Control Engineering
Experiment - 5

State feedback control for a linearised multi-state system

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Objective

To set up a simple gain scheduling adaptive state feedback scheme for the inverted pendulum system.
To generate the control signal by full state feedback to try and realise a dual control objective:
a. Both rotary variables must be stable, critically damped at a common eigenvalue of multiplicity two.
b. Both translatory variables must be stable, critically damped at a common eigenvalue of multiplicity two.

Our System

In its simplest form, the inverted pendulum consists of a carriage of mass M that can move on wheels of friction coefficient k - its horizontal displacement denoted as variable y. On the carriage is a horizontally oriented pivot, on which a solid arm of mass m (with the centre of mass at a distance L from the pivot; the net moment of inertia I) can rotate - its angle to the vertical at the pivot denoted as variable θ.

Where

M = mass of the cart

m = pendulum mass

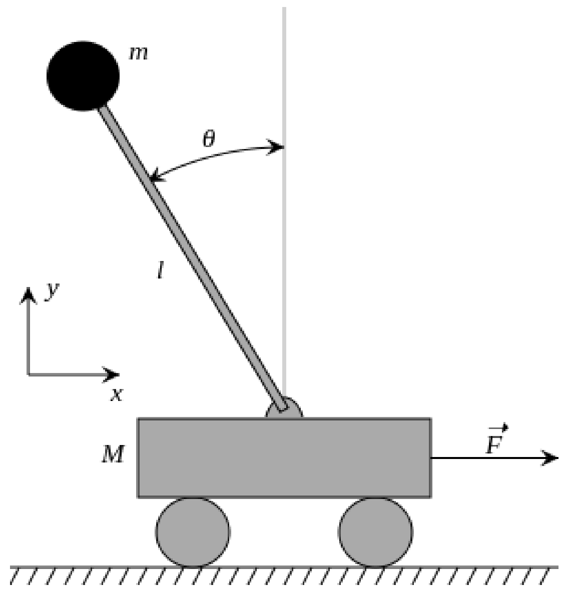
K = coefficient of friction

I = moment of inertia of the pendulum

l = length of pendulum com

g = acceleration due to gravity

We assume the inverted pendulum setup to be provided with a standard set of parameters, that is: M = 2 kg, m = 0.1 kg, L = 0.5 m, I = 0.0333 kg-m2 , k = 0.01 N-s/m



The equations of our system are :

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)} \cdot \left[\begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix}\right] \cdot \begin{bmatrix} mgL\sin\theta \\ F + mL\,\dot{\theta}^2 \sin\theta - k\,\dot{y} \end{bmatrix}$$

where $\Delta(\theta) = (I + mL^2)(m+M) - m^2L^2\cos^2\theta$

With full state feedback as

$$u = -\begin{bmatrix} k_\theta & k_{\theta'} & k_y & k_{y'} \end{bmatrix} \begin{bmatrix} \theta \\ \theta' \\ y \\ y' \end{bmatrix}$$

We use Taylor's series to linearise the system to get the following equations in terms of A, B, C, D. Here X denotes the state vector [θ, θ', y, y']. Also, note capital Y denotes the output vector and is different from small y which denotes displacement.

X' = AX + Bu

Y = CX + Du

Linearised about a general point $x_0 = [\, x_1, x_2, x_3, x_4 \,]$ and $u_0 = F$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{23} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & 0 & a_{44} \end{bmatrix}$$

$$a_{21} = \frac{\left(-m^2l^2sin(2x_1)\right).\left[(m+M)mglsin(x_1) - mlcos(x_1)\left(F + mlx_2^2sin(x_1) - kx_4\right)\right]}{\left(\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)\right)^2} + \frac{(m+M)mglcos(x_1) - m^2l^2x_2^2cos(2x_1)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)}$$

$$a_{22} = \frac{-m^2l^2sin(2x_1)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)}$$

$$a_{24} = \frac{-kmlcos(x_1)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)}$$

$$a_{41} = \frac{\left(-m^2l^2sin(2x_1)\right).\left[\left(-m^2l^2sin(x_1)cos(x_1)\right) + \left(I+ml^2\right)\left(F + mlx_2^2sin(x_1) - kx_4\right)\right]}{\left(\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)\right)^2} + \frac{-m^2l^2gcos(2x_1) + \left(I+ml^2\right)mlx_2^2cos(x_1)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)}$$

$$a_{42} = \frac{2mlx_2\left(I+ml^2\right)sin(x_1)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)}$$

$$a_{44} = \frac{-k\left(I + ml^2\right)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)}$$

$$B = \begin{bmatrix} 0 \\ \dfrac{-mlcos(x_1)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)} \\ 0 \\ \dfrac{-k\left(I+ml^2\right)}{\left(I+ml^2\right)(M+m) - m^2l^2cos^2(x_1)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad and \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Gain Scheduling

Gain scheduling is used to control nonlinear systems and is generally used when a single set of gains cannot provide the desired results and stability to the system. The system dynamics can be changed from one operating point to another(depending upon condition) using this technique of gain scheduling.

The way gain scheduling works is that we have a set of multiple gain values which can be used in the control system according to the operating point. For different ranges of operating points, we apply different gains which help the system to behave optimally.

Plots and Analysis

We linearise the above equation about different values of θ. Also note that the above terms of A, B, C, D are independent of displacement y (or state x_3) thus we do not have a variation of displacement y.

Case 1: $\theta = 0^0$ i.e. x_0 = [0 , 0 , 0 , 0] and u_0= 0

A =

0	1.00	0	0
8.58	0	0	0.0042
0	0	0	1.00
-0.2043	0	0	-0.0049

B =

[0
-0.4169
0
0.4861]

Rank of Controllability matrix =4

Rank of Observability matrix =4

Eigenvalues / pole placement at(for critically damped) = [-1.2, -1.2, -1.5, -1.5]

Gain matrix by pole placement= [-47.6254 -15.7264 -0.7930 -2.3890]

Gain by LQR control = [-265.8088 -135.1327 -10.0000 -24.5144]

Eigenvalues by LQR control = [-0.8269 + 0.0000i, -0.7132 + 0.8142i, -0.7132 - 0.8142i, -42.1729 + 0.0000i]

Zeta = [1.0000 0.6589 0.6589 1.0000]

Case 2: $\theta = 30^0$ i.e. x_0 = [2pi/12 , 0 , 0 , 0] and u_0= 0

A =

0	1.00	0	0
7.3153	0	0	0.0036
0	0	0	1.00
-0.1761	0	0	-0.0048

B =

[0
-0.3592
0
0.4836]

Rank of Controllability matrix =4

Rank of Observability matrix =4

Eigenvalues / pole placement at(for critically damped) = [-1.2, -1.2, -1.5, -1.5]

Gain matrix = [-51.9312 -18.7714 -0.9253 -2.7858]

Gain by LQR control = [-269.6804 -141.0329 -10.0000 -24.7376]

Eigenvalues by LQR control = [-0.8238 + 0.0000i, -0.7183 + 0.8066i, -0.7183 - 0.8066i, -36.4380 + 0.0000i]

Zeta = [1.0000 0.6650 0.6650 1.0000]

Case 4: $\theta = 45^0$ i.e. x_0 = [3pi/12 , 0 , 0 , 0] and u_0= 0

A =

0	1.00	0	0
5.88	0	0	0.003
0	0	0	1.00
-0.2103	0	0	-0.0049

B =

[0
-0.2918
0
0.4811]

Rank of Controllability matrix =4
Rank of Observability matrix =4
Eigenvalues / pole placement at(for critically damped) = [-1.2, -1.2, -1.5, -1.5]
Gain matrix = [-59.3688 -24.1726 -1.1450 -3.4449]
Gain by LQR control = [-265.8088 -135.1327 -10.0000 -24.5144]
Eigenvalues by LQR control= [-0.8192 + 0.0000i, -0.7276 + 0.7945i, -0.7276 - 0.7945i, -29.7602 + 0.0000i]
Zeta = [1.0000 0.6754 0.6754 1.0000]

Case 5: $\theta = 60^0$ i.e. x_0 = [4pi/12 , 0 , 0 , 0] and u_0= 0

A =

0	1.00	0	0
4.0939	0	0	0.0020
0	0	0	1.00
-0.1021	0	0	-0.0040

B =
[0
-02052
0
0.4784]
Rank of Controllability matrix =4
Rank of Observability matrix =4
Eigenvalues / pole placement at(for critically damped) = [-1.2, -1.2, -1.5, -1.5]
Gain matrix = [-34.1193 -15.1258 -0.1022 -0.6462]
Gain by LQR control = [-293.7468 -175.0499 -10.0000 -25.8463]
Eigenvalues by LQR control = [-0.8098 + 0.0000i -0.7489 + 0.7680i -0.7489 - 0.7680i -21.2546 + 0.0000i]
Zeta = [1.0000 0.698 0.6982 1.0000]

Case 6: $\theta = 90^0$ i.e. x_0 = [6pi/12, 0 , 0 , 0] and u_0= 0

A =

0	1.00	0	0
0.0	0	0	0.00

0	0	0	1.00
-0.2061	0	0	-0.004

B =

[0
-0.4021
0
0.4854]

Rank of Controllability matrix =2

Rank of Observability matrix =4

Eigenvalues / pole placement at(for critically damped) = [-1.2, -1.2, -1.5, -1.5]

Gain matrix = [-48.5964 -16.4095 -0.8233 -2.4798]

Here clearly the system is not controllable as can be seen from the rank of controllability matrix.

Observation and Discussion

In the below table we observe that we require pole placement closer to zero to gat valid results for higher range of theta. Also the magnitude of gain required increases for the states x_1, x_2 ,x_4 as the range of theta increases i.e. more gain is required to stabilize the system.

Range of θ	Eigenvalues / pole placement at	K matrix by pole placement	EigenValues for LQR control	K matrix by LQR control
Θ = 0 to 30 (Θ = 15)	[-1.2, -1.2, -1.5, -1.5]	[-48.5964 -16.4095 -0.8233 -2.4798]	[-0.8262 + 0.0000i, -0.7143 + 0.8124i, -0.7143 - 0.8124i, -40.7047 + 0.0000i]	[-266.6617 -136.4695 -10.0000 -24.5666]
Θ = 30 to 60 (Θ = 45)	[-1.2, -1.2, -1.5, -1.5]	[-59.3688 -24.1726 -1.1450 -3.4449]	[-0.8192 + 0.0000i, -0.7276 + 0.7945i, -0.7276 - 0.7945i, -29.7602 + 0.0000i]	[-265.8088 -135.1327 -10.0000 -24.5144]
Θ = 60 to 90 (Θ = 75)	[-.5, -.5, -.9, -.9]	[-47.7118 -32.0324 -0.1990 -1.2483]	[-0.7815 + 0.0000i, -0.8025 + 0.6813i, -0.8025 - 0.6813i, -11.7469 + 0.0000i]	-352.0403 -260.2533 -10.0000 -28.1390]