A VCG Adaptation for Participatory Budgeting

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ABSTRACT

Participatory Budgeting is a democratic procedure in which residents of a municipal authority collectively decide via voting on the allocation of a given available budget between various public expenditures alternatives or 'projects'. We study a unique mechanism that also allows voters to specify the amount of tax they are willing to pay to fund the chosen allocation, meaning that they also decide the volume of the available budget. That mechanism is constructed as a relatively simple adaptation of a VCG mechanism to a non-monetary utility functions environment, thus it satisfies strategy proofness (in strictly dominant strategies in our case), and it is the first mechanism to achieve that under the commonly adopted assumption of additive utilities. For the special case of logarithmic utilities, we also show that prices vanish in large populations under mild assumptions on preferences.

KEYWORDS

Participatory Budgeting, Additive utilities, VCG

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1 INTRODUCTION

Participatory Budgeting (PB) is a direct-democracy process in which residents of a municipal authority vote on how to allocate some available budget among different alternatives or projects. Established in Puerto Alegre, Brazil in 1989, it has since been practiced in over 7,000 cities worldwide.[1] Being a process of aggregation of collective preferences regarding resource allocation, PB has naturally drawn attention in the Social Choice and Algorithmic Game Theory academic fields of research [2]. Different works in that literature offer various approaches to different aspects of PB, mainly:

- The settings of the procedure itself, i.e. what are the feasible allocations and how voting ballots are designed.
- Modeling of preferences.
- Objectives, e.g. welfare-maximizing, allocation fairness (and different concepts of it), incentive-compatibility etc.

Typically in practice, voters are asked to mark down projects that they approve of out of a given list, each of them assigned with a fixed cost.[2] Much of the literature is dedicated to this setup ([1, 19] among many others). In some cases though, projects can be implemented fractionally, for instance rentable bicycle facilities scattered across the city,[3] the amount of which depends on the amount of money raised. Indeed, PB sometimes allows for different degrees of implementation in each project.[4] Another possibility is that we ask voters to fully describe their preferred allocation of the city budget (or some part of it) among its different departments such as education, health, leisure activities and so on. In these cases the set of feasible allocations is continuous (or is just assumed to be, as an approximation at the limit) and it is usually referred to as '(unbounded) divisible Participatory Budgeting'. Divisible PB models also naturally assume continuous preferences over allocations. A common model assumes ℓ_1-norm preferences where the disutility of agent i in budget allocation x is $u_i(x) = \|x_i^* - x\|_1$ for some x_i^* that is the preferred allocation of agent i. Two notable works [12, 14] have introduced a welfare-maximizing and truthful PB mechanisms under ℓ_1-norm preferences. ℓ_p norms [13] or more generally single-peacked preferences [3, 18], are well studied in social choice literature and known to allow for mechanisms with strong strategyproofness guarantees. That approach assumes that the utility of an agent depends solely on the 'distance' between the accepted outcome and her favoured alternative. Indeed, participants in any democratic process are probably more *satisfied* with results that are closer to their personal opinion. For that reason, minmizing the sum of ℓ_1 disutilities over all agents is not a bad idea for a solution concept.

However, we argue that when voters have concrete measurable utility from decisions, as in PB, the distance approach may not adequately capture the preferences of a voter, as illustrated in the following example.

Example 1.1. Consider a divisible PB instance with 3 projects and a voter i with an optimal allocation $x_i^* = (2, 5, 0)$. Also let $x = (3, 4, 0)$ and $x' = (2, 4, 1)$. Then $\|x_i^* - x_1\|_1 = \|x_i^* - x_2\|_1 = 2$, meaning that under the ℓ_1 norm i is indifferent between x and x'. Now, imagine that i is a senior lady and the projects (p_1, p_2, p_3) are a park nearby her house, improving public transportation and a youth center (that she has no use of) respectively. Given that p_2 is allocated only with 4 and p_1 with 2, she would probably like it better if the 1 unit of budget left is allocated the park, because money spent there does benefit her somehow, whereas money spent on p_3 yields no value for her at all.

Meaning, ℓ_1 preference model cannot capture full preferences in general. In particular, the truthfulness of mechanisms designed for the distance model (such as in [12, 14]) no longer holds under

[1] 'The Participatory Budget Project': https://www.participatorybudgeting.org/

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[2] See for example https://pbstanford.org/cambridgepb7/approval
[3] https://www.tel-o-fun.co.il/en/
[4] See for example https://pbstanford.org/boston16internal/knapsack

more realistic assumptions on preferences. For instance, if the voter described in the above example faces an opportunity to manipulate the outcome from x' to x she is incentivised to do that, while under the distance model she is not. Explicit utility functions (that are prevalent in other economic settings) provide a better abstraction of voters' preferences. The 'additive utility' model assumes $u_i(x) = \sum_j \alpha_{i,j}\theta_j(x_j)\ \forall i$ where x_j is the amount spent on project or alternative j, θ_j is the utility from that project that is increasing in x_j, and $\alpha_{i,j}$ are scalars that vary between agents (see e.g. [4, 9] for such models in PB). [5] In that area, Fain et al. [9] offer an 'additive-approximately-truthful' (randomized) mechanism, meaning that agents might benefit from misreporting but only up to some upper bound.

In this paper we suggest a new model of divisible PB mechanism with additive utilities, where we additionally let agents decide on the level of collected taxes that determine the volume of the budget. The closest thing to that we know of, at least in PB context, is in a recent paper by Brandl et al. [6], where agents can voluntarily donate to the budget any amount of their choice. In our mechanism, the tax (that might be also negative) that is collectively decided is binding for everyone.

1.1 Contribution

Conceptually, we propose a new Participatory Budgeting environment where voters also submit their preferred level of tax, which in turn determines the total available budget.

On the technical level, we introduce a truthful mechanism in strictly dominant strategies for general additive utility functions, that is essentially an adapted VCG mechanism. That adaptation is not straight-forward, and its difficulties as well as our solutions to them are discussed below in sections 1.2 and 2.3. The strict dominance of equilibrium strategies is a significant advantage, compared to the weak dominance of truthful strategies in general VCG mechanisms, in terms of disincentivizing group manipulations.

Unlike the mechanisms in [12] and [14] designed for ℓ_1 preferences, it is not welfare-maximizing but only Pareto-efficient. However, the reason for that is that we dismiss the quite abundant assumption that utilities are normalized across agents. Roughly speaking, that is the assumption that all agents' maximum possible utility over all allocations is the same or close to that. (The ℓ_1 norm implicitly assumes that as the maximum utility for every agent is zero.) For additive utilities that translates as $\sum_j \alpha_{i,j} = 1 \quad \forall i$, and subject to that assumption our mechanism would be also welfare maximizing.

1.2 Difficulties in applying VCG to Participatory Budgeting

The Vickrey-Clarke-Groves (VCG) mechanism is a very general approach [7] that allows us to obtain both social optimality and truthfulness as a weakly dominant strategy in a very broad class of economic situations. This is by essentially charging each player with the externality she has on the other players. Thus a naïve application of VCG to the PB scenario would elicit all voters' preferences, implement the socially optimal outcome (that includes both the budget allocation and the tax), and then charge some amount from each voter to align incentives. Unfortunately, some features of the PB scenario we consider present obstacles to this approach. First, a crucial underlying assumption of VCG is that utilities are quasi-linear, namely that participants' utility is linear in the payment [17, 20]. This is not the case in our model, as we allow more general treatment for utility following Prospect theory [21]. The second impediment is that the PB outcome already includes the tax collected from voters. Adding an unbounded external payment on top of that (and one that differentiates among voters) misses the point. We would therefore like to collect payments that are minimal, if at all. We should note that some of the literature on VCG deals with ways to reduce the payments collected from agents in other situations [15]. We elaborate more on this in Section 2.3 after presenting the model.

2 THE MODEL

A city of population n needs to decide on the volume and use of its public budget. The total available budget is $B = B_0 + nt$ where $B_0 \geq 0$ is an exogenous source and $t \in \mathbb{R}$ is collectively decided by voters. Meaning, voters can decide to either fund an enlargement of the budget with individual equal payments collected from everyone or allocate some of it directly to them as cash. Hereinafter we refer to t as "tax", bearing in mind that it could also be negative. The budget is then allocated between m different alternatives or "projects". A *budget decision* is a pair (x, t) where $t \in R$ is the tax raised and $x \in \Delta^m$ is the allocation of the available budget $B_0 + nt$. Every agent submits her vote $v_i = (v_i^x, v_i^t)$ that is her proposed budget decision and v denotes the profile of all votes.

2.1 Utility Functions

The utility of an individual i from a budget decision (x, t) is

$$U_i(x,t) = \sum_{j=1}^{m} A_{i,j}\theta_j(x_j \cdot (B_0 + nt)) - A_{i,f} f(t) \tag{1}$$

where $\theta_j : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a non-decreasing concave function for all $j \in [m]$, $A_{i,j} \in \mathbb{R}_+\ \forall i \in [n], j \in [m]$, and $A_{i,f} > 0$. For each j, $A_{i,j}\theta_j(\cdot)$ is the utility function of voter i for money spent on project j. We Occasionally write $A_{i,m+1}$ instead of $A_{i,f}$. $f(\cdot)$ expresses the disutility (utility) from any negative (positive) monetary payment. In this work we formulate it as the Kahneman-Tversky [21] 'value function'

$$f(t) = -\mathbb{1}_{\{t \leq 0\}} t^q + \lambda \cdot \mathbb{1}_{\{t>0\}} t^r \tag{2}$$

for some $0 < q, r < 1$, $\lambda > 0$. (here it is taken with a sign opposite to the original as we refer to t mainly as tax collected from individuals). We denote (x^i, t^i) the optimal (i.e. utility maximizing) budget decision for agent i.

Normalized Utilities. The normalized utility of agent i is

$$u_i(x,t) = \frac{1}{\sum_{j=1}^{m} A_{i,j}} U_i(x,t) \tag{3}$$

and $\alpha_{i,j} = \frac{A_{i,j}}{\sum_{j=1}^{m} A_{i,j}}$ are the normalized coefficients for all i, j. $\alpha_i := (\alpha_{i,1}, \ldots, \alpha_{i,m+1})$ is the *preference vector* of agent i. Note that for any set of utility functions $\vec{\theta} := \{\theta_j\}_{j=1}^{m}$ given explicitly, α_i consists the

[5] Having said that, implementing a mechanism that assumes additive utilities in practice is a much more difficult task, because what exactly should the assumed utility functions be is not at all clear.

full description of i's normalized utility. $\vec{\alpha} := (\alpha_1, \dots, \alpha_n)$ denotes *preferences profile* of all agents.

Remark 2.1. Conventionally, additive utility models assume some normalizing of utilities across agents ([9], [10], [5]) but in our context that assumption should be treated with caution. In general, the total impact of budget decisions might differ, and probably is different a lot of the times, from one person to another. For example, wealthier people's quality of life is likely to be less affected by public expenditures as their ability to finance their needs on their own is relatively higher. Still, taking the normalized coefficients as input is justified because we want all voters to have equal (ex-ante) "weight" in the process. Moreover, if agents submit their preferred allocation (and not the utility function explicitly, which seems highly impractical) we can only infer the coefficients $\{\alpha_{i,j}\}_{i,j}$ up to a normalizing factor. However, applying a VCG mechanism on such voting ballots is not straight forward as it's truthfulness relies crucially on the fact that it outputs the social welfare maximizing allocation, that is computed differently for normalized and non-normalized coefficients. Our mechanism indeed takes the normalized utilities as input, but since the non-normalized utility function is merely the normalized utility multiplied by a constant factor, normalizing does not affect incentives. We show that in more details below.

Utility with payments. The mechanism we introduce next outputs a budget decision (x, t) along with a price vector $p \in \mathbb{R}^n$ that assigns a payment p_i to every agent i. These payments are charged in addition to the tax t . Thus, the utility in outcome (x, t, p) becomes

$$U_i(x, t, p) = \sum_{j=1}^{m} A_{i,j}\theta_j(x_j \cdot (B_0 + nt)) - A_{i,f} f(t + p_i) \tag{4}$$

2.2 Mechanisms

Definition 2.2. For all $n \in \mathbb{N}$, a *mechanism* for Participatory Budgeting with tax is a function $\phi : (\Delta^m \times \mathbb{R})^n \mapsto (\Delta^m \times \mathbb{R}) \times \mathbb{R}^n$, $\phi(v) = (\phi(v)_x, \phi(v)_t, \phi(v)_p)$ that takes the voting profile of all proposed budget decisions $v \in (\Delta^m \times \mathbb{R})^n$ and outputs a budget decision $(\phi(v)_x, \phi(v)_t) \in \Delta^m \times \mathbb{R}$ along with a price vector $\phi(v)_p \in \mathbb{R}^n$ that assigns a price p_i to every agent i.

Definition 2.3.

i. A *strategy* of agent i is a function $s_i : (\Delta^m \times \mathbb{R}) \mapsto (\Delta^m \times \mathbb{R})$ that takes i's preference vector α_i and outputs a vote $v_i = s_i(\alpha_i)$.
ii. A strategy s_i is called a *(strictly) dominant strategy* for agent i in mechanism ϕ if
$$U_i(\phi(s_i(\alpha_i), v_{-i})) \overset{(>)}{\geq} U_i(\phi(v_i, v_{-i}))$$
for all $\alpha_i \in \Delta^m \times \mathbb{R}, v_{-i} \in (\Delta^m \times \mathbb{R})^{n-1}$ and $v_i \in (\Delta^m \times \mathbb{R})$.
iii. We say that a voter is "truthful" if she votes her true optimal budget decision, i.e. if $v_i = (x^i, t^i)$.
iv. A PB with tax mechanism ϕ is called *truthful* or *strategy-proof* if $s_i = (x^i, t^i)$ is a dominant strategy in ϕ for all agents $i \in [n]$.
v. A truthful mechanism is *Pareto-efficient* if for any voting profile v, no budget decision (x', t') exists such that
   a. $U_i(x', t') \geq U_i(\phi(v)_x, \phi(v)_t) \quad \forall i \in [n]$
   b. $\exists i \in [n] \; s.t. \; U_i(x', t') > U_i(\phi(v)_x, \phi(v)_t)$

2.3 Necessary Modifications in VCG

A VCG mechanism accepts as input the full preferences of the voters (it is a direct revelation mechanism), selects the socially optimal outcome, and sets a payment p_i for each voter. Formally,

$$p_i = \sum_{k \neq i} u_k(x^*) + h(v_{-i})$$

where x^* is the outcome selected by the mechanism and $h(v_{-i})$ is any function that is independent of i's reported preferences. That makes the overall utility (i.e. including the payment) $\mathcal{U}_i$ of agent i

$$\begin{aligned}\mathcal{U}_i(x^*) &= u_i(x^*) + \sum_{k \neq i} u_k(x^*) + h(v_{-i}) \\ &= \sum_{k \in [n]} u_k(x^*) + h(v_{-i})\end{aligned} \tag{5}$$

The mechanism always outputs $x^* \in argmax \sum_{k \in [n]} u_k(x)$, and thus any misreporting by i that changes the outcome can only harm her.

Non quasi-linearity. The problem in trying to apply that straightforwardly as a PB mechanism is that utilities are not expressed in monetary terms, thus if we naively decide to charge i with $\sum_{k \neq i} u_k(x^*) + h(v_{-i})$ dollars , the $+$ sign in (5) has no meaningful sense. If we want to keep truthfulness, we need to find a way of translating $\sum_{k \neq i} u_k(x^*) + h(v_{-i})$ into a monetary payment $P\Big(\sum_{k \neq i} u_k(x^*) + h(v_{-i})\Big)$, in such way that the overall utility in outcome x

$$\mathcal{U}_i(x) = u_i(x) + u_i\Big(P\Big(\sum_{k \neq i} u_k(x) + h(v_{-i})\Big)\Big)$$

is still increasing in $\sum_{k \in [n]} u_k(x)$. That way, truthfulness is retained due to fact that the mechanism outputs
$x^* \in argmax \sum_{k \in [n]} u_k(x)$.

We do that by utilizing the last component in the utility function (4) that is the (dis)utility $f(\cdot)$ from money (losses) gains.

Adding $f(\cdot)$ to the utility function implicitly expresses the relation between utility gained from monetary payments to that of consuming public products, and thus enabling the appropriate construction of P. It is defined in section 2.4 below and Lemma 2.5 proves how it meets our demands.

Payments on top of taxes. Beyond the technical difficulties, charging prices from participants in a PB process does not really make sense. It seems very unjustifiable if people would have to pay for participating in such democratic procedure. Moreover, as any other process of direct democracy, PB's biggest shortcoming is it's relatively low rates of participation as it is, and charging prices would further diminish the incentives to participate. Therefore, we consider crucial the possibility of vanishing the payments somehow. In Mechanism Design literature that desire is referred to as 'redistribution' [15]. In section 4, we show that under fairly weak assumptions on the preferences profile, prices do vanish to zero in large populations for logarithmic utilities. We see this special case as a benchmark case and hope to expand that result to other forms of utility function in the future.

2.4 The PB-VCG Mechanism

Definition 2.4 (direct generalized VCG function). For any $n \in \mathbb{N}$ and utility functions $\vec{\theta}$, the function $b^n_\theta : (\Delta^m \times \mathbb{R})^n \mapsto (\Delta^m \times \mathbb{R})$ maps every preference profile $\vec{\alpha}$ to a budget decision

$$b^n_\theta(\vec{\alpha}) = (x^{[n]}, t^{[n]}) \in argmax_{(x,t)} \sum_{i \in [n]} u_i(x,t)$$

where for all $i \in [n]$, $u_i(x,t)$ is the (unique) normalized utility function defined by $\vec{\theta}$ and α_i.
For any function $h : (\Delta^m \times \mathbb{R})^{n-1} \mapsto \mathbb{R}$, the function $p^n_{\theta,h} : (\Delta^m \times \mathbb{R})^n \mapsto \mathbb{R}^n$ maps every preferences profile $\vec{\alpha}$ to a payment vector $p^n_{\theta,h}(\vec{\alpha}) = p^{[n]}$ defined by:

$$p_i^{[n]} = f^{-1}\Big(-\frac{1}{\alpha_{i,f}} \sum_{k \neq i} u_k(b^n_\theta(\vec{\alpha})) + f(t^{[n]}) + \frac{1}{\alpha_{i,f}} h(\vec{\alpha}_{-i})\Big) - t^{[n]}$$

where $u_i(x,t)$ is the normalized utility function defined by $\vec{\theta}$ and α_i for all $i \in [n]$. We define the *direct generalized VCG function* $\mathcal{D}^n_{\theta,h} : (\Delta^m \times \mathbb{R})^n \mapsto (\Delta^m \times \mathbb{R}) \times \mathbb{R}^n$ as

$$\mathcal{D}^n_{\theta,h}(\vec{\alpha}) = (b^n_\theta(\vec{\alpha}), p^n_{\theta,h}(\vec{\alpha}))$$

(In a lot of places we drop the specification of n, h and $\vec{\theta}$ as they are clear from context or we are just making general statements.)

In words, $\mathcal{D}$ selects the welfare maximizing budget decision with respect to the normalized utilities $u_i(x,t)$, $i \in [n]$, just as any other VCG mechanism. The payments, as shown in the next lemma, are generalized VCG payments in the sense that the resulting overall utility for an agent is increasing in $\sum_{k \in [n]} u_k(x^{[n]}, t^{[n]})$.

Lemma 2.5. *For all preferences profiles $\vec{\alpha}$ and all $i \in [n]$,*

$$U_i(\mathcal{D}(\vec{\alpha})) = \sum_{j=1}^{m} A_{i,j}\Big(\sum_{k \in [n]} u_k(x^{[n]}, t^{[n]}) - h(\vec{\alpha}_{-i})\Big)$$

Proof. Note that

$$\begin{aligned} & f(p_i^{[n]} + t^{[n]}) = \\ & f\Bigg[f^{-1}\Big(f(t^{[n]}) - \frac{1}{\alpha_{i,f}} \sum_{k \neq i} u_k(x^{[n]}, t^{[n]}) + \frac{1}{\alpha_{i,f}} h(\vec{\alpha}_{-i})\Big) \\ & \qquad - t^{[n]} + t^{[n]}\Bigg] \\ =& f(t^{[n]}) - \frac{1}{\alpha_{i,f}} \sum_{k \neq i} u_k(x^{[n]}, t^{[n]}) + \frac{1}{\alpha_{i,f}} h(\vec{\alpha}_{-i}) \end{aligned}$$

Therefore,

$$\begin{aligned} U_i(\mathcal{D}(\vec{\alpha})) &= \sum_{j=1}^{m} A_{i,j}\theta_j(x_j^{[n]} \cdot (B_0 + nt^{[n]})) - A_{i,f} f(t^{[n]} + p_i^{[n]}) \\ &= \sum_{j=1}^{m} A_{i,j}\theta_j(x_j^{[n]} \cdot (B_0 + nt^{[n]})) \\ &\quad - A_{i,f}\Big(f(t^{[n]}) - \frac{1}{\alpha_{i,f}} \sum_{k \neq i} u_k(x^{[n]}, t^{[n]}) + \frac{1}{\alpha_{i,f}} h(\vec{\alpha}_{-i})\Big) \\ &\overset{(*)}{=} U_i(x^{[n]}, t^{[n]}) + \sum_{j=1}^{m} A_{i,j}\Big(\sum_{k \neq i} u_k(x^{[n]}, t^{[n]}) - h(\vec{\alpha}_{-i})\Big) \\ &= \sum_{j=1}^{m} A_{i,j}\Big(\sum_{k \in [n]} u_k(x^{[n]}, t^{[n]}) - h(\vec{\alpha}_{-i})\Big) \end{aligned}$$

where in $(*)$ we used $\frac{A_{i,f}}{\alpha_{i,f}} = \sum_{j=1}^{m} A_{i,j}$. □

Since agents submit their optimal budget decisions and not the explicit preferences, $\mathcal{D}$ alone cannot operate as a voting mechanism. For that purpose we need to map votes - that are perceived as every voter's optimal budget decision- into preferences.

Definition 2.6. For every set of utility functions $\vec{\theta} := \{\theta_j\}_{j=1}^m$, if there exists an invertible function $g : (\Delta^m \times \mathbb{R}) \mapsto (\Delta^m \times \mathbb{R})$ such that $g(\alpha) = argmax_{(x,t)} u_\alpha(x,t)$ where $u_\alpha(x,t)$ is the normalized utility function defined uniquely by $\vec{\theta}$ and $\alpha \in \Delta^m$, we call that function *the optimization mapping* for the set $\vec{\theta}$ and denote it with g_θ. For $n \in \mathbb{N}$ we define $g^n_\theta : (\Delta^m \times \mathbb{R})^n \mapsto (\Delta^m \times \mathbb{R})^n$ accordingly as $g^n_\theta(\alpha)_i = g_\theta(\alpha_i)$ for all $i \in [n]$.

We now have everything we need to define our PB mechanism that uses g_θ to interpret preferences from budget proposals (votes) and then implements $\mathcal{D}$ on these preferences.

Definition 2.7. [The PB-VCG Mechanism] For every $n \in \mathbb{N}$, $h : (\Delta^m)^{n-1} \mapsto \mathbb{R}$ and $\vec{\theta}$ such that g_θ exists, the PB-VCG mechanism is defined for all $v \in (\Delta^m \times \mathbb{R})^n$ as

$$\mathcal{M}^n_{\theta,h}(v) := \mathcal{D}^n_{\theta,h}\big((g^n_\theta)^{-1}(v)\big).$$

3 RESULTS

The main result in this section is the truthfulness of the PB-VCG mechanism. While VCG mechanisms in general only provide for weakly-dominant truthful strategies, in the PB-VCG mechanism these are strictly dominant for all agents, a difference that has significant implications on enabling group manipulations (See Remark 3.4 below). Lemma 2.5 is sufficient for weak-dominance truthfulness that stems from the fact that $(x^{[n]}, t^{[n]}) \in argmax_{(x,t)} \sum_{k \in [n]} u_k(x,t)$ and that $h(v_{-i})$ is independent of i's vote. The reason why that only provides for weak-dominance in general VCG mechanisms is twofold. First and most importantly, situations where an agent is not pivotal are very generic in VCG mechanisms. (In a 2^{nd} price auction for example, any bid of a loosing agent lower then the winning bid has no effect on the winner's identity). Second, even if that is not the case, $argmax_{(x,t)} \sum_{k \in [n]} u_k(x,t)$ might not be unique, meaning that there could be an opportunity for a voter of manipulating the outcome from one optimal point to another, without affecting her overall utility. The next lemma rules out both in the PB-VCG mechanism.

Definition 3.1. For any preference profile $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha^{[n]}$ denotes the mean of $(\alpha_1, \ldots, \alpha_n)$ i.e. $\alpha_j^{[n]} = \frac{1}{n} \sum_{i \in [n]} \alpha_{i,j}$ $\forall 1 \leq j \leq m+1$.

LEMMA 3.2. *For any function set $\vec{\theta}$ such that g_θ exists,*

$$b_\theta^n(\vec{\alpha}) = g_\theta(\alpha^{[n]})$$

Meaning, the social welfare maximizing budget decision for a given preferences profile $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is the same as the optimal budget decision for an individual whose preferences vector is the average preferences $\alpha^{[n]}$.

PROOF.

$$\sum_{i \in [n]} u_i(x,t) = \sum_{i \in [n]} \left(\sum_{j=1}^{m} \alpha_{i,j} \theta_j (x_j \cdot (B_0 + nt)) - \alpha_{i,f} f(t) \right)$$
$$= n \sum_{j=1}^{m} \left(\alpha_j^{[n]} \theta_j (x_j \cdot (B_0 + nt)) - \alpha_f^{[n]} f(t) \right)$$

that equals $n \cdot u_i(x,t)$ if $\alpha_i = \alpha^{[n]}$. □

Due to Lemma 3.2 and g_θ being 1:1, a deviation in any agent's vote while others are kept fixed inevitably changes $\alpha^{[n]}$ and consequently the outcome $b(\vec{\alpha}) = g_\theta(\alpha^{[n]})$, and to a sub-optimal in terms of social welfare. That guarantees truthfulness in strictly dominant strategies.

PROPOSITION 3.3. *The truthful voting strategy $s(\alpha_i) = (x^i, t^i)$ is a strictly dominant strategy in $\mathcal{M}$ for all $i \in [n]$.*

PROOF. Let $\tilde{\alpha}^{[n]}(v) := (g_\theta^n)^{-1}(v)$ be the mean of preferences as they are perceived from the voting profile v by $(g_\theta^n)^{-1}$, and $\tilde{u}_k$ the corresponding utility function of agent k for all $k \in [n]$. That is, if all agents vote truthfully then $\tilde{\alpha}^{[n]}(v) = \alpha^{[n]}$ and $\tilde{u}_k = u_k \forall k$.
Let:

- $v_i = (x^i, t^i)$ and $\hat{v}_i \neq v_i$.
- $(x^{[n]}, t^{[n]}, p^{[n]}) := \mathcal{M}(v_i, v_{-i})$
- $(\hat{x}^{[n]}, \hat{t}^{[n]}, \hat{p}^{[n]}) := \mathcal{M}(\hat{v}_i, v_{-i})$

Then By Lemma 3.2 and Definition 2.6, $(x^{[n]}, t^{[n]}) = g_\theta\big(\tilde{\alpha}^{[n]}(v_i, v_{-i})\big)$ is the unique maximum point of $u_i(x,t) + \sum_{k \neq i} \tilde{u}_k(x,t)$ and

$$(x^{[n]}, t^{[n]}) \neq (\hat{x}^{[n]}, \hat{t}^{[n]}) = g_\theta\big(\tilde{\alpha}^{[n]}(\hat{v}_i, v_{-i})\big)$$

because any change in i's vote when all other votes are fixed inevitably changes $\tilde{\alpha}^{[n]}(v)$ and g_θ is $1:1$. Thus, By Lemma 2.5 (in particular the line before the last one in the proof),

$$U_i(\mathcal{M}(\hat{v}_i, v_{-i})) - U_i(\mathcal{M}((x^i, t^i), v_{-i})) =$$
$$\sum_{j=1}^{m} A_{i,j} \Big(u_i(\hat{x}^{[n]}, \hat{t}^{[n]}) + \sum_{k \neq i} \tilde{u}_k(\hat{x}^{[n]}, \hat{t}^{[n]})$$
$$- u_i(x^{[n]}, t^{[n]}) - \sum_{k \neq i} \tilde{u}_k(x^{[n]}, t^{[n]}) \Big) < 0$$

□

Remark 3.4 (A Comment on Group Strategy-Proofness). Group Strategy-Proofness, that we do not get into formally define here, roughly means that not only no agent can benefit from misreporting her preferences, but also no subset of agents can. It is sometimes possible that while each member best response is the truthful strategy, their coordinated misreporting benefits at least some of them, comparing to when they are all being truthful. (It happens because agents in that group benefit each from the other's misreport). In general, VCG mechanisms are not group strategy proof [11], and nor is our mechanism. However, we do wish to emphasize the fact that while truthfulness in strictly dominant strategies does not imply group strategy proof by definition, it does serve as quite an impediment for manipulating groups in a more realistic sense. That is because in any case, an agent is strictly better off when reports truly. Thus, given that all other members deviated to the agreed false reporting, an agent has a strict incentive to "betray" them and deviate back to being truthful.

PROPOSITION 3.5. *$\mathcal{M}$ is Pareto-efficient.*

This result is immediate from the fact that $\mathcal{D}$ and consequently $\mathcal{M}$ maximize the "social welfare" with respect to the normalized utilities, making it Pareto-efficient with respect to those in particular, and the actual utility U_i of every agent is just the normalized one multiplied by a constant factor.

4 SPECIAL CASE : LOGARITHMIC UTILITIES WITH BALANCED BUDGET

In this section we apply $\mathcal{M}$ to the specific setup of $\theta_j = \ln$ $\forall 1 \leq j \leq m$, $B_0 = 0$ and $h = h^{\ln}$ given in 4.2 below. We demonstrate how the mechanism is constructed and show an additional desired property special to this case—namely that prices vanish in large populations (under some plausible assumptions on people's preferences). We consider this result essential, as charging money (meaning, beyond the collectively decided tax payment) from participants in a participatory budgeting procedure, as in any other democratic voting procedure, is not customary and would likely be considered unacceptable in most places. We do hope that it could be expanded to a broader class of utility models in further research.

Apart from their computational convenience, ln utilities seem a natural first choice as they are a monotone transform of the broadly used Cobb-Douglas utilities $u_i(xB) = \Pi_{j=1}^{m} (x_j B)^{\alpha_{i,j}}$ (in Social Choice literature as well as in other fields- see for example [8, 9, 16]. In particular they serve serves as a "lower limit" for the 'NON-SATURATING' class of utility functions introduced in [9]). Being a monotone transform of it, the logarithmic model implies the exact same preferences over different allocations, albeit expressed in an additive form of utility that fit our model (and also arguably more appropriate for the consumption of different public goods). The main difference between the two is that if $\alpha_{i,j} > 0$ for some $i \in [n]$ $j \in [m]$, then having $x_j = 0$ is infinitely bad in the ln model, while only vanishes to zero in the Cobb-Douglas utility. That difference becomes significant in our model as we add the $\alpha_{i,f} f(t)$ term into the utility function, expressing the relation between utility gained from monetary grants versus consuming public products. The ln model therefore might be most suitable (if we further assume $\alpha_{i,j} > 0$ $\forall i, j$) to the case of allocating the budget between “classic" public expenditures such as education, infrastructures, urban renewal etc. In that case, it seems plausible to assume that investing no money at all on such things is "infinitely bad" or at least way worse than not getting any monetary payment, which we assume to yield zero utility.

Applying $\mathcal{M}$

Let v be a voting profile of n agents. The the first step in executing $\mathcal{M}$ is to extract the full description of voters' (normalized) utility functions from v.

Proposition 4.1. *For $\theta_j(\cdot) = ln(\cdot)\ \forall 1 \le j \le m$,*

$$g_\theta(\alpha_j) = \begin{cases} \alpha_j & 1 \le j \le m \\ (\alpha_j \lambda r)^{-\frac{1}{r}} & j = m+1 \end{cases}$$

Proof.

$$(x^i, t^i) = argmax_{(x,t)} \sum_{j=1}^{m} \alpha_{i,j} \ln(x_j \cdot (nt)) - \alpha_{i,f} \lambda t^r$$

(t must be positive here). Moreover, note that it must be that

$$x^i = argmax_x \sum_{j=1}^{m} \alpha_{i,j} \ln(x_j B)$$

for any budget B, implying that $x^i_j = \alpha_{i,j}$. Thus, t^i optimizes

$$\sum_{j=1}^{m} \alpha_{i,j} \ln(\alpha_{i,j} \cdot (nt)) - \alpha_{i,f} \lambda t^r$$

$$\implies t^i = (\alpha_{i,f} \lambda r)^{-\frac{1}{r}}$$

□

Hence we can infer the preferences of every voter i using

$$\alpha_{i,j} = \left(g_\theta^{-1}(v_i)\right)_j = \begin{cases} v_{i,j} & 1 \le j \le m \\ \frac{(v_{i,j})^{-r}}{\lambda r} & j = m+1 \end{cases}$$

and then, relying on Lemma 3.2, we compute the mean of preferences $\alpha^{[n]}$ and choose the welfare maximizing budget decision

$$b^n_\theta(\vec{\alpha}) = g_\theta(\alpha^{[n]}_j) = \begin{cases} \alpha^{[n]}_j & 1 \le j \le m \\ (\alpha^{[n]}_j \lambda r)^{-\frac{1}{r}} & j = m+1 \end{cases}$$

Now we need to set prices, and we do that using the following definition for h:

Definition 4.2. For every $n \in \mathbb{N}$ and a preferences profile $\vec{\alpha} \in (\Delta^m \times \mathbb{R})^n$, define for all $i \in [n]$

$$h^{\ln}(\vec{\alpha}_{-i}) := \sum_{k \neq i} \sum_{j=1}^{m} \alpha_{k,j} \ln(\alpha^{[n]\setminus\{i\}}_j n t^{[n]\setminus\{i\}}) - \sum_{k \neq i} \alpha_{k,f} f(t^{[n]\setminus\{i\}})$$

where $\alpha^{[n]\setminus\{i\}}_j$ is the mean preference vector of all agents excluding i and $t^{[n]\setminus\{i\}}$ is the tax determined by $b(\vec{\alpha}_{-i})$ for that population.

That is, $h^{\ln}$ is defined almost identically to the familiar Clarke pivote-rule function [7] $h(\vec{\alpha}_{-i}) = \sum_{k \neq i} u_k(b^{(n-1)}_\theta(\vec{\alpha}_{-i}))$, that is the social welfare achieved if i was excluded from the population $[n]$. The only difference here is that we put n in the ln argument and not $n-1$, meaning that instead of taking the social welfare in i's absence, we are putting the social welfare of $[n] \setminus \{i\}$ when i is excluded from voting but still pays the tax $t^{[n]\setminus\{i\}}$. The following Lemma shows how this implies non-negative payments for all agents, which means that no outer source is required to implement the budget decision determined by $\mathcal{M}$. We should note that although the proof of Theorem 4.5 uses this fact to save some technical burden, we could also do without it. Moreover, if $h = h_c - 1$ where h_c is the Clarck pivote-rule function, prices are no longer non-negative but Theorem 4.5 still holds.

Lemma 4.3. *If $\theta_j(\cdot) = ln(\cdot)\ \forall 1 \le j \le m$ and $\mathcal{D}$ is defined with the $h^{\ln}$ given in Definition 4.2,*

$$p^{[n]}_i \ge 0 \quad \forall i \in [n]$$

Proof. As mentioned above, $h^{\ln}(\vec{\alpha}_{-i})$ equals the welfare of all agents but i if i was excluded from voting but still pays the tax determined by the others. Therefore,

$$h^{\ln}(\vec{\alpha}_{-i}) - \sum_{k \neq i} u_k(x^{[n]}, t^{[n]}) \ge 0$$

because moving from $b(\vec{\alpha}_{-i})$ to $b(\vec{\alpha}) = (x^{[n]}, t^{[n]})$ without increasing the available budget could only be suboptimal for $[n] \setminus i$. Due to f^{-1} being monotonically increasing,

$$\begin{aligned} p^n_i =& f^{-1}\Big(-\frac{1}{\alpha_{i,f}} \sum_{k \neq i} u_k(x^n, t^n) + f(t^n) + \frac{1}{\alpha_{i,f}} h^{\ln}(\vec{\alpha}_{-i})\Big) - t^{[n]} \\ \ge& f^{-1}\big(f(t^n)\big) - t^{[n]} = 0 \end{aligned}$$

□

Our next definition describes some limitations on the preferences profile $\vec{\alpha}$ needed for the theorem that follows.

Definition 4.4. We say that a preferences profile $\vec{\alpha} \in (\Delta^m \times \mathbb{R})^n$ is μ-bounded if

(1) $\frac{\alpha_{i,j} - \alpha^{[n]\setminus\{i\}}_j}{\alpha^{[n]\setminus\{i\}}_j} \le \mu \quad \forall i \in [n], j \in [m+1]$

(2) $\alpha_{i,f} \ge \frac{1}{\mu} \quad \forall i \in [n]$

where $\alpha^{[n]\setminus\{i\}}_j = \frac{1}{n-1} \sum_{k \in [n]\setminus\{i\}} \alpha_{k,j}$

In fact, we wish to point out that μ-boundedness is not much to ask for, in other words that Theorem 4.5 holds for quite generic preferences profiles. It should be expected that α^n_j are bounded away from zero in any population, as projects that draw negligible interest would probably not have been brought up in the first place. For large n that imposes a lower strictly positive bound also for $\alpha^{(n\setminus i)}_j$, and consequently an upper bound on $(\alpha_{i,j} - \alpha^{(n/i)}_j)/\alpha^{(n/i)}_j$. The assumption that $\alpha_{i,f}$ is bounded away from zero is reasonable because, well, people tend to have more than a negligible interest in money.

We end this section with its main result, namely that prices vanish in large enough populations with μ-bounded preferences.

Theorem 4.5. *Assume $\theta_j(\cdot) = ln(\cdot)\ \forall 1 \le j \le m$ and that $\mathcal{D}$ is defined with the $h^{\ln}$ given in Definition 4.2. Then, for any $\mu, \epsilon > 0$ there exists $n(\mu, \epsilon) \in \mathbb{N}$ such that in every population $[n]$ of size $\ge n(\mu, \epsilon)$ with μ-bounded preferences,*

$$|p^{[n]}_i| \le \epsilon \quad \forall i \in [n]$$

Proof. For abbrevience, the superscripts n, $(n \setminus i)$ substitute for $[n]$, $[n] \setminus \{i\}$ respectively . Recall that

$$p_i^n = f^{-1}\Big(- \frac{1}{\alpha_{i,f}} \sum_{k \neq i} u_k(x^n, t^n) + f(t^n) + \frac{1}{\alpha_{i,f}} h^{\ln}(\vec{\alpha}_{-i}) \Big) - t^n$$

and that $t^{(\cdot)} = (\alpha_f^{(\cdot)} \lambda r)^{-\frac{1}{r}}$.

$$h^{\ln}(\vec{\alpha}_{-i}) - \sum_{k \neq i} u_k(x^n, t^n) =$$

$$(*) \sum_{k \neq i} \sum_{j=1}^{m} \alpha_{k,j} \Big(\ln(\alpha_j^{(n/i)} n (\alpha_f^{(n/i)} \lambda r)^{-\frac{1}{r}}) - \ln(\alpha_j^n n (\alpha_f^n \lambda r)^{-\frac{1}{r}}) \Big) +$$

$$(**) \sum_{k \neq i} \alpha_{k,f} \Big(\lambda\big((\alpha_f^n \lambda r)^{-\frac{1}{r}}\big)^r - \lambda\big((\alpha_f^{(n/i)} \lambda r)^{-\frac{1}{r}}\big)^r \Big)$$

Put $\alpha_j^n = \alpha_j^{(n/i)} + \frac{1}{n}(\alpha_i, j - \alpha_j^{(n/i)})$ for all $j \in [m+1]$ to get

$$\begin{aligned}
(*) &= \sum_{k \neq i} \sum_{j=1}^{m} \alpha_{k,j} \ln \left[\left(1 + \frac{1}{n} \frac{\alpha_{i,j} - \alpha_j^{(n/i)}}{\alpha_j^{(n/i)}} \right)^{-1} \right] + \\
&\quad \sum_{k \neq i} \sum_{j=1}^{m} \alpha_{k,j} \ln \left[\left(1 + \frac{1}{n} \frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}} \right)^{1/r} \right] \\
&= \sum_{j=1}^{m} -(n-1)\alpha_j^{(n/i)} \ln \left[1 + \frac{1}{n} \frac{\alpha_{i,j} - \alpha_j^{(n/i)}}{\alpha_j^{(n/i)}} \right] + \qquad \text{(i)} \\
&\quad \frac{1}{r}(n-1) \ln \left[1 + \frac{1}{n} \frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}} \right] \qquad \text{(ii)}
\end{aligned}$$

where we used $\sum_{k \neq i} \alpha_{k,j} = (n-1)\alpha_j^{(n/i)}$ in the first summation and $\sum_{j=1}^{m} \alpha_{k,j} = 1 \ \forall k$ in the second. Doing the same with $(**)$ makes

$$\begin{aligned}
(**) &= \frac{1}{r}(n-1)\alpha_f^{(n/i)} \Big((\alpha_f^n)^{-1} - (\alpha_f^{(n/i)})^{-1} \Big) \\
&= -\frac{1}{r} \left(\frac{(n-1) \frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}}}{n + \frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}}} \right) \quad \text{(iii)}
\end{aligned}$$

It is not difficult to check that while keeping everything else as fixed parameters, (i)+(ii)+(iii) vanishes if we let $n \to \infty$. What we want to show next is that if we only consider populations with μ-bounded preferences for some $\mu \in \mathbb{R}_+$, it converge uniformly for all i. So now we fix μ and in everything that follows we assume that preferences are μ-bounded.
Consider the function $F_n(x) = x - (n-1)\ln(1 + \frac{x}{n})$. For any integer $n > 1$ and $x \in \mathbb{R}$, $F_n(x)$ satisfies:

- $\frac{d}{dx} F_n(x) \geq 0$ for $x \geq -1$.
- $\lim_{n \to \infty} F_n(x) = 0$

Since $-1 \leq \frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}} \ \forall i, j$,

$$\begin{aligned}
\text{(i)} \quad & \sum_{j=1}^{m} -(n-1)\alpha_j^{(n/i)} \ln \left[1 + \frac{1}{n} \frac{\alpha_{i,j} - \alpha_j^{(n/i)}}{\alpha_j^{(n/i)}} \right] \\
&= \sum_{j=1}^{m} \alpha_j^{(n/i)} \left(F_n\Big(\frac{\alpha_{i,j} - \alpha_j^{(n/i)}}{\alpha_j^{(n/i)}} \Big) - \Big(\frac{\alpha_{i,j} - \alpha_j^{(n/i)}}{\alpha_j^{(n/i)}} \Big) \right) \\
&\leq \sum_{j=1}^{m} \alpha_j^{(n/i)} \left(F_n(\mu) - \Big(\frac{\alpha_{i,j} - \alpha_j^{(n/i)}}{\alpha_j^{(n/i)}} \Big) \right) \\
&\xrightarrow[n \to \infty]{} \sum_{j=1}^{m} -\alpha_j^{(n/i)} \Big(\frac{\alpha_{i,j} - \alpha_j^{(n/i)}}{\alpha_j^{(n/i)}} \Big) = -(1-1) = 0 \\
\text{(ii)} \quad & \frac{1}{r}(n-1) \ln \left[1 + \frac{1}{n} \frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}} \right] \\
&= \frac{1}{r} \left(\frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}} - F_n\Big(\frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}} \Big) \right) \\
&\leq \frac{1}{r} \left(\frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}} - F_n(-1) \right) \xrightarrow[n \to \infty]{} \frac{1}{r} \frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}}
\end{aligned}$$

For (iii), define $G_n(x) = x - \frac{(n-1)x}{n+x}$. for all $n > 1, \ x \in R$:

- $G_n(x) \leq 0$ for $x \in [-1, 0]$.
- $G_n(x) > 0$ for $x > 0$.
- $\frac{d}{dx} G_n(x) > 0$ for $x > 0$.
- $\lim_{n \to \infty} G_n(x) = 0$

Thus, (iii) equals

$$-\frac{1}{r} \left(\frac{(n-1) \frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}}}{n + \frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}}} \right) = \frac{1}{r} \left(G_n\Big(\frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}} \Big) - \frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}} \right)$$

$$\leq \frac{1}{r} \left(G_n(\mu) - \frac{\alpha_{i,f} - \alpha^{(n/i)}}{\alpha^{(n/i)}} \right) \xrightarrow[n \to \infty]{} -\frac{1}{r} \frac{\alpha_{i,f} - \alpha_f^{(n/i)}}{\alpha_f^{(n/i)}}$$

To conclude, for any arbitrary small $\delta > 0$ there exists large enough n_δ so that $h^{\ln}(\vec{\alpha}_{-i}) - \sum_{k \neq i} u_k(x^n, t^n) < \delta$ for every agent i in a population of size $n \geq n_\delta$ with μ-bounded preferences.
Now fix ϵ and let $\delta > 0$ small enough so that

$$f^{-1}(f(t^{[n]} + x) < t^{[n]} + \epsilon \quad \forall x \in [f(t^{[n]}), f(t^{[n]}) + \delta]$$

Then for $n(\mu, \epsilon)$ large enough so that
$h^{\ln}(\vec{\alpha}_{-i}) - \sum_{k \neq i} u_k(x^n, t^n) < \frac{\delta}{\mu}$ we have

$$\begin{aligned}
p_i^n &= f^{-1}\Big(- \frac{1}{\alpha_{i,f}} \sum_{k \neq i} u_k(x^n, t^n) + f(t^n) + \frac{1}{\alpha_{i,f}} h^{\ln}(\vec{\alpha}_{-i}) \Big) - t^n \\
&\leq f^{-1}\Big(f(t^n) + \frac{1}{\alpha_{i,f}} \frac{\delta}{\mu} \Big) - t^n \leq f^{-1}\Big(f(t^n) + \delta \Big) - t^n \leq \epsilon
\end{aligned}$$

and Lemma 4.3 completes the proof. □

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