title

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Theorem 0.0.1 ([**HTT**])**.** *Let X be locally compact and Hausdorff and $\mathcal{C}$ an ∞-category with small limits and colimits and left exact filtered colimits. Let $\mathcal{F} : \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \to \mathcal{C}$. The following conditions are equivalent:*

1. *The presheaf $\mathcal{F}_{\mathcal{K}} := \mathcal{F}|\mathrm{N}(\mathcal{K}(X))^{op}$ is a $\mathcal{K}$-sheaf, and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$.*
2. *The presheaf $\mathcal{F}_{\mathcal{U}} := \mathcal{F}|\mathrm{N}(\mathcal{U}(X))^{op}$ is a sheaf, and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$.*

We will split the theorem into a few lemmas for readability.

Lemma 0.0.2. *Let $\mathcal{F}$ be as in the theorem statement. If $\mathcal{F}_{\mathcal{K}} := \mathcal{F}|\mathrm{N}(\mathcal{K}(X))^{op}$ is a $\mathcal{K}$-sheaf, then F is a left Kan extension of $\mathcal{F}_{\mathcal{U}} := \mathcal{F}|\mathrm{N}(\mathcal{U}(X))^{op}$.*

Proof. By definition we want to show that

$$\mathrm{N}(\mathcal{U}(X)^{op}_{/K})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$. The assumption that $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf means that for each $K \in \mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}(K)$ is a colimit of $\mathcal{F}|\mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}$. We will "transfer" this colimit to the colimit we want by cofinal maps

$$\mathrm{N}((\mathcal{U}(X)^{op})_{/K}) \xrightarrow{p} \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K}) \xleftarrow{p'} \mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}.$$

Recall that by **??** checking cofinality reduces to checking weak contractibility of certain simplicial sets. For p we must check $\mathrm{N}((\mathcal{U}(X))^{op}_{/K}) \times_{\mathrm{N}((\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{/K})} \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})_{K'/}$ is weakly contractible for every $K' \in \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})$. This is the simplicial set obtained by taking the nerve of the partially ordered set $\{U \in \mathcal{U}(X) | \; K \subseteq U \subseteq K'\}$. By [**HTT**] filtered ∞-categories are weakly contractible, and our partially ordered set is filtered as it is nonempty, stable under finite union, and taking nerves preserve the property of being filtered.

TODO: Make this a result in cofinality chapter instead.

The simplicial set $\mathrm{N}(\{K'' \in \mathcal{K}(X) | \; K \Subset K'' \subseteq K'\})$[1] is weakly contractible by exactly the same argument, and hence p and p' are cofinal maps. By cofinality of p and p', the diagram

$$\mathrm{N}((\mathcal{U}(X))^{op}_{/K})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram if and only if

$$\mathrm{N}((\mathcal{K}(X)_{\Subset K})^{op})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X) \cup \mathcal{K}(X))^{op}_{/K})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X) \cup \mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram, which it is by the assumption that $F_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf. $\square$

[1]

TODO: This should probably be strict inclusion. Double check that.

Lemma 0.0.3. *If $\mathcal{F}_{\mathcal{K}} := \mathcal{F}|\mathrm{N}(\mathcal{K}(X))^{op}$ is a $\mathcal{K}$-sheaf, and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$, then $\mathcal{F}_{\mathcal{U}}$ is a sheaf.*

Proof. By definition **??** we must show that for every $U \in \mathcal{U}(X)$ and every covering sieve $\mathcal{W}$ covering U,

$$\mathrm{N}(\mathcal{W})^{\triangleright} \hookrightarrow \mathrm{N}(\mathcal{U}(X)_{/U})^{\triangleright} \to \mathrm{N}(\mathcal{U}(X)) \xrightarrow{\mathcal{F}} \mathcal{C}^{op}$$

is a colimit diagram, or equivalently that

$$\mathrm{N}(\mathcal{W})^{op,\triangleleft} \hookrightarrow \mathrm{N}(\mathcal{U}(X)_{/U})^{op,\triangleleft} \to \mathrm{N}(\mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a limit diagram. We will once again use cofinality by observing that **??** implies cofinality of the inclusion

$$\mathrm{N}(\mathcal{W}) \subseteq \mathrm{N}(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))$$

where $\mathcal{K}_{\mathcal{W}}(X)$ is the set $\{K \in \mathcal{K}(X) | (\exists W \in \mathcal{W})[K \subseteq W]\}$, so it is enough to show the limit starting from $\mathrm{N}(\mathcal{W} \cup \mathcal{K}_{\mathcal{W}}(X))^{op}$.

TODO: Write out the details of why this is contractible and the inclusion is cofinal.

Recall that this is equivalent to showing that $\mathcal{F}|_{\mathrm{N}(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X))^{op,\triangleleft}}$ is right Kan extended from $\mathcal{F}|_{\mathrm{N}(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X))^{op}}$.

TODO: Write out the result in the Kan chapter.

By the assumption that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ and the observation that

$$\mathcal{F}(U) = \lim_{K \in \mathcal{K}(X)^{op}_{U/}} \mathcal{F}(K) = \lim_{K \in \mathcal{K}_{\mathcal{W}}(X)^{op}_{U/}} \mathcal{F}(K)$$

we see that $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}})^{op}}$ is a right Kan extension of $\mathcal{F}|_{(K_{\mathcal{W}}(X))^{op}}$. Hence, it suffices to prove that $\mathcal{F}|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. Outside of U this is clear from the fact that $F|_{(\mathcal{W}\cup\mathcal{K}_{\mathcal{W}}^{op})}$ is right Kan extended from $\mathcal{K}_{\mathcal{W}}(X)^{op}$. This means we only need to prove $\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. Observe that by assumption

$$\mathcal{F}(U) = \lim_{K \in \mathcal{K}(X)^{op}_{/U}} \mathcal{F}(K) = \lim_{K \in (\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \mathcal{F}(K)$$

so $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}}$ is right Kan extended from $\mathcal{K}(X)^{op}_{/U}$. Lemma **??** tells us that $\mathcal{F}|_{(\mathcal{K}(X)_{/U})^{op}}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}$. We have $\mathrm{N}(\mathcal{K}_{\mathcal{W}}(X))^{op} \subseteq \mathrm{N}(\mathcal{K}(X)_{/U})^{op} \subseteq \mathrm{N}(\mathcal{K}(X)_{/U} \cup \{U\})^{op}$, with Kan extensions as in proposition **??**, so we get that $\mathcal{F}|_{\mathrm{N}(\mathcal{K}(X)_{/U}\cup\{U\})^{op}}$ is right Kan extended from $\mathrm{N}(\mathcal{K}_{\mathcal{W}}(X))^{op}$. To summarize, we have the following square of inclusions

$$\begin{tikzcd}
\mathcal{K}_{\mathcal{W}}(X)^{op} \arrow[r, hook, "i"] \arrow[d, hook, "a"] & (\mathcal{K}_{\mathcal{W}}(X) \cup \{U\})^{op} \arrow[d, hook, "b"] \\
\mathcal{K}(X)^{op}_{/U} \arrow[r, hook, "j"] & (\mathcal{K}(X)_{/U} \cup \{U\})^{op}
\end{tikzcd}$$

where $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \simeq b_*(\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}\cup\{U\})^{op}})$ and $\mathcal{F}|_{(\mathcal{K}(X)_{/U}\cup\{U\})^{op}} \simeq (j \circ a)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$. We want to show $\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}} \simeq i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}})$. Since b is fully faithful (it is the inclusion of a full subcategory), we know $b^*b_* \simeq \mathrm{id}$, so we get

$$\begin{aligned}
\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}} &\simeq b^*b_*(\mathcal{F}|_{(\mathcal{K}_{\mathcal{W}}(X)\cup\{U\})^{op}}) && (1)\\
&\simeq b^*(j\circ a)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (2)\\
&\simeq b^*(b\circ i)_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (3)\\
&\simeq b^*b_*i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (4)\\
&\simeq i_*(\mathcal{F}|_{\mathcal{K}_{\mathcal{W}}(X)^{op}}) && (5)
\end{aligned}$$

□

Lemma 0.0.4. *If $\mathcal{F}_{\mathcal{U}} := \mathcal{F}|\mathrm{N}(\mathcal{U}(X))^{op}$ is a sheaf, and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$, then $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf.*

Proof. By definition we need to show three things: Firstly, observe that $\mathcal{F}_{\mathcal{K}}(\emptyset) = \mathcal{F}_U(\emptyset)$ and since $\mathcal{F}_U$ is a sheaf $\mathcal{F}_{\mathcal{K}}(\emptyset)$ is terminal. Secondly, we need the following diagram to be a pullback in $\mathcal{C}$ for any $K, K' \in \mathcal{K}(X)$.

$$\begin{tikzcd}
\mathcal{F}(K\cup K') \arrow[r] \arrow[d] & \mathcal{F}(K) \arrow[d] \\
\mathcal{F}(K') \arrow[r] & \mathcal{F}(K\cap K')
\end{tikzcd} \tag{6}$$

We will do this by using that $\mathcal{F}_U$ is a sheaf. Let $P = \{(U,U')|K \subseteq U, K' \subseteq U'\}$ and $\sigma : \Delta^1\times\Delta^1 \to \mathcal{C}$ denote diagram **??**. Now $\mathcal{F}$ induces a map $\sigma_P : \mathrm{N}(P^{op})^{\triangleright} \to \mathcal{C}^{\Delta^1\times\Delta^1}$ taking each (U,U') to

$$\begin{tikzcd}
\mathcal{F}(U\cup U') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}(U) \arrow[d] \\
\mathcal{F}(U') \arrow[r] & \mathcal{F}(U\cap U')
\end{tikzcd}$$

and the cone point is sent to σ. This is a pullback by the fact that $\mathcal{F}_U$ is a sheaf. Evaluating σ_P in each of the four vertices of $\Delta^1\times\Delta^1$ we get four maps $\mathrm{N}(P^{op})^{\triangleright} \to \mathcal{C}$. We now check that evaluating in the final vertex yields a colimit diagram. By assumption $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_U$ which by definition means that the following is a colimit diagram:

$$\mathrm{N}((\mathcal{U}(X)_{/(K\cap K')})^{op})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{/(K\cap K')})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

Observe that for every $U'' \in \mathcal{U}(X)_{/(K\cap K')}$, the set $P_{U''} = \{(U,U') \in P | U\cap U' \subseteq U''\}$ is nonempty and stable under finite intersections, which implies it is filtered and hence its nerve is contractible.

TODO: Reference this result.

By **??** this implies $\mathrm{N}(P^{op}) \to \mathrm{N}((\mathcal{U}(X)_{/(K\cap K')})^{op})$ is cofinal and we get a colimit diagram

$$\mathrm{N}(P^{op})^{\triangleright} \to \mathrm{N}((\mathcal{U}(X)_{/(K\cap K')})^{op})^{\triangleright} \hookrightarrow \mathrm{N}((\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{/(K\cap K')})^{\triangleright} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}.$$

We can show that evaluating the three other vertices also yields colimit diagrams by similar arguments. Since σ_P yields a colimit diagram when evaluated in each of the four vertices of $\Delta^1\times\Delta^1$, we conclude that σ_P is itself a colimit diagram by [**HTT**]. Observe now that σ_P is a filtered colimit in $\mathcal{C}$ and hence it is left exact. This concludes the argument that **??** is a pullback. Finally, we need to show that for each $K\in\mathcal{K}(X)$, $\mathcal{F}_{\mathcal{K}}$ is a colimit of $\mathcal{F}_{\mathcal{K}}|_{\mathrm{N}(\mathcal{K}_{K\Subset}(X))^{op}}$. We do this by showing

$$\mathrm{N}(\mathcal{K}_{K\Subset}(X)^{op})^{\triangleright} \to \mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. We use Proposition **??** to show that $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}}$ and $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}\cup\{K\}}$ are left Kan extensions of $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X))^{op}}$ which again implies $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}\cup\{K\}}$ is a left Kan extension of $\mathcal{F}|_{\mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}_{K\Subset}(X))^{op}}$. Now observe that

$$\mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))^{op,\triangleright}_{/K} = \mathrm{N}(\mathcal{K}_{K\Subset}(X)\cup\mathcal{U}(X)_{/K})^{op,\triangleright},$$

so in particular

$$\mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))^{op,\triangleright}_{/K} \to \mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram, and the statement is reduced to showing that $\mathrm{N}(\mathcal{K}_{K\Subset}(X)) \subseteq \mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))^{op}_{/K}$ is cofinal. Let $Y\in\mathrm{N}(\mathcal{K}(X)\cup\mathcal{U}(X))_{/K}$ and let R be the partially ordered set $\{K'\in\mathcal{K}(X)|\ K\Subset K'\subseteq Y\}$. Since R is nonempty and stable under intersections, R^{op} is filtered and hence $\mathrm{N}(R)$ is weakly contractible. By Lemma **??** the inclusion is cofinal and we have shown that $\mathcal{F}_{\mathcal{K}}$ is a $\mathcal{K}$-sheaf. $\square$

Lemma 0.0.5. *If $\mathcal{F}_{\mathcal{U}}$ is a sheaf, then $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$.*

Proof. We will show $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$ in a similar manner to how we showed $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathcal{U}}$ in the start of the proof, but we will consider the partial order on $\mathcal{U}(X)$ given by writing $V\Subset U$ whenever $V\in\mathcal{U}(X)$ and its closure $\overline{V}$ is compact and contained in U. Writing $\mathcal{U}(X)_{U/}$ for the set $\{V\in\mathcal{U}(X)|\ V\Subset U\}$, we need to show that

$$\mathrm{N}(\mathcal{K}(X)^{op}_{U/})^{\triangleleft} \hookrightarrow \mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{U/} \xrightarrow{c} \mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}(X))^{op} \xrightarrow{\mathcal{F}} \mathcal{C}$$

is a colimit diagram. As earlier we do this by finding cofinal inclusions

$$\mathrm{N}(\mathcal{K}(X)^{op}_{U/}) \xrightarrow{f} \mathrm{N}(\mathcal{U}(X)\cup\mathcal{K}(X))^{op}_{U/} \xleftarrow{f'} \mathrm{N}(\mathcal{K}(X)_{/U})^{op}.$$

By Lemma **??** f and f' are cofinal inclusions if for any $Y\in(\mathcal{U}(X)\cup\mathcal{K}(X))_{U/}$ the partially ordered sets

$$\{V\in\mathcal{U}(X)|Y\subseteq V\Subset U\}$$

and

$$\{K \in \mathcal{K}(X) | Y \subseteq K \subseteq U\}$$

have weakly contractible nerves, which they have by the usual argument; they are nonempty and stable under unions, hence filtered.

TODO: Give this argument a name and discuss it in an earlier section.

Since $\mathcal{U}(X)_{U/}$ is a sieve covering U and $\mathcal{F}_{\mathcal{U}}$ is a sheaf,

$$\mathrm{N}(\mathcal{U}(X)_{U/})^{op} \to \mathrm{N}(\mathcal{U}(X)_{U/})^{op,\triangleleft} \to \mathcal{C}$$

is a colimit diagram and this completes the proof that $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathcal{K}}$. $\square$