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Problem 1. 使用 \uparrow and \downarrow 表示下面的 proposition.

1. $\neg p \vee q$

2. $p \wedge \neg q$

3. $\neg p \vee \neg q$

4. $p \leftrightarrow q$

Solution. to 1.

$$\begin{aligned} &\Leftrightarrow \neg \neg (\neg p \vee q) \\ &\Leftrightarrow \neg (\neg p \downarrow q) \\ &\Leftrightarrow \neg ((p \downarrow p) \downarrow q) \\ &\Leftrightarrow ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q) \end{aligned}$$

而后

$$\begin{aligned} &\Leftrightarrow \neg p \vee \neg \neg q \\ &\Leftrightarrow p \uparrow (\neg q) \\ &\Leftrightarrow p \uparrow (q \uparrow q) \end{aligned}$$

Solution. to 2.

$$\begin{aligned} &p \wedge \neg q \\ &\Leftrightarrow \neg \neg p \wedge \neg q \\ &\Leftrightarrow \neg p \downarrow q \\ &\Leftrightarrow ((p \downarrow p) \downarrow q) \end{aligned}$$

而后

$$\begin{aligned}
 & p \wedge \neg q \\
 \Leftrightarrow & \neg \neg (p \wedge \neg q) \\
 \Leftrightarrow & \neg (p \uparrow \neg q) \\
 \Leftrightarrow & \neg (p \uparrow (q \uparrow q)) \\
 \Leftrightarrow & (p \uparrow (q \uparrow q)) \uparrow (p \uparrow (q \uparrow q))
 \end{aligned}$$

Solution. to 3.

$$\begin{aligned}
 & \neg p \vee \neg q \\
 \Leftrightarrow & \neg \neg (\neg p \vee \neg q) \\
 \Leftrightarrow & \neg (\neg p \downarrow \neg q) \\
 \Leftrightarrow & \neg ((p \downarrow p) \downarrow (q \downarrow q)) \\
 \Leftrightarrow & ((p \downarrow p) \downarrow (q \downarrow q)) \downarrow ((p \downarrow p) \downarrow (q \downarrow q))
 \end{aligned}$$

而后

$$\begin{aligned}
 & \neg p \vee \neg q \\
 \Leftrightarrow & \neg (p \wedge q) \\
 \Leftrightarrow & p \uparrow q
 \end{aligned}$$

Solution. to 4.

$$\begin{aligned}
 & p \leftrightarrow q \\
 \Leftrightarrow & (\neg p \vee q) \wedge (\neg q \vee p) \\
 \Leftrightarrow & \neg \neg ((\neg p \vee q) \wedge (\neg q \vee p)) \\
 \Leftrightarrow & \neg ((\neg (\neg p \vee q)) \vee (\neg (\neg q \vee p))) && \text{(de Morgan's law)} \\
 \Leftrightarrow & \neg ((\neg p \downarrow q) \vee (\neg q \downarrow p)) \\
 \Leftrightarrow & (\neg p \downarrow p) \downarrow (\neg q \downarrow p) \\
 \Leftrightarrow & ((p \downarrow p) \downarrow q) \downarrow ((q \downarrow q) \downarrow p)
 \end{aligned}$$

而后

$$\begin{aligned} p &\leftrightarrow q \\ \Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge q) &\quad (\text{经过简单的化简}) \\ \Leftrightarrow (\neg(\neg p \wedge \neg q)) \uparrow (\neg(p \wedge q)) \\ \Leftrightarrow (\neg p \uparrow \neg q) \uparrow (p \uparrow q) \\ \Leftrightarrow ((p \uparrow p) \uparrow (q \uparrow q)) \uparrow (p \uparrow q) \end{aligned}$$

Problem 2. 在 Propositional Calculus 中证明下面这些事实.

1. $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
2. $A \rightarrow B, \neg(B \rightarrow C) \rightarrow \neg A \vdash A \rightarrow C$
3. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((C \rightarrow D) \rightarrow (A \rightarrow (B \rightarrow D)))$
4. $\vdash ((A \rightarrow B) \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow A)$
5. $\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$
6. $\vdash ((A \rightarrow B) \rightarrow C) \rightarrow ((A \rightarrow C) \rightarrow C)$
7. $\vdash (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (((A \rightarrow B) \rightarrow B) \rightarrow C))$

Comment. 形如 $A \rightarrow B \rightarrow C \rightarrow D$ 的式子, 默认从右往左加括号, viz. 原本这个式子应为

$$A \rightarrow (B \rightarrow (C \rightarrow D))$$

Solution. to 1.

$$\begin{aligned} (A \rightarrow B) &\rightarrow (A \rightarrow B) && (1) \text{ (thm 1)} \\ A &\rightarrow ((A \rightarrow B) \rightarrow B) && (2) \\ &&& (\text{前件互换, thm 8}) \\ A &\rightarrow ((A \rightarrow B) \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B) && (3) \quad (\text{A2}) \\ (A &\rightarrow (A \rightarrow B)) &\rightarrow (A \rightarrow B) && (4) \\ &&& ((2) (3) \text{ thm 8}) \end{aligned}$$

Solution. to 2.

$$\begin{array}{ll}
(\neg(B \rightarrow C) \rightarrow \neg A) \rightarrow (A \rightarrow (B \rightarrow C)) & (1) \quad (\text{A3}) \\
\neg(B \rightarrow C) & (2) \\
(A \rightarrow (B \rightarrow C)) & (3) \quad ((1) (2) \text{ rmp}) \\
(A \rightarrow B) \rightarrow (A \rightarrow C) & (4) \quad (\text{A2, thm 8, rmp}) \\
A \rightarrow B & (5) \\
A \rightarrow C & (5) \quad ((4) (5) \text{ rmp})
\end{array}$$

Solution. to 3.

$$\begin{array}{ll}
(1) \neg A \rightarrow (A \rightarrow ((C \rightarrow D) \rightarrow (B \rightarrow D))) & (\text{thm 6}) \\
(2) \neg A \rightarrow ((C \rightarrow D) \rightarrow (A \rightarrow (B \rightarrow D))) & (\text{thm 3, rmp}) \\
(3) (B \rightarrow C) \rightarrow ((C \rightarrow D) \rightarrow (B \rightarrow D)) & (\text{thm 5}) \\
(4) A \rightarrow ((B \rightarrow C) \rightarrow ((C \rightarrow D) \rightarrow (B \rightarrow D))) & (\text{thm 1, rmp}) \\
(5) (B \rightarrow C) \rightarrow (A \rightarrow ((C \rightarrow D) \rightarrow (B \rightarrow D))) & (\text{thm 3, rmp}) \\
(6) (B \rightarrow C) \rightarrow ((C \rightarrow D) \rightarrow (A \rightarrow (B \rightarrow D))) & (\text{thm 8, thm 3}) \\
(7) (A \rightarrow (B \rightarrow C)) \rightarrow ((C \rightarrow D) \rightarrow (A \rightarrow (B \rightarrow D))) & ((2), (6)\text{thm 18})
\end{array}$$

Solution. to 4.

$$\begin{array}{ll}
(1) (B \rightarrow A) \rightarrow (B \rightarrow A) & (\text{thm 1}) \\
(2) \neg A \rightarrow (A \rightarrow B) & (\text{thm 6}) \\
(3) B \rightarrow (\neg A \rightarrow (A \rightarrow (A \rightarrow B))) & (\text{thm 1, rmp}) \\
(4) B \rightarrow (\neg(A \rightarrow B) \rightarrow A) & (\text{thm 14, thm 8}) \\
(5) \neg(A \rightarrow B) \rightarrow (B \rightarrow A) & (\text{thm 3, rmp}) \\
(6) ((A \rightarrow B) \rightarrow (B \rightarrow A)) \rightarrow (B \rightarrow A) & (\text{thm 18, (1)(6)})
\end{array}$$

Solution. to 5.

$$\begin{array}{ll}
(1) A \rightarrow A & (\text{thm 1}) \\
(2) \neg A \rightarrow (A \rightarrow B) & (\text{thm 6}) \\
(3) \neg(A \rightarrow B) \rightarrow A & (\text{thm 14, rmp}) \\
(4) ((A \rightarrow B) \rightarrow A) \rightarrow A & (\text{thm 18, (1) (3)})
\end{array}$$

Solution. to 6.

- (1) $\neg C \rightarrow (C \rightarrow B)$ (thm 6)
- (2) $A \rightarrow (\neg C \rightarrow (C \rightarrow B))$ (thm 1, rmp)
- (3) $\neg C \rightarrow (A \rightarrow (C \rightarrow B))$ (thm 3)
- (4) $\neg C \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B))$ (A2, thm 8)
- (5) $\neg C \rightarrow (\neg(A \rightarrow B) \rightarrow \neg(A \rightarrow C))$ (thm 13, thm 8)
- (6) $(\neg C \rightarrow (\neg(A \rightarrow B))) \rightarrow (\neg C \rightarrow \neg(A \rightarrow C))$ (A2, rmp)
- (7) $((A \rightarrow B) \rightarrow C) \rightarrow (\neg C \rightarrow \neg(A \rightarrow B))$ (thm 13)
- (8) $(\neg C \rightarrow \neg(A \rightarrow C)) \rightarrow ((A \rightarrow C) \rightarrow C)$ (A3)
- (9) $((A \rightarrow B) \rightarrow C) \rightarrow ((A \rightarrow C) \rightarrow C)$ (thm 21, (7)(8))

Solution. to 7. We shall use thm 18 to prove.

- (1) $\neg C \rightarrow (C \rightarrow B)$ (thm 6)
- (2) $A \rightarrow (\neg C \rightarrow (C \rightarrow B))$ (thm 1, rmp)
- (3) $\neg C \rightarrow A \rightarrow (C \rightarrow B)$ (thm 3)
- (4) $\neg C \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B))$ (A2, rmp)
- (5) $(A \rightarrow C) \rightarrow (\neg C \rightarrow (A \rightarrow B))$ (thm 3, rmp)
- (6) $(A \rightarrow C) \rightarrow (\neg(A \rightarrow B) \rightarrow C)$ (thm 14, thm 8)
- (7) $(B \rightarrow C) \rightarrow ((A \rightarrow C) \rightarrow (\neg(A \rightarrow B) \rightarrow C))$ (thm 1, rmp)
- (8) $\neg(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C))$
(多用几次 thm 3)

接下来证明另一半.

- (9) $(B \rightarrow C) \rightarrow (B \rightarrow C)$ (thm 1)
- (10) $(B \rightarrow (B \rightarrow C) \rightarrow C)$ (thm 3)
- (11) $(A \rightarrow C) \rightarrow (B \rightarrow (B \rightarrow C) \rightarrow C)$ (thm 1, rmp)
- (12) $B \rightarrow (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow C$ (thm 3, rmp)

一上面这两部分的结果作为 thm 18 的理由:

$$(13) \quad ((A \rightarrow B) \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow C) \\ \text{(thm 18 (12)(8))}$$

$$(14) \quad (A \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow C \\ \text{(thm 3, rmp)}$$

$$(15) \quad (((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow B) \rightarrow C) \\ \text{(thm 3)}$$

$$(16) \quad (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow B) \rightarrow C \quad \text{(thm 8)}$$