First and Second Order Stochastic Dominance

Portfolio optimisation techniques are basically done on the various characteristics of the returns such as mean,covariance etc.Whereas the stochastic dominance takes the entire distribution of return into account.At first we will define the important terminology we have used in this project.

First Order Stochastic Dominance (FSD):

The First order stochastic dominance(FSD) is used when the cumulative distribution of return of portfolio(or asset) x lies above the cumulative distribution of return of another portfolio (or asset) y.

The portfolio x is preferred over portfolio y with respect to FSD if and only if one of the following conditions hold.

1. $F_{Rx}(n) \le F_{Ry}(n)$ for all $n \in \mathbf{R}$ where $F_x(n) = \mathbf{P}(x \le n)$ be the probability distribution function of random variable x.
2. For any non-decreasing and integrable utility function U, $E(U(R_x)) \ge E(U(R_y))$.
3. For discrete variables R_x and R_y with T equally probable outcomes $(z_{11}, z_{12} \dots z_{1T})$ and $(z_{21}, z_{22} \dots z_{2T})$, such that $z_{11} \le z_{12} \dots \le z_{1T}$ and $z_{21} \le z_{22} \dots \le z_{2T}$, we have $z_{1i} \ge z_{2i}$, i = 1 to T, with at least one strict inequality.

Second order Stochastic Dominance(SSD):

The Second order stochastic dominance(SSD) is used when the cumulative distribution of return of portfolio(or asset) x crosses the cumulative distribution of return of another portfolio (or asset) y.

The portfolio x is preferred over portfolio y with respect to SSD if and only if one of the following conditions hold.

1. $F_{Rx}^{2}(n) \leq F_{Ry}^{2}(n)$ for all $n \in \mathbf{R}$ where $F_{Rx}^{(2)}(n) = \int_{-\infty}^{n} F_{Rx}(s)\, ds = E((n-R_x)_+)$
2. For any non-decreasing , concave and integrable utility function U, $E(U(R_x)) \geq E(U(R_y))$.
3. For discrete variables R_x and R_y with T equally probable outcomes $(z_{11}, z_{12} \ldots z_{1T})$ and $(z_{21}, z_{22} \ldots z_{2T})$, such that $z_{11} \leq z_{12} \ldots \leq z_{1T}$ and $z_{21} \leq z_{22} \ldots \leq z_{2T}$, we have $\Sigma_j z_{1i} \geq \Sigma_j z_{2i}$, i = 1 to T, with at least one strict inequality.

Optimisation with Stochastic Dominance Constraints:

Let the total no of stocks available for selection be n.The weights of each stock chosen be $X = \{x_1, x_2, x_3 \ldots x_n\}$ where $\Sigma_{j=1:n}\, x_j = 1$.With no short selling allowed all the $x_j \geq 0$.Let the random variable for return of each stock be r_i respectively.Then the total return random variable will be $R = \Sigma_{i=1:n}\, r_i x_i$. Expected value of return of the portfolio is $E(R) = \Sigma_{i=1:n}\, E(r_i) x_i$.

As the returns belong to a particular distribution we take some T scenarios for which the realisation of the random variables for i^{th} stock in t^{th}

scenario is r_{it} with probability p_t.Then the expected value of return of the portfolio is $E(R) = \Sigma_{i=1:n} E(r_i)x_i = \Sigma_{t=1:T}\Sigma_{i=1:n} (r_{it}\, p_t)x_i$.

Here as we are simulating data from the real stock returns we consider the scenarios as the returns at each time and take all of them as equal probability.Here we are using discrete variables for all the random variables.

For comparing the dominance of the portfolio we need a benchmark portfolio for that.In our codes we took all the 30 stocks of BSE and used it's index SENSEX as the benchmark.We downloaded the data using the yahoo finance package.Let the returns of the index for T scenarios be $\{b_1,b_2,b_3,...b_T\}$.

$F^{(2)}(X;n)$ can be written as $E[(n - X)+]$.
So from the SSD dominance conditions we can write the equation for optimising portfolio as

$$\max\ (\Sigma_{t=1:T}\Sigma_{i=1:n} (r_{it}\, p_t)x_i)$$

$$\text{Subject to: } E((b-R)^+) \leq E((b-B)^+) \text{ where } b \in \{b_1,b_2,b_3,...b_T\}$$

We can convert the above equations into linear problem as given below

$$\max\ (\Sigma_{t=1:T}\Sigma_{i=1:n} (r_{it}\, p_t)x_i)$$

$$d_{tk} \geq b_k - \Sigma_{i=1:n} (r_{it}\, x_i) \quad t,k = 1,2.....T$$

$$\Sigma_{t=1:T} (d_{tk}\, p_{ti}) \leq E((b_k-B)^+) \quad k = 1,2.....T$$

$$d_{tk} \geq 0 \quad t,k = 1,2.....T$$

Similarly from the FSD dominance constraints we can write the equation for optimising portfolio as

$$\max \left(\Sigma_{t=1:T}\Sigma_{i=1:n} (r_{it}\, p_t) x_i\right)$$

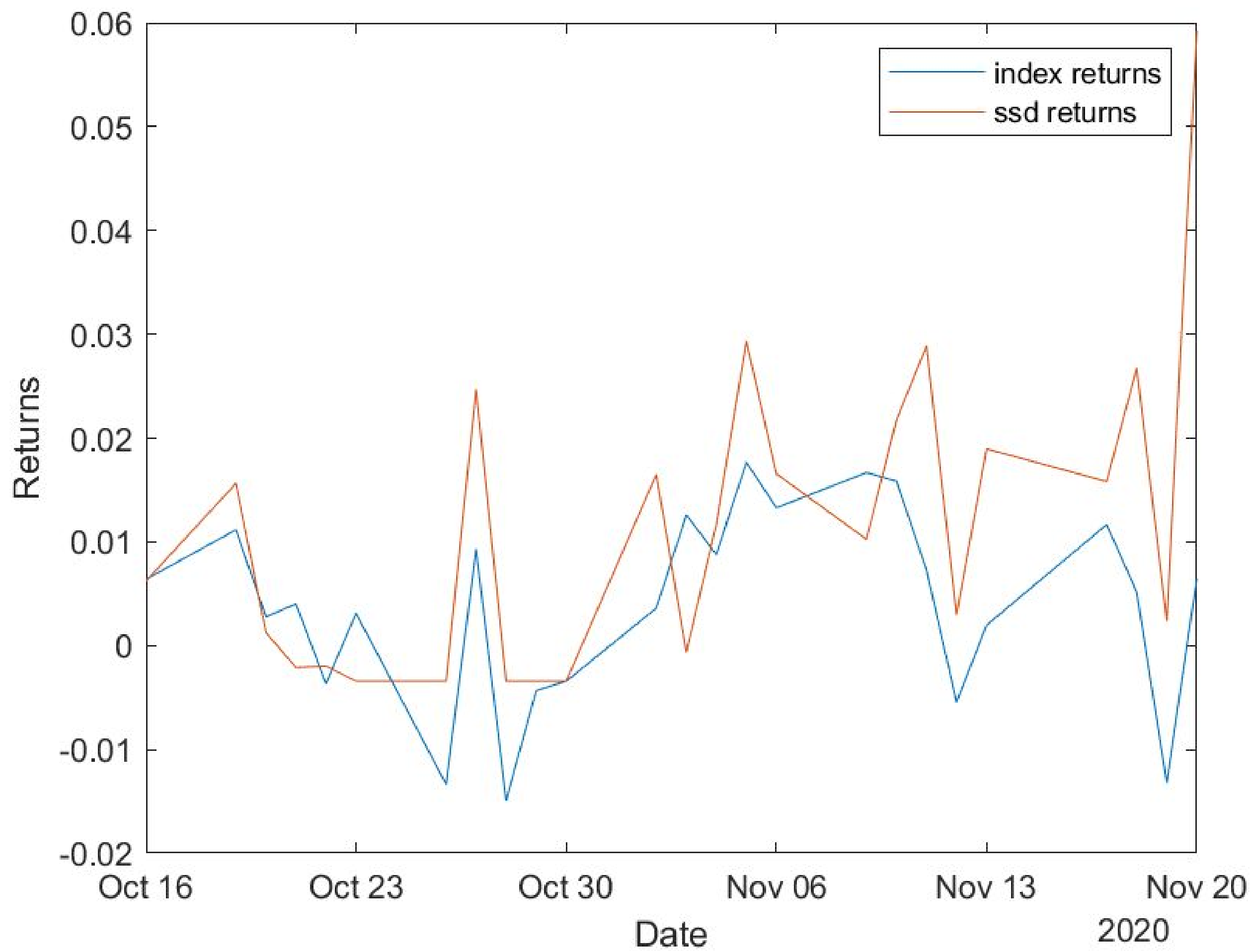
$$\text{Subject to: } F_{\mathbf{R}}(b) \leq F_{\mathbf{B}}(b) \text{ where } b \in \{b_1, b_2, b_3, \ldots b_T\}$$

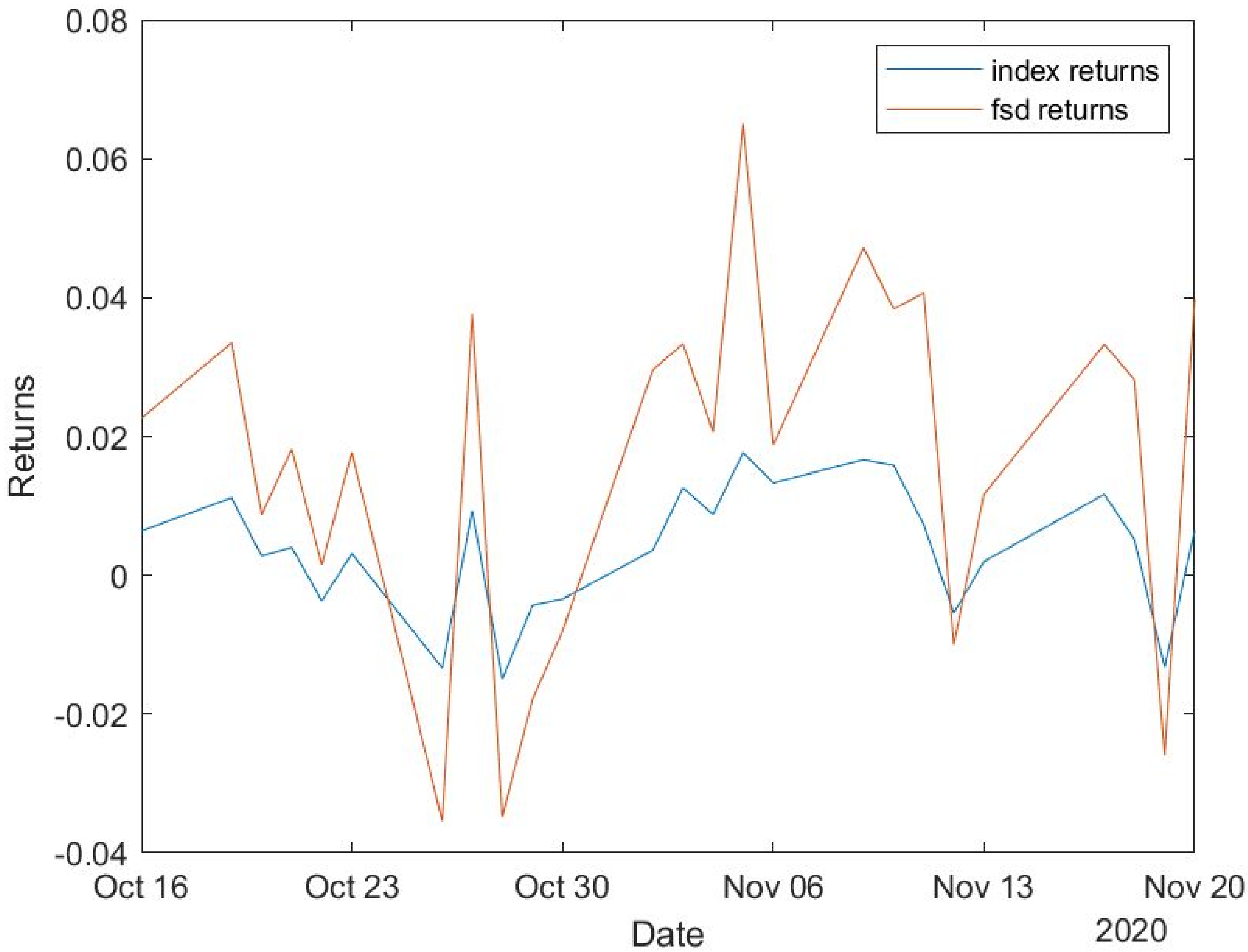
We solved the optimization problems for SSD with linear programming and FSD with non-linear programming techniques.

We collected the data from 15th Oct 2020 and optimised with the help of Optimisation Toolbox. The weights for optimised portfolio are

Infosys	0	L&T	0.006879263419
Axis Bank	0	Mahindra & Mahindra	0
Asian Paints	0	Maruti Suzuki	0
Bajaj Auto	0	Nestle	0.269672841644
Bajaj Finance	0	NTPC	0
Bajaj Finserv	0.506424979557	ONGC	0
Bharti Airtel	0.077890720099	Power Grid Corporation of India	0
HCL	0	Reliance Industries	0
HDFC Bank	0	SBI	0
HDFC	0	Sun Pharmaceutical Industries'	0

Hindustan Unilever	0	Tata Steel	0.034527024919
ICICI	0	TCS	0
IndusInd Bank	0	Tech Mahindra	0
ITC	0	Titan	0
Kotak Mahindra Bank	0.104605170360	UltraTech Cement	0





These are the 2 graphs corresponding to the optimal portfolio returns of ssd and fsd dominant to index returns.