Control Engineering
Experiment - 6

Features of a linearised multi-state system

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Aim of the experiment

To study the linear stability, controllability, and observability, for a given nonlinear multi-state system as linearised about different operating points covered by the state/input vector space

Theory

Eigenvalues and Stability

We say that the linear state-space model of the system is stable if all eigenvalues of the matrix Are having negative real parts to complex number eigenvalues. If the condition is satisfied then the system is stable, and that any initial condition converges exponentially to a stable attracting point. If any real parts are zero then the system will not converge to a point. Similarly, if the eigenvalues are positive the system is unstable and will exponentially diverge.

Controllability

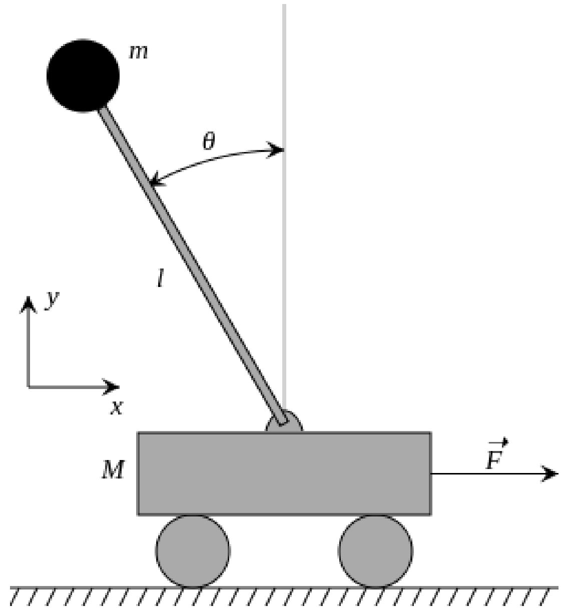
We know that a system is said to be controllable at time t if it is possible by means of a control vector to transfer the system from any initial state to any other desired state in a finite interval of time. We say that the system is controllable if every state of the system is controllable. Essentially on MATLAB, we will compare the rank of the Controllability Matrix and the system's "A matrix" to find if the system is controllable and also to find the number of uncontrollable states.

Observability

At time t0, a system is said to be observable if it is possible to determine its state from the output over a finite time interval with the system in state x(t0). We say that the system is controllable if every state of the system is controllable. Essentially on MATLAB, we will compare the rank of the Observability Matrix and the system's "Amatrix '' to find if the system is Observable and also to find the number of unobservable states.

Our System

In its simplest form, the inverted pendulum consists of a carriage of mass M that can move on wheels of friction coefficient k - its horizontal displacement denoted as variable y. On the carriage is a horizontally oriented pivot, on which a solid arm of mass m (with the centre of mass at a distance L from the pivot; a net moment of inertia I) can rotate - its angle to the vertical at the pivot denoted as variable θ.



The control objective is to keep the carriage in motion (changing y) by a horizontal force F so that the bar is maintained vertically at the pivot (θ maintained at zero) without any physical support by contact! The well known nonlinear dynamic equations of the system are given by (note that both elements in the LHS vector are double derivatives in time)
Now we will linearize the equations about the vertically upward equilibrium position, $\theta = 0^{o}$, and assume that the system stays within a small angle of this equilibrium. For this assumption to be valid the pendulum should not deviate more than 20 degrees from the vertically upward position. Let Φ represent the deviation of the pendulum's position from equilibrium, that is, $\Theta = \Phi$, presuming a small deviation (Φ) from equilibrium, we can use the following small-angle approximations of the nonlinear functions in our system equations:

$$cos\,\theta \;=\; cos(\,\Phi\,) \;\;\simeq\;\; 1$$

$$sin\,\theta \;=\; sin(\,\Phi\,) \;\;\simeq\;\; \Phi$$

$$\theta'^{\,2} \;=\; \Phi'^{\,2} \simeq 0$$

After substituting the above approximations into our nonlinear governing equations, we arrive at the two linearized equations of motion. Note u has been substituted for the input F.

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{y} \end{bmatrix} = \frac{1}{\Delta(\theta)} \cdot \left[\begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & I+mL^2 \end{bmatrix} \right] \cdot \begin{bmatrix} mgL\sin\theta \\ F+mL\,\dot{\theta}^2\,\sin\theta - k\,\dot{y} \end{bmatrix}$$

where $\Delta(\theta) = (I+mL^2)(m+M) - m^2L^2\cos^2\theta$

$$(I + ml^2)\ddot{\phi} - mgl\phi = ml\ddot{x}$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

Here,

M = mass of the cart

m = pendulum mass

K = coefficient of friction

I = moment of inertia of the pendulum

l = length of pendulum com

g = acceleration due to gravity

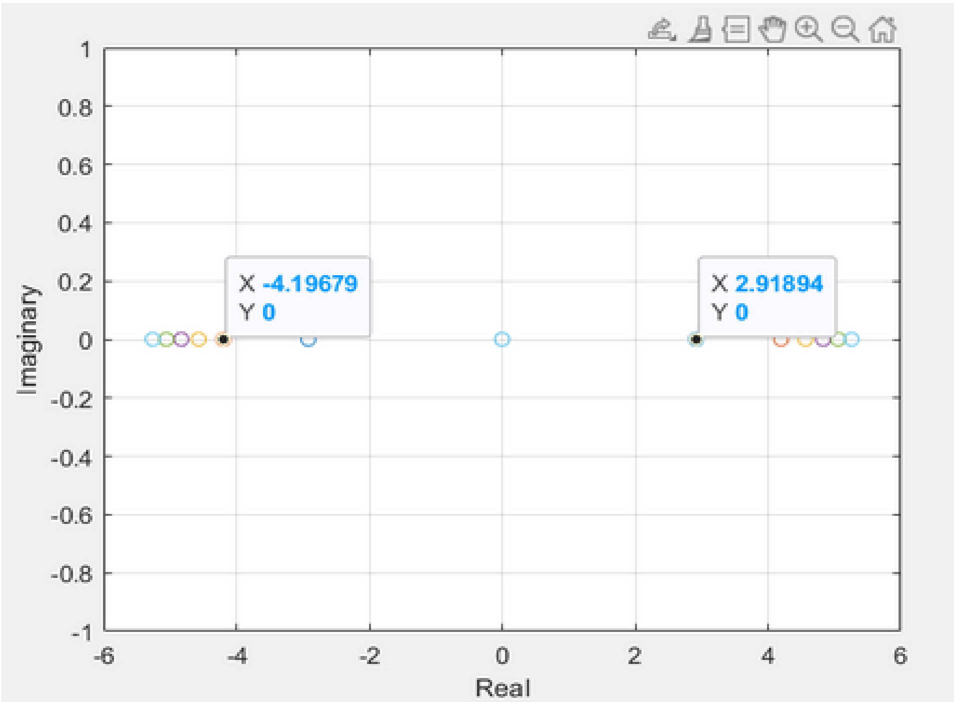
The linearized equations of motion by using the above assumption can be represented in the state-space form if they are rearranged into a series of first-order differential equations. Since the equations are linear, they can then be put into the standard matrix form shown below.

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} u$$

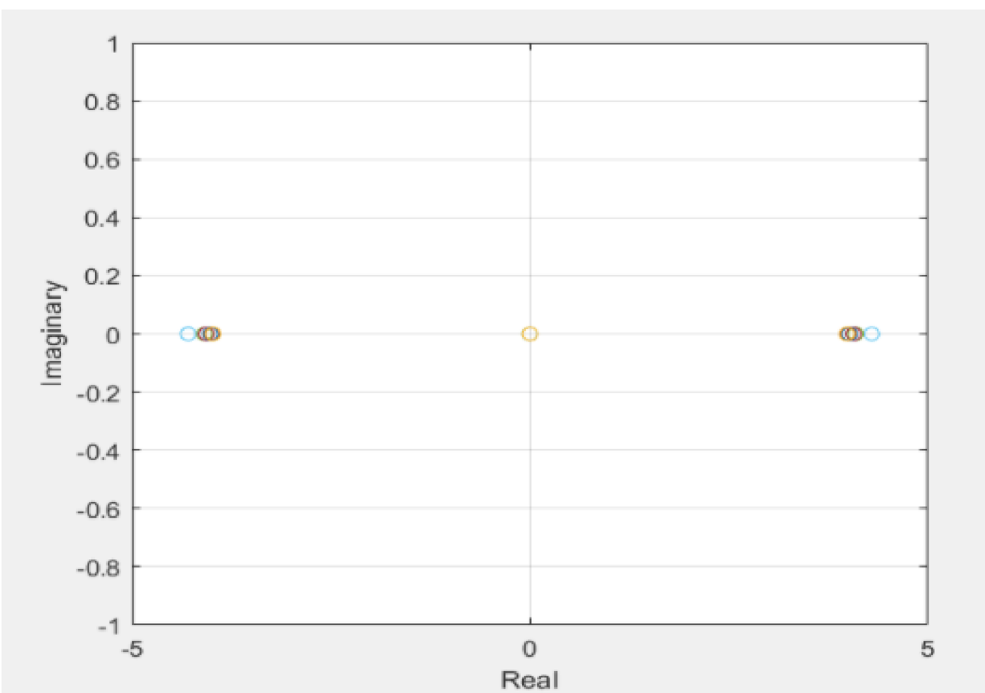
$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

Plots and Analysis

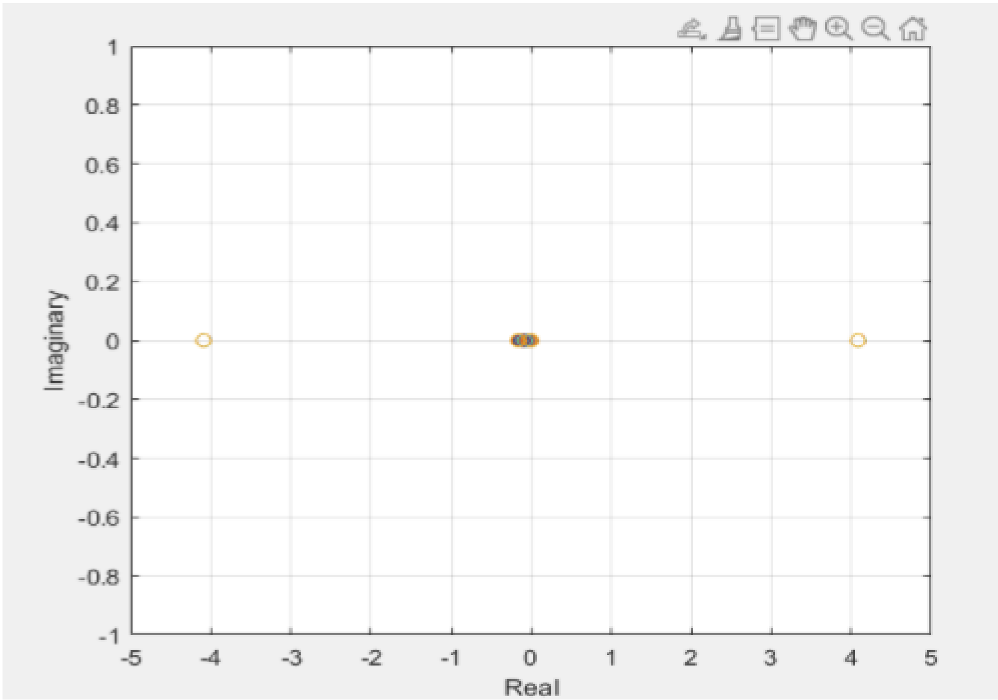
1) On increasing the value of m = 0.1:0.5:3. Two eigenvalues are close to 0(almost 0), whereas the other two eigenvalues increase.

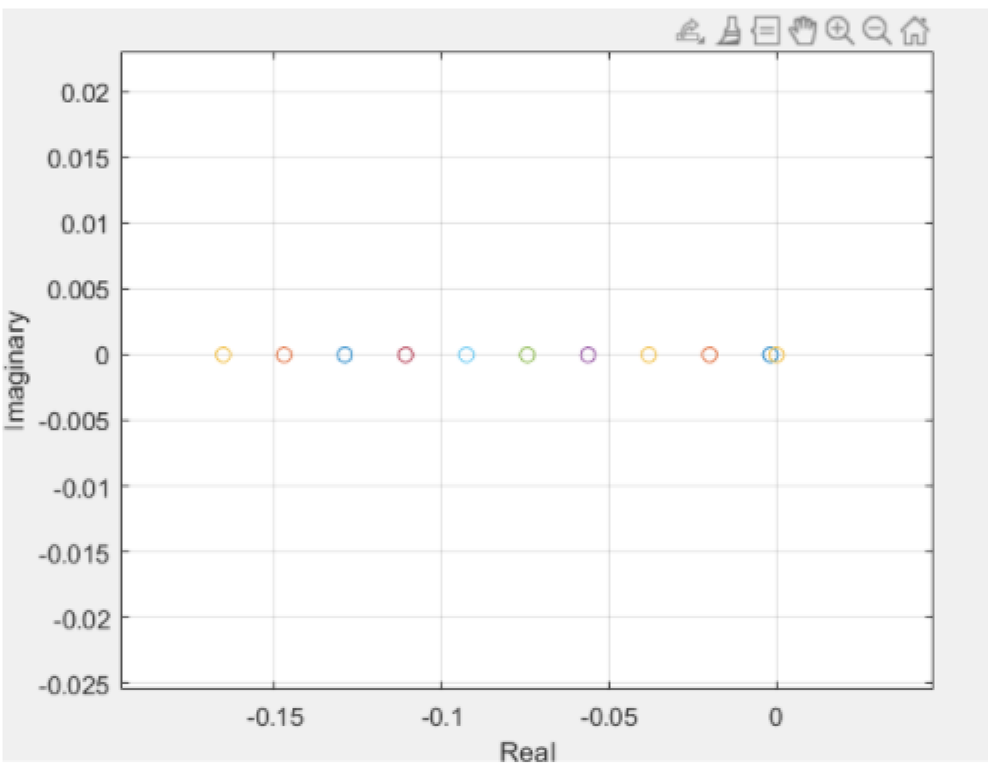


2) On increasing the value of m = 2:4:20. Minimal change in eigenvalues.



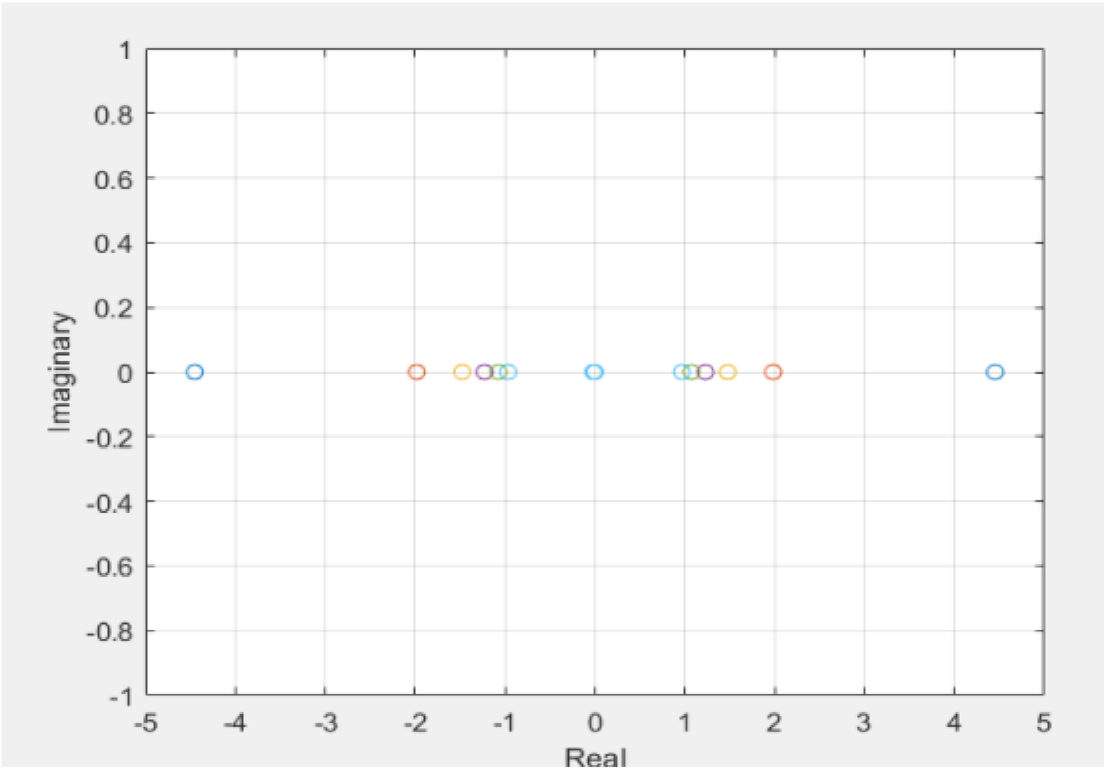
3) On increasing k = 0.01: 0.1: 1. Change in one of eigenvalues close to 0.



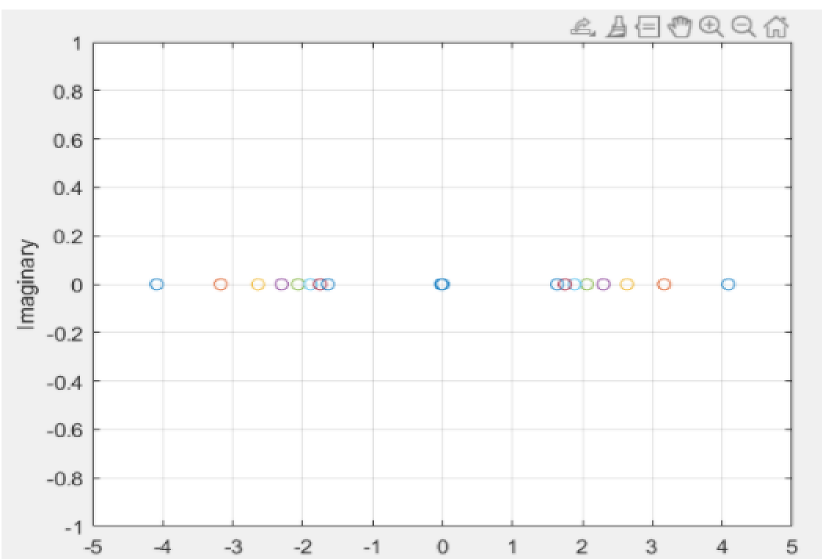


In the second magnified graph, we can observe that eigenvalues become more -ve.

4) Increasing moment of inertia I = 0.01: 0.5: 3. No change in eigenvalues at 0 and close to 0 whereas the other two eigenvalues decrease in magnitude.



5) On increasing the length **l = 0.5: 0.5: 4.** There is no change to the eigen value at 0 and close to 0 whereas the other two eigenvalues decrease in magnitude.



For

(a) M = 2;
m = 0.5;
b = 0.1; % k
I = 0.033;
l = 0.5;
g = 9.8;

Eigenvalues =

0
-0.0400
-4.2958
4.2883

Observability matrix =

1.0000	0	0	0
0	0	1.0000	0
0	1.0000	0	0
0	0	0	1.0000
0	-0.0475	1.8421	0
0	-0.0752	18.4211	0
0	0.0023	-0.0875	1.8421
0	0.0036	-0.1385	18.4211

Controllability matrix =

0	0.4752	-0.0226	1.3861
0.4752	-0.0226	1.3861	-0.1317
0	0.7519	-0.0357	13.8521
0.7519	-0.0357	13.8521	-0.7624

We also find the rank for both the matrices to be 4 which is the max rank thus the system is both controllable and observable.

(b) M = 10;

m = 2;

b = 0.3; % k

I = 0.05;

l = 0.7;

g = 9.8;

Eigen values:

0

-0.0250

-3.9812

3.9764

Observability matrix:

1.0000	0	0	0
0	0	1.0000	0
0	1.0000	0	0
0	0	0	1.0000
0	-0.0297	1.8469	0
0	-0.0404	15.8308	0
0	0.0009	-0.0549	1.8469
0	0.0012	-0.0746	15.8308

Controllability matrix:

0	0.0990	-0.0029	0.2487
0.0990	-0.0029	0.2487	-0.0148
0	0.1346	-0.0040	2.1312
0.1346	-0.0040	2.1312	-0.0734

For part (b) also, we find the rank for both the matrices to be 4 which is the max rank thus the system is both controllable and observable.

Observation and Discussion

- With an increase in m, L, I, that there is no change in the eigenvalue at 0 and close to 0, whereas the other two eigenvalues increase on increasing m but decrease on increasing L or I.
- As the point of linearisation is about θ close to 0, it is a point of unstable equilibrium. This results in at least one of the eigenvalues being positive. Whereas from the above two general cases we observe that both controllability and observability matrices are full-rank matrices(rank=4). Thus the system is both controllable and observable(considering $\theta$to be small).