Day 3: Conditional Probability

Terms you'll find helpful in completing today's challenge are outlined below.

Conditional Probability

This is defined as the probability of an event occurring, assuming that one or more other events have already occurred. Two events, A and B are considered to be independent if event A has no effect on the probability of event B (i.e. $P(B \mid A) = P(B)$). If events A and B are not independent, then we must consider the probability that both events occur. This can be referred to as the intersection of events A and B, defined as $P(A \cap B) = P(B \mid A) \cdot P(A)$. We can then use this definition to find the conditional probability by dividing the probability of the intersection of the two events ($A \cap B$) by the probability of the event that is assumed to have already occurred (event A):

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Bayes' Theorem

Let A and B be two events such that $P(A \mid B)$ denotes the probability of the occurrence of A given that B has occurred and $P(B \mid A)$ denotes the probability of the occurrence of B given that A has occurred, then:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} = \frac{P(B \mid A) \cdot P(A)}{P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c)}$$

Question 1

If the probability of student A passing an exam is $\frac{2}{7}$ and the probability of student B failing the exam is $\frac{3}{7}$, then find the probability that at least 1 of the 2 students will pass the exam.

We are given $P(A) = \frac{2}{7}$ and $P(B^c) = \frac{3}{7}$.

There are 4 possible events in our sample space:

1. A passes the exam and B fails ($A \cap B^c$).
2. B passes the exam and A fails ($A^c \cap B$).
3. A and B both pass the exam ($A \cap B$).
4. A and B both fail the exam ($A^c \cap B^c$).

Approach 1:

We are only concerned with the first 3 events ($A \cup B$). First, we calculate $P(B) = 1 - \frac{3}{7} = \frac{4}{7}$. Because one student's test grade does not depend on another student's test grade, A and B are independent and we can say that the probability of both students passing the exam is $P(A \cap B) = P(A) \cdot P(B) = \frac{2}{7} \cdot \frac{4}{7} = \frac{8}{49}$. Now that we know the probability of both students passing the exam, we can determine the probability of at least 1 student passing the exam:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A) \cdot P(B)$$

$$\Rightarrow P(A \cup B) = \frac{2}{7} + \frac{4}{7} - \frac{8}{49} = \frac{34}{49}$$

Approach 2:

Another way to approach this would be to just find the probability of event 4 (in which both A and B fail the exam) and subtract it from $P(S)$ (which is 1) to get the same answer:

$$P(A \cup B) = 1 - P(A^c) \cdot P(B^c) = 1 - \frac{5}{7} \cdot \frac{3}{7} = \frac{49}{49} - \frac{15}{49} = \frac{34}{49}$$

Question 2

Historical data shows that it has only rained 5 days per year in some desert region (assuming a 365 day year). A meteorologist predicts that it will rain today. When it actually rains, the meteorologist correctly predicts rain 90% of the time. When it doesn't rain, the meteorologist incorrectly predicts rain 10% of the time. Find the probability that it will rain today.

In this question, the probability of rain tomorrow depends on whether or not it rains today. We define the following events:

1. Event R: It rains today. $P(R) = \frac{5}{365} = \frac{1}{73}$
2. Event R^c: It doesn't rain today. $P(R^c) = \frac{360}{365} = \frac{72}{73}$
3. Event M: The meteorologist predicted it will rain today:
   - $P(M \mid R) = \frac{9}{10}$
   - $P(M \mid R^c) = \frac{1}{10}$.

Now we want to find the value of $P(R \mid M)$:

$$P(R \mid M) = \frac{P(R) \cdot P(M \mid R)}{P(R) \cdot P(M \mid R) + P(R^c) \cdot P(M \mid R^c)} = \frac{\frac{9}{730}}{\frac{9}{730} + \frac{72}{730}} = \frac{9}{81} = \frac{1}{9}$$