Covariance and correlation :

Let X denote a random d-dimensional vector :

$$X = (\underset{\substack{\downarrow \\ \text{random} \\ \text{variables}}}{x_1}, \ldots, x_d) \qquad , \text{ e.g. } x = (\underline{temp}, \underline{humidity})$$

The statistics of a random vector taking continuous values , can be characterized by

the joint distribution $p(x_1, x_2, \ldots, x_d)$.

Def :

The covariance between two random vectors $X = (x_1, \ldots, x_d)$ and $Y = (y_1, \ldots, y_d)$ measures the degree to which X and Y are linearly related.

$$\underset{\substack{d \times d \\ \text{matrix}}}{\text{cov}[X,Y]} := \mathbb{E}\Big[\underbrace{(X - \mathbb{E}[X])}_{d \times 1} \underbrace{(Y - \mathbb{E}[Y])^T}_{1 \times d} \Big] \quad \checkmark$$

$$\Sigma_{ij} = \text{cov}[x_i, y_j] = \mathbb{E}\Big[\underset{1 \times 1}{(x_i - \mathbb{E}[x_i])} \underset{1 \times 1}{(y_j - \mathbb{E}[y_j])} \Big] \quad , \quad i,j = 1, \ldots$$

We can also compute the (auto)-covariance of the r.v. X :

$$\text{cov}[X] = \text{cov}[X, X] = \begin{bmatrix} \text{Var}[x_1] & \text{cov}[x_1, x_2] & \cdots & \text{cov}[x_1, x_d] \\ & \ddots & & \\ & & \ddots & \\ \text{cov}[x_d, x_1] & & & \text{Var}[x_d] \end{bmatrix} =$$

$$= \underset{\substack{\downarrow \\ \text{covariance} \\ \text{matrix}}}{\Sigma} = \underset{\substack{\downarrow \\ \text{precision} \\ \text{matrix}}}{\Lambda^{-1}}$$

(*) Covariances can take values between zero and infinity.

Sometimes ...

Sometimes it might be preferable to work with a normalized measure that has a finite upper bound:

Pearson correlation coefficient:

$$\mathrm{corr}[X,Y] = \frac{\mathrm{cov}[X,Y]}{\sqrt{\mathbb{V}\mathrm{ar}[X]\,\mathbb{V}\mathrm{ar}[Y]}}, \quad \boxed{-1 \le \mathrm{corr}[X,Y] \le 1}$$

Specifically, one can show that $\mathrm{corr}[X,Y]=1$ if and only if:

$$Y = aX + b, \text{ for some } a, b.$$

If X and Y are <u>independent</u> then $\mathrm{cov}[X,Y]=0$, hence they are also uncorrelated.

<u>Caution</u>: The opposite is not necessarily true!

- <u>Empirical mean and covariance</u> (i.e. how to compute the mean and covariance of a r.v. X given N observed realizations)

— empirical mean: $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$

— empirical covariance: $\Sigma = \frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{x})(x_i-\bar{x})^T$

A covariance matrix should always be a symmetric and positive-definite matrix, i.e. $z^T\Sigma z > 0$, for any non-zero vector z.

All eigenvalues of Σ should be positive.

<u>Sums and linear transformations of random variables</u>

<u>Given</u> $X \sim p(x)$ with mean $\mathbb{E}[X]$
$Y \sim p(y)$ with mean $\mathbb{E}[Y]$, what can we say about:

$$Z = X + Y, \quad \mathbb{E}[Z] = ? \quad \mathbb{V}\mathrm{ar}[Z] = ?$$

The following holds:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] \qquad \mathbb{V}\mathrm{ar}[X+Y] = \mathbb{V}\mathrm{ar}[X] + \mathbb{V}\mathrm{ar}[Y] + \mathrm{cov}[X,Y]$$

$+ \mathrm{cov}[Y, X]$

$\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y]$ (due to linearity of expectation), $\mathbb{V}ar[X-Y] = \mathbb{V}ar[X] + \mathbb{V}ar[Y] - \mathrm{cov}[X,Y] - \mathrm{cov}[$

- Linear transformations :

Assume a r.v. $X \in \mathbb{R}^d$ with mean $\mathbb{E}[X] := \mu \in \mathbb{R}^d$ and covariance $\Sigma \in \mathbb{R}^{d \times d}$, and $Y = AX + b$, $Y \in \mathbb{R}^m$

e.g. $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$

Then :

$$\mathbb{E}[Y] = \mathbb{E}[AX+b] = A\,\mathbb{E}[X] + b = A\mu + b$$

$$\mathbb{V}ar[Y] = \mathbb{V}ar[AX+b] = \mathbb{V}ar[AX] = A\,\mathbb{V}ar[X]\,A^T = A\Sigma A^T$$

$$\mathrm{cov}[X,Y] = \Sigma A^T$$ (try to derive this on your own).