MATHEMATICS ANALYSIS AND APPROACHES INTERNAL INVESTIGATION

Exploring the time taken for an object to descend from one point to another

Introduction

Going through YouTube, I came across a short about the Brachistochrone problem. The problem posed in the video was "*What is the path connecting two points from which when a particle is released at the point on the top, the time taken to slide down the path and reach the bottom point is minimized?"* I initially thought that the answer was obvious, a straight line. However, the video proved me wrong and stated that there were other curves connecting the two points which resulted in a faster descent.[1]

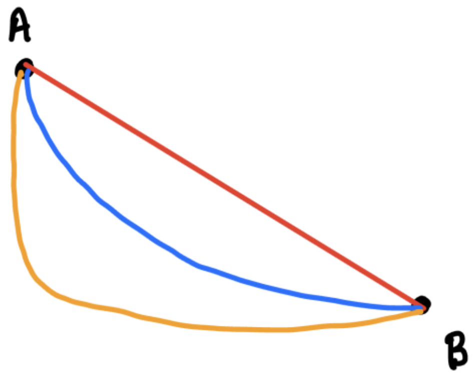


Figure 1: The Brachistochrone Problem (self-made using Notability)

I decided to explore this problem myself and tried to understand why a straight line wouldn't result in the fastest descent. The time taken for a particle to slide down either one of the three curves in Figure 1 from point A to point B is given by:

$$Time = \frac{Distance}{Average\ speed}$$

Since the red curve (straight line) has the shortest distance out of all the curves, one might jump to the conclusion that it would have the shortest time as well. However, the other two curves speed up the particle initially by dropping it down further, increasing the average speed, which decreases the time taken. Qualitatively, there is a tradeoff between short

[1] 3Blue1Brown. "The Brachistochrone, With Steven Strogatz." YouTube, 1 Apr. 2016, www.youtube.com/watch?v=Cld0p3a43fU. Accessed 2 Oct 2023.

distance and slow speed (red curve) and long distance and fast speed (orange curve). A curve in the middle of these two extremes (the blue curve) minimizes the time taken for a particle to go from A to B and is known as the *brachistochrone curve*.

Exploring this problem further, I learnt about the fascinating history behind it, with it being explored by Galileo, the Bernoulli brothers and Isaac Newton.[2] Each of these mathematicians had a different take on this problem. Wanting to dig into this problem deeper, I decided to explore it for my Mathematics Internal Assessment.

Aim and Approach

The aim of this exploration is to find a solution to the Brachistochrone problem. However, before considering the solution to the problem, I decided to test some curves connecting the two points to see how they compare against each other and to gain some intuition for the problem.

Instead of considering two generalized points A and B, I decided to only consider the points $A(0, 1)$ and $B(1, 0)$ for this investigation. This is because considering general coordinates just translates and stretches the entire system and would not provide any new insight into the problem (unless the ratio of height and width is different). Using general coordinates like the sources I have referred to have would just make calculations and equations unnecessarily long and tedious. Furthermore, performing this investigation with general coordinates would not provide an easy way to see how the time taken between curves compare against each

[2] Babb, Jeffry, and James D. Currie. "The Brachistochrone Problem: Mathematics for a Broad Audience via a Large Context Problem." The Mathematics Enthusiast, vol. 5, no. 2–3, July 2008, pp. 169–84. https://doi.org/10.54870/1551-3440.1099. Accessed 5 Nov 2023.

other, since the time would be expressions in terms of the coordinates of A and B rather than numerical values.

Further, for the sake of simplicity, some assumptions made in this problem are that there is no friction acting on the particle (making gravity the only force). Secondly, the particle is dropped at point A with no initial velocity.

The straight line

Finding the time taken for a particle to slide from A to B with a straight-line connecting A and B is a standard problem that is a part of the IBDP Physics course.

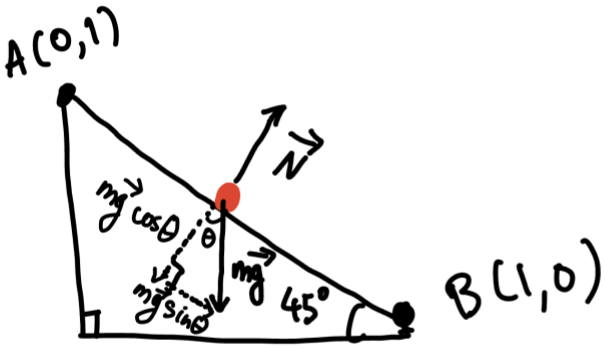


Figure 2: Free body diagram of scenario (self-made using Notability)

I sketched a *Free body diagram* in Figure 2 showing this scenario with the forces being exerted on the particle depicted as vectors pointing out of the particle. The mass of the particle is denoted by m and the acceleration due to gravity is denoted by $\vec{g}$. Angle B is clearly $45°$ using right-angle trigonometry. $m\vec{g}$ is the vector of the force of gravity on the particle, acting downwards. This vector can be resolved into two components, each with magnitude $mg \sin \theta$ and $mg \cos \theta$ respectively. By the similarity of the small and large right triangles, $\theta = 45°$. Since the cosine component of the gravitational force gets cancelled by the normal force ($\vec{N}$), the net force acting on the particle is given by the vector $m\vec{g}\ \sin\theta$. This vector has magnitude $mg\sin(45°) = \frac{mg}{\sqrt{2}}$. The acceleration of the particle can be obtained by substituting the magnitude of the net force into Newton's second law:

$$F_{net} = ma$$

$$\frac{mg}{\sqrt{2}} = ma$$

$$a = \frac{g}{\sqrt{2}}$$

Since acceleration is the derivative of velocity, I integrated with respect to time on both sides to obtain an expression for velocity:

$$\int a\,dt = \int \frac{g}{\sqrt{2}}dt$$

$$v = \frac{g}{\sqrt{2}}t + C_1$$

Since the initial velocity is zero, substituting $v = 0$ and $t = 0$, we get that $C_1 = 0$. Then, I integrated again to obtain an expression for displacement:

$$\int v\,dt = \int \frac{g}{\sqrt{2}}t\,dt$$

$$s = \frac{g}{\sqrt{2}} \times \frac{1}{2}t^2 + C_2$$

If we measure displacement starting from point A, $s = 0$ when $t = 0$, so we get $C_2 = 0$. So:

$$s = \frac{g}{2\sqrt{2}}t^2$$

To find how much time the particle takes to reach point B, we can set s to be the length of the line segment AB and solve for t.

$$\sqrt{1^2 + 1^2} = \frac{g}{2\sqrt{2}}t^2$$

$$t = \frac{2}{\sqrt{g}}$$

General expression for the time

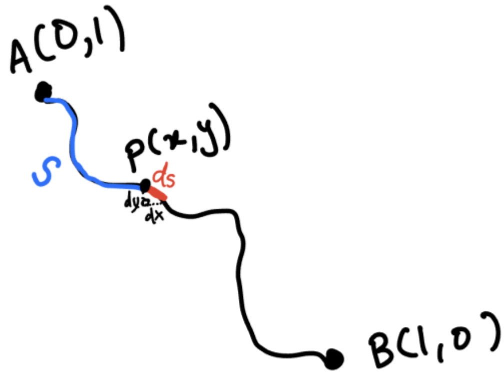


Figure 3: Particle moving along arbitrary curve (self-made using Notability)

Now that I have derived the time it takes for a particle to slide down a straight line, I will proceed to compare this to other curves. However, I realized that obtaining an expression for the time it takes for a particle to slide down isn't as easy for other curves as it was for the line. This is because the angle between the motion of the velocity of the particle and the direction of gravity was constant for the straight line, which is not the case for all curves. Figure 3 shows an arbitrary curve connecting points A and B, with the particle at point P. We can find the total time taken T by adding small snippets of time dt.[3] That is:

$$T = \int dt$$

Referring to Figure 3, we can write dt as:

$$dt = \frac{Distance}{Average\ speed} = \frac{ds}{v}$$

Furthermore, since ds is infinitesimally small, it can be considered a straight-line segment. Using the Pythagorean Theorem, $ds = \sqrt{dx^2 + dy^2}$, as shown in Figure 3. The velocity of

[3] Good Vibrations with Freeball. "The Brachistochrone Problem." YouTube, 8 Apr. 2021, www.youtube.com/watch?v=3HXCv4dmR7A. Accessed 16 Oct 2023.

the particle can be obtained using a fact from Physics regarding the conservation of energy. At all points on the curve, the particle must satisfy:

$$mgh = \frac{1}{2}mv^2$$

Where h is the vertical distance between the particle and point A, which is simply $1 - y$. Solving for v:

$$v = \sqrt{2g(1-y)}$$

Substituting all this into the expression for T, we obtain:

$$T = \int \sqrt{\frac{dx^2 + dy^2}{2g(1-y)}}$$

Since we want a differential to be able to evaluate the integral, we factor out dx^2:

$$T = \int \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2g(1-y)}}\,dx$$

In our scenario, $0 \leq x \leq 1$ since the particle goes from A to B. We can add this to the bounds of the integral, and write $\frac{dy}{dx}$ as $y'(x)$ to obtain:

$$T = \int_0^1 \sqrt{\frac{1 + y'(x)^2}{2g(1 - y(x))}}\,dx$$

I then decided to verify this formula by checking if I get the same result when using it to calculate the descent time for a straight line. The equation of the line connecting A and B can be easily derived using the slope-intercept form of the line:

$$(y - 1) = \frac{-1}{1}(x - 0)$$

$$y(x) = 1 - x$$

$$y'(x) = -1$$

Substituting into the formula for the descent time:

$$T = \int_0^1 \sqrt{\frac{1+(-1)^2}{2g(1-(1-x))}}\,dx$$

$$T = \int_0^1 \sqrt{\frac{2}{2gx}}\,dx$$

$$T = \frac{1}{\sqrt{g}}\int_0^1 x^{\frac{-1}{2}}\,dx$$

Using the reverse power rule and evaluating,

$$T = \frac{2}{\sqrt{g}}$$

which is the same as what was previously obtained.

Descent time for the quarter circle

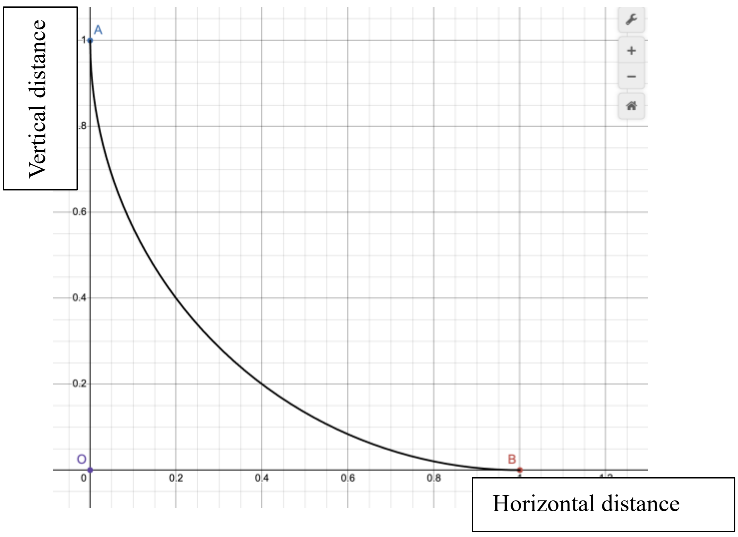


Figure 4: Quarter-circle connecting A and B (screenshot from self-made Desmos graph)

Galileo had proposed the quarter circle as a curve which minimizes the time of descent.[4] This makes sense qualitatively to me since this quarter circle has an initial drop increasing the speed of the particle for most of its path along the curve. While the distance has increased over the straight line, I still intuitively predict that the particle will reach point B quicker.

[4] Two New Sciences, Pp. 160-243. galileoandeinstein.physics.virginia.edu/tns_draft/tns_160to243.html. Accessed 3 Dec 2023.

The expression from the previous part can be used to find the time of descent. We will need to find y explicitly in terms of x for this arc of the circle to be able to use the equation. The general form of the equation of a circle with radius r centered at (h, k) is given by:

$$(x-h)^2 + (y-k)^2 = r^2$$

Referring to Figure 4, $r = h = k = 1$. Substituting and rearranging:

$$(y-1)^2 = 1 - (x-1)^2$$

Referring to Figure 4, $y - 1$ is negative in the domain being considered. So, we take the negative square root. Solving for y, we get:

$$y = 1 - \sqrt{1-(x-1)^2}$$

We also need an expression for $y'(x)$. Using the power and chain rules:

$$y'(x) = \frac{-1}{2\sqrt{(1-(x-1)^2}} \times -2(x-1) \times 1 = \frac{x-1}{\sqrt{1-(x-1)^2}}$$

Substituting $y'(x)$ and $y(x)$ into the expression for T:

$$T = \int_0^1 \sqrt{\frac{1+\left(\frac{x-1}{\sqrt{1-(x-1)^2}}\right)^2}{2g(1-(1-\sqrt{1-(x-1)^2}))}}\,dx$$

Making a common denominator in the fraction in the numerator and simplifying, we obtain:

$$T = \frac{1}{\sqrt{2g}} \int_0^1 (1-(x-1)^2)^{\frac{-3}{4}}\,dx$$

When I tried to evaluate this integral myself using trigonometric substitution, I was unable to proceed further and that this integral which had no closed form solution (other than using special functions) and could only be approximated.[5] Using WolframAlpha:

[5] Hypergeometric2F1: Gauss Hypergeometric Function—Wolfram Documentation. reference.wolfram.com/language/ref/Hypergeometric2F1.html. Accessed 3 Dec 2023.

$$\frac{1}{\sqrt{2}}\int_0^1 (1-(x-1)^2)^{\frac{-3}{4}}\,dx \approx 1.854$$

So, $T = \frac{1.854}{\sqrt{g}}$, which is less than the time for the straight line $(\frac{2}{\sqrt{g}})$, as I had hypothesized.

Descent time for a parabola

Another curve that I thought would be worth exploring would be a parabola, since it has the property of a steep drop initially that might make it a good contender. I decided to use the parabola $y = (x-1)^2$, since it passes through A and B and is decreasing and concave up.

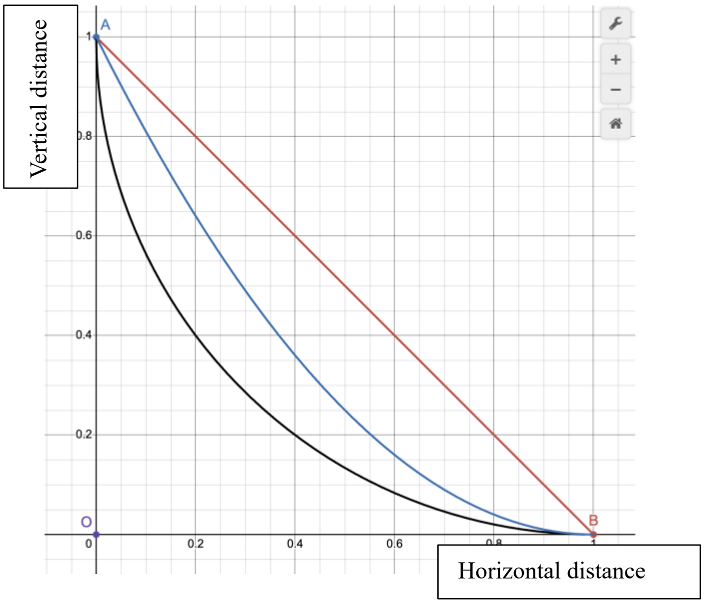


Figure 5: Parabola (blue), circle (black) and line (red) (screenshot from self-made Desmos graph)

Looking at figure 5, the parabola lies between the circular arc and the line. Calculating the time for the parabola would tell me where the optimal curve lies. If the parabola takes less time than the circular arc and the line, that will mean that the optimal tradeoff between small distance and fast speed is somewhere in the middle of the red and black curves. Substituting $y(x) = (x-1)^2$ and $y'(x) = 2(x-1)(1)$ into the expression for T:

$$T = \frac{1}{\sqrt{g}}\int_0^1 \sqrt{\frac{1+4(x-1)^2}{2(1-(x-1)^2)}}\,dx$$

This integral also could not be easily evaluated exactly. Using WolframAlpha:

$$T \approx \frac{1.863}{\sqrt{g}}$$

Which is more than the time for the quarter circle (the black curve). This tells me that the curve minimizing the time is steeper than the blue curve, since the time is decreasing as the curve is moved towards the left of the blue curve.

Finding the Brachistochrone curve

The curve minimizing the time is also known as the *Brachistochrone curve*. While mathematicians, like me, had found curves better than a straight line, the real challenge was to find the absolute best curve. This problem had been posed by Johann Bernoulli and was attempted by himself, his brother as well as several other mathematicians of the time including Sir Isaac Newton. Johann Bernoulli's solution involved making an analogy to the behavior of light, since light finds the path that minimizes the time it takes to go from one point to another.[6] I decided to explore Johann's solution specifically due to its beauty and brilliance.

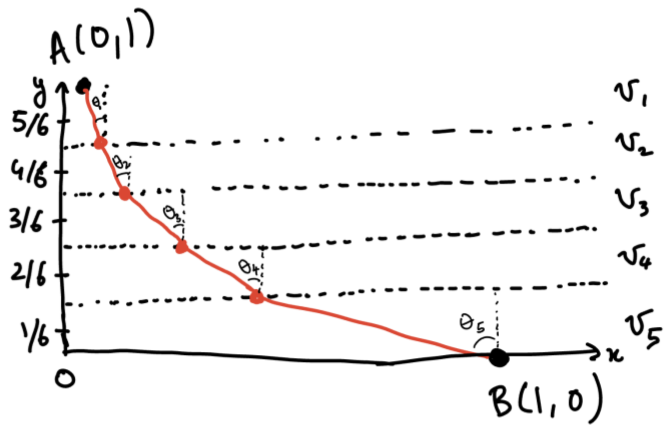


Figure 6: Dividing up the brachistochrone into linear pieces (self-made using Notability)

[6] 3Blue1Brown. "The Brachistochrone, With Steven Strogatz." YouTube, 1 Apr. 2016, www.youtube.com/watch?v=Cld0p3a43fU. Accessed 2 Oct 2023.

Bernoulli thought of the Brachistochrone curve as made up of n linear pieces where the velocity is assumed to be constant for each piece. Later, we let $n \to \infty$ to obtain the Brachistochrone curve. Figure 6 is shown with $n = 5$. The idea is that the initial pieces are steep and eventually become flat, as was the case with the other curves that were explored.

Each of these angles can be calculated by Snell's Law, which is obtained by the behavior of light trying to minimize its time of travel in media with different velocities.[7] This behavior essentially emulates the behavior we want our particle to have, assuming that it travels at a constant velocity for each partition.

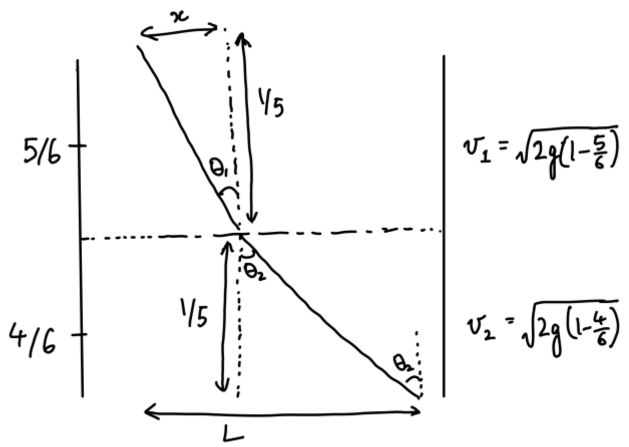


Figure 7: Derivation of Snell's Law (self-made using Notability)

Figure 7 zooms in on the first two partitions for the case when $n = 5$, but the deductions made here apply generally. The length of the first hypotenuse is given by $\sqrt{x^2 + \frac{1}{25}}$, and similarly $\sqrt{(L-x)^2 + \frac{1}{25}}$ for the second hypotenuse. Using $time = \frac{distance}{speed}$, the total time in these two partitions is given by:

[7] Reflection and Fermat's Principle. hyperphysics.phy-astr.gsu.edu/hbase/phyopt/Fermat.html. Accessed 24 Nov 2023.

$$T = \frac{\sqrt{x^2 + \frac{1}{25}}}{v_1} + \frac{\sqrt{(L-x)^2 + \frac{1}{25}}}{v_2}$$

To find the value of x (and hence the relationship between the angles) that minimize the time, we can set its derivative equal to 0.

$$\frac{dT}{dx} = \frac{2x}{2v_1\sqrt{x^2 + \frac{1}{25}}} + \frac{2(L-x)(-1)}{2v_2\sqrt{(L-x)^2 + \frac{1}{25}}} = 0$$

Since the time obviously increases when $x = 0$ and is increasing when x is large, this stationary point must be a minima. Simplifying:

$$\frac{x}{v_1\sqrt{x^2 + \frac{1}{25}}} = \frac{L-x}{v_2\sqrt{(L-x)^2 + \frac{1}{25}}}$$

Recognizing $\sin\theta_1 = \frac{x}{\sqrt{x^2+\frac{1}{25}}}$ and $\sin\theta_2 = \frac{L-x}{\sqrt{(L-x)^2+\frac{1}{25}}}$ from Figure 7, we conclude:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

Since this process can be continued inductively for layers below the first two:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} = \frac{\sin\theta_3}{v_3} = \cdots = c$$

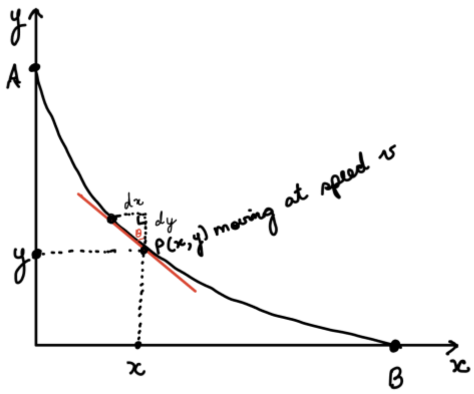


Figure 8: Limiting case of pieces (self-made using Notability)

Now comes the ingenious part of Bernoulli's solution, which involves considering the case where $n \to \infty$. There are a few motivations for performing this step. Firstly, the assumption that the velocity is constant in each piece is constant becomes true, since each piece is infinitesimally small. Secondly, the angle between each piece and the normal (each θ_i in Figure 6) intuitively becomes the angle θ between the vertical and the tangent to the curve in Figure 8. Considering this continuous limiting case simplifies the problem, just like integrals become easier to evaluate than Reimann sums. Intuitively, despite this being a limiting case, the fact that $\frac{\sin\theta_i}{v_i} = c$ from the finite case can still be used here by simply writing it as $\frac{\sin\theta}{v} = c$. All this simplifies the problem considerably and results in a clever solution to the problem involving intuitive ideas from calculus, which is why I love this solution so much. Since we want to obtain an expression describing this optimum curve, we are compelled to write v and $\sin\theta$ in terms of x and y. By the right triangle in Figure 8 and the Pythagorean Theorem:

$$\sin\theta = \frac{dx}{\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

From conservation of energy:

$$v = \sqrt{2g(1-y)}$$

Substituting into the Snell's Law equation:

$$\frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = c\sqrt{2g(1-y)}$$

Now, we have a differential equation describing the Brachistochrone curve. Bernoulli stopped here, recognizing this as the differential equation describing a shape known as a *cycloid*.[8] However, since I want to plot this curve and compare its shape and descent time with the other curves, I proceeded to solve this differential equation. By observation, I realized that

[8] Babb, Jeffry, and James D. Currie. "The Brachistochrone Problem: Mathematics for a Broad Audience via a Large Context Problem." The Mathematics Enthusiast, vol. 5, no. 2–3, July 2008, pp. 169–84. https://doi.org/10.54870/1551-3440.1099. Accessed 5 Nov 2023.

since the differential equation is first order and does not explicitly involve x, this differential equation is separable. Hence, I simplified and isolated $\frac{dy}{dx}$:

$$\frac{1}{1+\left(\frac{dy}{dx}\right)^2} = 2c^2g(1-y)$$

$$1 = 2c^2g(1-y) + \left(\frac{dy}{dx}\right)^2 2c^2g(1-y)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1-2c^2g(1-y)}{2c^2g(1-y)} = \frac{1}{2c^2g}\frac{1}{1-y} - 1$$

For the sake of brevity, I let $k = \frac{1}{2c^2g}$. In our scenario, looking at Figure 8, $\frac{dy}{dx}$ is clearly always negative. So, taking the negative square root:

$$\frac{dy}{dx} = -\sqrt{\frac{k}{1-y} - 1}$$

Separating variables and integrating:

$$\int \frac{-dy}{\sqrt{\frac{k}{1-y} - 1}} = \int dx$$

The integral on the left hand side was a bit tricky for me to evaluate. Firstly, I decided to make a common denominator and simplify:

$$I = -\int \sqrt{\frac{1-y}{k-1+y}}\, dy$$

Since there is a rational function with radicals involved, I thought that trigonometric substitution might be a good idea. However, y couldn't be directly substituted in since this wouldn't result in a simplification of the denominator. After some messing around, I wrote the denominator as follows:

$$I = -\int \sqrt{\frac{1-y}{k-(1-y)}}\, dy$$

Now, I realized that the substitution $1 - y = k\sin^2\theta$ would work since the denominator would simplify using the Pythagorean identity and the numerator would clearly simplify, cancelling out the radicals. Applying this substitution:

$$-dy = 2k\sin\theta\cos\theta\,d\theta$$

$$I = \int \sqrt{\frac{k\sin^2\theta}{k - k\sin^2\theta}}\,2k\sin\theta\cos\theta\,d\theta$$

$$I = 2k\int \sin^2\theta\,d\theta$$

This integral can be easily evaluated using the double angle formula for cosine:

$$\int \sin^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$$

We have:

$$I = k\theta - \frac{k}{2}\sin 2\theta = x + C$$

We can now reverse the substitution to obtain a relationship between x and y. Since I had an expression for $\sin^2\theta$ as part of the substitution, I decided to apply the double angle formula to get rid of the 2θ and have all trig functions in terms of θ.

$$k(\theta - \sin\theta\cos\theta) = x + C$$

Now, since $1 - y = k\sin^2\theta$:

$$\sin\theta = \pm\sqrt{\frac{1-y}{k}}$$

Since arcsin is an odd function:

$$\theta = \arcsin\left(\pm\sqrt{\frac{1-y}{k}}\right) = \pm\arcsin\left(\sqrt{\frac{1-y}{k}}\right)$$

Since both θ and $\sin\theta$ have the same sign, I factored out the sign (factoring out a -1 if the sign is – and a $+1$ if the sign is $+$) and absorbed it into the constant k to simplify the calculations. I also applied the Pythagorean identity to find an expression for $\cos\theta$, but this introduced another $\pm$ case. Finally, substituting everything in:

$$k\left(\arcsin\left(\sqrt{\frac{1-y}{k}}\right)\pm\sqrt{\frac{1-y}{k}}\sqrt{\frac{k-1+y}{k}}\right)=x+C$$

I realized that the information about the initial and final points had not been used, which is why there were two unknown constants in the equation (k and C). Therefore, I decided to plug in the coordinates of points A and B which must satisfy the equation. Plugging in $(0,1)$:

$$k\left(\arcsin\left(\sqrt{\frac{1-1}{k}}\right)\pm\sqrt{\frac{1-1}{k}}\sqrt{\frac{k-1+1}{k}}\right)=0+C$$

$$k(\arcsin 0\ \pm 0)=C$$

$$C=0$$

Plugging in $(1,0)$:

$$k\left(\arcsin\left(\sqrt{\frac{1}{k}}\right)\pm\sqrt{\frac{1}{k}}\sqrt{\frac{k-1}{k}}\right)=1$$

Since this equation involved both inverse trigonometric and algebraic functions (square root and arcsin), I knew from experience that this equation could not be solved manually and only approximated. Further, I still had not been able to figure out whether the $+$ or the $-$ corresponded to our scenario. Therefore, I plugged in both choices into WolframAlpha. The $+$ choice yielded no real solution for k, which means it is an extraneous solution not applicable to our scenario and can be discarded. The $-$ sign, however, did yield a solution of:

$$k\approx 1.14583$$

Now, knowing value of k and the sign, I obtained the explicit expression for x in terms of y:

$$x=\ 1.14583\left(\arcsin\left(\sqrt{\frac{1-y}{1.14583}}\right)-\sqrt{\frac{1-y}{1.14583}}\sqrt{\frac{0.14583+y}{1.14583}}\right)$$

Finally, I can graph this solution (in green) alongside the other curves on Desmos.

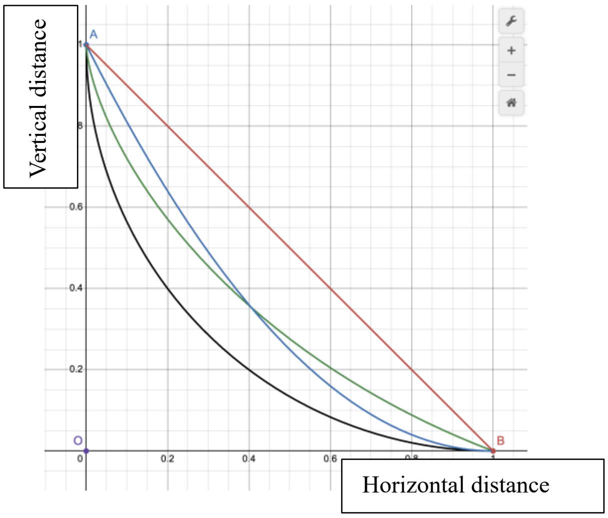


Figure 9: Parabola (blue), circle (black), line (red) and brachistochrone (green) (screenshot from self-made Desmos graph)

This graph surprised me initially. I had predicted that the brachistochrone curve would be steeper than the blue curve. I had made this prediction because the circle had a quicker time of descent than the parabola, so I thought that the optimal curve would be to the left of the blue curve. While this is true initially, the brachistochrone appears to eventually flatten out more. After analyzing the situation a bit more, this made some intuitive sense to me since it's better to make the curve steeper at the start and pick up some speed before making it flatter later to minimize the distance as well.

This result of the optimum curve derived amazed me since it wasn't a simple curve like a parabola or circular arc, but a non-standard function known as a cycloid. The appearance of this new function in this problem inspired me to learn more about it.

Descent time for Brachistochrone

I now calculated the descent time for the Brachistochrone curve to see how much better it is compared to the other simpler curves. However, the formula derived initially works for situations in which y is a function of x, but the equation I derived for the Brachistochrone is the opposite. I went through the derivation and found the step right before y is considered a function of x:

$$T = \int \sqrt{\frac{dx^2 + dy^2}{2g(1-y)}}$$

I realized the instead of factoring out the dx^2, we can factor out the dy^2 and assume x to be a function of y. Since $0 < y < 1$, we can add bounds to the integral as well:

$$T = \int_0^1 \sqrt{\frac{\left(\frac{dx}{dy}\right)^2 + 1}{2g(1-y)}}\,dy$$

Now, we need to compute $\frac{dx}{dy}$ for the Brachistochrone:

$$\frac{d}{dy}\left(1.14583\left(\arcsin\left(\sqrt{\frac{1-y}{1.14583}}\right) - \sqrt{\frac{1-y}{1.14583}}\sqrt{\frac{0.14583+y}{1.14583}}\right)\right)$$

Simplifying and using linearity:

$$1.14583\frac{d}{dy}\arcsin\left(\sqrt{\frac{1-y}{1.14583}}\right) - \frac{d}{dy}\sqrt{(1-y)(0.14583+y)}$$

The first derivative can be calculated using the chain rule:

$$\frac{1}{\sqrt{1-\sqrt{\frac{1-y}{1.14583}}^{2}}}\frac{d}{dy}\left(\frac{(1-y)^{\frac{1}{2}}}{\sqrt{1.14583}}\right) = \frac{1}{\sqrt{1-\frac{1-y}{1.14583}}} \times \frac{1}{2\sqrt{1.14583}} \times (1-y)^{\frac{-1}{2}} \times -1$$

The expression for the second derivative can be expanded and then chain rule can be used:

$$\frac{d}{dy}\left(\sqrt{0.14583 + y - 0.14583y - y^2}\right) = \frac{1 - 0.14583 - 2y}{2\sqrt{0.14583 + y - 0.14583y - y^2}}$$

Substituting these back in and simplifying:

$$\frac{dx}{dy} = \frac{-1.14583}{2\sqrt{1.14583(1-y)\left(1-\frac{1-y}{1.14583}\right)}} - \frac{0.85417 - 2y}{2\sqrt{0.14583 + 0.85417y - y^2}}$$

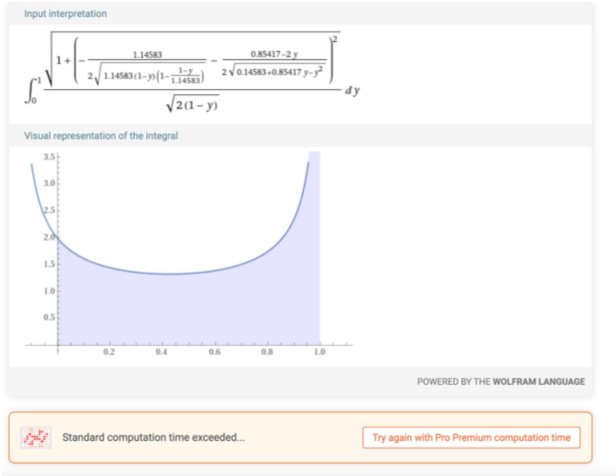


Figure 10: Putting integral into WolframAlpha (screenshot from WolframAlpha)

This expression was then substituted into the integral and typed into WolframAlpha, as shown in Figure 10. However, it showed an error of the computation time exceeding. From the graph, I realized that this might be because of the asymptote when $y = 1$, since the denominator becomes $1 - 1 = 0$. When I put 0.99 instead of 1, it worked. Therefore, I decided to input values closer and closer to 1 to see what the integral is converging to.

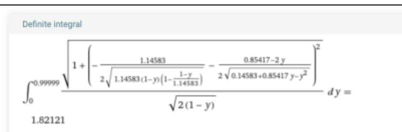
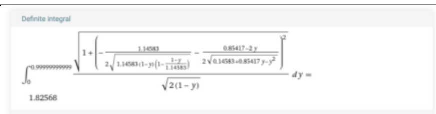
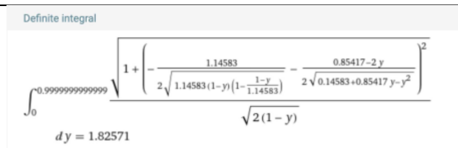
Precision	Result	Screenshot
0.99999	1.82121	
0.99999999999	1.82568	
0.9999999999999	1.82571	

Table 1: Self-generated using values from WolframAlpha

Using the data from Table 1, I can confidently say that, for the Brachistochrone:

$$T \approx \frac{1.826}{\sqrt{g}}$$

Conclusion

I explored the problem of finding the path with the shortest descent time to get from one point to another. I explored some regular functions and their descent time. From the functions explored, excluding the constant factor of $\frac{1}{\sqrt{g}}$, the straight line was the slowest with a time of 2, then came the parabola with 1.863, followed by the Quarter-circle with 1.854. As expected, the Brachistochrone was the fastest with 1.826. I also looked at the original derivation of the curve, which also proves that the Brachistochrone is indeed the fastest curve.

This exploration gave me a glimpse into optimization problems involving finding an optimal curve rather than an optimal value, extending the kind of optimization problems I have learnt about in my Mathematics AA class, which are further explored in the mathematical field of the calculus of variations.[9] Exploring the different curves, plotting them, and comparing the calculations to my intuition from graphs was a fruitful process which gave me a deeper understanding of this problem. Exploring and understanding Bernoulli's solution, then extending it by solving the differential equation, plotting the curve, and calculating the time of descent was a satisfying way to understand the solution. A limitation of this investigation was that specific coordinates were used and different ratios of width and height and how they affect the result wasn't considered. As an extension, I could also design an experiment with all these curves and see whether the result materializes in real life, though friction and drag will likely make the experimental results slightly off. However, there are models that take friction into account, which could be explored to find a theoretical expression for the optimal curve when taken friction into consideration.[10]

[9] Good Vibrations with Freeball. "The Brachistochrone Problem." YouTube, 8 Apr. 2021, www.youtube.com/watch?v=3HXCv4dmR7A. Accessed 16 Oct 2023.
[10] Golubev, Yu. F. "Brachistochrone With Friction." Journal of Computer and Systems Sciences International, vol. 49, no. 5, Oct. 2010, pp. 719–30. https://doi.org/10.1134/s1064230710050060. Accessed 3 Jan 2024.

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