Terms you'll find helpful in completing today's challenge are outlined below.

Poisson Random Variables

We've already learned that we can break many problems down into terms of n, x, and p and use the following formula for binomial random variables:

$$p(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$$

But what do we do when $p(x)$ cannot be calculated using that formula? Enter the Poisson random variable.

Poisson Experiment

A Poisson experiment is a statistical experiment that has the following properties:

- The outcome of each trial is either success or failure.
- The average number of successes (λ) that occurs in a specified region is known.
- The probability that a success will occur is proportional to the size of the region.
- The probability that a success will occur in an extremely small region is virtually zero.

Poisson Distribution

A Poisson random variable is the number of successes that result from a Poisson experiment. The probability distribution of a Poisson random variable is called a Poisson distribution:

$$P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Here,

- $e = 2.71828$
- λ is the average number of successes that occur in a specified region.
- k is the actual number of successes that occur in a specified region.
- $P(k, \lambda)$ is the Poisson probability, which is the probability of getting exactly k successes when the average number of successes is λ.

Example

Acme Realty company sells an average of 2 homes per day. What is the probability that exactly 3 homes will be sold tomorrow?

Here, $\lambda = 2$ and $k = 3$, so $P(k = 3, \lambda = 2) = \frac{\lambda^k e^{-\lambda}}{k!} = 0.180$

Example

Suppose the average number of lions seen by tourists on a one-day safari is 5. What is the probability that tourists will see fewer than 4 lions on the next one-day safari?

$P(k \leq 3, \lambda = 5) = \sum_{r=0}^{3} \frac{\lambda^r e^{-\lambda}}{r!} = 0.2650$

Special Case

Consider some Poisson random variable, X. Let $E[X]$ be the expectation of X. Find the value of $E[X^2]$.

Let $\text{Var}(X)$ be the variance of X. Recall that if a random variable has a Poisson distribution, then:

- $E[X] = \lambda$
- $\text{Var}(X) = \lambda$

Now, we'll use the following property of expectation and variance for any random variable, X:

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

$$\Rightarrow E[X^2] = \text{Var}(X) + (E[X])^2$$

So, for any random variable X having a Poisson distribution, the above result can be rewritten as:

$$\Rightarrow E[X^2] = \lambda + \lambda^2$$