SUST Final Year Thesis: Homework 3

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Problem 1

Given $V_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, V_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, V_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}$ and $M = \begin{bmatrix}1 & -3 & 3\\3 & -5 & 3\\6 & -6 & 4\end{bmatrix}$

Now,

$$\begin{aligned}
M \star V_1 &= \begin{bmatrix}1 & -3 & 3\\3 & -5 & 3\\6 & -6 & 4\end{bmatrix} \star \begin{bmatrix}1\\1\\0\end{bmatrix}\\
&= \begin{bmatrix}-2\\-2\\0\end{bmatrix}\\
&= -2 \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}\\
&= -2 \cdot V_1
\end{aligned}$$

Again,

$$\begin{aligned}
M \star V_2 &= \begin{bmatrix}1 & -3 & 3\\3 & -5 & 3\\6 & -6 & 4\end{bmatrix} \star \begin{bmatrix}1\\0\\-1\end{bmatrix}\\
&= \begin{bmatrix}-2\\0\\2\end{bmatrix}\\
&= -2 \cdot \begin{bmatrix}1\\0\\-1\end{bmatrix}\\
&= -2 \cdot V_2
\end{aligned}$$

And,

$$\begin{aligned}
M \star V_3 &= \begin{bmatrix}1 & -3 & 3\\3 & -5 & 3\\6 & -6 & 4\end{bmatrix} \star \begin{bmatrix}1\\1\\2\end{bmatrix}\\
&= \begin{bmatrix}4\\4\\8\end{bmatrix}\\
&= 4 \cdot V_3
\end{aligned}$$

Hence, the eigenvalues are -2, -2, and 4.

Problem 2

Given $A = \begin{bmatrix} 7 & 6+5i \\ 6-5i & -3 \end{bmatrix}$

Now,

$$\begin{aligned} A^\dagger = & (\bar{A})^T \\ = & \begin{bmatrix} 7 & 6-5i \\ 6+5i & -3 \end{bmatrix}^T \\ = & \begin{bmatrix} 7 & 6+5i \\ 6-5i & -3 \end{bmatrix} \\ = & A \end{aligned}$$

$\therefore A$ is a hermitian matrix.

Problem 3

$$\begin{aligned} LHS = & \langle AV, V' \rangle \\ = & (AV)^T \star V' \\ = & A^T V^T V' \\ = & V^T \star AV' \\ = & \langle V, AV' \rangle \\ = & RHS \end{aligned}$$

Problem 4

Let c is the eigenvalue of the symmetric matrix A. Symmetric matrices are simply Hermitian matrices with all entries real. So,

$$\begin{aligned} c\langle V, V \rangle = & \langle cV, V \rangle \\ = & \langle AV, V \rangle \\ = & \langle V, AV \rangle \\ = & \langle V, cV \rangle \\ = & \bar{c}\langle V, V \rangle \end{aligned}$$

As c and V are non-zero, c = $\bar{c}$ and hence c must be real.

Problem 5

Let $U = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now,

$$\begin{aligned}
U \star U^{\dagger} = & \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \star \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= & I_3 \\
U^{\dagger} \star U = & \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \star \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
= & I_3
\end{aligned}$$

$\therefore U \star U^{\dagger} = U^{\dagger} \star U = I_n$ and hence U is unitary.

Problem 6

$$\begin{aligned}
LHS = & \; d(UV_1, UV_2) \\
= & \; |UV_1 - UV_2| \\
= & \; \sqrt{\langle UV_1 - UV_2, UV_1 - UV_2 \rangle} \\
= & \; \sqrt{\langle U(V_1 - V_2), U(V_1 - V_2) \rangle} \\
= & \; \sqrt{\{U(V_1 - V_2)\}^{\dagger} \star \{U(V_1 - V_2)\}} \\
= & \; \sqrt{(V_1 - V_2)^{\dagger} U^{\dagger} \star U(V_1 - V_2)} \\
= & \; \sqrt{(V_1 - V_2)^{\dagger} \star I \star (V_1 - V_2)} \\
= & \; \sqrt{\langle (V_1 - V_2), (V_1 - V_2) \rangle} \\
= & \; |(V_1 - V_2)| \\
= & \; d(V_1, V_2) \\
= & \; RHS
\end{aligned}$$

Problem 7

The identity matrix I_n is symmetric. Since the identity matrix consists of all 1s along the main diagonal and 0s elsewhere, it remains unchanged when transposed. Therefore, $I_n = I_n^\dagger$, and I_n is Hermitian.

Taking the conjugate transpose of $-1 \cdot I_n$, we have: $(-1 \cdot I_n)^\dagger = -1 \cdot (I_n)^\dagger = -1 \cdot I_n$ Since $-1 \cdot I_n$ equals its conjugate transpose, $-1 \cdot I_n$ is Hermitian.

The conjugate transpose of I_n is $I_n^\dagger = I_n$. Multiplying $I_n^\dagger$ with I_n gives us: $I_n \cdot I_n = I_n$, which is the identity matrix. Therefore, I_n is unitary.

The conjugate transpose of $-1 \cdot I_n$ is $(-1 \cdot I_n)^\dagger = -1 \cdot (I_n)^\dagger = -1 \cdot I_n$. Multiplying $(-1 \cdot I_n)^\dagger$ with $-1 \cdot I_n$, we get: $(-1 \cdot I_n) \cdot (-1 \cdot I_n) = I_n \cdot I_n = I_n$, which is the identity matrix. Thus, $-1 \cdot I_n$ is unitary.

Therefore, both I_n and $-1 \cdot I_n$ satisfy the properties of being Hermitian and unitary.

Problem 8

$$\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ 4 \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ 7 \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 6 \\ -4 \\ 8 \\ -7 \\ 14 \end{bmatrix}$$

Problem 9

Let $A = \begin{bmatrix} 3+2i & 5-i & 2i \\ 0 & 12 & 6-3i \\ 2 & 4+4i & 9+3i \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3+4i & 5-7i \\ 10+2i & 6 & 2+5i \\ 0 & 1 & 2+9i \end{bmatrix}$

Now, $A \otimes B = \begin{bmatrix} 3+2i & 5-i & 2i \\ 0 & 12 & 6-3i \\ 2 & 4+4i & 9+3i \end{bmatrix} \otimes \begin{bmatrix} 1 & 3+4i & 5-7i \\ 10+2i & 6 & 2+5i \\ 0 & 1 & 2+9i \end{bmatrix}$

$$
= \begin{bmatrix}
(3+2i)\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix} & (5-i)\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix} & 2i\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix}\\
0\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix} & 12\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix} & (6-3i)\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix}\\
2\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix} & (4+4i)\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix} & (9+3i)\cdot\begin{bmatrix}1 & 3+4i & 5-7i\\ 10+2i & 6 & 2+5i\\ 0 & 1 & 2+9i\end{bmatrix}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
3+2i & 1+18i & 29-11i & 5-i & 19+17i & 18-40i & 2i & -8+6i & 14+10i\\
26+26i & 18+12i & -4+19i & 52 & 30-6i & 15+23i & -4+20i & 12i & -10+4i\\
0 & 3+2i & -12+31i & 0 & 5-i & 19+43i & 0 & 2i & -18+4i\\
0 & 0 & 0 & 12 & 36+48i & 60-84i & 6-3i & 30+15i & 9-57i\\
0 & 0 & 0 & 120+24i & 72 & 24+60i & 66-18i & 36-18i & 27+24i\\
0 & 0 & 0 & 0 & 12 & 24+108i & 0 & 6-3i & 39+48i\\
2 & 6+8i & 10-14i & 4+4i & -4+28i & 48-8i & 9+3i & 15+45i & 66-48i\\
20+4i & 12 & 4+10i & 32+48i & 24+24i & -12+28i & 84+48i & 54+18i & 3+51i\\
0 & 2 & 4+18i & 0 & 4+4i & -28+44i & 0 & 9+3i & -9+87i
\end{bmatrix}
$$

Ans.

Problem 10

Let A be a $p \times q$ matrix and B be a $r \times s$ matrix. The tensor product of A and B, denoted as $A \otimes B$, is a $(pr) \times (qs)$ matrix defined as follows:

$$
A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B\\ a_{21}B & a_{22}B & \dots & a_{2q}B\\ \vdots & \vdots & \ddots & \vdots\\ a_{p1}B & a_{p2}B & \dots & a_{pq}B \end{bmatrix}
$$

$$
= \begin{bmatrix}
a_{11}\begin{bmatrix} b_{11} & \cdots & b_{1s}\\ \vdots & \ddots & \vdots\\ b_{r1} & \cdots & b_{rs}\end{bmatrix} & \cdots & a_{1q}\begin{bmatrix} b_{11} & \cdots & b_{1s}\\ \vdots & \ddots & \vdots\\ b_{r1} & \cdots & b_{rs}\end{bmatrix}\\
\vdots & \ddots & \vdots\\
a_{p1}\begin{bmatrix} b_{11} & \cdots & b_{1s}\\ \vdots & \ddots & \vdots\\ b_{r1} & \cdots & b_{rs}\end{bmatrix} & \cdots & a_{pq}\begin{bmatrix} b_{11} & \cdots & b_{1s}\\ \vdots & \ddots & \vdots\\ b_{r1} & \cdots & b_{rs}\end{bmatrix}
\end{bmatrix}
$$

Here, each element of $A \otimes B$ is of the form $a_{ij} \cdot b_{kl}$, and this particular element is in row $(i-1)r + k$ of the Kronecker product, and in column $(j-1)s + l$.

Now,

$$
\begin{aligned}
((A \otimes B)^\dagger)_{(j-1)s+l,(i-1)r+k} &= ((A \otimes B)_{(i-1)r+k,(j-1)s+l})^* \\
&= (a_{ij}b_{kl})^* \\
&= a_{ij}^* b_{kl}^* \\
&= (A^\dagger)_{ji}(B^\dagger)_{lk} \\
&= (A^\dagger \otimes B^\dagger)_{(j-1)s+l,(i-1)r+k}
\end{aligned}
$$

That is, each element of $(A \otimes B)^\dagger$ and $A^\dagger \otimes B^\dagger$ are equals.
Therefore, we can conclude that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.