Seminar
grupa 131
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Tutoriat III.

Ex 1:

a) Metoda Gauss-Jordan:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \underset{\sim}{L_2 = -3L_1 + L_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -5 & -3 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{L_3 = L_3 + L_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -5 & -3 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right) \underset{\sim}{L_1 = L_1 + L_3}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 5 & -5 & -3 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right) \underset{\sim}{L_3 = 5L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 5 & -5 & -3 & 1 & 0 \\ 0 & -5 & 15 & 5 & 0 & 5 \end{array}\right)$$

$$\underset{\sim}{L_3 = L_3 + L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 5 & -5 & -3 & 1 & 0 \\ 0 & 0 & 10 & 2 & 1 & 5 \end{array}\right) \underset{\sim}{L_3 = \frac{L_3}{2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 5 & -5 & -3 & 1 & 0 \\ 0 & 0 & 5 & 1 & \frac{1}{2} & \frac{5}{2} \end{array}\right)$$

$$\underset{\sim}{L_2 = L_2 + L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & -2 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 5 & 1 & \frac{1}{2} & \frac{5}{2} \end{array}\right) \underset{\sim}{L_1 = L_1 - L_3}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 5 & 0 & -2 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 5 & 1 & \frac{1}{2} & \frac{5}{2} \end{array}\right) \underset{\sim}{L_2 = \frac{L_2}{5},\ L_3 = \frac{L_3}{5}}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{10} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{10} & \frac{1}{2} \end{array}\right)$$

Ex2:

a) $B = \{(2, 4, 8, -4), (4, -2, -1, 3), (3, 5, 2, -2), (-5, 1, 4, -6)\}$

1) Arătăm că e SLI:

Fie $a, b, c, d \in \mathbb{R}$

$a(2, 4, 8, -4) + b(4, -2, -1, 3) + c(3, 5, 2, -2) + d(-5, 1, 4, -6) =$

$= (0, 0, 0, 0) \Leftrightarrow \begin{cases} 2a + 4b + 3c - 5d = 0 \\ 4a - 2b + 5c + d = 0 \\ 8a - b + 2c + 4d = 0 \\ -4a + 3b - 2c - 6d = 0 \end{cases} \Leftrightarrow \dots \quad a = b = c = d = 0$

$\Rightarrow B$ e SLI

sau

1) $\Delta = \begin{vmatrix} 2 & 4 & 3 & -5 \\ 4 & -2 & 5 & 1 \\ 8 & -1 & 2 & 4 \\ -4 & 3 & -2 & -6 \end{vmatrix} = \dots \quad \neq 0 \xRightarrow{CLI} B$ e SLI

2) Arătăm că e SG:

$\forall x \in \mathbb{R}^4, \quad x = (x_1, x_2, x_3, x_4) = a(2, 4, 8, -4) + b(4, -2, -1, 3) +$

$\forall a, b, c, d \in \mathbb{R} \quad + c(3, 5, 2, -2) + d(-5, 1, 4, -6) \Leftrightarrow$

$\Leftrightarrow (*) \begin{cases} 2a + 4b + 3c - 5d = x_1 \\ 4a - 2b + 5c + d = x_2 \\ 8a - b + 2c + 4d = x_3 \\ -4a + 3b - 2c - 6d = x_4 \end{cases} \Rightarrow A = \begin{pmatrix} 2 & 4 & 3 & -5 \\ 4 & -2 & 5 & 1 \\ 8 & -1 & 2 & 4 \\ -4 & 3 & -2 & -6 \end{pmatrix}$

$\det A = \Delta \neq 0 \Rightarrow (*)$ compatibil determinist $\Rightarrow B$ este SG

sau

2) $\dim_{\mathbb{R}} \mathbb{R}^4 = 4 = |B|$

1), 2) $\Rightarrow B =$ Bază.

b) Matricea de trecere : notăm A

$B_0 = \{ (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1) \}$ — e_1, e_2, e_3, e_4

$B = \{ (2,4,8,-4), (4,-2,-1,3), (3,5,2,-2), (-5,1,7,-6) \}$ — v_1, v_2, v_3, v_4

$A = (a_{i,j})_{i,j=\overline{1,4}}$ a.î.

$v_i = a_{1i} e_1 + a_{2i} e_2 + a_{3i} e_3 + a_{4i} e_4$

$v_1 = (2,4,8,-4) = 2e_1 + 4e_2 + 8e_3 - 4e_4$

$\vdots$

$v_4 = (-5, 1, 7, -6) = -5e_1 + 1e_2 + 7e_3 - 6e_4$

$$A = \begin{pmatrix} 2 & 4 & 3 & -5 \\ 4 & -2 & 5 & 1 \\ 8 & -1 & 2 & 7 \\ -4 & 3 & -2 & -6 \end{pmatrix}$$

Obs: $A = A^{-1}$ (inversa)

[Obs: Matricea resp. B_0 care are coloanele e_1, e_2, e_3, e_4 este I_4. Matricea, respectivă bazei B este A.

Matricea de trecere ar fi o necunoscută X a.î.:

$I \cdot X = A$

$X = A$, fiindcă $IA = A \cdot I = A$.]

c) $(1,2,1,2) = a(2,4,8,-4) + b(4,-2,-1,3) + c(3,5,2,-2) + d(-5,1,7,6) \iff \begin{cases} 2a + 4b + 3c - 5d = 1 \\ 4a - 2b + 5c + d = 2 \\ 8a - b + 2c + 7d = 1 \\ -4a + 3b - 2c + 6d = 2 \end{cases}$ ....

Aflăm a, b, c, d..

Ex3:

a) $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ <=> Matricea de trecere de la baza canonică la o bază $B = \{(1,1,1), (1,-1,1), (1,0,1)\}$

b) Matricea lui f în raport cu o bază <=> Matricea de trecere de la M la baza dată:

$(1,1,1) = a(1,1,1) + b(2,1,4) + c(-1,1,3)$

$\begin{cases} a + 2b - c = 1 \\ a + b + c = 1 \\ a + 4b + 3c = 1 \end{cases} \overset{\cdot(-3)}{\Leftrightarrow} \begin{cases} 2a + 3b = 2 \\ -2a + b = -2 \\ a + b + c = +1 \end{cases} \Leftrightarrow \begin{cases} 4b = 0 \\ 2a + 3b = 2 \\ a + b + c = 1 \end{cases}$

$\Leftrightarrow \begin{cases} b = 0 \\ a = 1 \\ c = 0 \end{cases} \Leftrightarrow \underline{(1, 0, 0)}$

$(1,-1,1) = a(1,1,1) + b(2,1,4) + c(-1,1,3)$

$\begin{cases} a + 2b - c = 1 \\ a + b + c = -1 \\ a + 4b + 3c = 1 \end{cases} \overset{\cdot 3}{\Leftrightarrow} \begin{cases} 2a + 3b = 0 \\ -2a + b = 4 \\ a + b + c = -1 \end{cases} \Leftrightarrow \begin{cases} 4b = 4 \\ 2a + 3b = 0 \\ a + b + c = -1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} b = 1 \\ 2a = -3 \\ a + 1 + c = -1 \end{cases} \Leftrightarrow \begin{cases} b = 1 \\ a = -\frac{3}{2} \\ c = -\frac{1}{2} \end{cases} \Leftrightarrow \left(-\frac{3}{2}, 1, -\frac{1}{2}\right)$

$(1,0,1) = a(1,1,1) + b(2,1,4) + c(-1,1,3)$

$\begin{cases} a + 2b - c = 1 \\ a + b + c = 0 \\ a + 4b + 3c = 1 \end{cases} \overset{\cdot(-3)}{\Leftrightarrow} \begin{cases} 2a + 3b = 1 \\ -2a + b = 1 \\ a + b + c = 0 \end{cases} \Leftrightarrow \begin{cases} 4b = 2 \\ 2a + 3b = 1 \\ a + b + c = 0 \end{cases}$

$\Leftrightarrow \begin{cases} b = \frac{1}{2} \\ a = -\frac{3}{4} \\ c = \frac{1}{4} \end{cases} \Leftrightarrow \left(-\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right)$

Obținem:

$$M' = \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{3}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \sim \begin{pmatrix} 4 & -6 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & 1 \end{pmatrix}$$

$L_1 = 4L_1$
$L_2 = 2L_2$
$L_3 = 4L_3$

Seria 14

Ex 1: $(\mathbb{R}^4, +, \cdot)/\mathbb{R}$

$V' = \{ x \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 - x_4 = 0 \}$

$V'' = \{ x \in \mathbb{R}^4 \mid x_1 - x_2 + x_3 - x_4 = 0 \}$

a) $V' + V'' = \mathbb{R}^4$ ($\dim_{\mathbb{R}}(V' + V'') = 4$)

b) $V = V' \cap V''$. Precizați un reper în V', V'', respectiv pentru V.

c) $\mathbb{R}^4 = V \oplus W$. $W = ?$ (subspațiul complementar lui V)

Rezolvare:

a) V': $x_1 + x_2 - x_3 - x_4 = 0$

$A' = (1 \; 1 \; -1 \; -1)$, $rg(A) = 1$

$V' = S(A')$

$\dim_{\mathbb{R}} V' = 4 - 1 = 3$ (=> V' este <u>hiperplan</u> în $\mathbb{R}^4$)
cu 1 dim. mai puțin decât spațiul ambient

V'': $x_1 - x_2 + x_3 - x_4 = 0$

$A'' = (1 \; -1 \; 1 \; -1)$, $rg(A'') = 1$

$V'' = S(A'')$

$\dim_{\mathbb{R}} V'' = 4-1=3 \Rightarrow V''$ este hiperplan

$V = V' \cap V''$

$V: \begin{cases} x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$ $A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

$rg(A) = 2$

$\dim_{\mathbb{R}} V = 4-2=2$

$\dim_{\mathbb{R}} (V'+V'') = \dim_{\mathbb{R}} V' + \dim_{\mathbb{R}} V'' - \dim_{\mathbb{R}} V = 3+3-2=4$

$\dim_{\mathbb{R}} \mathbb{R}^4 = 4$

$\Rightarrow V' + V'' = \mathbb{R}^4$

$\quad\quad \shortparallel$

$\langle V' \cup V'' \rangle$.

b) $V': \; x_1 = -x_2 + x_3 + x_4 \Rightarrow V' = \{(-x_2 + x_3 + x_4, x_2, x_3, x_4) \mid x_2, x_3, x_4 \in \mathbb{R}\} \Rightarrow V' = \{x_2(-1, 1, 0, 0) + x_3(1, 0, 1, 0) + x_4(1, 0, 0, 1) \mid x_2, x_3, x_4 \in \mathbb{R}\}$

(1) $R' = \{(-1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\} \Rightarrow$ SG pentru V'

(2) $rg\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$ (maxim) $\overset{CLI}{\Rightarrow} R'$ este SLI

(1) și (2) $\Rightarrow R'$ este reper. / bază

$V'': \; x_1 = x_2 - x_3 + x_4$

$V'' = \{(x_2 - x_3 + x_4, x_2, x_3, x_4) \mid x_2, x_3, x_4 \in \mathbb{R}\} =$
$= \{x_2(1, 1, 0, 0) + x_3(-1, 0, 1, 0) + x_4(1, 0, 0, 1) \mid x_2, x_3, x_4 \in \mathbb{R}\}$

$R'' = \{(1, 1, 0, 0), (-1, 0, 1, 0), (1, 0, 0, 1)\} \Rightarrow$ SG pentru V''

$\dim_{\mathbb{R}} V'' = |R''| = 3 \Rightarrow R''$ e SLI

$\Rightarrow R''$ e bază

$V: \begin{cases} x_1 + x_2 = x_3 + x_4 \\ x_1 - x_2 = -x_3 + x_4 \end{cases} \Leftrightarrow \begin{cases} 2x_1 = 2x_4 \\ x_1 - x_2 = -x_3 + x_4 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x_1 = x_4 \\ -x_2 = -x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_4 \\ x_2 = x_3 \end{cases}$

$\Rightarrow V = \{(x_4, x_3, x_3, x_4) \mid x_3, x_4 \in \mathbb{R}\} = \{x_4(1,0,0,1) + x_3(0,1,1,0) \mid x_3, x_4 \in \mathbb{R}\} \Rightarrow R = \{(1,0,0,1), (0,1,1,0)\}$ SG pentru V (1)

$$rg\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = 2 \Rightarrow R \text{ este SLI } (2)$$

(1) și (2) $\Rightarrow R$ este reper

c) Extindem R la un reper în $\mathbb{R}^4$:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

(coloanele 3 și 4: vectorii adăugați)

$$W = \langle \underbrace{\{(1,0,0,0), (0,1,0,0)\}}_{\substack{\shortparallel \\ \text{reper în } W = R_W}} \rangle \Rightarrow \mathbb{R}^4 = V \oplus W.$$

$W = \langle R_W \rangle$.

d) Fie $x = (1,1,2,0)$. Să se descompună x în raport cu $\mathbb{R}^4 = V \oplus W$.

Aflăm coordonatele lui x în raport cu reperul

$R' = \{(1,0,0,1), (0,1,1,0), (1,0,0,0), (0,1,0,0)\}$

$$x = (1,1,2,0) = \underbrace{a(1,0,0,1) + b(0,1,1,0)}_{\in V} + \underbrace{c(1,0,0,0) + d(0,1,0,0)}_{\in W}$$

$$\begin{cases} a + c = 1 \\ b + d = 1 \\ b = 2 \\ a = 0 \end{cases} \Leftrightarrow \begin{cases} c = 1 \\ d = -1 \\ b = 2 \\ a = 0 \end{cases} \Rightarrow (0, 2, 1, -1) \text{ sunt coord. lui } x \text{ în raport cu reperul } R.$$

$$\underset{V}{\overset{\in}{v}} = a(1,0,0,1) + b(0,1,1,0) = 2(0,1,1,0) = (0,2,2,0)$$

$$w = c(1,0,0,0) + d(0,1,0,0) = (1,0,0,0) + (0,-1,0,0) = (1,-1,0,0) \in W.$$

$$(1,1,2,0) = (0,2,2,0) + (1,-1,0,0).$$

Ex 2: $(\mathbb{R}^3, +, \cdot)/_{\mathbb{R}}$

$U = \langle \{u_1 = (1,1,1),\ u_2 = (0,3,1),\ u_3 = (2,-1,1)\} \rangle$

$V = \langle \{v_1 = (1,-2,4),\ v_2 = (-2,4,-8)\} \rangle$

a) $\dim_{\mathbb{R}} U$

b) Precizați o ecuație pentru U.

c) $\dim_{\mathbb{R}} V$

d) $\dim_{\mathbb{R}} (U \cap V)$

e) $\dim_{\mathbb{R}} (U + V)$

Rezolvare:

a) $M = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$\det M = 0 \Rightarrow rg\, M = 2.$

$\{u_1, u_2, u_3\} = SLD$

$\{u_1, u_2\} = SLI$

$\mathcal{U} = \langle \{u_1, u_2\} \rangle = \{a \cdot u_1 + b \cdot u_2 \mid a, b \in \mathbb{R}\} =$
$= \{a(1,1,1) + b(0,3,1) \mid a, b \in \mathbb{R}\} = \{(a, a+3b, a+b) \mid$
$a, b \in \mathbb{R}\}$, $\{u_1, u_2\}$ e SG ($\forall$ submulțime a unui SG este SG).

$\{u_1, u_2\}$ = bază $\Rightarrow \dim_{\mathbb{R}} \mathcal{U} = 2$

b) $\begin{cases} x_1 = a \\ x_2 = a+3b \\ x_3 = a+b \end{cases} \Rightarrow A = \left(\begin{array}{cc|c} 1 & 0 & x_1 \\ 1 & 3 & x_2 \\ 1 & 1 & x_3 \end{array}\right)$, $rg(A) = 2$

$\Delta C = \begin{vmatrix} 1 & 0 & x_1 \\ 1 & 3 & x_2 \\ 1 & 1 & x_3 \end{vmatrix} = 0 \Rightarrow 3x_3 + x_1 - 3x_1 - x_2 = 0 \Rightarrow$

$\Rightarrow -2x_1 - x_2 + 3x_3 = 0$

$\Rightarrow \mathcal{U} = \{x \in \mathbb{R}^3 \mid -2x_1 - x_2 + 3x_3 = 0\}$

M_{II}:

$\begin{cases} x_1 = a \\ x_2 = a + 3b \\ x_3 = a + b \end{cases} \Leftrightarrow \begin{cases} x_1 = a \\ x_2 = x_1 + 3(x_3 - a) \\ b = x_3 - a \end{cases} \Leftrightarrow \begin{cases} x_1 = a \\ b = x_3 - a \\ x_2 = x_1 + 3x_3 - 3x_1 \end{cases}$

$\Rightarrow \mathcal{U} = \{x \in \mathbb{R}^3 \mid x_2 = 3x_3 - 2x_1\}$

c) $v_2 = -2v_1 \Rightarrow \{v_1, v_2\}$ este SLD.

$V = \langle \{v_1 = (1, -2, 4)\} \rangle$, $\{v_1\}$ este SLI și SG

$\Rightarrow \{v_1\}$ este bază în V

$\dim_{\mathbb{R}} V = 1$

d) $\mathcal{U} \cap V$; verificăm dacă $v_1 \in \mathcal{U}$, sau $u_1, u_2 \in V$.

$\{u_1, u_2, v_1\}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 4 \end{pmatrix} \Rightarrow \det A = 12 \neq 0 \Rightarrow \{u_1, u_2, v_1\} \in$$

SLI $\Rightarrow v_1 \notin \mathcal{U}$, (dacă $v_1 \in \mathcal{U} \Rightarrow \{u_1, u_2, v_1\} \in$ SLD)

$\mathcal{U} \cap V = \{0_{\mathbb{R}^3}\} = \{(0, 0, 0)\}$

$\Rightarrow \dim_{\mathbb{R}}(\mathcal{U} \cap V) = 0$

e) $\dim(\mathcal{U} + V) = \dim(\mathcal{U}) + \dim(V) - \dim(\mathcal{U} \cap V) =$

$= 2 + 1 - 0 = 3 = \dim(\mathcal{U} \oplus V)$

$\Rightarrow \mathcal{U} + V = \mathbb{R}^3 = \mathcal{U} \oplus V.$