Self Balancing Robot: LQR Control

By E-Yantra.

PID Controller Advantages

- We do not need the mathematical model of the plant to implement it.
- Easy to understand and use.

PID Controller Disadvantages.

- Works only for SISO type systems.
- Not Robust: often fails under ~~rough~~ / high demand working conditions.

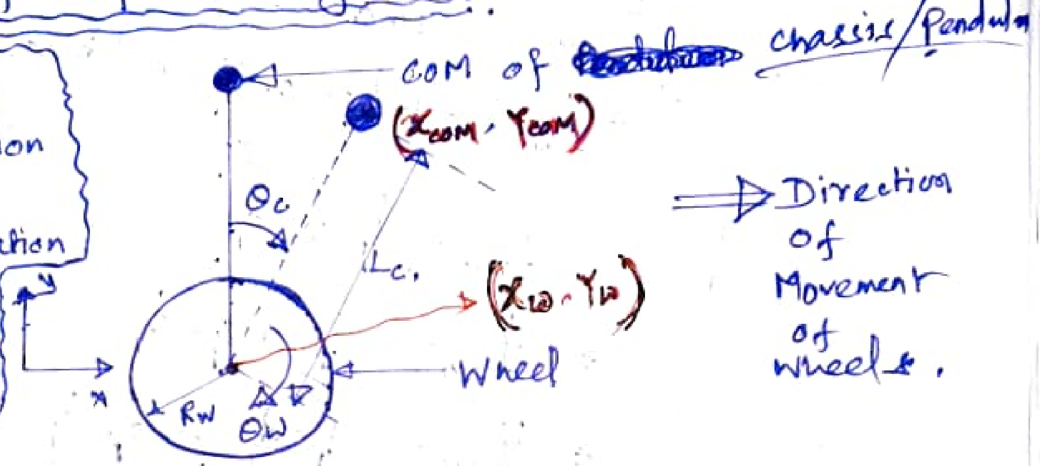
~~At~~ Deriviation of Self-Balancing Robot

r_w = radius of Wheel

θ_c = angle of rotation of pendulum

θ_w = angle of rotation of wheel.

L_c = Distance of COM from wheel centre.



- State Vector, $X(t) = [\theta_c, \dot{\theta}_c, \theta_w, \dot{\theta}_w]$

- Coordinates of COM $= \begin{bmatrix} x_{com} \\ y_{com} \end{bmatrix} = \begin{bmatrix} r_w\theta_w + L_c \sin\theta_c \\ L_c \cos\theta_c \end{bmatrix}$

- Coordinates of Wheel $= \begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} r_w\theta_w \\ 0 \end{bmatrix}$

- Now we will try to Formulate the ~~[crossed out]~~ Lagrangian Function: $\mathcal{L} = KE - PE$.

$$L = KE - PE$$

M_c = Mass of body/pendulum com
V_c = Linear velocity of pendulum COM
I_c = Moment of Inertia of pendulum ~~com abt~~ COM abt wheel centre.
$\dot{\theta}_c$ = Angular velocity of com of pendulum/body.
M_w = mass of wheel/cart
V_w = Linear velocity of wheel/cart

$$= (KE_c + KE_w) - (PE_c + PE_w)$$

$$= (KE_c + \underbrace{2KE_w}) - (PE_c + \underbrace{2PE_w})$$

Because there are 2-wheels.

KE part of the Equation

$$\Rightarrow L = \underbrace{\frac{1}{2} M_c V_c^2}_{\text{Translational KE of body/ Pendullum COM}} + \underbrace{\frac{1}{2} I_c (\dot{\theta}_c)^2}_{\text{Rotational KE of body/pendulum COM}} + 2\left(\underbrace{\frac{1}{2} M_w \cdot V_w^2}_{\text{Translational KE of wheel}} + \underbrace{\frac{1}{2} I_w \cdot \dot{\theta}_w^2}_{\text{Rotational KE of wheel.}}\right)$$

I_w = Moment of Inertia of wheel.
$\dot{\theta}_w$ = Angular velocity of wheel/cart.

$$- \left(\underbrace{M_c g y_{com}}_{\text{P.E of COM of body /pendulum.}} + \right.$$ PE of wheels will be zero because wheels will always stay on the ground.)

$$\Rightarrow L = \underbrace{\left(\frac{1}{2} M_c V_c^2 + \frac{1}{2} I_c (\dot{\theta}_c)^2 + M_w V_w^2 + I_w \dot{\theta}_w^2\right)}_{KE} - \underbrace{(M_c g y_{com})}_{PE}$$

Now, V_c has 2 components : $V_c = \dot{x}_{com} + \dot{y}_{com}$

$$\Rightarrow V_c^2 = r_w \dot{\theta}_w^2 + L_c^2 \dot{\theta}_c^2 + 2R_w L_c \dot{\theta}_w \dot{\theta}_c \cos\theta_c$$

$$\Rightarrow L = \underbrace{\left\{\frac{1}{2} M_c \left(R^2 \dot{\theta}_w^2 + L_c^2 \dot{\theta}_c^2 + 2R_w L_c \dot{\theta}_w \dot{\theta}_c \cos\theta_c\right) + \frac{1}{2} I_c \dot{\theta}_c^2 + M_w \cdot R_w^2 \cdot \dot{\theta}_w^2 + I_w \cdot \dot{\theta}_w^2\right\} - M_c g L_c \cos\theta_c}_{\text{Lagrangian Expression written in form of STATE Variables } x(t) = [\theta_c, \theta_w, \dot{\theta}_c, \dot{\theta}_w]}$$

- Our next objective is to find the Euler-Lagrange Equation : F or $\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$ where,

F = constraint force in case of linear motion

τ = Torque in case of rotational motion

q = position vector

$\dot{q}$ = velocity vector.

~~Step 1:~~ $\cancel{\frac{\partial L}{\partial \theta_w}}$ Step 1

- First of all we will find partial derivatives of the Lagrangian Expression w.r.t each of the state variables i.e $\theta_c, \dot{\theta}_c, \theta_w, \dot{\theta}_w$

- $\frac{\partial L}{\partial \theta_w} = 0$ (There are no terms which contain θ_w in the equation. Hence every term of the equation is counted as a constant.)

- $\frac{\partial L}{\partial \theta_c} = -M_c L_c \left(R_w \dot{\theta}_w \dot{\theta}_c - g\right) \sin\theta_c = -M_c L_c R_w \dot{\theta}_w \dot{\theta}_c \sin\theta_c + M_c L_c g \sin\theta_c$

- $\frac{\partial L}{\partial \dot{\theta}_w} = 2\dot{\theta}_w \left(I_w + M_w R_w^2 + \frac{M_c R_w^2}{2}\right) + M_c L_c R_w \dot{\theta}_c \cos\theta_c$

- $\frac{\partial L}{\partial \dot{\theta}_c} = \dot{\theta}_c \left(I_c + M_c L_c^2\right) + M_c L_c R_w \dot{\theta}_w \cos\theta_c$

Step 2

• Next we need to calculate

① $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_w}\right)$ ② $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_c}\right)$

$\ddot{\theta}_c$ = Angular acceleration of the pendulum body

$\ddot{\theta}_w$ = Angular acceleration of the wheels (each)

• $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_w}\right) = 2\ddot{\theta}_w\left(I_w + M_w R_w^2 + \frac{M_c R_w^2}{2}\right) + M_c L_c R_w\left(\ddot{\theta}_c \cos\theta_c - \dot{\theta}_c^2 \sin\theta_c\right)$$

• $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_c}\right) = \ddot{\theta}_c\left(J_c + M_c L_c^2\right) + M_c L_c R_w\left(\ddot{\theta}_w \cos\theta_c - \dot{\theta}_w \dot{\theta}_c \sin\theta_c\right)$$

Step 3

Assembling the Euler-Lagrange Equation

• $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_w}\right) - \frac{\partial L}{\partial \theta_w} = T_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_c}\right) - \frac{\partial L}{\partial \theta_c} = -T_1$$

(T_1 = Torque applied on the wheels)

When the wheels rotate in forward direction, the pendulum body rotates in backward direction. Hence

Torque applied on body = − Torque applied on wheels

Note: Torque applied to the wheels T_1 will be our system input

~~Special Note:~~

~~$\ddot{\theta}_w =$~~

Deriving the State-Space Equations

• $$\ddot{\theta}_w = \frac{K_2 T_1 + K_2 K_3 \dot{\theta}_c^2 \sin\theta_c - K_3 T_2 \cos\theta_c - K_3 K_4 (0.5)\sin(2\theta_c)}{K_1 K_2 - K_3^2 \cos^2\theta_c}$$

Here we have taken the inputs (T_1, T_2) and state variables as it is, while K_1, K_2, K_3, K_4 are constant expressions.

- $$\ddot{\theta}_c = \frac{K_1 T_2 + K_1 K_4 \sin\theta_c - K_3 T_1 \cos\theta_c - K_3^2 (0.5) \dot{\theta}_c^2 \sin(2\theta_c)}{K_1 K_2 - K_3^2 \cos^2\theta_c}$$

- Now,

$$X(t) = \begin{bmatrix} \theta_c \\ \dot{\theta}_c \\ \theta_\omega \\ \dot{\theta}_\omega \end{bmatrix} \Rightarrow \dot{X}(t) = \begin{bmatrix} \dot{\theta}_c \\ \ddot{\theta}_c \\ \dot{\theta}_\omega \\ \ddot{\theta}_\omega \end{bmatrix}$$

Linearization:

- Now, for finding the equillibrium point, equate $\dot{x}(t) = 0 \Rightarrow \begin{bmatrix} \dot{\theta}_c \\ \ddot{\theta}_c \\ \dot{\theta}_\omega \\ \ddot{\theta}_\omega \end{bmatrix} = 0$

Also, for the equillibrium point, inputs should be zero. $\Rightarrow T_1 = 0$ & $T_2 = -T_1 = 0$

From the above set of assumptions the equations for $\ddot{\theta}_\omega$ & $\ddot{\theta}_c$ reduces to :

$$\ddot{\theta}_\omega = \frac{-K_3 K_4 (0.5) \sin(2\theta_c)}{K_1 K_2 - K_3^2 \cos^2\theta_c} = 0 \Rightarrow \theta_c = n\pi \quad (0, \pi, 2\pi \ldots)$$

$$\ddot{\theta}_c = \frac{K_1 K_4 \sin\theta_c}{K_1 K_2 - K_3^2 \cos^2\theta_c} = 0 \Rightarrow \theta_c = n\pi$$

Hence our equillibrium points are

$$\cancel{x(t) = \begin{bmatrix} \theta_w \\ \dot{\theta}_w \\ \theta_c \\ \dot{\theta}_c \end{bmatrix}}$$

$$x(t)_{eq} = \begin{bmatrix} \theta_c \\ \dot{\theta}_c \\ \theta_w \\ \dot{\theta}_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \pi \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Finding the Jacobians

The state equation

$$\dot{x}(t) = \begin{bmatrix} \dot{\theta}_c \\ \ddot{\theta}_c \\ \dot{\theta}_w \\ \ddot{\theta}_w \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \dot{\theta}_c}{\partial \theta_c} & \frac{\partial \dot{\theta}_c}{\partial \dot{\theta}_c} & \frac{\partial \dot{\theta}_c}{\partial \theta_w} & \frac{\partial \dot{\theta}_c}{\partial \dot{\theta}_w} \\ \frac{\partial \ddot{\theta}_c}{\partial \theta_c} & \frac{\partial \ddot{\theta}_c}{\partial \dot{\theta}_c} & \frac{\partial \ddot{\theta}_c}{\partial \theta_w} & \frac{\partial \ddot{\theta}_c}{\partial \dot{\theta}_w} \\ \frac{\partial \dot{\theta}_w}{\partial \theta_c} & \frac{\partial \dot{\theta}_w}{\partial \dot{\theta}_c} & \frac{\partial \dot{\theta}_w}{\partial \theta_w} & \frac{\partial \dot{\theta}_w}{\partial \dot{\theta}_w} \\ \frac{\partial \ddot{\theta}_w}{\partial \theta_c} & \frac{\partial \ddot{\theta}_w}{\partial \dot{\theta}_c} & \frac{\partial \ddot{\theta}_w}{\partial \theta_w} & \frac{\partial \ddot{\theta}_w}{\partial \dot{\theta}_w} \end{bmatrix}_{4\times4}}_{\text{Jacobian A } (J_A)} \times \begin{bmatrix} \theta_c \\ \dot{\theta}_c \\ \theta_w \\ \dot{\theta}_w \end{bmatrix}_{4\times1}$$

$$+ \underbrace{\begin{bmatrix} \frac{\partial \dot{\theta}_c}{\partial T_1} \\ \frac{\partial \ddot{\theta}_c}{\partial T_1} \\ \partial \dot{\theta}_w / \partial T_1 \\ \partial \ddot{\theta}_w / \partial T_1 \end{bmatrix}}_{\text{Jacobian B } (J_B)} \begin{bmatrix} T_1 \end{bmatrix}$$

$$\Rightarrow \dot{x}(t) = \begin{bmatrix} \dot{\theta}_c \\ \ddot{\theta}_c \\ \dot{\theta}_w \\ \ddot{\theta}_w \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial \ddot{\theta}_c}{\partial \theta_c} & \frac{\partial \ddot{\theta}_c}{\partial \dot{\theta}_c} & \frac{\partial \ddot{\theta}_c}{\partial \theta_w} & \frac{\partial \ddot{\theta}_c}{\partial \dot{\theta}_w} \\ 0 & 0 & 0 & 1 \\ \frac{\partial \ddot{\theta}_w}{\partial \theta_c} & \frac{\partial \ddot{\theta}_w}{\partial \dot{\theta}_c} & \frac{\partial \ddot{\theta}_w}{\partial \theta_w} & \frac{\partial \ddot{\theta}_w}{\partial \dot{\theta}_w} \end{bmatrix}}_{J_A} \times \begin{bmatrix} \theta_c \\ \dot{\theta}_c \\ \theta_w \\ \dot{\theta}_w \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ \partial \dot{\theta}_w / \partial T_1 \\ \partial \ddot{\theta}_w / \partial T_1 \end{bmatrix}}_{J_B} \times [T_1]$$

Finding the A & B matrices.

On substituting the values of equillibrium state vectors $x(t)_{eq} = [0\ 0\ 0\ 0]^T$ & $[0\ \pi\ 0\ 0]^T$ to the matrices J_A & J_B we get the Linearized A & B matrices as:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -K_3K_4/K_5 & 0 & 0 \\ 0 & K_1K_4/K_5 & 0 & 0 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 0 \\ 0 \\ K_2+K_3/K_5 \\ K_1+K_3/K_5 \end{bmatrix}$$

Note: While calculating A & B, the author changed the order of state vector matrix to $[\theta_w\ \theta_c\ \dot{\theta}_w\ \dot{\theta}_c]$.

Checking the controllability of the system:

$$C = \begin{bmatrix} B & AB & A^3B \end{bmatrix}$$

Addressing some gaps of above Derivation.

Starting Point: Euler-Lagrange Torque Equations

- $$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_w}\right) - \frac{\partial L}{\partial \theta_w}$$

$$\Rightarrow T_1 = \underbrace{2\ddot{\theta}_w\left(I_w + M_w R_w^2 + \frac{M_c R_w^2}{2}\right)}_{K_2} + \underbrace{M_c L_c R_w}_{K_3}\left(\ddot{\theta}_c \cos\theta_c - \dot{\theta}_c^2 \sin\theta_c\right)$$

- $$-T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_c}\right) - \frac{\partial L}{\partial \theta_c}$$

$$\Rightarrow -T_1 = \ddot{\theta}_c\left(I_c + M_c L_c^2\right) + M_c L_c R_w\left(\ddot{\theta}_w \cos\theta_c - \dot{\theta}_w \dot{\theta}_c \sin\theta_c\right) + M_c L_c\left(R_w \dot{\theta}_w \dot{\theta}_c - g\right)\sin\theta_c.$$

$$\Rightarrow -T_1 = \ddot{\theta}_c\left(I_c + M_c L_c^2\right) + M_c L_c R_w \ddot{\theta}_w \cos\theta_c - \cancel{M_c L_c R_w \dot{\theta}_w \dot{\theta}_c \sin\theta_c} + \cancel{M_c L_c R_w \dot{\theta}_w \dot{\theta}_c \sin\theta_c} - M_c L_c g \sin\theta_c$$

$$\Rightarrow -T_1 = \ddot{\theta}_c \underbrace{\left(I_c + M_c L_c^2\right)}_{K_1} + \underbrace{M_c L_c R_w}_{K_3} \ddot{\theta}_w \cos\theta_c - \underbrace{M_c L_c g}_{K_4} \sin\theta_c$$

Defining some constant terms for simplification

$$K_1 = M_c L_c^2 + I_c$$

$$K_2 = M_c R_w^2 + 2 M_w R_w^2 + 2 I_w$$

$$K_3 = M_c R_w L_c$$

$$K_4 = M_c L_c g$$

** $K_5 = K_1 K_2 - K_3^2 \cos^2\theta_c \approx K_1 K_2 - K_3^2$ (at small angles)

* Rewriting Equations using K constants.

$$T_1 = K_2 \ddot{\theta}_w + K_3 \left(\ddot{\theta}_c \cos\theta_c - \dot{\theta}_c^2 \sin\theta_c \right) \quad \text{— ①}$$

$$-T_1 = K_1 \ddot{\theta}_c + K_3 \ddot{\theta}_w \cos\theta_c - K_4 \sin\theta_c \quad \text{— ②}$$

* Solve the coupled system for $\ddot{\theta}_c$ and $\ddot{\theta}_w$.

from eq. ① isolate $\ddot{\theta}_w$:

$$\ddot{\theta}_w = \frac{T_1 - K_3 \ddot{\theta}_c \cos\theta_c + K_3 \dot{\theta}_c^2 \sin\theta_c}{K_2} \quad \text{— ③}$$

substituting ③ in ②:

$$-T_1 = K_1 \ddot{\theta}_c + K_3 \left[\frac{T_1 - K_3 \ddot{\theta}_c \cos\theta_c + K_3 \dot{\theta}_c^2 \sin\theta_c}{K_2} \right] \cos\theta_c - K_4 \sin\theta_c$$

multiply by K_2:

$$-K_2 T_1 = K_1 K_2 \ddot{\theta}_c + K_3 \cos\theta_c \left[T_1 - K_3 \ddot{\theta}_c \cos\theta_c + K_3 \dot{\theta}_c^2 \sin\theta_c \right] - K_4 K_2 \sin\theta_c$$

expand:

$$-K_2 T_1 = K_1 K_2 \ddot{\theta}_c + T_1 K_3 \cos\theta_c - K_3^2 \ddot{\theta}_c \cos^2\theta_c + K_3^2 \dot{\theta}_c^2 \sin\theta_c \cos\theta_c - K_4 K_2 \sin\theta_c.$$

Collect $\ddot{\theta}_c$ terms:

$$\ddot{\theta}_c \left(K_1 K_2 - K_3^2 \cos^2\theta_c \right) = -K_2 T_1 - T_1 K_3 \cos\theta_c - K_3^2 \dot{\theta}_c^2 \sin\theta_c \cos\theta_c + K_4 K_2 \sin\theta_c$$

$$\Rightarrow \ddot{\theta}_c = \frac{-K_2 T_1 - T_1 K_3 \cos\theta_c + K_4 K_2 \sin\theta_c - K_3^2 \cdot \dot{\theta}_c^2 \sin\theta_c \cos\theta_c}{K_1 K_2 - K_3^2 \cos^2\theta_c}$$

$$\Rightarrow \boxed{\ddot{\theta}_c = \frac{-K_2 T_1 - T_1 K_3 \cos\theta_c + K_4 K_2 \sin\theta_c - K_3^2 \cdot \dot{\theta}_c^2 \sin\theta_c \cos\theta_c}{K_5}}$$

Similarly, $\ddot{\theta}_w$ from ②

$$\ddot{\theta}_w = \frac{-T_1 - K_1 \ddot{\theta}_c + K_4 \sin\theta_c}{K_3 \cos\theta_c}$$

Substitute value of $\ddot{\theta}_c$:

$$\boxed{\ddot{\theta}_w = \frac{K_1 T_1 + K_3 T_1 \cos\theta_c - K_1 K_4 \sin\theta_c + K_2^2 \dot{\theta}_c^2 \sin\theta_c \cos\theta_c}{K_5 \cos\theta_c}}$$

* First Deriviative of State Variable θ_c & θ_w

$$\boxed{\dot{\theta}_c = \dot{\theta}_c}$$

$$\boxed{\dot{\theta}_w = \dot{\theta}_w}$$

* Linearization

Currently, the derived equations for $\dot{\theta}_c$, $\ddot{\theta}_c$, $\dot{\theta}_w$, $\ddot{\theta}_w$ are all non-linear in nature. This non-linearity is introduced in these equations due to the $\sin(\theta_c)$ and $\cos\theta_c$ terms, as well as non-linear coupling terms like $(\dot{\theta}_c)^2 \sin\theta_c$.

In order to use these equations for state-space modelling and apply linear control techniques (like PID and LQR), we must neutralize the effect of these non-linearity inducing terms and transform the overall equations into linear form.

To achieve this, we assume the operating point is near the equillibrium where $\theta_c \to 0$ (the robot stands upright). Under this small-angle approximation:

- $\sin\theta_c \approx \theta_c$
- $\cos\theta_c \approx 1$
- $(\dot{\theta}_c)^2 \times \sin\theta_c \times \cos\theta_c \approx 0$ (second and higher order terms are negligible)

Applying these approximations, the linearized equations becomes:

$$\boxed{\ddot{\theta}_c = \frac{K_2 K_4}{K_5}\theta_c - \frac{K_2 + K_3}{K_5} T_1}$$

$$\boxed{\ddot{\theta}_w = \frac{-K_1 K_4}{K_5}\theta_c + \frac{K_1 + K_3}{K_5} T_1}$$

These equations are now purely linear, allowing us to use linear controllers.

* Derivation of the state space equations

$$\dot{X}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_c \\ \ddot{\theta}_c \\ \dot{\theta}_w \\ \ddot{\theta}_w \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{K_2 K_4}{K_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_1 K_4}{K_5} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_c \\ \dot{\theta}_c \\ \theta_w \\ \dot{\theta}_w \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{K_2 + K_3}{K_5} \\ 0 \\ \frac{K_1 + K_3}{K_5} \end{bmatrix} [T_1]$$

$\dot{\theta}_c$ = Chasis angular velocity (body tilt rate)
$\ddot{\theta}_c$ = Chasis angular accelaration (body tilt accelaration)
$\dot{\theta}_w$ = Wheel angular velocity (wheel speed)
$\ddot{\theta}_w$ = wheel angular acceleration (wheel accelaration

$$\left.\begin{aligned} x_1 &= \theta_c \\ x_2 &= \dot{\theta}_c \\ x_3 &= \theta_w \\ x_4 &= \dot{\theta}_w \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= \dot{\theta}_c = x_2 \\ \dot{x}_2 &= \ddot{\theta}_c \\ \dot{x}_3 &= \dot{\theta}_w = x_4 \\ \dot{x}_4 &= \ddot{\theta}_w \end{aligned}$$

Assume: $C = I$ (Identity Matrix) & $D = 0$ & $Y(t) = X(t)$

$$Y(t) = C \cdot X(t) + D\,U(t)$$

$$\Rightarrow Y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_c \\ \dot{\theta}_c \\ \theta_w \\ \dot{\theta}_w \end{bmatrix} + [0][T_1]$$

Deriviation of the Transfer Function for application of the Basic PID Controller.

* **Linearized Torque Equations**

$$T_1 = K_2 \ddot{\theta}_w + K_3 \ddot{\theta}_c \quad \text{— (1)}$$

$$-T_1 = K_1 \ddot{\theta}_c + K_3 \ddot{\theta}_w - K_4 \theta_c \quad \text{— (2)}$$

$\left.\begin{array}{l}\text{Reference} \\ \text{Input}\end{array}\right\} = \theta_{c,ref} = 0$

$\left.\begin{array}{l}\text{Plant Input} \\ \text{or Control} \\ \text{Signal}\end{array}\right\} = T_1 = \text{Motor Torque}$

$\left.\begin{array}{l}\text{Plant output} \\ \text{(Feedback)}\end{array}\right\} = \theta_c = \text{Chasis Tilt Angle.}$

* **Eliminate θ_w from the Equations**

$$(1) \Rightarrow \ddot{\theta}_w = \frac{T_1 - K_3 \ddot{\theta}_c}{K_2}$$

$$\Rightarrow (2) \Rightarrow -T_1 = K_1 \ddot{\theta}_c + K_3 \left[\frac{T_1 - K_3 \ddot{\theta}_c}{K_2}\right] - K_4 \theta_c$$

$$\Rightarrow -T_1 = K_1 \ddot{\theta}_c + \frac{K_3 T_1 - K_3^2 \ddot{\theta}_c}{K_2} - K_4 \theta_c$$

$$\Rightarrow -K_2 T_1 = K_1 K_2 \ddot{\theta}_c + K_3 T_1 - K_3^2 \ddot{\theta}_c - K_4 \theta_c K_2$$

$$\Rightarrow -K_2 T_1 - K_3 T_1 = K_1 K_2 \ddot{\theta}_c - K_3^2 \ddot{\theta}_c - K_4 \theta_c K_2$$

$$\Rightarrow (-K_2 - K_3) T_1 + K_4 \theta_c K_2 = (K_1 K_2 - K_3^2) \ddot{\theta}_c$$

$$\Rightarrow K_2 K_4 \theta_c - (K_2 + K_3) T_1 - K_5 \ddot{\theta}_c = 0$$

$$\Rightarrow \frac{K_2 K_4}{K_5} \theta_c - \left(\frac{K_2 + K_3}{K_5}\right) T_1 = \ddot{\theta}_c$$

* Applying Laplace Equation

$$s^2\theta_c(s) = \frac{K_2K_4}{K_5}\theta_c(s) - \frac{K_1+K_3}{K_5}T_1(s)$$

$$\Rightarrow s^2\theta_c(s) - \frac{K_2K_4}{K_5}\theta_c(s) = -\frac{K_1+K_3}{K_5}T_1(s)$$

$$\Rightarrow \theta_c(s)\left(s^2 - \frac{K_2K_4}{K_5}\right) = -\frac{K_1+K_3}{K_5}T_1(s)$$

$$\Rightarrow \boxed{\frac{\theta_c(s)}{T_1(s)} = \frac{-\frac{K_1+K_3}{K_5}}{s^2 - \frac{K_2K_4}{K_5}}}$$