# zuoye2

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**Problem 1.** 使用 $\uparrow$ and $\downarrow$ 表示下面的 proposition.

1. $\neg p \vee q$
2. $p \wedge \neg q$
3. $\neg p \vee \neg q$
4. $p \leftrightarrow q$

**Solution.** to 1.

$$\begin{aligned}
&\Leftrightarrow \neg\neg\left(\neg p \vee q\right)\\
&\Leftrightarrow \neg\left(\neg p \downarrow q\right)\\
&\Leftrightarrow \neg\left(\left(p \downarrow p\right) \downarrow q\right)\\
&\Leftrightarrow \left(\left(p \downarrow p\right) \downarrow q\right) \downarrow \left(\left(p \downarrow p\right) \downarrow q\right)
\end{aligned}$$

而后

$$\begin{aligned}
&\Leftrightarrow \neg p \vee \neg\neg q\\
&\Leftrightarrow p \uparrow \left(\neg q\right)\\
&\Leftrightarrow p \uparrow \left(q \uparrow q\right)
\end{aligned}$$

**Solution.** to 2.

$$\begin{aligned}
&p \wedge \neg q\\
\Leftrightarrow &\neg\neg p \wedge \neg q\\
\Leftrightarrow &\neg p \downarrow q\\
\Leftrightarrow &\left(\left(p \downarrow p\right) \downarrow q\right)
\end{aligned}$$

而后

$$\begin{aligned}
& p \wedge \neg q \\
\Leftrightarrow & \neg\neg (p \wedge \neg q) \\
\Leftrightarrow & \neg (p \uparrow \neg q) \\
\Leftrightarrow & \neg (p \uparrow (q \uparrow q)) \\
\Leftrightarrow & (p \uparrow (q \uparrow q)) \uparrow (p \uparrow (q \uparrow q))
\end{aligned}$$

**Solution.** to 3.

$$\begin{aligned}
& \neg p \vee \neg q \\
\Leftrightarrow & \neg\neg (\neg p \vee \neg q) \\
\Leftrightarrow & \neg (\neg p \downarrow \neg q) \\
\Leftrightarrow & \neg ((p \downarrow p) \downarrow (q \downarrow q)) \\
\Leftrightarrow & ((p \downarrow p) \downarrow (q \downarrow q)) \downarrow ((p \downarrow p) \downarrow (q \downarrow q))
\end{aligned}$$

而后

$$\begin{aligned}
& \neg p \vee \neg q \\
\Leftrightarrow & \neg (p \wedge q) \\
\Leftrightarrow & p \uparrow q
\end{aligned}$$

**Solution.** to 4.

$$\begin{aligned}
& p \leftrightarrow q \\
\Leftrightarrow & (\neg p \vee q) \wedge (\neg q \vee p) \\
\Leftrightarrow & \neg\neg ((\neg p \vee q) \wedge (\neg q \vee p)) \\
\Leftrightarrow & \neg ((\neg (\neg p \vee q)) \vee (\neg (\neg q \vee p))) && \text{(de Morgan's law)} \\
\Leftrightarrow & \neg ((\neg p \downarrow q) \vee (\neg q \downarrow p)) \\
\Leftrightarrow & (\neg p \downarrow p) \downarrow (\neg q \downarrow p) \\
\Leftrightarrow & ((p \downarrow p) \downarrow q) \downarrow ((q \downarrow q) \downarrow p)
\end{aligned}$$

而后

$$
\begin{aligned}
& p \leftrightarrow q \\
\Leftrightarrow & (\neg p \wedge \neg q) \vee (p \wedge q) && \text{(经过简单的化简)} \\
\Leftrightarrow & (\neg (\neg p \wedge \neg q)) \uparrow (\neg (p \wedge q)) \\
\Leftrightarrow & (\neg p \uparrow \neg q) \uparrow (p \uparrow q) \\
\Leftrightarrow & ((p \uparrow p) \uparrow (q \uparrow q)) \uparrow (p \uparrow q)
\end{aligned}
$$

**Problem 2.** 在 Propositonal Calculus 中证明下面这些事实.

1. $\vdash (A \to (A \to B)) \to (A \to B)$
2. $A \to B, \neg (B \to C) \to \neg A \vdash A \to C$
3. $\vdash (A \to (B \to C)) \to ((C \to D) \to (A \to (B \to D)))$
4. $\vdash ((A \to B) \to (B \to A)) \to (B \to A)$
5. $\vdash ((A \to B) \to A) \to A$
6. $\vdash ((A \to B) \to C) \to ((A \to C) \to C)$
7. $\vdash (A \to C) \to ((B \to C) \to (((A \to B) \to B) \to C))$

**Comment.** 形如 $A \to B \to C \to D$ 的式子, 默认从右往左加括号, viz. 原本这个式子应为

$$A \to (B \to (C \to D))$$

**Solution.** to 1.

$$
\begin{aligned}
& (A \to B) \to (A \to B) && (1) && \text{(thm 1)} \\
& A \to \big( (A \to B) \to B \big) && (2) \\
& && \text{(前件互换, thm 8)} \\
& A \to \big( (A \to B) \to B \big) \to \big( A \to (A \to B) \big) \to (A \to B) && (3) && \text{(A2)} \\
& (A \to (A \to B)) \to (A \to B) && (4) \\
& && \text{((2) (3) thm 8)}
\end{aligned}
$$

**Solution.** to 2.

| | | |
|---|---|---|
| $\big(\neg(B \to C) \to \neg A\big) \to (A \to (B \to C))$ | (1) | (A3) |
| $(\neg(B \to C))$ | (2) | |
| $(A \to (B \to C))$ | (3) | ((1) (2) rmp) |
| $(A \to B) \to (A \to C)$ | (4) | (A2, thm 8, rmp) |
| $A \to B$ | (5) | |
| $A \to C$ | (5) | ((4) (5) rmp) |

**Solution.** to 3.

| | |
|---|---|
| (1) $\neg A \to (A \to ((C \to D) \to (B \to D)))$ | (thm 6) |
| (2) $\neg A \to ((C \to D) \to (A \to (B \to D)))$ | (thm 3, rmp) |
| (3) $(B \to C) \to ((C \to D) \to (B \to D))$ | (thm 5) |
| (4) $A \to ((B \to C) \to ((C \to D) \to (B \to D)))$ | (thm 1, rmp) |
| (5) $(B \to C) \to (A \to ((C \to D) \to (B \to D)))$ | (thm 3, rmp) |
| (6) $(B \to C) \to ((C \to D) \to (A \to (B \to D)))$ | (thm 8, thm 3) |
| (7) $(A \to (B \to C)) \to ((C \to D) \to (A \to (B \to D)))$ | ((2), (6)thm 18) |

**Solution.** to 4.

| | |
|---|---|
| (1) $(B \to A) \to (B \to A)$ | (thm 1) |
| (2) $\neg A \to (A \to B)$ | (thm 6) |
| (3) $B \to (\neg A \to (A \neg (A \to B)))$ | (thm 1, rmp) |
| (4) $B \to (\neg(A \to B) \to A)$ | (thm 14, thm 8) |
| (5) $\neg(A \to B) \to (B \to A)$ | (thm 3, rmp) |
| (6) $((A \to B) \to (B \to A)) \to (B \to A)$ | (thm 18, (1)(6)) |

**Solution.** to 5.

| | | |
|---|---|---|
| (1) | $A \to A$ | (thm 1) |
| (2) | $\neg A \to (A \to B)$ | (thm 6) |
| (3) | $\neg(A \to B) \to A$ | (thm 14, rmp) |
| (4) | $((A \to B) \to A) \to A$ | (thm 18, (1) (3)) |

**Solution.** to 6.

| | | |
|---|---|---|
| (1) | $\neg C \to (C \to B)$ | (thm 6) |
| (2) | $A \to (\neg C \to (C \to B))$ | (thm 1, rmp) |
| (3) | $\neg C \to (A \to (C \to B))$ | (thm 3) |
| (4) | $\neg C \to ((A \to C) \to (A \to B))$ | (A2, thm 8) |
| (5) | $\neg C \to (\neg (A \to B) \to \neg (A \to C))$ | (thm 13, thm 8) |
| (6) | $\Big(\neg C \to (\neg (A \to B))\Big) \to \Big(\neg C \to \neg (A \to C)\Big)$ | (A2, rmp) |
| (7) | $\Big((A \to B) \to C\Big) \to (\neg C \to \neg (A \to B))$ | (thm 13) |
| (8) | $(\neg C \to \neg (A \to C)) \to ((A \to C) \to C)$ | (A3) |
| (9) | $((A \to B) \to C) \to ((A \to C) \to C)$ | (thm 21, (7)(8)) |

**Solution.** to 7. We shall use thm 18 to prove.

| | | |
|---|---|---|
| (1) | $\neg C \to (C \to B)$ | (thm 6) |
| (2) | $A \to (\neg C \to (C \to B))$ | (thm 1, rmp) |
| (3) | $\neg C \to A \to (C \to B)$ | (thm 3) |
| (4) | $\neg C \to ((A \to C) \to (A \to B))$ | (A2, rmp) |
| (5) | $(A \to C) \to (\neg C \to (A \to B))$ | (thm 3, rmp) |
| (6) | $(A \to C) \to (\neg (A \to B) \to C)$ | (thm 14, thm 8) |
| (7) | $(B \to C) \to ((A \to C) \to (\neg (A \to B) \to C))$ | (thm 1, rmp) |
| (8) | $\neg (A \to B) \to \big((A \to C) \to \big((B \to C) \to C\big)\big)$ | (多用几次 thm 3) |

接下来证明另一半.

| | | |
|---|---|---|
| (9) | $(B \to C) \to (B \to C)$ | (thm 1) |
| (10) | $(B \to (B \to C) \to C)$ | (thm 3) |
| (11) | $(A \to C) \to (B \to (B \to C) \to C)$ | (thm 1, rmp) |
| (12) | $B \to (A \to C) \to (B \to C) \to C$ | (thm 3, rmp) |

一上面这两部分的结果作为 thm 18 的理由:

(13) $\big((A \to B) \to B\big) \to ((A \to C) \to (B \to C) \to C)$

(thm 18 (12)(8))

(14) $(A \to C) \to \big((A \to B) \to B\big) \to (B \to C) \to C$

(thm 3, rmp)

(15) $\Big(((A \to B) \to B) \to (B \to C) \to C\Big) \to \Big((B \to C) \to ((A \to B) \to B) \to C\Big)$

(thm 3)

(16) $(A \to C) \to (B \to C) \to ((A \to B) \to B) \to C$ (thm 8)