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1. 1. 没有无知的教授

$$\neg\exists x\,(P_1x\wedge P_2x)$$

其中 P_1x 表示 x 是教授; P_2x 表示 x 无知.

2. 所有无知者均爱虚荣

$$\forall x\,(P_3x\to P_4x)$$

P_3x 表示 x 无知, P_4x 表示 x 爱虚荣.

3. 没有爱虚荣的教授

$$\neg\exists x\,(P_5x\wedge P_6x)$$

其中 P_5x 表示 x 是教授; P_6x 表示 x 爱虚荣.

2. 1. $\vdash(A\to\exists vB)\to\exists v\,(A\to B)$

(1)	$\neg A\to(A\to B)$	(PC 定理 6)
(2)	$\neg\,(A\to B)\to A$	(逆否命题)
(3)	$\forall v\neg\,(A\to B)\to\neg(A\to B)$	(去全称)
(4)	$\forall v\neg\,(A\to B)\to A$	((2)(3) 三段论)
(5)	$B\to(A\to B)$	(A1)
(6)	$\neg\,(A\to B)\to\neg B$	(逆否命题)
(7)	$\forall v\neg\,(A\to B)\to\neg B$	((3)(6) 三段)
(8)	$\forall v\neg\,(A\to B)\vdash\forall v\neg B$	(全称推广)
(9)	$\forall v\neg\,(A\to B)\vdash\neg\,(A\to\neg\forall x\neg B)$	((4)(8))
(10)	$\vdash\forall v\neg\,(A\to B)\to\neg\,(A\to\neg\forall v\neg B)$	(演绎定理)
(11)	$\vdash(A\to\neg\forall v\neg B)\to\neg\forall v\neg\,(A\to B)$	(逆否)
(12)	$\vdash(A\to\neg\forall v\neg B)\to\exists v\,(A\to B)$	

2. $\vdash\exists v\,(A\to B)\to(A\to\exists vB)$

(1)	$\{A,\forall v\neg B\}\vdash\neg B$	(去全称)
(2)	$\{A,\forall v\neg B\}\vdash\neg\,(A\to B)$	((1) , 已知条件)
(3)	$\{A,\forall v\neg B\}\vdash\forall v\neg\,(A\to B)$	(全称推广)
(4)	$\{A,\neg\forall v\neg\,(A\to B)\}\vdash\neg\forall v\neg B$	(演绎定理和逆否)
(5)	$\{\neg\forall v\neg\,(A\to B)\}\vdash A\to\neg\forall v\neg B$	(演绎)
(6)	$\vdash\neg\forall v\neg\,(A\to B)\to(A\to\neg\forall v\neg B)$	(演绎)

3. $\vdash (\forall v B \to A) \to \exists v\,(B \to V)$

$$
\begin{array}{lll}
(1) & \forall v \neg\,(B \to A) \vdash \neg\,(B \to A) & (\text{去全称}) \\
(2) & \forall v \neg\,(B \to A) \vdash \neg A & ((1)) \\
(3) & \forall v \neg\,(B \to A) \vdash B & ((1)) \\
(4) & \forall v \neg\,(B \to A) \vdash \forall v B & (\text{全称推广}) \\
(5) & \forall v \neg\,(B \to A) \vdash \neg\,(\forall v B \to A) & ((2)\ (4)) \\
(6) & \forall v B \to A \vdash \neg \forall v \neg\,(B \to A) & (\text{逆否}) \\
(7) & \forall v B \to A \vdash \exists v\,(B \to A) &
\end{array}
$$

4. $\vdash \exists v\,(B \to A) \to (\forall v B \to A)$

$$
\begin{array}{lll}
(1) & \neg\,(\forall v B \to A) \vdash \forall v B & (\text{定理}) \\
(2) & \neg\,(\forall v B \to A) \vdash \neg A & (\text{定理}) \\
(3) & \neg\,(\forall v B \to A) \vdash \forall v B \to B & (\text{去全称}) \\
(4) & \neg\,(\forall v B \to A) \vdash B & ((1)(3)\mathrm{rmp}) \\
(5) & \neg\,(\forall v B \to A) \vdash \neg\,(B \to A) & ((2)(4)\ \text{定理}) \\
(6) & \neg\,(\forall v B \to A) \vdash \forall v \neg\,(B \to A) & (\text{全称推广})
\end{array}
$$

3. 1. $\forall x\,(A \to B) \vdash\dashv A \to \forall x B$, x 在 A 中无自由出现.

先是证明 $\vdash \forall x\,(A \to B) \to (A \to \forall x B)$, 使用演绎定理转化一下:

$$
\begin{array}{ll}
(1) & \forall x\,(A \to B) \vdash (A \to \forall x B) \\
(2) & \{\forall x\,(A \to B)\,, A\} \vdash \forall x B
\end{array}
$$

所以我们要证明 $\{\forall x\,(A \to B)\,, A\} \vdash \forall x B$

$$
\begin{array}{lll}
(1) & \{\forall x\,(A \to B)\,, A\} \vdash \forall x A \to \forall x B & (\mathrm{A5}) \\
(2) & \{\forall x\,(A \to B)\,, A\} \vdash \forall x A & (\text{全称推广}) \\
(3) & \{\forall x\,(A \to B)\,, A\} \vdash \forall x B & (\text{三段})
\end{array}
$$

然后证明 $(A \to \forall x B) \vdash \forall x\,(A \to B)$

$$
\begin{array}{lll}
(1) & \{A \to \forall x B\} \vdash (\forall x B \to B) & (\text{去全称}) \\
(2) & \{A \to \forall x B\} \vdash (A \to B) & ((1),\ \text{已知},\ \text{三段论}) \\
(3) & \{A \to \forall x B\} \vdash \forall x\,(A \to B) & (\text{全称推广})
\end{array}
$$

2. $\forall x\,(A \to B) \vdash\dashv \exists x A \to B$, x 在 B 中无自由出现.

先是证明: $\forall x\,(A \to B) \vdash \exists x A \to B$, 使用演绎定理转化一下:

$$
\begin{aligned}
&(1) \quad \forall x\,(A \to B) \vdash (\neg\forall x \neg A) \to B \\
&(2) \quad \forall x\,(A \to B) \vdash \neg B \to \forall x \neg A \\
&(3) \quad \{\forall x\,(A \to B)\,, \neg B\} \vdash \forall x \neg A
\end{aligned}
$$

$$
\begin{array}{lll}
(1) & \{\forall x\,(A \to B)\,, \neg B\} \vdash \forall x\,(A \to B) \to (A \to B) & \\
(2) & \{\forall x\,(A \to B)\,, \neg B\} \vdash (A \to B) & (\text{rmp}) \\
(3) & \{\forall x\,(A \to B)\,, \neg B\} \vdash (A \to B) \to (\neg B \to \neg A) & (\text{逆否}) \\
(4) & \{\forall x\,(A \to B)\,, \neg B\} \vdash (\neg B \to \neg A) & (\text{rmp}) \\
(5) & \{\forall x\,(A \to B)\,, \neg B\} \vdash \neg A & ((4)\text{ 已知条件 rmp}) \\
(6) & \{\forall x\,(A \to B)\,, \neg B\} \vdash \forall x \neg A & (\text{全称推广})
\end{array}
$$

然后证明: $\exists x A \to B \vdash \forall x\,(A \to B)$

$$
\begin{array}{lll}
(1) & \forall x \neg A \vdash \neg A & \\
(2) & \forall x \neg A \vdash \neg A \to (A \to B) & (\text{PC 定理 6}) \\
(3) & \forall x \neg A \vdash (A \to B) & (\text{rmp}) \\
(4) & \forall x \neg A \vdash \forall x\,(A \to B) & (\text{全称推广}) \\
(5) & \neg\forall x\,(A \to B) \vdash \neg\forall x \neg A & (\text{逆否}) \\
(6) & \vdash B \to (A \to B) & (\text{A1}) \\
(7) & B \vdash A \to B & (\text{演绎}) \\
(8) & B \vdash \forall x\,(A \to B) & (\text{全称推广}) \\
(9) & \neg\forall x\,(A \to B) \vdash \neg B & (\text{逆否}) \\
(10) & \neg\forall x\,(A \to B) \vdash \neg\,(\exists x A \to B) & ((5)\ (9)) \\
(11) & (\exists x A \to B) \vdash \forall x\,(A \to B) &
\end{array}
$$

3. $\forall\,(A \wedge B) \vdash\dashv \forall x A \wedge \forall x B$

我们知道 $A \wedge B$ 实际上就是 $\neg\,(A \to \neg B)$, 于是我们就是要证明

$$\forall x \neg\,(A \to \neg B) \vdash\dashv \neg\,(\forall x A \to \neg\forall x \neg B)$$

这里先证明 $\forall x \neg(A \to \neg B) \vdash \neg(\forall x A \to \neg \forall x B)$

$$\begin{array}{lll}
(1) & \forall x \neg(A \to \neg B) \vdash \neg(A \to \neg B) & (\text{去全称}) \\
(2) & \forall x \neg(A \to \neg B) \vdash \neg(A \to \neg B) \to A & (\text{PC 定理 6 的逆否}) \\
(3) & \forall x \neg(A \to \neg B) \vdash A & (\text{rmp}) \\
(4) & \forall x \neg(A \to \neg B) \vdash \forall x A & (\text{全程推广}) \\
(5) & \forall x \neg(A \to \neg B) \vdash \neg B \to (A \to \neg B) & (\text{A1}) \\
(6) & \forall x \neg(A \to \neg B) \vdash \neg(A \to \neg B) \to B & (\text{逆否}) \\
(7) & \forall x \neg(A \to \neg B) \vdash B & (\text{rmp, (6), 已知}) \\
(8) & \forall x \neg(A \to \neg B) \vdash \forall x B & \\
(9) & \forall x \neg(A \to \neg B) \vdash \neg(\forall x A \to \neg \forall x B) & ((4),(8))
\end{array}$$

然后证明另一半:

$$\begin{array}{lll}
(1) & \neg(\forall x A \to \neg \forall x B) \vdash \forall x A & (\text{定理 6 的逆否, rmp}) \\
(2) & \neg(\forall x A \to \neg \forall x B) \vdash \neg\neg \forall x B & (\text{A1 的逆否, rmp}) \\
(3) & \neg(\forall x A \to \neg \forall x B) \vdash \forall x A \to A & (\text{去全称}) \\
(4) & \neg(\forall x A \to \neg \forall x B) \vdash A & (\text{rmp}) \\
(5) & \neg(\forall x A \to \neg \forall x B) \vdash \neg\neg \forall x B \to \forall x B & (\text{否定的否定}) \\
(6) & \neg(\forall x A \to \neg \forall x B) \vdash \forall x B & (\text{rmp}) \\
(7) & \neg(\forall x A \to \neg \forall x B) \vdash B & (\text{去全称}) \\
(8) & \neg(\forall x A \to \neg \forall x B) \vdash \neg(A \to \neg B) & ((7),\ (4)) \\
(9) & \neg(\forall x A \to \neg \forall x B) \vdash \forall x \neg(A \to \neg B) & (\text{全称推广})
\end{array}$$

4. $\exists x (A \vee B) \vdash\dashv \exists x A \vee \exists x B$

等价于证明: $\vdash \exists x (A \vee B) \leftrightarrow \exists x A \vee \exists x B$
注意到:

$$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$$

下面证明这个:

$$\vdash \forall x \neg(A \vee B) \leftrightarrow \forall x (\neg A \wedge \neg B)$$

$$
\begin{array}{lll}
(1) & \forall x\neg(A\vee B)\vdash\neg(A\vee B) & (\text{去全称}) \\
(2) & \forall x\neg(A\vee B)\vdash\neg(A\vee B)\to\neg A\wedge\neg B & \\
(3) & \forall x\neg(A\vee B)\vdash\neg A\wedge\neg B & (\text{rmp}) \\
(4) & \forall x\neg(A\vee B)\vdash\forall x(\neg A\wedge\neg B) & (\text{全称推广}) \\
(5) & \vdash\forall x\neg(A\vee B)\to\forall x(\neg A\wedge\neg B) & (\text{演绎}) \\
(6) & \forall x(\neg A\wedge\neg B)\vdash\neg A\wedge\neg B & (\text{去全称}) \\
(7) & \forall x(\neg A\wedge\neg B)\vdash\neg A\wedge\neg B\to\neg(A\vee B) & \\
(8) & \forall x(\neg A\wedge\neg B)\vdash\neg(A\vee B) & (\text{rmp}) \\
(9) & \forall x(\neg A\wedge\neg B)\vdash\forall x\neg(A\vee B) & (\text{全称推广}) \\
(10) & \vdash\forall x(\neg A\wedge\neg B)\to\forall x\neg(A\vee B) & (\text{演绎}) \\
(11) & \vdash\forall x(\neg A\wedge\neg B)\leftrightarrow\forall x\neg(A\vee B) & ((5)(10))
\end{array}
$$

接下来证明 $\vdash\exists xA\vee\exists xB\to\exists x(A\vee B)$

$$
\begin{array}{lll}
(1) & \vdash\forall x(\neg A\wedge\neg B)\to\forall x\neg A\wedge\forall x\neg B & (\text{上一问的结论}) \\
(2) & \vdash\forall x\neg(A\vee B)\to\forall x(\neg A\wedge\neg B) & (\text{刚刚证明的结论}) \\
(3) & \vdash\forall x\neg(A\vee B)\to\forall x\neg A\wedge\forall x\neg B & ((1)(2)\ \text{三段论}) \\
(4) & \vdash\neg(\forall x\neg A\wedge\forall x\neg B)\to\neg\forall x\neg(A\vee B) & (\text{逆否}) \\
(5) & \vdash(\neg\forall x\neg A\vee\neg\forall x\neg B)\to\neg(\forall x\neg A\wedge\forall x\neg B) & \\
(6) & \vdash(\neg\forall x\neg A\vee\neg\forall x\neg B)\to\neg\forall x\neg(A\vee B) & ((4)(5)\ \text{三段论}) \\
(7) & \vdash\exists xA\vee\exists xB\to\exists x(A\vee B) &
\end{array}
$$

证明 $\vdash\exists x(A\vee B)\to\exists xA\vee\exists xB$ 也是完全类似的. 本题就相当于上一题的对偶的版本, 使用 deMorgan 律转化而来.