title

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May 10, 2023

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Abstract

In 1965, Jean-Louis Verdier introduced Verdier duality for locally compact topological spaces, thus generalizing the classical theory of Poincaré duality for manifolds. Verdier Duality is a cohomological duality allowing exchanging cohomology for cohomology with compact support. More precisely it states that the derived functor of the compactly supported direct image functor has a right adjoint in the derived category of sheaves. By using sheaf cohomology one can derive the classical Poincaré duality as a special case. In his book "Higher Algebra" Jacob Lurie extends the theory to the ∞-categorical setting by showing there is an equivalence between sheaves and cosheaves valued in ∞-categories. This thesis follows this proof closely, expanding and adding details where necesarry. To introduce the relevant background on sheaves and $\mathcal{K}$-sheaves valued in stable ∞-categories we introduce and utilize Kan extensions, an ubiquitous concept in category theory.

Chapter 1

∞-categories

TODO: I have used same notation for 1-cats and ∞-cats. Consider changing.

What Lurie [Lur09] calls ∞-categories were originally called restricted Kan complexes by Boardman and Vogt [BV73], but without the intent of using them for ∞-categories. The first development of such a theory was done by Joyal in [Joy02] who called them quasicategories. As most of this thesis follows Lurie's works very closely, we will follow his convention and use the name ∞-categories.[1] While [Lur09] gives a good introduction to ∞-categories extending on the work of Joyal, his web-project [Lur22] reworks a lot of the foundations and we take a lot of inspiration from this presentation.

1.1 Simplicial sets

Originally, simplicial sets was used to rephrase the homotopy theory of spaces in combinatorial terms. There are many good introductions to simplicial sets, depending on what you want to use them for, but Friedman's [Fri21] was enlightening for the author of this thesis. For algebraic topologists, Peter May's [May92] is a good introduction to semi-simplicial topology.[2]

Definition 1.1.1. Usually denoted by Δ, the simplex category or the simplicial category is the category with linearly ordered sets $[n] = \{0, 1, 2, \cdots, n\}$ as its objects and order-preserving maps between them as its morphisms. That is, for a map $\varphi : [m] \to [n]$ we have that $0 \leq \varphi(i) \leq \varphi(j) \leq n$ for each $0 \leq i \leq j \leq m$.

We denote by δ^i the elementary face operator $[n-1] \to [n]$ and by σ^i the elementary degeneracy

[1] It should be noted that in other sources "∞-categories" might refer to other models than the one we use.

[2]

Preben: Is this even relevant?

operator $[n+1] \to [n]$ given by

$$\delta^i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}, \quad \sigma^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

Remark 1.1.2. All morphisms in Δ are finite compositions of such morphisms.

Definition 1.1.3. We define the category sSet, also denoted Set_Δ by Lurie, of simplicial sets as Set-valued presheaves on Δ, i.e. functors $\Delta^{op} \to \mathrm{Set}$.

Let $X \in \mathrm{sSet} := \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$. We will denote by X_n the set $X([n])$ of n-simplices (also called n-cells) of X. We define the standard n-simplex as $\Delta^n := y([n])$ where y is the Yoneda embedding, meaning Δ^n is the presheaf $\mathrm{Hom}_\Delta(-, [n])$. By the Yoneda lemma $\mathrm{Hom}_{\mathrm{Fun}(\Delta^{op}, \mathrm{Set})}(\Delta^n, X) \simeq X_n$, so we can identify each simplex $x \in X_n$ with a map $x : \Delta^n \to X$. This application of the Yoneda lemma is a crucial part of the theory of simplicial sets and we will more often than not consider n-simplices of a simplicial set X as maps of simplicial sets instead. Observe, moreover, that composition with the elementary face operator gives us a map $\Delta^{n-1} \to \Delta^n$.

Definition 1.1.4. For a simplicial set X, we define the face and degeneracy maps

$$d_i := X(\delta^i) : X_n \to X_{n-1}, \quad s_i := X(\sigma^i) : X_n \to X_{n+1}$$

where both maps are given by composition with δ^i and σ^i respectively.

Example 1.1.5. The standard 0-simplex $\Delta^0 := \mathrm{Hom}(-, [0])$ is a terminal object in sSet, meaning it maps any $[m] \in \Delta$ to a singleton. This is usually just referred to as the point and denoted $*$.

Example 1.1.6 ([Lur22, Remark 000P]). Let $X \in \mathrm{sSet}$ and suppose we have subsets $T_n \subseteq X_n$ for every $n \geq 0$ such that $d_i(T_n) \subseteq T_{n-1}$ and $s_i(T_n) \subseteq T_{n+1}$. Then the collection $\{T_n\}_{n \geq 0}$ is a simplicial set we will call a simplicial subset $T \subseteq X$.

Definition 1.1.7. We define the boundary $\partial\Delta^n$ of Δ^n as the simplicial set

$$(\partial\Delta^n)_m = (\partial\Delta^n)([m]) := \{\alpha \in \mathrm{Hom}_\Delta([m], [n]) | \; [n] \not\subseteq \mathrm{im}(\alpha)\}.$$

Observe that $\partial\Delta^0 = \emptyset$ because every map $[m] \to [0]$ is surjective.

Definition 1.1.8. For $0 \leq i \leq n$, we define the horn Λ^n_i as the simplicial set

$$(\Lambda^n_i)_m = (\Lambda^n_i)([m]) := \{\alpha \in \mathrm{Hom}_\Delta([m], [n]) | \; \delta^i[n] \not\subseteq \mathrm{im}(\alpha)\}.$$

Observe that the horn is inside the boundary. We usually refer to Λ^n_i as the ith horn in Δ^n and we will call the horns such that $0 < i < n$ the inner horns.

Example 1.1.9.

TODO: Write about $\mathrm{Sing}(X)$ for a topological space X. Rezk page 28 is useful.

Example 1.1.10. We define the nerve $\mathrm{N}(\mathcal{C})$ of a 1-category $\mathcal{C}$ by

$$\mathrm{N}(\mathcal{C}) := \mathrm{Hom}_{\mathsf{Cat}}([-], \mathcal{C})$$

where we view the sets $[n]$ as categories (posets with a map i to j whenever $i \leq j$). Observe that for any order-preserving morphism $\alpha : [m] \to [n]$ we get a map

$$\mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C}) \xrightarrow{-\circ\alpha} \mathrm{Hom}_{\mathsf{Cat}}([m], \mathcal{C})$$

and it is clear that the nerve is a simplicial set with $\mathrm{N}(\mathcal{C})_n = \mathrm{Hom}_{\mathsf{Cat}}([n], \mathcal{C})$.
Observe furthermore that for a functor $F : \mathcal{C} \to \mathcal{D}$ we get a simplicial map $\mathrm{N}(F) : \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ by sending n-cells $\varphi : [n] \to \mathcal{C}$ in $\mathrm{N}(\mathcal{C})_n$ to n-cells $F(\varphi) : [n] \to \mathcal{D}$ in $\mathrm{N}(\mathcal{D})_n$, so the construction is functorial. It should also be clear that the set of objects of $\mathcal{C}$ is identified with the 0-cells $\mathrm{N}(\mathcal{C})_0$ and the morphisms with the 1-cells $\mathrm{N}(\mathcal{C})_1$. Additionally, the 2-cells $\mathrm{N}(\mathcal{C})_2$ is in bijection with the set of composable pairs of morphisms in $\mathcal{C}$ and likewise the n-cells is the set of n-composable morphisms. We will talk more about composition of morphisms in the next section.

Definition 1.1.11. We define the connected components $\pi_0 X$ of a simplicial set X as the colimit of X.

This is equivalently the quotient of X_0 by the equivalence relation on X_0 generated by the relation $x \sim y$ if and only if there exists an edge $f \in X_1$ such that $f_0 = x$ and $f_1 = y$.

1.2 ∞-categories

Before we give a precise definition, we will take a closer look at the nerve construction. Clearly, we want the nerve of a 1-category to give us an ∞-category and most of this thesis will revolve around nerves of certain poset-categories of topological spaces. Nerves of categories are not just any ordinary simplicial sets, but simplicial sets with some more structure inherited from the underlying 1-category. For instance, 1-categories have composition of morphisms. Take for example

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

in some ordinary 1-category $\mathcal{C}$. This diagram gives us a morphism $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ of simplicial sets, but in $\mathcal{C}$ f and g can be composed to a morphism $h : X \to Z$ which in turn gives a unique way to extend the simplicial map $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ to a map $\Delta^2 \to \mathrm{N}(\mathcal{C})$. If we instead look at the outer horns Λ^2_0 and Λ^2_2 we will not necessarily have a way to extend morphisms to Δ^2 in general. For example the diagram

$$\begin{tikzcd} X \arrow[dr, "\mathrm{id}_X"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

gives a map $\Lambda^2_2 \to \mathrm{N}(\mathcal{C})$, but extending this to a morphism $\Delta^2 \to \mathrm{N}(\mathcal{C})$ would amount to finding a right inverse to g, which of course is not something we can always do in general, unless $\mathcal{C}$ was a groupoid. This property of extending a morphism from a horn to the standard n-simplex is sometimes also called filling the horn, and we will see that it is a defininig property for ∞-categories. In fact, the existence of horn fillings completely classifies the simplicial sets which are nerves of categories:

Proposition 1.2.1 ([Lur09, Proposition 1.1.2.2])**.** *Let $X \in$ sSet. Then the following conditions are equivalent:*

1. *There exists a small category $\mathcal{C}$ with an isomorphism $X \simeq \mathrm{N}(\mathcal{C})$.*
2. *Every inner horn $\Lambda^n_i \to X$ of X can be filled in a unique way. Or, in other words, for any solid diagram as below, there is a unique dotted arrow making it commute:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Simplicial sets which admits extensions for all horn inclusions are called Kan complexes:

Definition 1.2.2. A simplicial set X is a Kan complex if it satisfies the following condition: For $0 \leq i \leq n$, any map $\sigma_0 : \Lambda^n_i \to X$ can be extended to a map $\sigma : \Delta^n \to X$.

Proposition 1.2.3. *Groupoids are Kan complexes.*

Proof. All morphisms are invertible, so all horns can be filled. □

As we saw in the example of a map $\Lambda^2_2 \to \mathrm{N}(\mathcal{C})$ above, whenever there's non-invertible morphisms around some outer horns will be impossible to fill. This motivates the definition of an ∞-category. The following definition is due to Boardman and Vogt [BV73] who defined weak Kan complexes as simplicial sets satisfying what they called the restricted Kan condition: [3]:

Definition 1.2.4 (Boardman and Vogt [BV73])**.** A simplicial set X is an ∞-category if it satisfies the following condition: For $0 < i < n$, any map $\sigma_0 : \Lambda^n_i \to X$ can be extended to a map $\sigma : \Delta^n \to X$.

This means that any Kan complex is an ∞-category, and in particular so is $\mathrm{Sing}(X)$ for a topological space X. Additionally, observe that the nerve $\mathrm{N}(\mathcal{C})$ of an ordinary category $\mathcal{C}$ is an ∞-category. Because the nerve functor is fully faithful (see example 2.5.8), many authors choose to omit its notation altogether.

Preben: Consider if we should omit it as well. Currently we have been using it, but it is fucking annoying to remember writing N all the time.

[3] Maybe more commonly known as the weak Kan extension condition.

In many ways ∞-categories behave similarly to ordinary categories and we will often write about them almost as if they were ordinary categories instead. For example, we will use the terminology of ordinary category theory and refer to the vertices and edges of our simplicial sets as objects and morphisms in our ∞-categories. There are however some obvious differences between ordinary categories and ∞-categories which needs adressing before adopting complete 1-categorical language. For example, and perhaps most crucially, we have higher-level maps given by simplices of dimension $n \geq 2$. While we have seen that nerves of categories admit unique horn extensions, this condition is dropped for general ∞-categories, and hence composition of morphisms in an ∞-category are not necessarily unique, but rather unique up to homotopy. Before we can make this precise, we must define what we mean by homotopy.

1.2.1 Homotopy

TODO: Here $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ means $\mathcal{C}_1$. Make this clear or change notation or something.

Definition 1.2.5 ([Lur22, Definition 003V]). Let $\mathcal{C}$ be an ∞-category and $f, g : X \to Y$ morphisms in $\mathcal{C}$. We define a homotopy between f and g as a 2-simplex $\sigma \in \mathcal{C}$ with boundary specified by $d_0(\sigma) = \mathrm{id}_Y$, $d_1(\sigma) = g$ and $d_2(\sigma) = f$ as illustrated in the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "\mathrm{id}_Y"] \\ & Y. & \end{tikzcd}$$

We say f and g are homotopic if such a homotopy σ exists.

Example 1.2.6. For a 1-category $\mathcal{C}$ two morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ are homotopic in $\mathrm{N}(\mathcal{C})$ if and only if $f = g$.

Proposition 1.2.7 ([Lur22, Proposition 003Z]). *Let $\mathcal{C}$ be an ∞-category and X, Y objects of $\mathcal{C}$. Then homotopy is an equivalence relation on the collection of morphisms from X to Y in $\mathcal{C}$.*

Proof.

TODO: Prove. Both Rune and Lurie have nice proofs. It is just drawing the correct horns and filling in.

□

Proposition 1.2.8. *$f, g \in \mathcal{C}_1$ are homotopic if and only if they are homotopic as morphisms in $\mathcal{C}^{op}$.*

TODO: See Run's comment.

Proof.

TODO: Draw horn.

□

TODO: Consider adding Corollary [Lur22, Corollary 00V0]

Now that we know what it means for morphisms of ∞-categories to be homotopic, we can define a composition of morphisms.

Definition 1.2.9 ([Lur22, Definition 0042]). Let $\mathcal{C}$ be an ∞-category with morphisms

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

We define h to be a composition of f and g if there exists some 2-simplex $\sigma \in \mathcal{C}$ such that $d_0(\sigma) = g, d_1(\sigma) = h$ and $d_2(\sigma) = f$. We say σ witnesses h as a composition of f and g and we will use the usual notation $h = g \circ f$.

Observe that we have only defined composition up to homotopy. We make this precise in the following proposition:

Proposition 1.2.10 ([Lur22, Proposition 0043]). *Let $\mathcal{C}$ be an ∞-category with morphisms f and g as follows:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Then there exists a composition h of f and g and any other morphism $X \to Z$ is a composition of f and g if and only if it is homotopic to h.

Preben: The proof seems kind of technical, so might not be worth doing. Might do it if there's time or Rune thinks I should. Considered adding [Lur22, Exercise 000Z] as this exercise is used in a lot of the proofs in this subsection, but seems too technical for this text.

Furthermore, compositions respect homotopy in the following sense:

Proposition 1.2.11 ([Lur22, Proposition 0048]). *Let $\mathcal{C}$ be an ∞-category with homotopic morphisms $f \sim f' : X \to Y$ and $g \sim g' : Y \to Z$. Let $h = f \circ g$ and $h' = f' \circ g'$. Then h is homotopic to h'.*

Remark 1.2.12. The nerve construction preserves compositions in the sense that for a 1-category $\mathcal{C}$ with morphisms f, g as above, there is a unique morphism $h : \mathrm{N}(X) \to \mathrm{N}(Z)$ in $\mathrm{N}(\mathcal{C})$ which is given by $f \circ g$ in $\mathcal{C}$.

One can show that the nerve construction $\mathrm{Cat} \xrightarrow{\mathrm{N}} \mathrm{sSet}$ admits a left adjoint h and moreover that the counit of this adjunction is an isomorphism which in turn means that the nerve is fully faithful. This can be shown directly and the interested reader can see for example [Rez22, Proposition 4.10.] or [Lur22, Subsection 002Y] for proofs, but we will instead delay the proof to section 2.5 to illustrate the usefulness of Kan extensions. We now construct this left adjoint directly, but delay the proof of the adjunction. Analogously to the construction of the fundamental groupoid $\pi_{\leq 1}(X)$ of a topological space X, we can construct the homotopy category $\mathrm{h}\mathcal{C}$ of an ∞-category $\mathcal{C}$.

Preben: Writing about the fundamental gruopoid could be smart.

Definition 1.2.13. Let $\mathcal{C}$ be an ∞-category. We denote by $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ homotopy classes of morphisms $X \to Y \in \mathcal{C}$ and for a morphism $f : X \to Y$ in $\mathcal{C}$, we denote by $[f]$ its equivalence class in $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$.

Proposition 1.2.14 ([Lur22, Proposition 004B]). *We have a unique composition of morphisms*

$$\circ : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Z)$$

such that $[h] = [f] \circ [g]$ for any $h = f \circ g \in \mathcal{C}$. This composition law is both

1. *associative in the sense that any triple $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{C}$ yields an equivalence*

$$([h] \circ [g]) \circ [f] = [h] \circ (g \circ f) \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(W, Z).$$

2. *unital in the sense that for any $X \in \mathcal{C}$ the homotopy class $[\mathrm{id}_X]$ of the identity on X is a two-sided identity with respect to the composition law. In other words, for every $W \xrightarrow{f} X$ and every $X \xrightarrow{g} Y$ in $\mathcal{C}$, we have $[id_X] \circ [f] = [f]$ and $g \circ [id_X] = [g]$. This means that $([h] \circ [g]) \circ\ [f] = [w]$ and $[h] \circ ([g] \circ [f]) = [h] \circ [u]$, so it remains to show that $[w] = [h] \circ [u]$.*

Proof. The existence of the composition law follows directly from the previous two propositions. To prove 1. we pick compositions $u = g \circ f$, $v = h \circ g$ and $w = v \circ f$. Choosing 2-cells $\sigma_0, \sigma_2, \sigma_3$ witnessing the compositions $v = g \circ h, u = g \circ f$ and $w = v \circ f$, respectively yields a map $\Lambda^3_1 \to \mathcal{C}$ as depicted in the following diagram:

$$\begin{tikzcd}
& X \arrow[dr, "g"] \arrow[drr, "v"] & & \\
W \arrow[ur, "f"] \arrow[rr, dotted, "u"'] \arrow[rrr, dotted, "w"] & & Y \arrow[r, dotted, "h"'] & Z
\end{tikzcd}$$

where the dotted lines represent the "missing" 2-cell. Since $\mathcal{C}$ is an ∞-category we can extend this map to a 3-cell $\Delta^3 \to \mathcal{C}$ essentially "filling" in the missing 2-cell witnessing the desired composition $w = h \circ u$.

To prove 2. pick $X \in \mathcal{C}$ and maps $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ and observe that the degenerate 2-cells with boundaries as in the following diagrams:

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, equal, "\mathrm{id}_X"] \arrow[rr, "g"'] & & Y \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dr, equal, "\mathrm{id}_X"] & \\ W \arrow[ur, "f"] \arrow[rr, "f"'] & & X \end{tikzcd}$$

witnesses the compositions $g \circ \mathrm{id}_X = g$ and $\mathrm{id}_X \circ f = f$. □

We can now define the homotopy category $\mathrm{h}\mathcal{C}$ of an ∞-category $\mathcal{C}$.

Definition 1.2.15. Let $\mathcal{C}$ be an ∞-category. Then we define $\mathrm{h}\mathcal{C}$ to be the 1-category with objects of $\mathcal{C}$ as its objects and homotopy classes of morphisms as defined in 1.2.13 as its morphisms. The previous proposition provides identity morphisms $[\mathrm{id}_X]$ for any object $X \in \mathcal{C}$ and composition law satisfying the axioms for being a 1-category.

Example 1.2.16.

1. $\mathrm{h}\Delta^n = [n] \simeq \{0 < 1 < \cdots < n\}$.
2. For a topological space X one can identify $\mathrm{hSing}(X)$ with $\pi_{\leq 1}(X)$.

1.2.2 Isomorphims

Definition 1.2.17. Let $\mathcal{C}$ be an ∞-category and let $X \xrightarrow{f} Y$ in $\mathcal{C}$. We say f is an isomorphism if $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$. Isomorphims are also often called equivalences.

Example 1.2.18. Let $\mathcal{C}$ be a 1-category. A morphism in $\mathcal{C}$ is an isomorphism if and only if it is an isomorphism in $\mathrm{N}(\mathcal{C})$.

Preben: Is this example too trivial?

TODO: Write about ∞-groupoids?

Definition 1.2.19. An ∞-groupoid is an ∞-category such that $\mathrm{h}\mathcal{C}$ is a groupoid, or in other words an ∞-category where every morphism is an isomorphism.

Example 1.2.20. Every Kan complex K is an ∞-groupoid because every horn can be filled and filling the horns $\Lambda^2_0 \to K$ and $\Lambda^2_1 \to K$ yields inverses for any morphisms in K.
In particular, this means the singular complex $\mathrm{Sing}X$ of a topological space X is an ∞-groupoid and one can show that $\mathrm{hSing}X$ is the fundamental groupoid of X.

As one should maybe expect, this works the other way around as well; ∞-groupoids are Kan complexes. Thankfully, this is true, but it is a non-trivial and technical theorem which is the main focus of [Joy02]. For a proof, see [Joy02][Corollary 1.4] or [Rez22][Section 34]. Inspired by [Gro20][Corollary 14.2.18.] we can write the following commutative diagram of fully faithful functors:

$$\begin{tikzcd}
\mathrm{Grpd} \arrow[r] \arrow[d, "\mathrm{N}"'] & \mathrm{Cat} \arrow[d, "\mathrm{N}"] \arrow[dr, "\mathrm{N}"] & \\
\mathrm{Kan} = \mathrm{Grpd}_\infty \arrow[r] & \mathrm{Cat}_\infty \arrow[r] & \mathrm{sSet}
\end{tikzcd}$$

Definition 1.2.21. The core of an ∞-category $\mathcal{C}$ is the ∞-groupoid $\mathcal{C}^{\simeq}$ (also written $\mathcal{C}^{\mathrm{core}}$ by some authors) consisting of the same objects as $\mathcal{C}$ but only the isomorphisms.

TODO: Be a bit more precise: you need a pullback over $\mathcal{C} \to \mathrm{h}\mathcal{C}$ (or say "all simplices s.t. ...")

We will say two objects in an ∞-category are isomorphic whenever there exists an isomorphism between them. Furthermore being isomorphic is an equivalence relation on the objects of an ∞-category which means we can sensibly speak of isomorphism classes.

Preben: Rezk mentions $\pi_0(\mathcal{C}^{\mathrm{core}})$. Do we care?

In addition to having a notion of isomorphism or equivalence of objects in an ∞-category we would like a notion of natural isomorphism or equivalence of ∞-categories, but first we define the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors between ∞-categories.

1.2.3 Mapping spaces

For ordinary 1-categories $\mathcal{C}$ and $\mathcal{D}$ we can create the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with functors its objects and natural transformations as its morphisms. We want to create an ∞-categorical analogue:

Definition 1.2.22. Let $X, Y \in \mathrm{sSet}$. We define $\mathrm{Fun}(X, Y)$ by $\mathrm{Fun}(X, Y)_n := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times X, Y)$.

If σ is some map $[m] \to [n]$ in Δ, the induced map

$$\sigma^* : \mathrm{Fun}(X, Y)_n \to \mathrm{Fun}(X, Y)_m$$

is defined by

$$(X \times \Delta^n \xrightarrow{f} Y) \mapsto (X \times \Delta^m \xrightarrow{\mathrm{id}_X \times \sigma} X \times \Delta^n \xrightarrow{f} Y).$$

In particular, this means that $\mathrm{Fun}(X, Y)_0$ is precisely the set of maps between the simplicial sets X and Y. Observe that Fun defines a functor $\mathrm{sSet}^{op} \times \mathrm{sSet} \to \mathrm{sSet}$ and for each n it is clear that we have a bijection between $\mathrm{Hom}(\Delta^n \times X, Y)$ and $\mathrm{Hom}(\Delta^n, \mathrm{Fun}(X, Y))$. Furthermore, we can extend the bijection to any simplicial set:

Proposition 1.2.23 ([Rez22][Proposition 15.3.]). *Let $X, Y, Z \in$ sSet, then there is a bijeciton*

$$\mathrm{Hom}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Fun}(Y, Z)).$$

This proposition yields a natural isomorphism of simplicial sets $\mathrm{Fun}(X \times Y, Z) \cong \mathrm{Fun}(X, \mathrm{Fun}(Y, Z))$.

It can be shown that applying the same construction to ∞-categories $\mathcal{C}$ and $\mathcal{D}$ yields a new ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with functors as objects (0-cells) and natural transformations as morphisms (1-cells). Proving this uses machinery that I will not introduce in this text, but there are lots of good texts on Joyal's lifting theorem. See for example

TODO: reference Joyal2008 and mention "certain stability properties of the class of categorical equivalences and the so-called inner anodyne maps" or the proof of [Lur09][1.2.7.3] on p.94.

This allows us to define an equivalence between ∞-categories:

Definition 1.2.24. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and $f, g \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ functors between them. We will say that a natural transformation φ between f and g is a natural isomorphism or a natural equivalence if it is an equivalence in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 1.2.25. Let $f \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be a functor of ∞-categories. Then f is a categorical equivalence if there exists a functor $g \in \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ and natural equivalences between gf and $\mathrm{id}_{\mathcal{C}}$ and between $\mathrm{id}_{\mathcal{D}}$ and fg.

TODO: Maybe mention Rezk and weak saturated classes etc.

Proposition 1.2.26 ([Rez22][Exercise 15.8.]). *Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary 1-categories. Then* $\mathrm{N}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}(\mathrm{N}(\mathcal{C}), \mathrm{N}(\mathcal{D}))$.

Proof. We will show they are the same on the level of n-cells for all n. First use $\Delta^n = \mathrm{N}([n])$ and that the nerve preserves finite products to observe the following:

$$\begin{aligned}
\mathrm{Fun}(\mathrm{N}(\mathcal{C}), \mathrm{N}(\mathcal{D}))_n &:= \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times \mathrm{N}(\mathcal{C}), \mathrm{N}(\mathcal{D})) && (1.1)\\
&= \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{N}([n]) \times \mathrm{N}(\mathcal{C}), \mathrm{N}(D)) && (1.2)\\
&= \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{N}([n] \times \mathcal{C}), \mathrm{N}(D)) && (1.3)
\end{aligned}$$

Finally, we use fully-faithfulness of the nerve to get

$$\mathrm{Hom}_{\mathrm{sSet}}(\mathrm{Fun}([-], [n] \times \mathcal{C}), \mathrm{Fun}([-], \mathcal{D})) = \mathrm{Fun}([n] \times \mathcal{C}, \mathcal{D}).$$

Now, we use closure of Fun to get

$$\begin{aligned}
\mathrm{N}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))_n &:= \mathrm{Fun}([n], \mathrm{Fun}(\mathcal{C}, \mathcal{D})) && (1.4)\\
&= \mathrm{Fun}([n] \times \mathcal{C}, \mathcal{D}) && (1.5)
\end{aligned}$$

□

Definition 1.2.27. Let $\mathcal{C}$ be an ∞-category with objects X and Y. Then we define the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \{(X, Y)\} \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

which we will see is an actual space.

For a proof that this is always a space/∞-groupoid/Kan complex, see [Rez22][Proposition 45.2].

Definition 1.2.28. We say a functor $f \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of ∞-categories is essentially surjective if the functor $\mathrm{h}f \in \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{h}\mathcal{D})$ is essentially surjective.

Definition 1.2.29. We say a functor $f \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of ∞-categories is fully faithful if the functor $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(f(X), f(Y))$ is an equivalence.

Charles Rezk names the following the "fundamental theorem" of ∞-categories (and he names the corresponding result for 1-categories the "fundamental theorem of category theory" though he mentions Yoneda's lemma might be more deserving of that name).

Theorem 1.2.30 ([Rez22][Theorem 48.2.]). *A functor $f : \mathcal{C} \to \mathcal{D}$ of ∞-categories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

We can also define a weaker notion of equivalence between simplicial sets.

Definition 1.2.31. We say a map $f : X \to Y$ of simplicial sets is a weak homotopy equivalence if for any space Z, the induced functor

$$\mathrm{Fun}(f, Z) : \mathrm{Fun}(Y, Z) \to \mathrm{Fun}(X, Z)$$

is a categorical equivalence.

An immmediate observation is that any categorical equivalence is also a weak homotopy equivalence.

Example 1.2.32. $\Delta^0 \hookrightarrow \Delta^1$ is a weak homotopy equivalence but not a categorical equivalence.

Chapter 2

Limits and Colimits

TODO: Write some general stuff.

TODO: Where should I write about adjunctions?

TODO: Write about joins and slices before defining limits and colimits.

2.1 Joins and slices

In this section we will introduce the join and slice constructions. We will start with a recollection of what these constructions are in the case of ordinary 1-categories before defining the appropriate ∞-categorical notions. For most people, at least for me, the slice construction is very familiar while the join maybe not so much. These two constructions will ultimately give us a way to talk about the right notions of limits and colimits in the world of ∞-categories.

Definition 2.1.1. Let $\mathcal{C}$ be a 1-category and $C \in \mathcal{C}$. The slice category, or over category, $\mathcal{C}_{/C}$ is the category with arrows $C' \to C$ in $\mathcal{C}$ as objects and commutative triangles in $\mathcal{C}$ as its morphisms. The coslice category, or under category, $\mathcal{C}_{C/}$ is the category with arrows $C \to C'$ in $\mathcal{C}$ as objects and commutative triangles in $\mathcal{C}$ as its morphisms.

Remark 2.1.2. We have pullbacks

$$\begin{tikzcd}
\mathcal{C}_{C/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],\mathcal{C}) \arrow[d, "\mathrm{ev}_0"', "\mathrm{ev}_1"] & \mathcal{C}_{/C} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\{C\} \arrow[r] & \mathcal{C} & \{C\} \arrow[l]
\end{tikzcd}$$

where $\mathrm{ev}_0 : \mathrm{Fun}([1], \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$ and $\mathrm{ev}_1 : \mathrm{Fun}([1], \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$.

The above remark says that we can identify the slice and coslice categories with fibers of the evaluation functors ev_0 and ev_1 and we will use this idea to define the notion of slicing over (and under) diagrams.

Definition 2.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be 1-categories. For each $C \in \mathcal{C}$, we let $\underline{C} : \mathcal{D} \to \mathcal{C}$ denote the constant functor sending each $D \in \mathcal{D}$ to C and each morphism to id_C. For each functor $F : \mathcal{D} \to \mathcal{C}$ we denote by $\mathcal{C}_{/F}$ the fiber product

$$\begin{tikzcd} \mathcal{C}_{/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{D}, \mathcal{C})_{/F} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \end{tikzcd}$$

where the bottom arrow is given by $C \mapsto \underline{C}$. Dually, we denote by $\mathcal{C}_{F/}$ the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D}, \mathcal{C})_{F/}$. Here $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{/F}$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{F/}$ are simply the slice and coslice categories of definition 2.1.1.

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be 1-categories. We define the join $\mathcal{C} \star \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ as the category with $\mathcal{C} \coprod \mathcal{D}$ as its object and for objects X, Y morphisms given by:

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X, Y) := \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathcal{C}, \\ \mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathcal{D}, \\ \{*\} & \text{if } X \in \mathcal{C}, Y \in \mathcal{D}, \\ \emptyset & \text{if } X \in \mathcal{D}, Y \in \mathcal{C}, \end{cases}$$

with composition defined such that $\mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D} \hookleftarrow \mathcal{D}$ are functors.

Remark 2.1.5. These inclusions are isomorphisms to full subcategories of the join. It is usual to abuse notation a bit and identify $\mathcal{C}$ and $\mathcal{D}$ with these subcategories.

Remark 2.1.6.

TODO: Joins and slices define an adjoint relationship.

Example 2.1.7. Maybe the most important examples of joins, at least in this text, are the left and right cone of a category. Letting $[0]$ denote the category with one object and one morphism, we denote by $\mathcal{C}^{\triangleleft} := [0] \star \mathcal{C}$ the right cone of a 1-category $\mathcal{C}$ and $\mathcal{C}^{\triangleright} := \mathcal{C} \star [0]$ the left cone of $\mathcal{C}$. In practice, the right cone of $\mathcal{C}$ is the category obtained by adjoining an additional object X_0 to $\mathcal{C}$ and for every $C \in \mathcal{C}^{\triangleright}$ a unique morphism $X_0 \to C$ so that X_0 becomes terminal in $\mathcal{C}^{\triangleright}$. Dually, the left cone is obtained by adjoining an additional object which becomes initial in $\mathcal{C}^{\triangleleft}$.

The usefulness of cones materializes when considering limits and colimits. Lurie denotes the category of functors extending F to the cones by $\mathrm{Fun}_F(\mathcal{C}^{\triangleright}, \mathcal{D}) := \{G \in \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D}) |\ G|_{\mathcal{C}} = F\}$ and $\mathrm{Fun}_F(\mathcal{C}^{\triangleleft}, \mathcal{D}) := \{G \in \mathrm{Fun}(\mathcal{C}^{\triangleleft}, \mathcal{D}) |\ G|_{\mathcal{C}} = F\}$ and colimits and limits of F can be identified with initial and terminal objects in $\mathrm{Fun}_F(\mathcal{C}^{\triangleright}, \mathcal{D})$ and $\mathrm{Fun}_F(\mathcal{C}^{\triangleleft}, \mathcal{D})$ respectively.

We now define the join of two simplicial sets.

Definition 2.1.8. Let $X, Y \in \mathrm{sSet}$. We define the join $X \star Y$ on n-cells:

$$(X \star Y)_n := \coprod_{[n]=[n_1]\sqcup[n_2]} X_{n_1} \times X_{n_2},$$

where $[n_1], [n_2] \in \Delta \cup [-1] = \emptyset := \Delta_+$ and $\sqcup : \Delta_+ \times \Delta_+ \to \Delta_+$ is the ordered disjoint union. That is, $[p] \sqcup [q] = [p + 1 + q]$. We allow $[n_1]$ and $[n_2]$ to be -1 with $[-1] = \emptyset$ and $X_{-1} = * = Y_{-1}$.

Remark 2.1.9. This is the left Kan extension of the product $X \times Y$ along the ordered disjoint union.

Example 2.1.10. We denote by $X^{\triangleleft}$ the left cone $\Delta^0 \star X$ and by $X^{\triangleright}$ the right cone $X \star \Delta^0$.

One can show that the nerve of a 1-category commute with joins in the sense that $\mathrm{N}(\mathcal{C} \star \mathcal{D}) = \mathrm{N}(\mathcal{C}) \star \mathrm{N}(\mathcal{D})$ which in particular means that the nerve commutes with the cone constructions:

$$\mathrm{N}(\mathcal{C}^{\triangleright}) \cong \mathrm{N}(\mathcal{C})^{\triangleright} \quad \text{and} \quad \mathrm{N}(\mathcal{C}^{\triangleleft}) \cong \mathrm{N}(\mathcal{C})^{\triangleleft}.$$

Furthermore, it can be shown that the join of ∞-categories is again an ∞-category. See for example [Lur22, Corollary 02QV] or [Rez22][Proposition 28.17.]. To define limits and colimits for ∞-categories we will need ∞-categorical versions of the slice functors introduced above and we do this by finding right adjoints to the functors $X \star - : \mathrm{sSet} \to \mathrm{sSet}_{X/}$ and $- \star X : \mathrm{sSet} \to \mathrm{sSet}_{X/}$.

Proposition 2.1.11. *For $X \in \mathrm{sSet}$, the functors $X \star -$ and $- \star X$ preserve colimits.*

Proof. Writing out definition 2.1.8, we get

$$(X \star Y)_n = X_n \coprod (X_{n-1} \times Y_0) \coprod \cdots \coprod (X_0 \times Y_{n-1}) \coprod Y_n,$$

which means that cell-wise we have a functor to $\mathrm{Set}_{X_n/}$ and here colimits and products commute. □

Consequently, we can find right adjoints going from $\mathrm{sSet}_{X/}$ to sSet which we call slice functors. For a map $f : X \to Y$ and simplicial set K, we have bijections

$$\mathrm{Hom}\left(K, Y_{f/}\right) \cong \mathrm{Hom}_{X/}(X \star K, Y)$$

and

$$\mathrm{Hom}\left(K, Y_{/f}\right) \cong \mathrm{Hom}_{X/}(K \star X, Y).$$

These universal properties will serve as our definitions, but we can define slices by what they do on n-cells by considering the special case $K = \Delta^n$:

$$(Y_{f/})_n \cong \mathrm{Hom}_{\mathrm{sSet}_{X/}}(X \star \Delta^n, X) \quad \text{and} \quad (Y_{/f})_n \cong \mathrm{Hom}_{\mathrm{sSet}_{X/}}(\Delta^n \star X, Y).$$

As one might expect, the nerve of a 1-category commute with taking slices as it does for joins, i.e. for a functor $F : \mathcal{C} \to \mathcal{D}$ between 1-categories, we have

$$\mathrm{N}(\mathcal{D}_{F/}) \cong \mathrm{N}(\mathcal{D})_{\mathrm{N}(F)/} \quad \text{and} \quad \mathrm{N}(\mathcal{D}_{/F}) \cong \mathrm{N}(\mathcal{D})_{/\mathrm{N}(F)}.$$

TODO: Define slice of object by considering morphism $\Delta^0 \to X$.

Furthermore, as one might expect – or at least desire – the slice of an ∞-category is again an ∞-category; see for example [Rez22][Proposition 30.2.] for a proof.

2.2 Iniitial and terminal objects

We now have what we need to define initial and terminal objects of ∞-categories.

Definition 2.2.1. Let $\mathcal{C}$ be an ∞-category. An object $c \in \mathcal{C}$ is initial if for all $n \geq 1$ every $f : \partial\Delta^n \to \mathcal{C}$ such that $f|_{\{0\}} = c$ can be extended to a map $f' : \Delta^n \to \mathcal{C}$. Dually, a terminal object is an object c' such that for all $n \geq 1$ every $f : \partial\Delta^n \to \mathcal{C}$ such that $f|_{\{n\}} = c'$ can be extended to a map $f' : \Delta^n \to \mathcal{C}$. Equivalently initial and terminal objects are objects such that we can always solve the respective extension problems:

$$
\begin{tikzcd}
\{0\} \arrow[r, hook] \arrow[rr, bend left, "c"] \arrow[dr] & \partial\Delta^n \arrow[r] & \mathcal{C} \\
& \Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{n\} \arrow[r, hook] \arrow[rr, bend left, "c'"] \arrow[dr] & \partial\Delta^n \arrow[r] & \mathcal{C} \\
& \Delta^n \arrow[ur, dashed, "\exists"] &
\end{tikzcd}
$$

Remark 2.2.2. It can be shown that initial and terminal objects are invariant under isomorphisms.

In his notes, Rezk shows that the slice categories give alternate definitions of initial and terminal objects:

Proposition 2.2.3 ([Rez22][Proposition 31.3.][1])**.** *Let $\mathcal{C}$ be an ∞-category and X an object in $\mathcal{C}$.*

1. *Then X is initial if and only if the projection $\mathcal{C}_{X/} \to \mathcal{C}$ is a categorical equivalence.*
2. *Then X is terminal if and only if the projection $\mathcal{C}_{/X} \to \mathcal{C}$ is a categorical equivalence.*

In ordinary category theory we define initial and terminal objects by contractibility of Hom-sets and after defining the mapping space between two ∞-categories one would hope for a ∞-categorical analogy of this classification.

Definition 2.2.4. We say an ∞-category is contractible if it is categorically equivalent to Δ^0.

This definition leads us to the desired result:

Proposition 2.2.5 ([Rez22][Proposition 63.7.])**.** *Let $\mathcal{C}$ be an ∞-category and X an object in $\mathcal{C}$.*

1. *Then X is initial if and only if $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is contractible for any object $Y \in \mathcal{C}$.*
2. *Then X is terminal if and only if $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is contractible for any object $Y \in \mathcal{C}$.*

While initial and terminal objects in 1-categories are unique up to unique isomorphism, initial and terminal objects in ∞-categories are unique up to equivalence in the sense that the full subcategories spanned by the initial and terminal objects are either empty or contractible. Furthermore, any object isomorphic to an initial (terminal) object is itself initial (terminal).

[1]Rezk shows that the maps are trivial fibrations rather than categorical equivalences but instead of introducing fibrations we remedy this by referring to [Rez22][40.8.] which states that isofibrations that are also categorical equivalences are trivial fibrations and claim the projections in question are isofibrations. This is essentially [Rez22][Remark 31.4.].

2.3 Limits and colimits

Now we can finally define limits and colimits.

Definition 2.3.1. Let $\mathcal{C}$ be an ∞-category and $f : K \to \mathcal{C}$ a map of simplicial sets. A colimit of f is an initial object in $\mathcal{C}_{f/}$, and a limit of f is a terminal object in $\mathcal{C}_{/f}$.

This means that a colimit of $f : K \to \mathcal{C}$ is a map $\bar{f}$ such that we can always solve the following extension problem

$$\begin{tikzcd} \{0\} \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, "\bar{f}"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{f/} \\ & \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Recalling that we defined the slice of a simplicial set as right adjoint to the join, we know that maps $\{0\} \to \mathcal{C}_{f/}$ is in bijection with maps $K \star \Delta^0 = K^{\triangleright} \to \mathcal{C}$. In summary the adjunction tells us that a colimit of f is a map $\bar{f} : K^{\triangleright} \to \mathcal{C}$ extending f such that we can always solve the following extension problem

$$\begin{tikzcd} K \star \{0\} \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, "\bar{f}"] & K \star \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & K \star \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Dually, a limit of f is a map $\bar{f} : K^{\triangleleft} \to \mathcal{C}$ such that we can always solve the following extension problem

$$\begin{tikzcd} \{n\} \star K \arrow[r, hook] \arrow[dr] \arrow[rr, bend left, "\bar{f}"] & \partial\Delta^n \star K \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^n \star K \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Lurie [Lur09] refers to the object $\operatorname{colim} f \in \mathcal{C}_{f/}$ as a colimit of f and the map $\bar{f} : K^{\triangleright} \to \mathcal{C}$ as a colimit diagram. We will probably be lazy and not care about this distinction. It follows immediately that limits and colimits are unique in the same sense that initial and terminal objects are: the full subcategory spanned by the (co)limits is either empty or contractible. When saying something is unique in ∞-category theory it is usually this notion we mean; it is unique up to a contractible space.

TODO: Be sure to mention in the intro somewhere that "space" refers to ∞-groupoids or equivalently Kan complexes.

Remark 2.3.2. As the slice and join constructions commute with the nerve construction so does initial and terminal objects and hence also limits and colimits.

2.4 Cofinality

A major part of the proofs in this thesis will revolve around calculating certain (co)limits and in particular certain Kan extensions. Following Lurie we will very often do these calculations by using cofinality of certain maps to replace our diagrams with simpler diagrams without changing the (co)limit. We start this section with defining the notion of cofinal maps in an ∞-category and state some results that will turn out to be very important in proving our main theorems.

Definition 2.4.1. A map $f : X \to Y$ in an ∞-category $\mathcal{C}$ is a right fibration if the following extension problem can be solved for any $0 < i \leq n$ and a left fibration if it can be solved for any $0 \leq i < n$:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

Definition 2.4.2. We say a simplicial set is weakly contractible if it is weakly homotopy equivalent to Δ^0.

Definition 2.4.3. [Lur22, Definition 02N1] Let $p : S \to Y \in$ sSet. We say p is right/left cofinal if, for any right/left fibration $X \to Y$, precomposition with p induces a homotopy equivalence

$$\mathrm{Fun}_{/Y}(Y, X) \xrightarrow{\simeq} \mathrm{Fun}_{/Y}(S, X).$$

Proposition 2.4.4 ([Lur09][Proposition 4.1.1.8.]). *Let $f : X \to Y$ be a map of simplicial sets. Then the following conditions are equivalent:*

1. *The map f is right cofinal.*
2. *For any ∞-category $\mathcal{C}$ and functor $p : Y \to \mathcal{C}$ the induced map $\mathcal{C}_{f/} \to \mathcal{C}_{(p\circ f)/}$ is a categorical equivalence.*
3. *For any ∞-category $\mathcal{C}$ and colimit $\overline{p} : Y^{\triangleright} \to \mathcal{C}$, the induced map $\overline{(p \circ f)} : X^{\triangleright} \to \mathcal{C}$ is a colimit.*

Corollary 2.4.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then the following conditions are equivalent:*

1. *F is right cofinal,*
2. *for any ∞-category $\mathcal{E}$ and functor $G : \mathcal{D} \to \mathcal{E}$, the colimit $\mathrm{colim}_{\mathcal{D}}\, G$ exists if and only if the colimit $\mathrm{colim}_{\mathcal{C}}\, GF$ exists, and when they exist they are equivalent in $\mathcal{E}$;*

and the following conditions are equivalent:

1. *F is left cofinal,*
2. *for any ∞-category $\mathcal{E}$ and functor $G : \mathcal{D} \to \mathcal{E}$, the limit $\lim_{\mathcal{D}} G$ exists if and only if the limit $\lim_{\mathcal{C}} GF$ exists, and when they exist they are equivalent in $\mathcal{E}$.*

This corollary makes it clear that cofinal maps are very useful for calculating (co)limits, but the definition is not the easiest to work with when determining wether a map is cofinal or not. The following theorem remedies this by giving a very convenient way of checking cofinality.

Theorem 2.4.5. *[Lur22, Theorem 02NY] Let $f : \mathcal{C} \to \mathcal{D}$ be a map of simplicial sets, where $\mathcal{D}$ is an ∞-category. Then the following conditions are equivalent:*

1. *The functor f is right cofinal,*
2. *for every $D \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible;*

and the following conditions are equivalent:

1. *The functor f is left cofinal,*
2. *for every $D \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible.*

2.5 Kan Extensions

We will start with a detour into the world of ordinary 1-categories. In the classic [Mac71] Saunders MacLane writes "The notion of Kan Extensions subsumes all the other fundamental concepts of category theory". In the fantastic introduction to category theory [Rie17] Emily Riehl devotes a whole chapter to the slogan "All concepts are Kan extensions".

Preben: This intro made me cringe.

Ubiquitous in the toolbox of any category theorist, Kan extensions are central for most everything we do in this thesis. As we have seen, a lot of ∞-categorical concepts can be thought of as if we are working with ordinary 1-categories, and we will therefore start by defining Kan extensions in ordinary categories.

Definition 2.5.1 ([Rie17, Definition 6.1.1.]). Given functors F and K as in the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, dashed, "K_!F"'] & \end{tikzcd}$$

a left Kan extension of F along K is a functor $K_!F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta : F \Rightarrow K_!F \circ K$ such that for any other pair $G : \mathcal{D} \Rightarrow \mathcal{E}, \gamma : F \to G \circ K$, γ factors uniquely through

η as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\swarrow\gamma"] & \mathcal{E} \\ \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\swarrow\eta"] & \mathcal{E} \\ \mathcal{D} \arrow[ur, bend left, "K_!F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\exists!\searrow"] & \end{tikzcd}$$

Dually, a right Kan extension of F along K is a functor $K_*F : \mathcal{D} \to \mathcal{E}$ with a natural transformation $\epsilon : K_*F \circ K \Rightarrow F$ such that any functor $G : \mathcal{D} \to E$ and any natural transformation $\delta : G \Rightarrow F$, δ factors uniquely through ϵ.

Preben: Diagram is exactly the same but all 2-cells go the other way.

The following result justifies the choice to denote left and right Kan extensions by lower shriek and star:

Proposition 2.5.2 ([Rie17, Proposition 6.1.5.]). *Let K be a functor $\mathcal{C} \to \mathcal{D}$ and $\mathcal{E}$ some category. If the left and right Kan extensions of any functor F along K exists, these define left and right adjoints to the pre-composition functor $K^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and by uniqueness of adjoints, any left or right adjoint to the pre-composition functor yields left or right Kan extensions.*

While the description of Kan extensions as certain adjoints are useful, there's usually another description available to us that turns out to be even more useful. Whenever $\mathcal{C}$ and $\mathcal{D}$ are respectively small and locally small, potential Kan extensions along $K : \mathcal{C} \to \mathcal{D}$ are what we call pointwise Kan extensions. More precisely, the existence of left or right Kan extensions along such functors coincides with the existence of certain colimits or limits, respectively. For a functor $K : \mathcal{C} \to \mathcal{D}$, [Mac71] denotes by $d \downarrow K$ the category $\mathcal{C} \times_{\mathcal{C}} \mathcal{D}_{d/}$ and we will choose to denote it by $K_{d/}$. Likewise the category $K \downarrow d = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d}$ will be denoted $K_{/d}$. These categories come with projection functors Π_d and Π^d, respectively, which send the objects $d \to Kc$ and $Kc \to d$ to the object $c \in \mathcal{C}$. The following theorem gives a formula for calculating certain left and right Kan extensions as colimits and limits.

Theorem 2.5.3 ([Rie17, Theorem 6.2.1.]). *Let $\mathcal{D} \xleftarrow{K} \mathcal{C} \xrightarrow{F} \mathcal{E}$ be functors. If it exists for every object $d \in D$, the following colimit defines the left Kan extension $K_!F$:*

$$K_!F(d) := \mathrm{colim}(K_{/d} \xrightarrow{\Pi^d} \mathcal{C} \xrightarrow{F} \mathcal{E})$$

and the unit transformation $\eta : F \to K_!F \circ K$ can be extracted from the colimit cone. Dually, if the following limit exists for every object $d \in D$, they define the right Kan extension K_*F:

$$K_*F(d) := \lim(K_{d/} \xrightarrow{\Pi_d} \mathcal{C} \xrightarrow{F} \mathcal{E})$$

and the counit transformation $\epsilon : K_*F \circ K \to F$ can be extracted from the limit cone.

Preben: Is it clear what it means to extract the 2-cells from the cones?

Proof. Proofs can be found in [Rie17, Theorem 6.2.1.] and [Mac71, Theorem X.3.1.] □

When $\mathcal{D}$ and $\mathcal{E}$ are locally small we call Kan extensions that can be caluclated by the colimit and limit formulae above pointwise Kan extensions. In [Rie17], Riehl says the consensus among category theorists is that the important Kan extensions are the pointwise Kan extensions and quotes [Kel05, §4]: "Our present choice of nomenclature is based on our failure to find a single instance where a weak Kan extension plays any mathematical role whatsoever." This thesis is no different, and we will only care about pointwise Kan extensions from here on out. We will see that there are analogous limit formulae for Kan extensions in ∞-categories which are central to most of the proofs in this thesis. Before we extend the theory Kan extensions from ordinary categories to the world of ∞-categories, we will consider some important examples, but first observe that Theorem 2.5.3 gives the following immmediate consequence:

Corollary 2.5.3.1. *If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then for a functor $K : \mathcal{C} \to \mathcal{D}$ we have:*

1. *If $\mathcal{E}$ is cocomplete, left Kan extensions of functors $\mathcal{C} \to \mathcal{E}$ along K exists and are given by the colimit formula of Theorem 2.5.3.*
2. *If $\mathcal{E}$ is complete, right Kan extensions of functors $\mathcal{C} \to \mathcal{E}$ along K exists and are given by the limit formula of Theorem 2.5.3.*

Example 2.5.4. n-truncation is given by restricting along the inclusion $i : \Delta_{\leq n} \hookrightarrow \Delta$, and since Set is both cocomplete and complete, we have both left and right Kan extensions to n-truncation.

$$\begin{tikzcd} \mathrm{sSet} \arrow[rr, "i_n^*" description] & & \mathrm{sSet}_{\leq n} \arrow[ll, bend right, "(i_n)_!"', "\perp"] \arrow[ll, bend left, "(i_n)_*", "\perp"'] \end{tikzcd}$$

where $\mathrm{sSet}_{\leq n} := \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathrm{Set})$.

Lemma 2.5.5 (Kan extension along fully faithful functors)**.** *Let $K : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Then, up to natural isomorphism, any pointwise Kan extension along K defines an actual on the nose extension.*

Proof. Observe that $\mathcal{C}_{/c} \simeq K_{/K(c)}$ is an equivalence of categories because K is fully faithful, so we can calculate the left Kan extension $K_!F$ on objects by the colimit formula:

$$K_!F(K(c)) = \mathrm{colim}(K_{/K(c)} \simeq \mathcal{C}_{/c} \xrightarrow{\Pi} \mathcal{C} \xrightarrow{F} \mathcal{E}).$$

Since the identity on c is terminal in $\mathcal{C}_{/c}$ the colimit reduces to evaluation at the terminal object $K(c) \xrightarrow{id} K(c)$ in $K_{/Kc}$, so $\eta_c : F(c) \cong K_!F(K(c))$ is an isomorphism. The proof for pointwise right Kan extensions is completely dual. □

Example 2.5.6 (Yoneda extension). Let $\mathcal{C}$ be small, $\mathcal{E}$ locally small and cocomplete. By the corollary above, any functor $F : \mathcal{C} \to \mathcal{E}$ admits a left Kan extension $y_!F$ along the Yoneda embedding y and since y is fully faithful the unit $F \cong y_!F \circ y$ is an identity natural transformation. In fact, $y_!F$ has a right adjoint R, defined on objects by

$$R(e) := \mathrm{Hom}_{\mathcal{E}}(F(-), e) : \mathcal{C}^{op} \to \mathsf{Set}.$$

The full proof showing that this is in fact right adjoint to $y_!F$ can be found in [Rie17, Remark 6.5.9.] from which we have taken this example.

The process of left Kan extending along the Yoneda embedding is called Yoneda extension in [KS06, pp.62-64] and it provides lots of interesting examples of Kan extensions. We will look at a couple examples in the special case $\mathcal{C} = \Delta$.

Preben: These are exercises in [Rie17]. Should I cite?

Example 2.5.7. Let $\Delta_{\mathsf{Top}} : \Delta \to \mathsf{Top}$ be the functor known as the standard topological n-simplex:

$$[n] \mapsto \Delta^n_{\mathsf{Top}} := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_i x_i = 1, x_i \geq 0\}.$$

By the discussion of Yoneda extension above we have an adjoint pair $(y_!\Delta_{\mathsf{Top}}, R)$ where the right adjoint is given by

$$R(e) = \mathrm{Hom}_{\mathsf{Top}}(\Delta_{\mathsf{Top}}(-), e)$$

and this is what is known as the total singular complex functor, also written Sing.

TODO: Check if Sing is mentioned in chapter 1 and reference it back.

Example 2.5.8 (The Nerve construction). Let F be the embedding $\Delta \hookrightarrow \mathsf{Cat}$. Yoneda extension yields an adjoint pair $(y_!F, R)$ where the right adjoint R is given by

$$R(\mathcal{C}) = \mathrm{Hom}_{\mathsf{Cat}}(F(-), \mathcal{C}).$$

Recall from Example 1.1.10 that this is the definition of the nerve of $\mathcal{C}$. The left adjoint $y_!F$ is the homotopy category h of a simplicial set. Observe furthermore that the counit $\mathrm{h}(\mathrm{N}(X)) \to X$ is an isomorphism which implies that the nerve construction is a fully faithful functor.

Yoneda extension produces even more examples of adjunctions. In [Rie17, Exercise 6.5.iii.], Riehl fixes a topological space X and constructs the inclusion $\mathsf{Open}(X) \to \mathsf{Top}_{/X}$ by sending open subsets $U \subseteq X$ to the inclusion map $U \hookrightarrow X$. Yoneda extension now yields an adjunction

$$\begin{tikzcd} \mathsf{Top}_{/X} \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\mathcal{U}(X)^{op}, \mathsf{Set}) \arrow[l, shift right] \end{tikzcd}$$

and as Riehl writes, all adjunctions restrict to an equivalence of subcategories which in this case yields the equivalence between the category $\mathsf{Shv}(X)$ of sheaves on X and the category $\mathsf{Et}(X)$ of étale spaces on X.

2.5.1 Kan extensions for ∞-categories

Preben: THIS IS CURRENTLY JUST ALONG INCLUSIONS

Definition 2.5.9.

TODO: This is chaos. Fix all slice notations in separate subsection of infcat chapter.

Let $\mathcal{A}$ be an ∞-category with a full subcategory $\mathcal{A}^0$ and $p : K \to \mathcal{A}$ a diagram. Following [Lur09, Notation 4.3.2.1] we write $\mathcal{A}^0_{/p}$ for the fiber product $\mathcal{A}_{/p} \times_{/\mathcal{A}} \mathcal{A}^0$. If $A \in \mathcal{A}$, then $\mathcal{A}^0_{/A}$ is the full subcategory of $\mathcal{A}_{/A}$ spanned by the morphisms $A' \to A$ where $A' \in \mathcal{A}^0$.
Analogously $\mathcal{A}^0_{p/}$ denotes $\mathcal{A}_{p/} \times_{\mathcal{A}} \mathcal{A}^0$ and $\mathcal{A}^0_{A/}$ is the full subcategory spanned by morphisms $A \to A'$.

Definition 2.5.10 ([Lur22, Definition 02YQ])**.** For a functor $F : \mathcal{A} \to \mathcal{C}$ between ∞-categories where $\mathcal{A}$ has a full subcategory $\mathcal{A}^0$, we say F is left Kan extended from $\mathcal{A}^0$ if

$$(\mathcal{A}^0_{/A})^{\triangleright} \hookrightarrow (\mathcal{A}_{/A})^{\triangleright} \xrightarrow{c} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$ for every object $A \in \mathcal{A}$. Here c is the slice contraction morphism of [Lur22, Tag 0188], i.e. $c|_{\mathcal{A}_{/A}}$ is the projection and $c|_{\Delta^0} = A$. Recalling the adjoint relationship between joins and slices, this is the counit of the adjunction.
Right Kan extensions are opposite to left Kan extensions, i.e. F is right Kan extended from $\mathcal{A}^0$ if

$$(\mathcal{A}^0_{A/})^{\triangleleft} \hookrightarrow (\mathcal{A}_{A/})^{\triangleleft} \xrightarrow{c'} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

is a limit diagram, where c' is the coslice contraction morphism

TODO: Should probably write more about what coslice contraction is. Plan is full chapter on slice ∞-cats.

.

Later on we will need the following result about Kan extensions of full subcategories.

Proposition 2.5.11 ([Lur09, Proposition 4.3.2.8])**.** *For a functor $F : \mathcal{A} \to \mathcal{C}$ of ∞-categories where $\mathcal{A}^0 \subseteq \mathcal{A}^1$ are full subcategories of $\mathcal{A}$, if $F|_{\mathcal{A}^1}$ is left Kan extended from $\mathcal{A}^0$, then F is left Kan extended from $\mathcal{A}^1$ if and only if it is left Kan extended from $\mathcal{A}^0$.*

2.6 Stable ∞-categories

TODO: Consider moving this to after the chapter on limits and colimits.

TODO: Define initial and terminal objects.

Definition 2.6.1. A zero-object, usually denoted 0, in an ∞-category is an object that is both initial and terminal. A pointed ∞-category is an ∞-category containing a zero-object.

Remark 2.6.2. An object $0 \in \mathcal{C}$ is a zero-object if $\mathrm{Fun}_{\mathcal{C}}(X, 0)$ and $\mathrm{Fun}_{\mathcal{C}}(0, X)$ are contractible for every object $X \in \mathcal{C}$ and such an object is determined up to equivalence.

TODO: Define contractibility earlier.

Chapter 3

Sheaves and K-sheaves

Captured and made a prisoner of war by the Germans in 1940, the french professor and officer Jean Leray spent his five years in captivity inventing the theory of sheaves and spectral sequences. Prior to being captured algebraic topology was merely a small interest and his real interests lied in using topological methods to prove the existence of solutions of certain differential equations, Leray feared that if the Germans learned of his competence as a "mechanic" ("mécanicien" in his own words) they would force him to "work for the German war machine, so he converted his minor interest to his major one, in fact to his essentially unique one, presented himself as a pure mathematician, and devoted himself mainly to algebraic topology.[Bor98][p. 1-21]." Originally, Leray defined sheaves as a functor from closed subspaces, and it was Cartan that reformulated the theory using open subspaces.[**H.Miller**] His first example was the sheaf assigning a space its pth cohomology group.

TODO: Cite Cartan's 1950 seminar?

We will closely follow Lurie [Lur09]. Let $\mathcal{U}(X)$ denote the partial order of open subsets of a topological space X.

3.1 Sheaves on topological spaces

TODO: Define covering sieves.

Definition 3.1.1 ([Lur09, Definition 7.3.3.1])**.** Let $X \in \mathbf{Top}$ and $\mathcal{C}$ an ∞-category. We define a $\mathcal{C}$-valued sheaf on X to be a presheaf $\mathcal{F} : \mathcal{U}(X)^{op} \to \mathcal{C}$ such that for every $U \subseteq X$ and every covering sieve $\mathcal{W} \subseteq \mathcal{U}(X)_{/U}$, the diagram

$$\mathrm{N}(\mathcal{W})^{\triangleright} \hookrightarrow \mathrm{N}(\mathcal{U}(X)_{/U})^{\triangleright} \to \mathrm{N}(\mathcal{U}(X)) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

is a colimit.

Remark 3.1.2. We will often utilize the fact that this is equivalent to the following limit diagram:

$$\mathrm{N}((\mathcal{W})^{op})^{\triangleleft} \hookrightarrow \mathrm{N}((\mathcal{U}(X)_{/U})^{op})^{\triangleleft} \to \mathrm{N}(\mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

TODO: Mention the "normal definitions using covers and Cech nerves and briefly discuss relation to classical definition of sheaves in a 1-category.

We write $\mathrm{Presh}(X, \mathcal{C})$ for the ∞-category $\mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ of $\mathcal{C}$-valued presheaves on X and $\mathrm{Shv}(X; \mathcal{C})$ for the full subcategory of $\mathrm{Presh}(X; \mathcal{C})$ spanned by the $\mathcal{C}$-valued sheaves on X. Whenever we write $\mathrm{Shv}(X)$ without specifying the target category $\mathcal{C}$, we will always mean sheaves valued in spaces, i.e. $\mathrm{Shv}(X; \mathcal{S})$.

TODO: Make sure $\mathcal{S}$ is introduced in chapter on stable infcats.

3.2 Sheaves on locally compact spaces

In this section we will show that for locally compact Hausdorff spaces there is an equivalence of ∞-categories between $\mathrm{Shv}(X; \mathcal{C})$ and $\mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})$ where the latter denote so-called $\mathcal{K}$-sheaves and $\mathcal{C}$ is a presentable ∞-category with left exact filtered colimits. These are sheaves defined on the collection of compact subsets instead of the opens. Classically it is known that sheaves of sets on such spaces are determined by compact subsets as well as the opens.

TODO: Expand on this with references.

Definition 3.2.1. For a locally compact Hausdorff space X, we write $\mathcal{K}(X)$ for its collection of compact subsets.

Definition 3.2.2. If $K, K' \subseteq X$, we write $K \Subset K'$ if there exists an open subset $U \subseteq X$ between K and K', i.e. $K \subseteq U \subseteq K'$.

Definition 3.2.3. If $K \subseteq X$ is compact, we write $\mathcal{K}_{K\Subset}(X)$ for the set $\{K' \in \mathcal{K}(X) | K \Subset K'\}$ which gives a poset category $\mathcal{K}(X)$.

Definition 3.2.4. A presheaf $\mathcal{F} : \mathrm{N}(\mathcal{K}(X))^{op} \to \mathcal{C}$ is a $\mathcal{K}$-sheaf if it satisfies the following:

1. $\mathcal{F}(\emptyset)$ is terminal.
2. For every pair $K, K' \in \mathcal{K}(X)$, the diagram

$$\begin{tikzcd}
\mathcal{F}(K \cup K') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(K) \arrow[d] \\
\mathcal{F}(K') \arrow[r] & \mathcal{F}(K \cap K')
\end{tikzcd}$$

is a pullback in $\mathcal{C}$.

3. For each $K \in \mathcal{K}(X)$, $\mathcal{F}(K)$ is a colimit of $\mathcal{F}|\mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}$.

Definition 3.2.5. We denote the full subcategory of $\mathrm{Presh}(\mathrm{N}(\mathcal{K}(X));\mathcal{C})$ spanned by the $\mathcal{K}$-sheaves by $\mathrm{Shv}_{\mathcal{K}}(X;\mathcal{C})$.

Lemma 3.2.6 ([Lur09, Lemma 7.3.4.8])**.** *Let X be locally compact and Hausdorff, and let $\mathcal{C}$ be a presentable ∞-category with left exact filtered colimits. Let $\mathcal{W}$ be an open cover of X and denote by $\mathcal{K}_{\mathcal{W}}(X)$ the compact subsets of X that are contained in some element of $\mathcal{W}$. Any $\mathcal{K}$-sheaf $\mathcal{F} \in \mathrm{Shv}_{\mathcal{K}}(X;\mathcal{C})$ is a right Kan extension of $\mathcal{F}|\mathrm{N}(\mathcal{K}_{\mathcal{W}}(X))^{op}$.*

Proof.

TODO: Proof is in HTT.

□

Theorem 3.2.7 ([Lur09, Theorem 7.3.4.9])**.** *Let X be locally compact and Hausdorff and $\mathcal{C}$ an ∞-category with small limits and colimits and left exact filtered colimits. Let $\mathcal{F} : \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \to \mathcal{C}$. The following conditions are equivalent:*

1. *The presheaf $\mathcal{F}_{\mathcal{K}} := \mathcal{F}|\mathrm{N}(\mathcal{K}(X))^{op}$ is a $\mathcal{K}$-sheaf, and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$.*
2. *The presheaf $\mathcal{F}_{\mathcal{U}} := \mathcal{F}|\mathrm{N}(\mathcal{U}(X))^{op}$ is a sheaf, and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$.*

We will split the theorem into a few lemmas for readability.

Lemma 3.2.8. *If $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf, then F is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$.*

Proof. By definition we want to show that

$$\mathrm{N}(\mathcal{U}(X)^{op}_{/K})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$. The assumption that $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf means that for each $K \in \mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}(K)$ is a colimit of $\mathcal{F}|\mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}$. We will "transfer" this colimit to the colimit we want by cofinal maps

$$\mathrm{N}((\mathcal{U}(X)^{op})_{/K}) \xrightarrow{p} \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K}) \xleftarrow{p'} \mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}.$$

Recall that by 2.4.5 checking cofinality reduces to checking weak contractibility of certain simplicial sets. For p we must check $\mathrm{N}((\mathcal{U}(X))^{op}_{/K}) \times_{\mathrm{N}((\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{/K})} \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})_{K'/}$ is weakly contractible for every $K' \in \mathrm{N}((\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{/K})$. This is the simplicial set obtained by taking the nerve of the partially ordered set $\{U \in \mathcal{U}(X) | \ K \subseteq U \subseteq K'\}$. By [Lur09, Lemma 5.3.1.20] filtered ∞-categories are weakly contractible, and our partially ordered set is filtered as it is nonempty, stable under finite union, and taking nerves preserve the property of being filtered.

TODO: Make this a result in cofinality chapter instead.

The simplicial set $\mathrm{N}(\{K'' \in \mathcal{K}(X) | \ K \Subset K'' \subseteq K'\})$[1] is weakly contractible by exactly the same argument, and hence p and p' are cofinal maps. By cofinality of p and p', the diagram

$$\mathrm{N}((\mathcal{U}(X))^{op}_{/K})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram if and only if

$$\mathrm{N}((\mathcal{K}(X)_{\Subset K})^{op})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram, which it is by the assumption that $F_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf. □

Lemma 3.2.9. *If $\mathcal{F}_{\mathcal{K}} := \mathcal{F}|\mathrm{N}(\mathcal{K}(X))^{op}$ is a $\mathcal{K}$-sheaf, and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$, then $\mathcal{F}_{\mathcal{U}}$ is a sheaf.*

Proof. By definition 3.1.1 we must show that for every $U \in \mathcal{U}(X)$ and every covering sieve $\mathcal{W}$ covering U,

$$\mathrm{N}(\mathcal{W})^{\triangleright} \hookrightarrow \mathrm{N}(\mathcal{U}(X)_{/U})^{\triangleright} \to \mathrm{N}(\mathcal{U}(X)) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

is a colimit diagram, or equivalently that

$$\mathrm{N}(\mathcal{W})^{op,\triangleleft} \hookrightarrow \mathrm{N}(\mathcal{U}(X)_{/U})^{op,\triangleleft} \to \mathrm{N}(\mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a limit diagram. We will once again use cofinality by observing that 2.4.5 implies cofinality of the inclusion

$$\mathrm{N}(\mathcal{W}) \subseteq \mathrm{N}(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))$$

where $\mathcal{K}_{\mathcal{W}}(X)$ is the set $\{K \in \mathcal{K}(X) | (\exists W \in \mathcal{W})[K \subseteq W]\}$, so it is enough to show the limit starting from $\mathrm{N}(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))^{op}$.

TODO: Write out the details of why this is contractible and the inclusion is cofinal.

Recall that this is equivalent to showing that $\mathcal{F}|_{\mathrm{N}(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))^{op,\triangleleft}}$ is right Kan extended from $\mathcal{F}|_{\mathrm{N}(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))^{op}}$.

TODO: Write out the result in the Kan chapter.

By the assumption that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ and the observation that

$$\mathcal{F}(U) = \lim_{K \in \mathcal{K}(X)^{op}_{U/}} \mathcal{F}(K) = \lim_{K \in \mathcal{K}_{\mathcal{W}}(X)^{op}_{U/}} \mathcal{F}(K)$$

---

[1] **TODO:** This should probably be strict inclusion. Double check that.

we see that $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}})^{op}}$ is a right Kan extension of $\mathcal{F}|_{(K_{\mathcal{W}}(X))^{op}}$. Hence, it suffices to prove that $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. Outside of U this is clear from the fact that $F|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}^{op})}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. This means we only need to prove $\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. Observe that by assumption

$$\mathcal{F}(U) = \lim_{K\in\mathcal{K}(X)^{op}_{/U}} \mathcal{F}(K) = \lim_{K\in(\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \mathcal{F}(K)$$

so $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}}$ is right Kan extended from $\mathcal{K}(X)^{op}_{/U}$. Lemma 3.2.6 tells us that $\mathcal{F}|_{(\mathcal{K}(X)_{/U})^{op}}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. We have $\mathrm{N}(\mathcal{K}_{\mathcal{W}}(X))^{op} \subseteq \mathrm{N}(\mathcal{K}(X)_{/U})^{op} \subseteq \mathrm{N}(\mathcal{K}(X)_{/U} \cup \{U\})^{op}$, with Kan extensions as in proposition 2.5.11, so we get that $\mathcal{F}|_{\mathrm{N}(\mathcal{K}(X)_{/U}\cup\{U\})^{op}}$ is right Kan extended from $\mathrm{N}(\mathcal{K}_{\mathcal{W}}(X))^{op}$. To summarize, we have the following square of inclusions

$$\begin{tikzcd}
\mathcal{K}_{\mathcal{W}}(X)^{op} \arrow[r, hook, "i"] \arrow[d, hook, "a"] & (\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op} \arrow[d, hook, "b"] \\
\mathcal{K}(X)^{op}_{/U} \arrow[r, hook, "j"] & (\mathcal{K}(X)_{/U} \cup \{U\})^{op}
\end{tikzcd}$$

where $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \simeq b_*(\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}\cup\{U\})^{op}})$ and $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \simeq (j\circ a)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$. We want to show $\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}} \simeq i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$. Since b is fully faithful (it is the inclusion of a full subcategory), we know $b^*b_* \simeq \mathrm{id}$, so we get

$$\begin{aligned}
\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}} &\simeq b^*b_*(\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}) && (3.1)\\
&\simeq b^*(j\circ a)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (3.2)\\
&\simeq b^*(b\circ i)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (3.3)\\
&\simeq b^*b_*i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (3.4)\\
&\simeq i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (3.5)
\end{aligned}$$

□

Lemma 3.2.10. *If $\mathcal{F}_{\mathcal{U}} := \mathcal{F}|\mathrm{N}(\mathcal{U}(X))^{op}$ is a sheaf, and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$, then $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf.*

Proof. By definition we need to show three things: Firstly, observe that $\mathcal{F}_{\mathcal{K}}(\emptyset) = \mathcal{F}_U(\emptyset)$ and since $\mathcal{F}_U$ is a sheaf $\mathcal{F}_{\mathcal{K}}(\emptyset)$ is terminal. Secondly, we need the following diagram to be a pullback in $\mathcal{C}$ for any $K, K' \in \mathcal{K}(X)$.

$$\begin{tikzcd}
\mathcal{F}(K \cup K') \arrow[r] \arrow[d] & \mathcal{F}(K) \arrow[d] \\
\mathcal{F}(K') \arrow[r] & \mathcal{F}(K \cap K')
\end{tikzcd} \tag{3.6}$$

We will do this by using that $\mathcal{F}_U$ is a sheaf. Let $P = \{(U, U')|K \subseteq U, K' \subseteq U'\}$ and $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ denote diagram 3.6. Now $\mathcal{F}$ induces a map $\sigma_P : \mathrm{N}(P^{op})^{\triangleright} \to \mathcal{C}^{\Delta^1 \times \Delta^1}$ taking each (U, U') to

$$\begin{tikzcd}
\mathcal{F}(U \cup U') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F}(U) \arrow[d] \\
\mathcal{F}(U') \arrow[r] & \mathcal{F}(U \cap U')
\end{tikzcd}$$

and the cone point is sent to σ. This is a pullback by the fact that $\mathcal{F}_U$ is a sheaf. Evaluating σ_P in each of the four vertices of $\Delta^1 \times \Delta^1$ we get four maps $\mathrm{N}(P^{op})^{\triangleright} \to \mathcal{C}$. We now check that evaluating in the final vertex yields a colimit diagram. By assumption $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_U$ which by definition means that the following is a colimit diagram:

$$\mathrm{N}((\mathcal{U}(X)_{/(K\cap K')})^{op})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/(K\cap K')})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

Observe that for every $U'' \in \mathcal{U}(X)_{/(K\cap K')}$, the set $P_{U''} = \{(U, U') \in P|U \cap U' \subseteq U''\}$ is nonempty and stable under finite intersections, which implies it is filtered and hence its nerve is contractible.

TODO: Reference this result.

By 2.4.5 this implies $\mathrm{N}(P^{op}) \to \mathrm{N}((\mathcal{U}(X)_{/(K\cap K')})^{op})$ is cofinal and we get a colimit diagram

$$\mathrm{N}(P^{op})^{\triangleright} \to \mathrm{N}((\mathcal{U}(X)_{/(K\cap K')})^{op})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/(K\cap K')})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}.$$

We can show that evaluating the three other vertices also yields colimit diagrams by similar arguments. Since σ_P yields a colimit diagram when evaluated in each of the four vertices of $\Delta^1 \times \Delta^1$, we conclude that σ_P is itself a colimit diagram by [Lur09, Proposition 5.1.2.2]. Observe now that σ_P is a filtered colimit in $\mathcal{C}$ and hence it is left exact. This concludes the argument that 3.6 is a pullback. Finally, we need to show that for each $K \in \mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}$ is a colimit of $\mathcal{F}_{\mathcal{K}}|_{\mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}}$. We do this by showing

$$\mathrm{N}(\mathcal{K}_{K\Subset}(X)^{op})^{\triangleright} \to \mathrm{N}(\mathcal{K}(X) \cup \mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. We use Proposition 2.5.11 to show that $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}}$ and $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}\cup\{K\}}$ are left Kan extensions of $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X))^{op}}$ which again implies $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}\cup\{K\}}$ is a left Kan extension of $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}}$. Now observe that

$$\mathrm{N}(\mathcal{K}(X) \cup \mathcal{U}(X))^{op,\triangleright}_{/K} = \mathrm{N}(\mathcal{K}_{K\Subset}(X) \cup \mathcal{U}(X)_{/K})^{op,\triangleright},$$

so in particular

$$\mathrm{N}(\mathcal{K}(X) \cup \mathcal{U}(X))^{op,\triangleright}_{/K} \to \mathrm{N}(\mathcal{K}(X) \cup \mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram, and the statement is reduced to showing that $\mathrm{N}(\mathcal{K}_{K\Subset}(X)) \subseteq \mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))^{op}_{/K}$ is cofinal. Let $Y \in \mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))_{/K}$ and let R be the partially ordered set $\{K' \in \mathcal{K}(X)|\ K \Subset K' \subseteq Y\}$. Since R is nonempty and stable under intersections, R^{op} is filtered and hence $\mathrm{N}(R)$ is weakly contractible. By Lemma 2.4.5 the inclusion is cofinal and we have shown that $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf. □

Lemma 3.2.11. *If $\mathcal{F}_{\mathcal{U}}$ is a sheaf, then $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$.*

Proof. We will show $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ in a similar manner to how we showed $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$ in the start of the proof, but we will consider the partial order on $\mathcal{U}(X)$ given by writing $V \Subset U$ whenever $V \in \mathcal{U}(X)$ and its closure $\overline{V}$ is compact and contained in U. Writing $\mathcal{U}(X)_{U/}$ for the set $\{V \in \mathcal{U}(X) |\ V \Subset U\}$, we need to show that

$$\mathrm{N}(\mathcal{K}(X)^{op}_{U/})^{\triangleleft} \hookrightarrow \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{U/} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. As earlier we do this by finding cofinal inclusions

$$\mathrm{N}(\mathcal{K}(X)^{op}_{U/}) \xrightarrow{f} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{U/} \xleftarrow{f'} \mathrm{N}(\mathcal{K}(X)_{/U})^{op}.$$

By Lemma 2.4.5 f and f' are cofinal inclusions if for any $Y \in (\mathcal{U}(X) \cup \mathcal{K}(X))_{U/}$ the partially ordered sets

$$\{V \in \mathcal{U}(X) | Y \subseteq V \Subset U\}$$

and

$$\{K \in \mathcal{K}(X) | Y \subseteq K \subseteq U\}$$

have weakly contractible nerves, which they have by the usual argument; they are nonempty and stable under unions, hence filtered.

TODO: Give this argument a name and discuss it in an earlier section.

Since $\mathcal{U}(X)_{U/}$ is a sieve covering U and $\mathcal{F}_{\mathcal{U}}$ is a sheaf,

$$\mathrm{N}(\mathcal{U}(X)_{U/})^{op} \to \mathrm{N}(\mathcal{U}(X)_{U/})^{op,\triangleleft} \to \mathcal{C}$$

is a colimit diagram and this completes the proof that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$. $\square$

Chapter 4

Verdier Duality

4.1 Classical Verdier Duality

TODO: I don't know, maybe write some shit about regular Verdier Duality for 1-categories and stuff.

Blablabla k a field and A the category of chain complexes of k-vector spaces. Vector space duality gives a limit preserving functor $\mathrm{N}(A^{op}) \to \mathrm{N}(A)$ which induces a functor

$$\mathrm{Shv}(X; \mathrm{N}(A)^{op}) \to \mathrm{Shv}(X; \mathrm{N}(A))$$

for any locally compact Hausdorff space. Composing with the equivalence below yields a functor

$$\mathbb{D}' : \mathrm{Shv}(X; \mathrm{N}(A))^{op} \to \mathrm{Shv}(X; \mathrm{N}(A))$$

and it is this functor that is usually called Verdier Duality. This is not necesarrily an equivalence of ∞-categories unless certain finiteness conditions are imposed.

4.2 Verdier Duality in infinity-categories

This chapter is all about proving the following theorem:

Theorem 4.2.1 ([Lur17, Theorem 5.5.5.1])**.** *Let X be a locally compact Hausdorff space and $\mathcal{C}$ be a stable ∞-category with small limits and colimits. Then we have the following equivalence of ∞-categories*

$$\mathbb{D} : \mathrm{Shv}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}(X; \mathcal{C}^{op}).$$

We will be using the theory of $\mathcal{K}$-sheaves set up in the previous chapter to prove the theorem. By corollary **??** we can rewrite theorem 4.2.1 in terms of $\mathcal{K}$-sheaves instead:

Theorem 4.2.2. *Let X be a locally compact Hausdorff space and $\mathcal{C}$ be a stable ∞-category with small limits and colimits. Then we have the following equivalence of ∞-categories:*

$$\mathbb{D}_{\mathcal{K}} : \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op} \simeq \mathrm{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}).$$

Definition 4.2.3 ([Lur17, Notation 5.5.5.5])**.** Let X be a locally compact Hausdorff space. We define a partially ordered set M as follows:

1. The objects of M are pairs (i, S) where $0 \leq i \leq 2$ and $S \subseteq X$ such that $i = 0$ implies S is compact and $i = 2$ implies $X - S$ is compact.
2. We have $(i, S) \leq (j, T)$ if either $i \leq j$ and $S \subseteq T$, or $i = 0$ and $j = 2$.

Remark 4.2.4 ([Lur17, Remark 5.5.5.6]). Observe that projecting $(i, S) \to i$ gives a map $\varphi : M \to [2]$ of partially ordered sets. For $0 \leq i \leq 2$ denote the fiber $\varphi^{-1}\{i\}$ by M_i. Also, observe that $M_0 \simeq \mathcal{K}(X)$, $M_2 \simeq \mathcal{K}(X)^{op}$ and M_1 is isomorphic to the powerset poset of X.

Definition 4.2.5. Let M' denote the partially ordered sets of pairs (i, S), where $0 \leq i \leq 2$ and $S \subseteq X$ such that $i = 0$ implies S is compact and $i = 2$ implies $X - S$ is either open or compact. Let $(i, S) \leq (j, T)$ if $i \leq j$ and $S \subseteq T$ or if $i = 0$ and $j = 2$. For $0 \leq i \leq 2$, let M'_i denote the subset $\{(j, S) \in M' | j = i\} \subseteq M'$.

TODO: Something about connecting this new notion of Verdier duality to the classical notion of exchanging cohomology with cohomology with compact support.

Definition 4.2.6 ([Lur17, Definition 5.5.5.9])**.** Let X be a locally compact Hausdorff space and $\mathcal{C}$ a pointed ∞-category with small limits and colimits. For a sheaf $\mathcal{F} \in \mathrm{Shv}(X; \mathcal{C})$ and K compact we denote by $\Gamma_K(X; \mathcal{F})$ the fiber product $\mathcal{F}(X) \times_{\mathcal{F}(X-K)} 0$. For U open, we denote by $\Gamma_c(U; \mathcal{F})$ the filtered colimit $\mathrm{colim}_{\mathcal{K}(X)_{/U}} \Gamma_K(X; \mathcal{F})$ where K ranges over all compact subsets of U.

Preben: Let's be consistent on wether we write $\mathrm{colim}_{K \subseteq U}$ or $\mathrm{colim}_{\mathcal{K}(X)_{/U}}$.

Preben: Lurie writes $\mathrm{colim}_{K \subseteq U} \Gamma_K(M; \mathcal{F})$, but I think that is a mistake.

The construction $U \mapsto \Gamma_c(U; \mathcal{F})$ determines a functor

$$\Gamma_c(-; \mathcal{F}) : \mathrm{N}(\mathcal{U}(X)) \to \mathcal{C}.$$

Remark 4.2.7. Observe that for K a compact subset of an open subset U we have $\Gamma_K(X; \mathcal{F}) = \Gamma_K(U; \mathcal{F})$.

Proof. We have pullbacks, since $\mathcal{F}$ is a sheaf.

$$\begin{tikzcd}
\mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Gamma(X; \mathcal{F}) \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner"] & \Gamma(U; \mathcal{F}) \arrow[d, "g"] \\
0 \arrow[r] & \Gamma(X - K; \mathcal{F}) \arrow[r] & \Gamma(U - K; \mathcal{F})
\end{tikzcd}$$

As the composition of pullbacks is again a pullback we get $\Gamma_K(X;\mathcal{F}) = \mathrm{fib}(f) = \mathrm{fib}(g) = \Gamma_K(U;\mathcal{F})$. □

Lemma 4.2.8 ([Lur17, Proposition 5.5.5.7]). *Let X be a locally compact Hausdorff space, $\mathcal{C}$ be a stable ∞-category with small limits and colimits and M be as in 4.2.3. Let $F : \mathrm{N}(M) \to \mathcal{C}$ be a functor. Then the following conditions are equivalent:*

1. *The restriction $(F|\mathrm{N}(M_0))^{op}$ determines a $\mathcal{K}$-sheaf $\mathrm{N}(\mathcal{K}(X))^{op} \to \mathcal{C}^{op}$, the restriction $F|\mathrm{N}(M_1)$ is zero, and F is left Kan extended from $\mathrm{N}(M_0 \cup M_1)$.*
2. *The restriction $F|\mathrm{N}(M_2)$ determines a $\mathcal{K}$-sheaf $\mathrm{N}(\mathcal{K}(X))^{op} \to \mathcal{C}$, the restriction $F|\mathrm{N}(M_1)$ is zero, and F is right Kan extended from $\mathrm{N}(M_1 \cup M_2)$.*

Proof. First observe that the map $(i, S) \mapsto (2-i, X-S)$ is an order-reversing bijection $M \to M$ which is moreover self-inverse. This means that is it is enough to show that condition 2. implies condition 1..

We start by assuming condition 2., and let $F : \mathrm{N}(M) \to \mathcal{C}$ be such a functor. Let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(M'), \mathcal{C})$ spanned by those functors F satisfying the following conditions:

1. $F|\mathrm{N}(M_2)$ is a $\mathcal{K}$-sheaf on X.
2. $F|\mathrm{N}(M_2')$ is a right Kan extension of $F|\mathrm{N}(M_2)$.
3. $F|\mathrm{N}(M_1')$ is zero.
4. $F|\mathrm{N}(M')$ is a right Kan extension of $F|\mathrm{N}(M_1' \cup M_2')$.

By [Lur09, Proposition 4.3.2.15] we can extend F to a functor $F' \in \mathcal{D}$.

Preben: This should just boil down to fully faithful Kan extensions along fully faithful functors give actual on the nose extensions, but might be smart to ref the result.

Observe that we have a bijection between $\mathcal{U}(X)^{op}$ and the partially ordered set of closed subsets of X by sending $\mathcal{U}(X) \ni U \mapsto (X - U)$ and we have a natural inclusion $\mathcal{U}(X)^{op} \hookrightarrow M_2'$. By Theorem 3.2.7 we can restrict $\mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ to $\mathrm{Shv}(X;\mathcal{C})$, so we can also restrict M_2' and even $\mathcal{D}$ to $\mathrm{Shv}(X;\mathcal{C})$. Let $\mathcal{F}$ be the sheaf obtained by restricting F'. We will first prove that $F|_{\mathrm{N}(M_0)}$ is given informally by the formula $F|_{\mathrm{N}(M_0)}(K) = \Gamma_K(X;\mathcal{F})$.

Preben: This should be made a separate lemma for readability.

Define $\varphi : \mathrm{N}(M_0) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{N}(M'))$ by sending an object $(0, K) \in M_0$ to the diagram

$$\begin{tikzcd} (0,K) \arrow[r] \arrow[d] & (1,K) \arrow[d] \\ (2,\emptyset) \arrow[r] & (2,K). \end{tikzcd}$$

We can regard $\varphi(0,K)$ as a map $i : \Lambda^2_2 \to (M'_1 \cup M'_2)_{(0,K)/}$:

$$\begin{tikzcd} & a \arrow[d] \\ b \arrow[r] & c \end{tikzcd} \longrightarrow \begin{tikzcd} & (0,K)\to(1,K) \arrow[d] \\ (0,K)\to(2,\emptyset) \arrow[r] & (0,K)\to(2,K) \end{tikzcd}$$

Here we have abused notation to write the fiber product $\mathrm{N}(M')_{(0,K)/} \times_{\mathrm{N}(M')} \mathrm{N}\left(M'_1 \cup M'_2\right)$ as $\mathrm{N}(M'_1 \cup M'_2)_{(0,K)/}$. By 2.4.5 i is cofinal if and only if for every $(m,A) \in \mathrm{N}(M'_1 \cup M'_2)_{(0,K)/}$ the fiber product

$$\begin{tikzcd} \mathrm{PB} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \left(\mathrm{N}(M'_1 \cup M'_2)_{(0,K)/}\right)_{/(m,A)} \arrow[d, hook, "j"] \\ \Lambda^2_2 \arrow[r, hook, "i"] & \mathrm{N}(M'_1 \cup M'_2)_{(0,K)/} \end{tikzcd}$$

is weakly contractible. As we have partially ordered sets j is just the inclusion, and we have $j((0,K) \to (r,B) \to (m,A)) = ((0,K) \to (r,B))$. Since $i(a) = (2,\emptyset), i(b) = (1,K)$ and $i(c) = (2,K)$ and the pullback of a mono is mono, PB has to be a subcategory of

$$\begin{tikzcd} & (b,(1,K)) \arrow[d] \\ (a,(2,\emptyset)) \arrow[r] & (c,(2,K)). \end{tikzcd}$$

Observe that such a subcategory fails to be contractible only if (m,A) is chosen such that the pullback is either empty or consists of two disjoint objects. If $r=1$ we know $(1,K) \le (1,B)$ and have no arrows from $(2,\emptyset)$ or $(2,K)$ to $(1,B)$. If $r=2$ we must have $(2,\emptyset) \le (2,K) \le (2,B)$, so the pullback is always weakly contractible. By condition 4., $F'|_{\mathrm{N}(M')}$ is right Kan extended from $F'|_{\mathrm{N}(M'_1 \cup M'_2)}$, which by definition 2.5.10 means

$$(M'_1 \cup M'_2)^{\triangleleft}_{(0,K)/} \hookrightarrow M'^{\triangleleft}_{(0,K)/} \to M' \xrightarrow{F'} \mathcal{C}$$

is a limit diagram. In other words, we have

$$\lim_{\mathrm{N}(M'_1 \cup M'_2)_{(0,K)/}} F' = F'(0,K)$$

and by the left cofinality of i we get

$$F'(0,K) = \lim_{\mathrm{N}(M'_1 \cup M'_2)_{(0,K)/}} F' = \lim_{\Lambda^2_2}(F' \circ i) = \lim F'((2,\emptyset) \to (2,K) \leftarrow (1,K))$$

which means condition 4. is equivalent to requiring that F' composed with $\varphi(0,K)$ yields another pullback diagram

$$\begin{tikzcd} F'(0,K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F'(1,K) \arrow[d] \\ F'(2,\emptyset) \arrow[r] & F'(2,K) \end{tikzcd}$$

Observe now that by condition 3. $F'(1,K) = 0$ and hence

$$F'(0,K) = F(0,K) \simeq \mathrm{fib}(F(2,\emptyset) \to F(2,K)).$$

Recall that we defined $\mathcal{F}$ as the restriction of F' to $\mathrm{Shv}(X;\mathcal{C})$ by identifying open sets U with their complements. This means that $F'(2,\emptyset) = \mathcal{F}(X)$ and $F'(2,K) = \mathcal{F}(X-K)$ which in turn means that

$$F'(0,K) = F(0,K) \simeq \mathrm{fib}(F(2,\emptyset) \to F(2,K)) \simeq \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(X-K)) = \Gamma_K(X;\mathcal{F})$$

which completes the proof that $F|_{\mathrm{N}(M_0)}(K)$ is given by $\Gamma_K(X;\mathcal{F})$.

Let us now denote $F|_{\mathrm{N}(M_0)}(K)$ by $\mathcal{G}$ and check that $\mathcal{G}^{op} \in \mathrm{Shv}_{\mathcal{K}}(X;C^{op})$. Here $\mathcal{G}^{op}$ does the same on objects as $\mathcal{G}$ but we think of it as a functor $\mathrm{N}(M_0)^{op} \to \mathcal{C}^{op}$. We must show that it satisfies the following three properties:

1. $\mathcal{G}(\emptyset)$ is a zero object, because $\mathcal{G}(\emptyset) = \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(X-\emptyset)) = 0$.
2. For any compact subsets K and K' of X we must show the following diagram is a pushout:

$$\begin{tikzcd} \mathcal{G}(K \cap K') \arrow[r] \arrow[d] & \mathcal{G}(K') \arrow[d] \\ \mathcal{G}(K) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{G}(K \cup K') \end{tikzcd}$$

Observe that the diagram can be identified with the fiber of the map

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(X) \arrow[d] \arrow[r, phantom, ""] & & \mathcal{F}(X-(K\cap K')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(X-K') \arrow[d] \\ \mathcal{F}(X) \arrow[r] & \mathcal{F}(X) & & \mathcal{F}(X-K) \arrow[r] & \mathcal{F}(X-(K\cup K')) \end{tikzcd} \longrightarrow$$

As $\mathcal{F}$ is a sheaf, this is a map between pullbacks, so our diagram is also a pullback.

3. For any compact subset K of X we must show that the map $\theta : \mathcal{G}(K) \to \lim_{K \Subset K'} \mathcal{G}(K')$ is an equivalence in $\mathcal{C}$. Observe now that θ gives us a map between two fiber sequences

$$\begin{tikzcd}
\mathcal{G}(K) \arrow[r, "\theta"] \arrow[d] & \lim_{K \Subset K'} \mathcal{G}(K') \arrow[d] \\
\mathcal{F}(X) \arrow[r, "\theta'"] \arrow[d] & \lim_{K \Subset K'} \mathcal{F}(X) \arrow[d] \\
\mathcal{F}(X-K) \arrow[r, "\theta''"] & \lim_{K \Subset K'} \mathcal{F}(X-K')
\end{tikzcd}$$

Since the partially ordered set $\{K' \in \mathcal{K}(X) | K \Subset K'\}$ is filtered it is weakly contractible and hence θ' is an equivalence. Since $\mathcal{F}$ is a sheaf and the set $\{X - K' | K \Subset K'\}$ is a covering sieve on $X - K'$, θ'' is also an equivalence. As we have shown that θ' and θ'' are equivalences, θ must also be an equivalence, and we have shown that $\mathcal{G}^{op}$ determines a $\mathcal{K}$-sheaf $\mathrm{N}(\mathcal{K}(X)^{op} \to \mathcal{C}^{op})$.

To complete the proof that 2. implies 1., we must show F is left Kan extended from $F|_{\mathrm{N}(M_0 \cup M_1)}$. Let $M'' = \{(i, S) \in M_0 \cup M_1 | (i, S) \in \mathcal{K}(X)\}$. We can observe that $F|_{\mathrm{N}(M_0 \cup M_1)}$ is left Kan extended from $F|_{\mathrm{N}(M'')}$ (F is zero on M_1). By Proposition 2.5.11 it is enough to show that F is a left Kan extension of $F|_{\mathrm{N}(M'')}$, and this is enough to check at every $(2, S) \in M_2$. We will instead show that $F'|_{\mathrm{N}(M'' \cup M_2')}$ is a left Kan extension of $F|_{\mathrm{N}(M'')}$ and for this we define

$$B := \{(2, X - U) \subseteq M_2' |\, U \in \mathcal{U}(X) | \overline{U} \in \mathcal{K}(X)\}.$$

By Proposition 2.5.11 it is enough to show that

(a) $F'|_{\mathrm{N}(M'' \cup M_2')}$ is a left Kan extension of $F'|_{\mathrm{N}(M'' \cup B)}$.
First observe that M'' and M_2' are disjoint so it is enough to check that for every $(2, X - K) \in M_2' - B$, the following

$$\mathrm{N}(M'' \cup B)^{\triangleright}_{/(2,X-K)} \hookrightarrow \mathrm{N}(M'' \cup M_2')^{\triangleright}_{/(2,X-K)} \to \mathrm{N}(M'' \cup M_2') \to \mathcal{C}$$

is a colimit diagram. According to Lemma 2.4.5 we can restrict the colimit from $(M'' \cup B)_{/(2,X-K)}$ to $B_{/(2,X-K)}$ if we can show that the pullback

$$\begin{tikzcd}
PB \arrow[r] \arrow[d] & \big((M'' \cup B)_{/(2,X-K)}\big)_{/(2,X-U)} \arrow[d] \\
\big(B_{/(2,X-K)}\big)_{/(2,X-U)} \arrow[r] & (M'' \cup B)_{/(2,X-K)}
\end{tikzcd}$$

is weakly contractible. As this is just the partially ordered set $\{(2, X - U) \in B |\, (i, S) \leq (2, X - U) \leq (2, X - K)\}$ it is weakly contractible by the usual argument (it is nonempty and stable under finite unions hence filtered). This means that it is enough to show that $F'|_{\mathrm{N}(M_2')}$ is left Kan extended from B. Assumption 2. says that $F|_{\mathrm{N}(M_2)}$ determines a $\mathcal{K}$-sheaf, $F|_{\mathrm{N}(M_1)} = 0$ and that F is a right Kan extension from $\mathrm{N}(M_1 \cup M_2)$. Identifying $M_2 = \{(2,S)|(X-S) \in \mathcal{K}(X)\}$ with $\mathcal{K}(X)^{op}$ we see that we are in the situation of Theorem 3.2.7. As $M_2' = \{(2,S)|(X-S)| \in \mathcal{U}(X) \cup \mathcal{K}(X)\}$ we can identify it with $(\mathcal{U}(X) \cup \mathcal{K}(X))^{op}$ and by Theorem 3.2.7 we get that $F'|_{\mathrm{N}(M_2')}$ is a left Kan extension of $F'|_{\mathrm{N}(\mathcal{U}(X)^{op})}$. By observing that for a $K \in \mathcal{K}(X)$ the collection of open neigbhorhoods of K with compact closure is cofinal in the collection of all open neigbhorhoods of K in X we get that $F'|_{\mathrm{N}(M_2')}$ is furthermore left Kan extended from B, which was what we wanted to show.

$$\begin{tikzcd} B \arrow[r, hook, "i"] & \mathcal{U}(X)^{op} \arrow[rr] \arrow[dr, hook] & & \mathcal{C} \\ & & M_2' \arrow[ur] & \end{tikzcd}$$

Preben: My thoughts here are that we calculate Kan extensions as colimits, so i being cofinal over some fixed K means restricting the colimit from $\mathcal{U}(X)^{op}$ back to B is an equivalence.

Preben: Lurie never states what he means with M''', but I think he might just mean B.

(b) $F'|_{\mathrm{N}(M'' \cup B)}$ is a left Kan extension of $F|_{\mathrm{N}(M'')}$ Fix $U \in \mathcal{U}(X)$ such that $\overline{U} \in \mathcal{K}(X)$. By 2.5.10 we want to show that $F'(2, X-U)$ is a colimit of the diagram $F'|_{\mathrm{N}(M'')/(2,X-U)}$. For $K \in \mathcal{K}(X)$ denote by M''_K the subset of M'' consisting of pairs (i,S) such that $(0,K) \leq (i,S) \leq (2, X-U)$. Now, observe that $\mathrm{N}(M'')_{/(2,X-U)}$ is a filtered colimit of $\{\mathrm{N}(M''_K)\}_{K \in \mathcal{K}(X)_{U/}}$. By [Lur09][4.2.3] we can identify $\mathrm{colim}(F'|_{\mathrm{N}(M'')})_{/(2,X-U)}$ with the filtered colimit of the diagram $\{\mathrm{colim}(F'|_{\mathrm{N}(M''_K)})\}_K$. This means that we are reduced to showing that for every $K \in \mathcal{K}(X)_{U/}$, F' exhibits $F'(2, X-U)$ as a colimit of $F'|_{\mathrm{N}(M''_K)}$. By Lemma 2.4.5 the diagram

$$\begin{tikzcd} (0, K-U) \arrow[r] \arrow[d] & (1, K-U) \\ (0,K) & \end{tikzcd}$$

is left cofinal in $\mathrm{N}(M''_K)$ and hence[1] it is enough to show that

$$\begin{tikzcd} F'(0, K-U) \arrow[r] \arrow[d] & F'(1, K-U) \arrow[d] \\ F'(0,K) \arrow[r] & F'(2, X-U) \end{tikzcd}$$

[1] **TODO:** Explain this step.

is a pushout in $\mathcal{C}$. We will show this by considering the larger diagram

$$\begin{tikzcd}
F'(0, K-U) \arrow[r] \arrow[d] & F'(1, K-U) = 0 \arrow[d] & \\
F'(0, K) \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(1, K) = 0 \arrow[d] \\
F'(2, \emptyset) \arrow[r] & F(2, K-U) \arrow[r] \arrow[d] & F(2, K) \arrow[d] \\
& F(2, X-U) \arrow[r] & F(2, X)
\end{tikzcd}$$

where we already know that the middle rectangle is a pullback (we have shown $F'(0,K)$ to be the fiber of the map $F'(2,\emptyset) \to F(2,K)$), so the middle left square is also a pullback. As we have shown $F'(0, K-U) = \mathrm{fib}(F(2,\emptyset) \to F(2, K-U))$ the left vertical rectangle is also a pullback, so the upper left must be as well. Since $\mathcal{C}$ is stable it is also a pushout. As $F(1,K) = 0$ and $F(2,X) = \mathcal{F}(\emptyset) = 0$ we have an equivalence $F(1,K) \to F(2,K) \to F(2,X)$ which means that if we can show the composite square

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & F(1, K) = 0 \arrow[d] \\
F(2, X-U) \arrow[r] & F(2, X) = \mathcal{F}(\emptyset) = 0
\end{tikzcd}$$

is a pullback, we have shown the desired equivalence $Z \to F(2, X-U)$ using the fact that pullback along an equivalence is again an equivalence.
To complete the proof it is therefore enough to show that the lower right square is a pullback. Replacing F by $\mathcal{F}$ we get

$$\begin{tikzcd}
\mathcal{F}((X-K) \cup U) \arrow[r] \arrow[d] & \mathcal{F}(X-K) \arrow[d] \\
\mathcal{F}(U) \arrow[r] & \mathcal{F}(\emptyset)
\end{tikzcd}$$

which is a pullback because $\mathcal{F}$ is a sheaf (U and $X-K$ are disjoint).

$\square$

We can now prove Verdier Duality (Theorem 4.2.1):

Proof. Let $\mathcal{E}(\mathcal{C}) \subseteq \text{Fun}(\text{N}(M))$ be the full subcategory spanned by those functors satisfying the conditions of Proposition 4.2.8 and observe that the inclusions $M_0 \hookrightarrow M \hookleftarrow M_2$ give restrictions

$$\text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op}) \xleftarrow{\theta} \mathcal{E}(\mathcal{C})^{op} \xrightarrow{\theta'} \text{Shv}_{\mathcal{K}}(X; \mathcal{C})^{op}.$$

Because we Kan extend along inclusions of full subcategories which are fully faithful these are equivalences of ∞-categories.

Preben: Note that $F \in \mathcal{E}(\mathcal{C})$ means $F|_{M_0} \in \text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})$, $F|_{M_0 \cup M_1} = RKAN(F|_{M_0})$ and $F = LKAN(F|_{M_0 \cup M_1})$.

This proves Theorem 4.2.2 and by Corollary **??** we have shown Theorem 4.2.1. □

Proposition 4.2.9 ([Lur17, Proposition 5.5.5.10])**.** *Let X be a locally compact Hausdorff space and $\mathcal{C}$ a pointed and stable ∞-category with small limits and colimits. Then the equivalence of ∞-categories*

$$\mathbb{D} : \text{Shv}(X; \mathcal{C})^{op} \simeq \text{Shv}(X; \mathcal{C}^{op}).$$

given in Theorem 4.2.1 is given by $\mathbb{D}(\mathcal{F})(U) = \Gamma_c(U; \mathcal{F})$, and this is the infinity-categorical generalization of the classical fact that conjugation by Verdier Duality exchanges cohomology and cohomology with compact support.

TODO: *Maybe worth trying to make the bridge between this statement and the classical fact even more concrete.*

Proof. It follows from the proof of Theorem 3.2.7 that the equivalence

$$\theta : \text{Shv}(X; \mathcal{C}^{op})^{op} \simeq \text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op}$$

is given by the formula $\theta(\mathcal{F})(U) = \text{colim}_{K \subseteq U} \mathcal{F}(K)$.

Preben: Make sure this colimit is taken in the correct category. It should probably just be a limit in $\mathcal{C}$.

Preben: Should consider writing out theta more explicitly in the proof of 3.2.7.

Let $\psi : \text{Shv}(X; \mathcal{C}) \to \text{Shv}_{\mathcal{K}}(X; \mathcal{C})$ be the equivalence of Corollary **??** and ψ' the equivalence $\text{Shv}_{\mathcal{K}}(X; \mathcal{C}) \to \text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op}$ of Theorem 4.2.2. Composing, we get a string of equivalences

$$\mathbb{D}^{op} : \text{Shv}(X; \mathcal{C}) \xrightarrow{\psi} \text{Shv}_{\mathcal{K}}(X; \mathcal{C}) \xrightarrow{\psi'} \text{Shv}_{\mathcal{K}}(X; \mathcal{C}^{op})^{op} \xrightarrow{\theta} \text{Shv}(X; \mathcal{C}^{op})^{op}.$$

Let $\mathcal{D}$ be as in the proof of Lemma 4.2.8. By Theorem 3.2.7 the restriction $\mathcal{D} \to \text{Fun}(\text{N}(\mathcal{U}(X))^{op}, \mathcal{C})$ is a trivial Kan fibration onto $\text{Shv}(X; \mathcal{C})$.

Preben: Here we have used that we Kan extend along the fully faithful inclusion of $\mathrm{Shv}(X;\mathcal{C})$ (it is a full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathcal{U}(X))^{op},\mathcal{C})$.)

In the other direction we restrict $\mathcal{D} \to \mathrm{Fun}(\mathrm{N}(M_0),\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{K}(X),\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{K}(X)^{op},\mathcal{C}^{op})^{op}$ and $\psi' \circ \psi$ is given by the composition $\mathrm{Shv}(X;C) \to \mathcal{D} \to \mathrm{Fun}(\mathcal{K}(X)^{op},\mathcal{C}^{op})^{op}$ and as we saw in the proof of Lemma 4.2.8, restriction from $\mathcal{D}$ to functors from $\mathrm{N}(M_0)$ is given by $\Gamma_K(X;\mathcal{F})$. This means that $\psi' \circ \psi : \mathcal{F} \mapsto (K \mapsto \Gamma_K(X;\mathcal{F}))$ so by Remark 4.2.7 we have

$$(\theta \circ \psi' \circ \psi)(\mathcal{F})(U) = \mathrm{colim}_{K \subseteq U}(\Gamma_K(X;F)) = \mathrm{colim}_{K \subseteq U}(\Gamma_K(U;F)) = \Gamma_c(U;\mathcal{F}).$$

$\square$

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