be a pullback in \mathcal{C} for any $K, K' \in \mathcal{K}(X)$.

$$\begin{array}{ccc} \mathcal{F}(K \cup K') & \longrightarrow & \mathcal{F}(K) \\ \downarrow & & \downarrow \\ \mathcal{F}(K') & \longrightarrow & \mathcal{F}(K \cap K') \end{array} \quad (3.7)$$

We will do this by using that \mathcal{F}_U is a sheaf. Let $P = \{(U, U') | K \subseteq U, K' \subseteq U'\}$ and $\sigma : \Delta^1 \times \Delta^1$ denote diagram 3.7. Now \mathcal{F} induces a map $\sigma_P : \mathbf{N}(P^{op})^\triangleright \rightarrow \mathcal{C}^{\Delta^1 \times \Delta^1}$ taking each (U, U') to

$$\begin{array}{ccc} \mathcal{F}(U \cup U') & \longrightarrow & \mathcal{F}(U) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(U') & \longrightarrow & \mathcal{F}(U \cap U') \end{array}$$

and the cone point is sent to σ . This is a pullback by the fact that \mathcal{F}_U is a sheaf. By evaluating σ_P in each of the four vertices of $\Delta^1 \times \Delta^1$ we get four maps $\mathbf{N}(P^{op})^\triangleright \rightarrow \mathcal{C}$. We now check that evaluating in the final vertex yields a colimit diagram. By assumption \mathcal{F} is a left Kan extension of \mathcal{F}_U which by definition means that the following is a colimit diagram:

$$\mathbf{N}((\mathcal{U}(X)/(K \cap K'))^{op})^\triangleright \hookrightarrow \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/(K \cap K')}^{op})^\triangleright \xrightarrow{c} \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

Observe that for every $U'' \in \mathcal{U}(X)_{/(K \cap K')}$, the set $P_{U''} = \{(U, U') \in P \mid U \cap U' \subseteq U''\}$ is nonempty and stable under finite intersections, which implies it is filtered and hence its nerve is contractible.

TODO: Reference this result.

By 2.2.2 this implies $\mathbf{N}(P^{op}) \rightarrow \mathbf{N}((\mathcal{U}(X)_{/(K \cap K')})^{op})$ is cofinal and we get a colimit diagram

$$\mathbf{N}(P^{op})^{\triangleright} \rightarrow \mathbf{N}((\mathcal{U}(X)_{/(K \cap K')})^{op})^{\triangleright} \hookrightarrow \mathbf{N}((\mathcal{U}(X) \cup \mathcal{K}(X))_{/(K \cap K')}^{op})^{\triangleright} \xrightarrow{c} \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}.$$

We can show that evaluating the three other vertices also yields colimit diagrams by similar arguments.

Preben: I should maybe do this for all four vertices, or at least check how different the argument is for another vertex, but it looks pretty similar.

Since σ_P yields a colimit diagram when evaluated in each of the four vertices of $\Delta^1 \times \Delta^1$, we conclude that σ_P is itself a colimit diagram by [Lur09, Proposition 5.1.2.2]. Observe now that σ_P is a filtered colimit in \mathcal{C} and hence it is left exact. This concludes the argument that 3.7 is a pullback. Finally, we need to show that for each $K \in \mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}$ is a colimit of $\mathcal{F}_{\mathcal{K}}|_{\mathbf{N}(\mathcal{K}_{K \in}(X))^{op}}$. We do this by showing

$$\mathbf{N}(\mathcal{K}_{K \in}(X))^{op} \rightarrow \mathbf{N}(\mathcal{K}(X) \cup \mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. We use Proposition 2.3.11 to show that $\mathcal{F}|_{\mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}_{K \in}(X))^{op}}$ and $\mathcal{F}|_{\mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}_{K \in}(X))^{op} \cup \{K\}}$ are left Kan extensions of $\mathcal{F}|_{\mathbf{N}(\mathcal{U}(X))^{op}}$ which again implies $\mathcal{F}|_{\mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}_{K \in}(X))^{op} \cup \{K\}}$ is a left Kan extension of $\mathcal{F}|_{\mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}_{K \in}(X))^{op}}$. Now observe

$$\mathbf{N}(\mathcal{K}(X) \cup \mathcal{U}(X))_{/K}^{op, \triangleright} = \mathbf{N}(\mathcal{K}_{K \in}(X) \cup \mathcal{U}(X)_{/K})^{op, \triangleright},$$

so

$$\mathbf{N}(\mathcal{K}(X) \cup \mathcal{U}(X))_{/K}^{op, \triangleright} \rightarrow \mathbf{N}(\mathcal{K}(X) \cup \mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram and the statement reduces to showing $\mathbf{N}(\mathcal{K}_{K \in}(X)) \subseteq \mathbf{N}(\mathcal{K}(X) \cup \mathcal{U}(X))_{/K}^{op}$ is cofinal. Let $Y \in \mathbf{N}(\mathcal{K}(X) \cup \mathcal{U}(X))_{/K}$ and let R be the partially ordered set $\{K' \in \mathcal{K}(X) \mid K \in K' \subseteq Y\}$. Since R is nonempty and stable under intersections R^{op} is filtered and hence $\mathbf{N}(R)$ is weakly contractible. By Lemma 2.2.2 the inclusion is cofinal and we have shown that $\mathcal{F}_{\mathcal{K}}$ is a \mathcal{K} -sheaf.

We will show \mathcal{F} is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ in a similar manner to how we showed \mathcal{F} is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$ in the start of the proof, but we will consider the partial order on $\mathcal{U}(X)$ given by writing $V \Subset U$ whenever $V \in \mathcal{U}(X)$ and its closure \overline{V} is compact and contained in U . Writing $\mathcal{U}(X)_{U/}$ for the set $\{V \in \mathcal{U}(X) \mid V \Subset U\}$, we need to show that

$$\mathbf{N}(\mathcal{K}(X)_{U/}^{op})^{\triangleleft} \hookrightarrow \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))_{U/}^{op} \xrightarrow{c} \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. As earlier we do this by finding cofinal inclusions

$$\mathbf{N}(\mathcal{K}(X)_{U/}^{op}) \xrightarrow{f} \mathbf{N}(\mathcal{U}(X) \cup \mathcal{K}(X))_{U/}^{op} \xleftarrow{f'} \mathbf{N}(\mathcal{K}(X)_{/U})^{op}.$$

By Lemma 2.2.2 f and f' are cofinal inclusions if for any $Y \in (\mathcal{U}(X) \cup \mathcal{K}(X))_{U/}$ the partially ordered sets

$$\{V \in \mathcal{U}(X) | Y \subseteq V \subseteq U\}$$

and

$$\{K \in \mathcal{K}(X) | Y \subseteq K \subseteq U\}$$

have weakly contractible nerves, which they have by the usual argument; they are nonempty and stable under unions, hence filtered. Since $\mathcal{U}(X)_{U/}$ is a sieve covering U and \mathcal{F}_U is a sheaf,

$$\mathbf{N}(\mathcal{U}(X)_{U/})^{op} \rightarrow \mathbf{N}(\mathcal{U}(X)_{U/})^{op, \triangleleft} \rightarrow \mathcal{C}$$

is a colimit diagram and this completes the proof that \mathcal{F} is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$. \square

Corollary 3.0.8.1. *Let X be a locally compact Hausdorff space and \mathcal{C} an ∞ -category with left exact filtered colimits, then $\mathrm{Shv}(X; \mathcal{C}) \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ is an equivalence of ∞ -categories.*

Proof. Let $\mathrm{Shv}_{\mathcal{KU}}(X; \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathbf{N}(\mathcal{K}(X) \cup \mathcal{U}(X))^{op}, \mathcal{C})$ spanned by those presheaves satisfying the equivalent conditions of theorem 3.0.8. We get restrictions

$$\mathrm{Shv}(X; \mathcal{C}) \leftarrow \mathrm{Shv}_{\mathcal{KU}}(X; \mathcal{C}) \rightarrow \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}),$$

and these are equivalences of ∞ -categories because inclusions of full subcategories are fully faithful and Kan extensions along fully faithful functors give isomorphisms.

(K fully faithful means $G \cong \mathrm{Lan}_K(G)K$.) \square

Chapter 4

Verdier Duality

4.1 Classical Verdier Duality

I don't know, maybe write some shit about regular Verdier Duality for 1-categories and stuff. Blablabla k a field and A the category of chain complexes of k -vector spaces. Vector space duality gives a limit preserving functor $N(A^{op}) \rightarrow N(A)$ which induces a functor

$$\mathrm{Shv}(X; N(A)^{op}) \rightarrow \mathrm{Shv}(X; N(A))$$

for any locally compact Hausdorff space. Composing with the equivalence below yields a functor

$$\mathbb{D}' : \mathrm{Shv}(X; N(A))^{op} \rightarrow \mathrm{Shv}(X; N(A))$$

and it is this functor that is usually called Verdier Duality. This is not necessarily an equivalence of ∞ -categories unless certain finiteness conditions are imposed.

4.2 Verdier Duality in infinity-categories

This chapter is all about proving the following theorem:

Theorem 4.2.1 ([Lur17, Theorem 5.5.5.1]). *Let X be a locally compact Hausdorff space and \mathcal{C} be a stable ∞ -category with small limits and colimits. Then we have the following equivalence of ∞ -categories*

$$\mathbb{D} : \mathrm{Shv}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}(X; \mathcal{C}^{op}).$$

We will be using the theory of \mathcal{K} -sheaves set up in the previous chapter to prove the theorem. By corollary 3.0.8.1 we can rewrite theorem 4.2.1 in terms of \mathcal{K} -sheaves instead:

Theorem 4.2.2. Let X be a locally compact Hausdorff space and \mathcal{C} be a stable ∞ -category with small limits and colimits. Then we have the following equivalence of ∞ -categories:

$$\mathbb{D}_{\mathcal{K}} : \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}).$$

Definition 4.2.3 ([Lur17, Notation 5.5.5.5]). Let X be a locally compact Hausdorff space. We define a partially ordered set M as follows:

1. The objects of M are pairs (i, S) where $0 \leq i \leq 2$ and $S \subseteq X$ such that $i = 0$ implies S is compact and $i = 2$ implies $X - S$ is compact.
2. We have $(i, S) \leq (j, T)$ if either $i \leq j$ and $S \subseteq T$, or $i = 0$ and $j = 2$.

Remark 4.2.4 ([Lur17, Remark 5.5.5.6]). Observe that projecting $(i, S) \rightarrow i$ gives a map $\varphi : M \rightarrow [2]$ of partially ordered sets. For $0 \leq i \leq 2$ denote the fiber $\varphi^{-1}\{i\}$ by M_i . Also, observe that $M_0 \simeq \mathcal{K}(X)$, $M_2 \simeq \mathcal{K}(X)^{op}$ and M_1 is isomorphic to the powerset poset of X .

Definition 4.2.5. Let M' denote the partially ordered sets of pairs (i, S) , where $0 \leq i \leq 2$ and $S \subseteq X$ such that $i = 0$ implies S is compact and $i = 2$ implies $X - S$ is either open or compact. Let $(i, S) \leq (j, T)$ if $i \leq j$ and $S \subseteq T$ or if $i = 0$ and $j = 2$. For $0 \leq i \leq 2$, let M'_i denote the subset $\{(j, S) \in M' \mid j = i\} \subseteq M'$.

Let's see if we can connect this new notion of Verdier duality to the classical notion of exchanging cohomology with cohomology with compact support.

Definition 4.2.6 ([Lur17, Definition 5.5.5.9]). Let X be a locally compact Hausdorff space and \mathcal{C} a pointed ∞ -category with small limits and colimits. For a sheaf $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ and K compact we denote by $\Gamma_K(X; \mathcal{F})$ the fiber product $\mathcal{F}(X) \times_{\mathcal{F}(X-K)_0}$. For U open, we denote by $\Gamma_c(U; \mathcal{F})$ the filtered colimit $\mathrm{colim}_{K \subseteq U} \Gamma_K(X; \mathcal{F})$ where K ranges over all compact subsets of U .

Preben: Let's be consistent on whether we write $\mathrm{colim}_{K \subseteq U}$ or $\mathrm{colim}_{\mathcal{K}(X)/U}$.

Preben: Lurie writes $\mathrm{colim}_{K \subseteq U} \Gamma_K(M; \mathcal{F})$, but I think that is a mistake.

The construction $U \mapsto \Gamma_c(U; \mathcal{F})$ determines a functor

$$\Gamma_c(-; \mathcal{F}) : \mathbf{N}(\mathcal{U}(X)) \rightarrow \mathcal{C}.$$

Remark 4.2.7. Observe that for K a compact subset of an open subset U we have $\Gamma_K(X; \mathcal{F}) = \Gamma_K(U; \mathcal{F})$.

Proof. Because \mathcal{F} is a sheaf we have pullbacks

$$\begin{array}{ccccc} \mathrm{fib}(f) & \longrightarrow & \Gamma(X; \mathcal{F}) & \longrightarrow & \Gamma(U; \mathcal{F}) \\ \downarrow & \lrcorner & \downarrow f & \lrcorner & \downarrow g \\ 0 & \longrightarrow & \Gamma(X - K; \mathcal{F}) & \longrightarrow & \Gamma(U - K; \mathcal{F}) \end{array}$$

and as the composition of pullbacks is again a pullback we get $\Gamma_K(X; \mathcal{F}) = \text{fib}(f) = \text{fib}(g) = \Gamma_K(U; \mathcal{F})$. \square

Lemma 4.2.8 ([Lur17, Proposition 5.5.5.7]). *Let X be a locally compact Hausdorff space, \mathcal{C} be a stable ∞ -category with small limits and colimits and M be as in 4.2.3. Let $F : \mathbf{N}(M) \rightarrow \mathcal{C}$ be a functor. Then the following conditions are equivalent:*

1. *The restriction $(F|_{\mathbf{N}(M_0)})^{op}$ determines a \mathcal{K} -sheaf $\mathbf{N}(\mathcal{K}(X))^{op} \rightarrow \mathcal{C}^{op}$, the restriction $F|_{\mathbf{N}(M_1)}$ is zero, and F is left Kan extended from $\mathbf{N}(M_0 \cup M_1)$.*
2. *The restriction $F|_{\mathbf{N}(M_2)}$ determines a \mathcal{K} -sheaf $\mathbf{N}(\mathcal{K}(X))^{op} \rightarrow \mathcal{C}$, the restriction $F|_{\mathbf{N}(M_1)}$ is zero, and F is right Kan extended from $\mathbf{N}(M_1 \cup M_2)$.*

Proof. First observe that the map $(i, S) \mapsto (2-i, X-S)$ is an order-reversing bijection $M \rightarrow M$ which is moreover self-inverse. This means that it is enough to show that condition 2. implies condition 1.. We start by assuming condition 2., so let $F : \mathbf{N}(M) \rightarrow \mathcal{C}$ be such a functor. Let \mathcal{D} denote the full subcategory of $\text{Fun}(\mathbf{N}(M'), \mathcal{C})$ spanned by those functors F satisfying the following conditions:

1. $F|_{\mathbf{N}(M_2)}$ is a \mathcal{K} -sheaf on X .
2. $F|_{\mathbf{N}(M'_2)}$ is a right Kan extension of $F|_{\mathbf{N}(M_2)}$.
3. $F|_{\mathbf{N}(M'_1)}$ is zero.
4. $F|_{\mathbf{N}(M')}$ is a right Kan extension of $F|_{\mathbf{N}(M'_1 \cup M'_2)}$.

By [Lur09, Proposition 4.3.2.15] we can extend F to a functor $F' \in \mathcal{D}$.

Preben: This should just boil down to fully faithful Kan extensions along fully faithful functors give actual on the nose extensions, but might be smart to ref the result.

Observe that we have a bijection between $\mathcal{U}(X)^{op}$ and the partially ordered set of closed subsets of X by sending $\mathcal{U}(X) \ni U \mapsto (X - U)$ and we have a natural inclusion $\mathcal{U}(X)^{op} \hookrightarrow M'_2$. By Theorem 3.0.8 we can restrict $\text{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ to $\text{Shv}(X; \mathcal{C})$, so we can also restrict M'_2 and even \mathcal{D} to $\text{Shv}(X; \mathcal{C})$. Let \mathcal{F} be the sheaf obtained by restricting F' . We will first prove that $F|_{\mathbf{N}(M_0)}$ is given informally by the formula $F|_{\mathbf{N}(M_0)}(K) = \Gamma_K(X; \mathcal{F})$.

Preben: Maybe have this as a separate lemma.

Define $\varphi : \mathbf{N}(M_0) \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathbf{N}(M'))$ by sending an object $(0, K) \in M_0$ to the diagram

$$\begin{array}{ccc} (0, K) & \longrightarrow & (1, K) \\ \downarrow & & \downarrow \\ (2, \emptyset) & \longrightarrow & (2, K). \end{array}$$

We can regard $\varphi(0, K)$ as a map $i : \Lambda_2^2 \rightarrow (M'_1 \cup M'_2)_{(0,K)}/$:

$$\begin{array}{ccc}
 & a & (0, K) \rightarrow (1, K) \\
 & \downarrow & \downarrow \\
 b & \longrightarrow & c & \xrightarrow{\quad} & (0, K) \rightarrow (2, \emptyset) & \longrightarrow & (0, K) \rightarrow (2, K)
 \end{array}$$

Here we have abused notation to write the fiber product $\mathbf{N}(M')_{(0,K)/} \times_{\mathbf{N}(M')} \mathbf{N}(M'_1 \cup M'_2)$ as $\mathbf{N}(M'_1 \cup M'_2)_{(0,K)}/$. By 2.2.2 i is cofinal if and only if for every $(m, A) \in \mathbf{N}(M'_1 \cup M'_2)_{(0,K)}/$ the fiber product

$$\begin{array}{ccc}
 \text{PB} & \longrightarrow & (\mathbf{N}(M'_1 \cup M'_2)_{(0,K)}/)_{/(m,A)} \\
 \downarrow & \lrcorner & \downarrow j \\
 \Lambda_2^2 & \xrightarrow{i} & \mathbf{N}(M'_1 \cup M'_2)_{(0,K)}/
 \end{array}$$

is weakly contractible. As we have partially ordered sets As j is just the inclusion we have $j((0, K) \rightarrow (r, B) \rightarrow (m, A)) = ((0, K) \rightarrow (r, B))$. Since $i(a) = (2, \emptyset)$, $i(b) = (1, K)$ and $i(c) = (2, K)$ and the pullbacks of monos are mono PB has to be a subcategory of

$$\begin{array}{ccc}
 & (b, (1, K)) & \\
 & \downarrow & \\
 (a, (2, \emptyset)) & \longrightarrow & (c, (2, K)).
 \end{array}$$

Observe that such a subcategory only fails to be contractible if (m, A) is such that the pullback is either empty or consist of two disjoint objects. If $r = 1$ we know $(1, K) \leq (1, B)$ and have no arrows from $(2, \emptyset)$ or $(2, K)$ to $(1, B)$ and if $r = 2$ we must have $(2, \emptyset) \leq (2, K) \leq (2, B)$, so the pullback is always weakly contractible. By condition 4. $F'|_{\mathbf{N}(M')}$ is right Kan extended from $F'|_{\mathbf{N}(M'_1 \cup M'_2)}$ which by definition 2.3.10 means

$$(M'_1 \cup M'_2)_{(0,K)}/^{\triangleleft} \hookrightarrow M'_{(0,K)}/^{\triangleleft} \rightarrow M' \xrightarrow{F'} \mathcal{C}$$

is a limit diagram. In other words we have

$$\lim_{\mathbf{N}(M'_1 \cup M'_2)_{(0,K)}/} F' = F'(0, K)$$

and by the left cofinality of i we get

$$F'(0, K) = \lim_{\mathbf{N}(M'_1 \cup M'_2)_{(0,K)}/} F' = \lim_{\Lambda_2^2} (F' \circ i) = \lim_{\Lambda_2^2} F'((2, \emptyset) \rightarrow (2, K) \leftarrow (1, K))$$

which means condition 4. is equivalent to requiring that F' composed with $\varphi(0, K)$ yields another pullback diagram

$$\begin{array}{ccc} F'(0, K) & \longrightarrow & F'(1, K) \\ \downarrow & \lrcorner & \downarrow \\ F'(2, \emptyset) & \longrightarrow & F'(2, K) \end{array}$$

Observe now that by condition 3. $F'(1, K) = 0$ and hence

$$F'(0, K) = F(0, K) \simeq \text{fib}(F(2, \emptyset) \rightarrow F(2, K)).$$

Recall that we defined \mathcal{F} as the restriction of F' to $\text{Shv}(X; \mathcal{C})$ by identifying open sets U with their complements. This means that $F'(2, \emptyset) = \mathcal{F}(X)$ and $F'(2, K) = \mathcal{F}(X - K)$ which in turn means that

$$F'(0, K) = F(0, K) \simeq \text{fib}(F(2, \emptyset) \rightarrow F(2, K)) \simeq \text{fib}(\mathcal{F}(X) \rightarrow \mathcal{F}(X - K)) = \Gamma_K(X; \mathcal{F})$$

which completes the proof that $F|_{\mathbf{N}(M_0)}(K)$ is given by $\Gamma_K(X; \mathcal{F})$.

Let us now denote $F|_{\mathbf{N}(M_0)}(K)$ by \mathcal{G} and check that $\mathcal{G}^{op} \in \text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})$. Here \mathcal{G}^{op} does the same on objects as \mathcal{G} but we think of it as a functor $\mathbf{N}(M_0)^{op} \rightarrow \mathcal{C}^{op}$. We must show that it satisfies the following three properties:

1. $\mathcal{G}(\emptyset)$ is a zero object, because $\mathcal{G}(\emptyset) = \text{fib}(\mathcal{F}(X) \rightarrow \mathcal{F}(X - \emptyset)) = 0$.
2. For any compact subsets K and K' of X we must show the following diagram is a pushout:

$$\begin{array}{ccc} \mathcal{G}(K \cap K') & \longrightarrow & \mathcal{G}(K') \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{G}(K) & \longrightarrow & \mathcal{G}(K \cup K') \end{array}$$

Observe that the diagram can be identified with the fiber of the map

$$\begin{array}{ccccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X) & & \mathcal{F}(X - (K \cap K')) & \longrightarrow & \mathcal{F}(X - K') \\ \downarrow & \lrcorner & \downarrow & \longrightarrow & \downarrow & \lrcorner & \downarrow \\ \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X) & & \mathcal{F}(X - K) & \longrightarrow & \mathcal{F}(X - (K \cup K')) \end{array}$$

and because \mathcal{F} is a sheaf this is a map between pullbacks so our diagram is also a pullback.

3. For any compact subset K of X we must show that the map $\theta : \mathcal{G}(K) \rightarrow \lim_{K \in K'} \mathcal{G}(K')$ is an equivalence in \mathcal{C} . Observe now that θ gives us a map between two fiber sequences

$$\begin{array}{ccc}
 \mathcal{G}(K) & \xrightarrow{\theta} & \lim_{K \in K'} \mathcal{G}(K') \\
 \downarrow & & \downarrow \\
 \mathcal{F}(X) & \xrightarrow{\theta'} & \lim_{K \in K'} \mathcal{F}(X) \\
 \downarrow & & \downarrow \\
 \mathcal{F}(X - K) & \xrightarrow{\theta''} & \lim_{K \in K'} \mathcal{F}(X - K')
 \end{array}$$

Since the partially ordered set $\{K' \in \mathcal{K}(X) | K \subseteq K'\}$ is filtered it is weakly contractible and hence θ' is an equivalence. Since \mathcal{F} is a sheaf and the set $\{X - K' | K \subseteq K'\}$ is a covering sieve on $X - K$, θ'' is also an equivalence. As we have shown that θ' and θ'' are equivalences, θ must also be an equivalence, and we have shown that \mathcal{G}^{op} determines a \mathcal{K} -sheaf $\mathbf{N}(\mathcal{K}(X)^{op} \rightarrow \mathcal{C}^{op})$.

To complete the proof that 2. implies 1., we must show F is left Kan extended from $F|_{\mathbf{N}(M_0 \cup M_1)}$. Let $M'' = \{(i, S) \in M_0 \cup M_1 | (i, S) \in \mathcal{K}(X)\}$. We can observe that $F|_{\mathbf{N}(M_0 \cup M_1)}$ is left Kan extended from $F|_{\mathbf{N}(M'')}$ (F is zero on M_1). By Proposition 2.3.11 it is enough to show that F is a left Kan extension of $F|_{\mathbf{N}(M'')}$ and this is enough to check at every $(2, S) \in M_2$. We will instead show that $F'|_{\mathbf{N}(M'' \cup M'_2)}$ is a left Kan extension of $F'|_{\mathbf{N}(M'')}$ and for this we define

$$B := \{(2, X - U) \subseteq M'_2 | U \in \mathcal{U}(X) | \bar{U} \in \mathcal{K}(X)\}.$$

By Proposition 2.3.11 it is enough to show that

- (a) $F'|_{\mathbf{N}(M'' \cup M'_2)}$ is a left Kan extension of $F'|_{\mathbf{N}(M'') \cup B}$.

First observe that M'' and M'_2 are disjoint so it is enough to check that for every $(2, X - K) \in M'_2 - B$, the following

$$\mathbf{N}(M'' \cup B)_{/(2, X - K)}^{\triangleright} \hookrightarrow \mathbf{N}(M'' \cup M'_2)_{/(2, X - K)}^{\triangleright} \rightarrow \mathbf{N}(M'' \cup M'_2) \rightarrow \mathcal{C}$$

is a colimit diagram. According to Lemma 2.2.2 we can restrict the colimit from $(M'' \cup B)_{/(2, X - K)}$ to $B_{/(2, X - K)}$ if we can show that the pullback

$$\begin{array}{ccc}
 PB & \longrightarrow & ((M'' \cup B)_{/(2, X - K)})_{/(2, X - U)} \\
 \downarrow & & \downarrow \\
 (B_{/(2, X - K)})_{/(2, X - U)} & \longrightarrow & (M'' \cup B)_{/(2, X - K)}
 \end{array}$$

is weakly contractible. As this is just the partially ordered set $\{(2, X - U) \in B \mid (i, S) \leq (2, X - U) \leq (2, X - K)\}$ it is weakly contractible by the usual argument (it is nonempty and stable under finite unions hence filtered). This means that it is enough to show that $F'|_{\mathbf{N}(M'_2)}$ is left Kan extended from B . Assumption 2. says that $F|_{\mathbf{N}(M_2)}$ determines a \mathcal{K} -sheaf, $F|_{\mathbf{N}(M_1)} = 0$ and that F is a right Kan extension from $\mathbf{N}(M_1 \cup M_2)$. Identifying $M_2 = \{(2, S) \mid (X - S) \in \mathcal{K}(X)\}$ with $\mathcal{K}(X)^{op}$ we see that we are in the situation of Theorem 3.0.8. As $M'_2 = \{(2, S) \mid (X - S) \in \mathcal{U}(X) \cup \mathcal{K}(X)\}$ we can identify it with $(\mathcal{U}(X) \cup \mathcal{K}(X))^{op}$ and by Theorem 3.0.8 we get that $F'|_{\mathbf{N}(M'_2)}$ is a left Kan extension of $F'|_{\mathbf{N}(\mathcal{U}(X)^{op})}$. By observing that for a $K \in \mathcal{K}(X)$ the collection of open neighborhoods of K with compact closure is cofinal in the collection of all open neighborhoods of K in X we get that $F'|_{\mathbf{N}(M'_2)}$ is furthermore left Kan extended from B , which was what we wanted to show.

$$\begin{array}{ccccc}
 B & \xhookrightarrow{i} & \mathcal{U}(X)^{op} & \xrightarrow{\quad} & \mathcal{C} \\
 & & \searrow & & \nearrow \\
 & & M'_2 & &
 \end{array}$$

Preben: My thoughts here are that we calculate Kan extensions as colimits, so i being cofinal over some fixed K means restricting the colimit from $\mathcal{U}(X)^{op}$ back to B is an equivalence.

Preben: Lurie never states what he means with M''' , but I think he might just mean B .

- (b) $F'|_{\mathbf{N}(M'' \cup B)}$ is a left Kan extension of $F|_{\mathbf{N}(M'')}$. Fix $U \in \mathcal{U}(X)$ such that $\bar{U} \in \mathcal{K}(X)$. By 2.3.10 we want to show that $F'(2, X - U)$ is a colimit of the diagram $F'|_{\mathbf{N}(M'')/(2, X - U)}$. For $K \in \mathcal{K}(X)$ denote by M''_K the subset of M'' consisting of pairs (i, S) such that $(0, K) \leq (i, S) \leq (2, X - U)$. Now, observe that $\mathbf{N}(M'')/(2, X - U)$ is a filtered colimit of $\{\mathbf{N}(M''_K)\}_{K \in \mathcal{K}(X)_{U/}}$. By [Lur09][4.2.3] we can identify $\text{colim}(F'|_{\mathbf{N}(M'')/(2, X - U)})$ with $\text{colim}\{\text{colim}(F'|_{\mathbf{N}(M''_K)})\}_K$. This means that we are reduced to showing that for every $K \in \mathcal{K}(X)_{U/}$, F' exhibits $F'(2, X - U)$ as a colimit of $F'|_{\mathbf{N}(M''_K)}$. By Lemma 2.2.2 the diagram

$$\begin{array}{ccc}
 (0, K - U) & \longrightarrow & (1, K - U) \\
 \downarrow & & \\
 (0, K) & &
 \end{array}$$

is left cofinal in $N(M_K'')$ and hence¹ it is enough to show that

$$\begin{array}{ccc} F'(0, K - U) & \longrightarrow & F'(1, K - U) \\ \downarrow & & \downarrow \\ F'(0, K) & \longrightarrow & F'(2, X - U) \end{array}$$

is a pushout in \mathcal{C} . We will show this by considering the larger diagram

$$\begin{array}{ccccccc} F'(0, K - U) & \longrightarrow & F'(1, K - U) = 0 & & & & \\ \downarrow & & \downarrow & & & & \\ F'(0, K) & \longrightarrow & Z & \xrightarrow{\quad \lrcorner \quad} & F(1, K) = 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ F'(2, \emptyset) & \longrightarrow & F(2, K - U) & \longrightarrow & F(2, K) & & \\ & & \downarrow & & \downarrow & & \\ & & F(2, X - U) & \longrightarrow & F(2, X) & & \end{array}$$

where we already know that the middle rectangle is a pullback (we have shown $F'(0, K)$ to be the fiber of the map $F'(2, \emptyset) \rightarrow F(2, K)$), so the middle left square is also a pullback. As we have shown $F'(0, K - U) = \text{fib}(F(2, \emptyset) \rightarrow F(2, K - U))$ the left vertical rectangle is also a pullback, so the upper left must be as well. Since \mathcal{C} is stable it is also a pushout. As $F(1, K) = 0$ and $F(2, X) = \mathcal{F}(\emptyset) = 0$ we have an equivalence $F(1, K) \rightarrow F(2, K) \rightarrow F(2, X)$ which means that if we can show the composite square

$$\begin{array}{ccc} Z & \longrightarrow & F(1, K) = 0 \\ \downarrow & & \downarrow \\ F(2, X - U) & \longrightarrow & F(2, X) = \mathcal{F}(\emptyset) = 0 \end{array}$$

is a pullback, we have shown the desired equivalence $Z \rightarrow F(2, X - U)$.

¹

TODO: Explain this step.

Preben: I think this follows from something like pullback along equiv is equiv or something. Probably smart to think about htpy pullbacks. Feels really obvious in this case though since we have zero objects.

To complete the proof it is therefore enough to show that the lower right square is a pullback. Replacing F by \mathcal{F} we get

$$\begin{array}{ccc} \mathcal{F}((X - K) \cup U) & \longrightarrow & \mathcal{F}(X - K) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(\emptyset) \end{array}$$

which is a pullback because \mathcal{F} is a sheaf (U and $X - K$ are disjoint).

□

We can now prove Verdier Duality (Theorem 4.2.1):

Proof. Let $\mathcal{E}(\mathcal{C}) \subseteq \text{Fun}(\mathbf{N}(M))$ be the full subcategory spanned by those functors satisfying the conditions of Proposition 4.2.8 and observe that the inclusions $M_0 \hookrightarrow M \hookleftarrow M_2$ give restrictions

$$\text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}) \xleftarrow{\theta} \mathcal{E}(\mathcal{C})^{op} \xrightarrow{\theta'} \text{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op}.$$

Because we can extend along inclusions of full subcategories which are fully faithful these are equivalences of ∞ -categories.

Preben: Note that $F \in \mathcal{E}(\mathcal{C})$ means $F|_{M_0} \in \text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})$, $F|_{M_0 \cup M_1} = RKAN(F|_{M_0})$ and $F = LKAN(F|_{M_0 \cup M_1})$.

This proves Theorem 4.2.2 and by Corollary 3.0.8.1 we have shown Theorem 4.2.1. □

Proposition 4.2.9 ([Lur17, Proposition 5.5.5.10]). *Let X be a locally compact Hausdorff space and \mathcal{C} a pointed and stable ∞ -category with small limits and colimits. Then the equivalence of ∞ -categories*

$$\mathbb{D} : \text{Shv}(X; \mathcal{C})^{op} \simeq \text{Shv}(X; \mathcal{C}^{op}).$$

given in Theorem 4.2.1 is given by $\mathbb{D}(\mathcal{F})(U) = \Gamma_c(U; \mathcal{F})$, and this is the infinity-categorical generalization of the classical fact that conjugation by Verdier Duality exchanges cohomology and cohomology with compact support.

TODO: Maybe worth trying to make the bridge between this statement and the classical fact even more concrete.

Proof. It follows from the proof of Theorem 3.0.8 that the equivalence

$$\theta : \mathbf{Shv}(X; \mathcal{C}^{op})^{op} \simeq \mathbf{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op}$$

is given by the formula $\theta(\mathcal{F})(U) = \operatorname{colim}_{K \subseteq U} \mathcal{F}(K)$.

Preben: Make sure this colimit is taken in the correct category. It should probably just be a limit in \mathcal{C} .

Preben: Should consider writing out theta more explicitly in the proof of 3.0.8.

Let $\psi : \mathbf{Shv}(X; \mathcal{C}) \rightarrow \mathbf{Shv}_{\mathcal{K}}(X; \mathcal{C})$ be the equivalence of Corollary 3.0.8.1 and ψ' the equivalence $\mathbf{Shv}_{\mathcal{K}}(X; \mathcal{C}) \rightarrow \mathbf{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op}$ of Theorem 4.2.2. Composing, we get a string of equivalences

$$\mathbb{D}^{op} : \mathbf{Shv}(X; \mathcal{C}) \xrightarrow{\psi} \mathbf{Shv}_{\mathcal{K}}(X; \mathcal{C}) \xrightarrow{\psi'} \mathbf{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op} \xrightarrow{\theta} \mathbf{Shv}(X; \mathcal{C}^{op})^{op}.$$

Let \mathcal{D} be as in the proof of Lemma 4.2.8. By Theorem 3.0.8 the restriction $\mathcal{D} \rightarrow \operatorname{Fun}(\mathbf{N}(\mathcal{U}(X))^{op}, \mathcal{C})$ is a trivial Kan fibration onto $\mathbf{Shv}(X; \mathcal{C})$.

Preben: Here we have used that we Kan extend along the fully faithful inclusion of $\mathbf{Shv}(X; \mathcal{C})$ (it is a full subcategory of $\operatorname{Fun}(\mathbf{N}(\mathcal{U}(X))^{op}, \mathcal{C})$.)

In the other direction we restrict $\mathcal{D} \rightarrow \operatorname{Fun}(\mathbf{N}(M_0), \mathcal{C}) \simeq \operatorname{Fun}(\mathcal{K}(X), \mathcal{C}) \simeq \operatorname{Fun}(\mathcal{K}(X)^{op}, \mathcal{C}^{op})^{op}$ and $\psi' \circ \psi$ is given by the composition $\mathbf{Shv}(X; \mathcal{C}) \rightarrow \mathcal{D} \rightarrow \operatorname{Fun}(\mathcal{K}(X)^{op}, \mathcal{C}^{op})^{op}$ and as we saw in the proof of Lemma 4.2.8, restriction from \mathcal{D} to functors from $\mathbf{N}(M_0)$ is given by $\Gamma_K(X; \mathcal{F})$. This means that $\psi' \circ \psi : \mathcal{F} \mapsto (K \mapsto \Gamma_K(X; \mathcal{F}))$ so by Remark 4.2.7 we have

$$(\theta \circ \psi' \circ \psi)(\mathcal{F})(U) = \operatorname{colim}_{K \subseteq U} (\Gamma_K(X; \mathcal{F})) = \operatorname{colim}_{K \subseteq U} (\Gamma_K(U; \mathcal{F})) = \Gamma_c(U; \mathcal{F}).$$

□

bibname

- [Lur22] Jacob Lurie. *Kerodon*. 2022. URL: <https://kerodon.net>.
- [Joy02] A. Joyal. ‘Quasi-categories and Kan complexes’. In: vol. 175. 1-3. Special volume celebrating the 70th birthday of Professor Max Kelly. 2002, pp. 207–222. DOI: 10.1016/S0022-4049(02)00135-4. URL: [https://doi.org/10.1016/S0022-4049\(02\)00135-4](https://doi.org/10.1016/S0022-4049(02)00135-4).
- [Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: <https://doi.org/10.1515/9781400830558>.
- [BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973, pp. x+257.
- [Mac71] Saunders MacLane. *Categories for the working mathematician*. Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971, pp. ix+262.
- [Rie17] E. Riehl. *Category theory in context*. Aurora: Dover modern math originals. Dover Publications, 2017. ISBN: 978-0-486-82080-4.
- [Kel05] G. M. Kelly. ‘Basic concepts of enriched category theory’. In: *Repr. Theory Appl. Categ.* 10 (2005). Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714], pp. vi+137.
- [KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*. Vol. 332. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, pp. x+497. ISBN: 978-3-540-27949-5; 3-540-27949-0. DOI: 10.1007/3-540-27950-4. URL: <https://doi.org/10.1007/3-540-27950-4>.
- [Lur17] Jacob Lurie. ‘Higher algebra’. Unpublished. Available online at <https://www.math.ias.edu/~lurie/>. Sept. 2017.