Chapter 12

QUANTIFYING UNCERTAINTY

12.1 Show from first principles that $P(a \mid b \wedge a) = 1$.

12.2 Using the axioms of probability, prove that any probability distribution on a discrete random variable must sum to 1.

12.3 For each of the following statements, either prove it is true or give a counterexample.

a. If $P(a \mid b, c) = P(b \mid a, c)$, then $P(a \mid c) = P(b \mid c)$

b. If $P(a \mid b, c) = P(a)$, then $P(b \mid c) = P(b)$

c. If $P(a \mid b) = P(a)$, then $P(a \mid b, c) = P(a \mid c)$

12.4 Would it be rational for an agent to hold the three beliefs $P(A) = 0.4$, $P(B) = 0.3$, and $P(A \vee B) = 0.5$? If so, what range of probabilities would be rational for the agent to hold for $A \wedge B$? Make up a table like the one in Figure 12.1, and show how it supports your argument about rationality. Then draw another version of the table where $P(A \vee B) = 0.7$. Explain why it is rational to have this probability, even though the table shows one case that is a loss and three that just break even. (*Hint:* what is Agent 1 committed to about the probability of each of the four cases, especially the case that is a loss?)

Agent 1		Agent 2		Outcomes and payoffs to Agent 1			
Proposition	Belief	Bet	Stakes	a, b	$a, \neg b$	$\neg a, b$	$\neg a, \neg b$
a	0.4	a	4 to 6	–6	–6	4	4
b	0.3	b	3 to 7	–7	3	–7	3
$a \vee b$	0.8	$\neg(a \vee b)$	2 to 8	2	2	2	–8
				–11	–1	–1	–1

Figure 13.2 Because Agent 1 has inconsistent beliefs, Agent 2 is able to devise a set of bets that guarantees a loss for Agent 1, no matter what the outcome of a and b.

Figure 12.1: Exercise 12.4

12.5 This question deals with the properties of possible worlds, defined in the textbook as assignments to all random variables. We will work with propositions that correspond to exactly one possible world because they pin down the assignments of all the variables. In probability theory, such propositions are called *atomic events*. For example, with Boolean variables X_1, X_2, X_3, the proposition $x_1 \wedge \neg x_2 \wedge \neg x_3$ fixes the assignment of the variables; in the language of propositional logic, we would say it has exactly one model.

a. Prove, for the case of n Boolean variables, that any two distinct atomic events are mutually exclusive; that is, their conjunction is equivalent to $false$.

b. Prove that the disjunction of all possible atomic events is logically equivalent to $true$.

c. Prove that any proposition is logically equivalent to the disjunction of the atomic events that entail its truth.

12.6 Given the full joint distribution shown in Figure 12.2, calculate the following:

a. $\mathbf{P}(toothache)$.

b. $\mathbf{P}(Cavity)$.

c. $\mathbf{P}(Toothache \mid cavity)$.

d. $\mathbf{P}(Cavity \mid toothache \vee catch)$.

492 Chapter 13. Quantifying Uncertainty

	$toothache$		$\neg toothache$	
	$catch$	$\neg catch$	$catch$	$\neg catch$
$cavity$	0.108	0.012	0.072	0.008
$\neg cavity$	0.016	0.064	0.144	0.576

Figure 13.3 A full joint distribution for the $Toothache$, $Cavity$, $Catch$ world.

Figure 12.2: Exercise 12.6

12.7 Given the full joint distribution shown in Figure 12.2, calculate the following:

a. $\mathbf{P}(toothache)$.

b. $\mathbf{P}(Catch)$.

c. $\mathbf{P}(Cavity \mid catch)$.

d. $\mathbf{P}(Cavity \mid toothache \vee catch)$.

12.8 In his letter of August 24, 1654, Pascal was trying to show how a pot of money should be allocated when a gambling game must end prematurely. Imagine a game where each turn consists of the roll of a die, player *E* gets a point when the die is even, and player *O* gets a point when the die is odd. The first player to get 7 points wins the pot. Suppose the game is interrupted with *E* leading 4–2. How should the money be fairly split in this case? What is the general formula? (Fermat and Pascal made several errors before solving the problem, but you should be able to get it right the first time.)

12.9 Deciding to put probability theory to good use, we encounter a slot machine with three independent wheels, each producing one of the four symbols BAR, BELL, LEMON, or CHERRY with equal probability. The slot machine has the following payout scheme for a bet of 1 coin (where "?" denotes that we don't care what comes up for that wheel):

BAR/BAR/BAR pays 20 coins
BELL/BELL/BELL pays 15 coins
LEMON/LEMON/LEMON pays 5 coins
CHERRY/CHERRY/CHERRY pays 3 coins
CHERRY/CHERRY/? pays 2 coins
CHERRY/?/? pays 1 coin

a. Compute the expected "payback" percentage of the machine. In other words, for each coin played, what is the expected coin return?

b. Compute the probability that playing the slot machine once will result in a win.

c. Estimate the mean and median number of plays you can expect to make until you go broke, if you start with 10 coins. You can run a simulation to estimate this, rather than trying to compute an exact answer.

12.10 We wish to transmit an n-bit message to a receiving agent. The bits in the message are independently corrupted (flipped) during transmission with ϵ probability each. With an extra parity bit sent along with the original information, a message can be corrected by the receiver if at most one bit in the entire message (including the parity bit) has been corrupted. Suppose we want to ensure that the correct message is received with probability at least $1 - \delta$. What is the maximum feasible value of n? Calculate this value for the case $\epsilon = 0.001$, $\delta = 0.01$.

12.11 Show that the three forms of independence in Equation 12.11are equivalent:

$$\mathbf{P}(X \mid Y) = \mathbf{P}(X) \text{ or } \mathbf{P}(Y \mid X) = \mathbf{P}(Y) \text{ or } \mathbf{P}(X, Y) = \mathbf{P}(X)\mathbf{P}(Y).$$

12.12 It is quite often useful to consider the effect of some specific propositions in the context of some general background evidence that remains fixed, rather than in the complete absence of information. The following question asks you to prove more general versions of the product rule and Bayes' rule, with respect to some background evidence **e**. Prove the conditionalized version of the general product rule:

$$\mathbf{P}(X, Y \mid \mathbf{e}) = \mathbf{P}(X \mid Y, \mathbf{e})\mathbf{P}(Y \mid \mathbf{e}) .$$

12.13 Show that the statement of conditional independence

$$\mathbf{P}(X, Y \mid Z) = \mathbf{P}(X \mid Z)\mathbf{P}(Y \mid Z)$$

is equivalent to each of the statements

$$\mathbf{P}(X \mid Y, Z) = \mathbf{P}(X \mid Z) \quad \text{and} \quad \mathbf{P}(Y \mid X, Z) = \mathbf{P}(Y \mid Z) .$$

12.14 Suppose you are given a bag containing n unbiased coins. You are told that $n - 1$ of these coins are normal, with heads on one side and tails on the other, whereas one coin is a fake, with heads on both sides.

a. Suppose you reach into the bag, pick out a coin at random, flip it, and get a head. What is the (conditional) probability that the coin you chose is the fake coin?

b. Suppose you continue flipping the coin for a total of k times after picking it and see k heads. Now what is the conditional probability that you picked the fake coin?

c. Suppose you wanted to decide whether the chosen coin was fake by flipping it k times. The decision procedure returns *fake* if all k flips come up heads; otherwise it returns $normal$. What is the (unconditional) probability that this procedure makes an error?

12.15 Let X, Y, Z be Boolean random variables. Label the eight entries in the joint distribution $\mathbf{P}(X, Y, Z)$ as a through h. Express the statement that X and Y are conditionally independent given Z, as a set of equations relating a through h. How many *nonredundant* equations are there?