CS7015 (Deep Learning): Lecture 4

Feedforward Neural Networks, Backpropagation

Mitesh M. Khapra

Department of Computer Science and Engineering
Indian Institute of Technology Madras

References/Acknowledgments

See the excellent videos by Hugo Larochelle on Backpropagation

Module 4.1: Feedforward Neural Networks (a.k.a. multilayered network of neurons)

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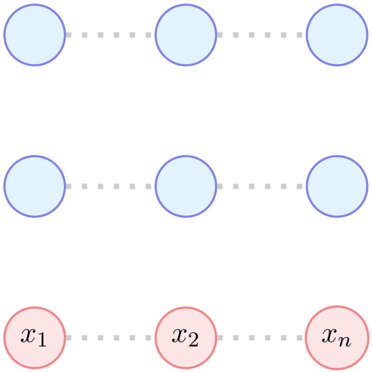

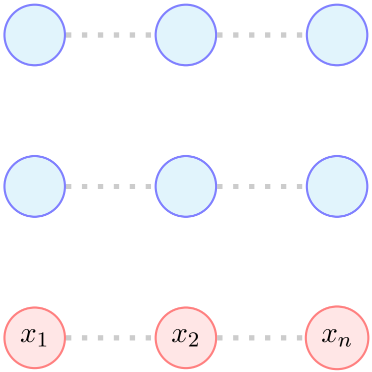

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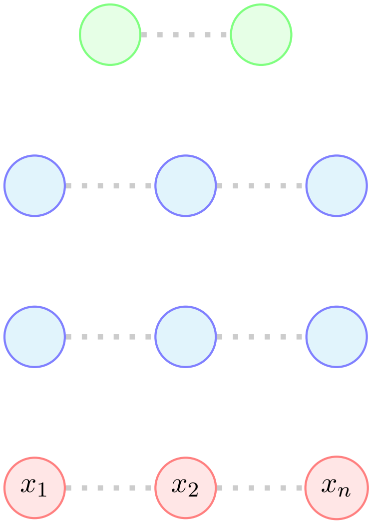

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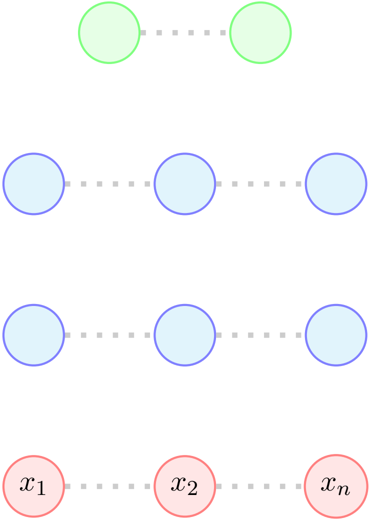

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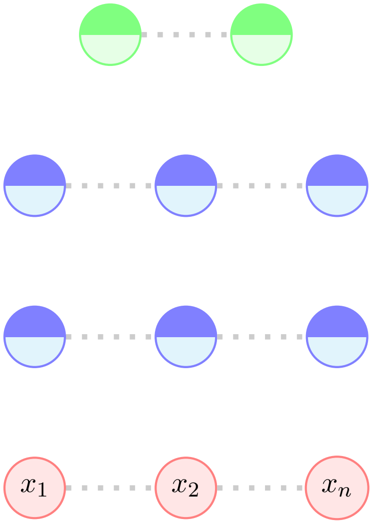

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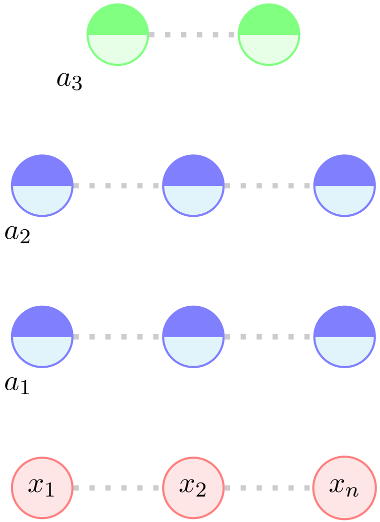

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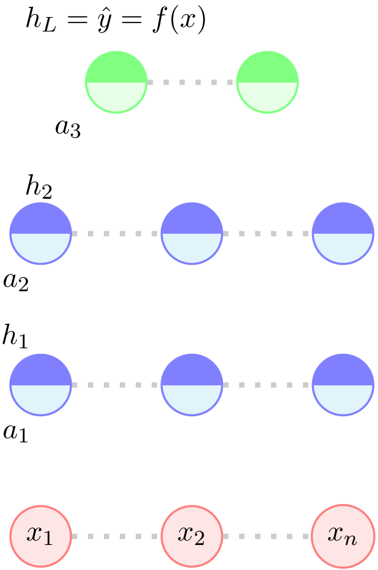

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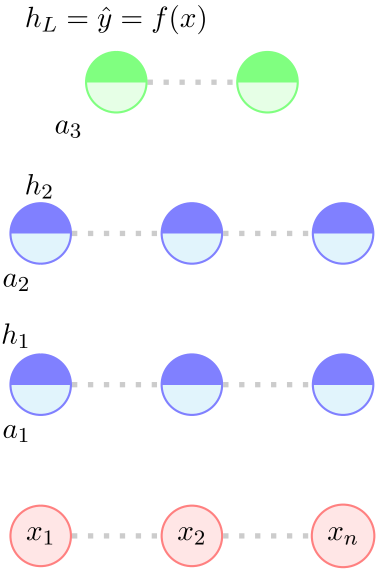

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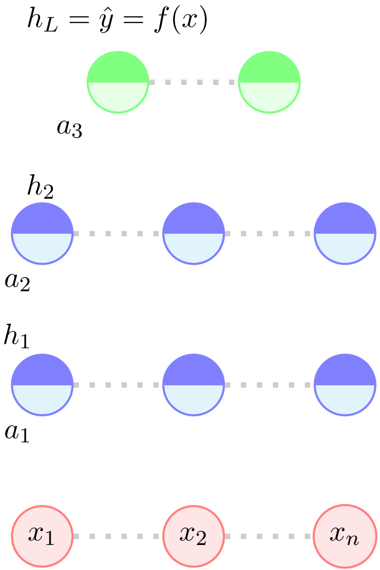

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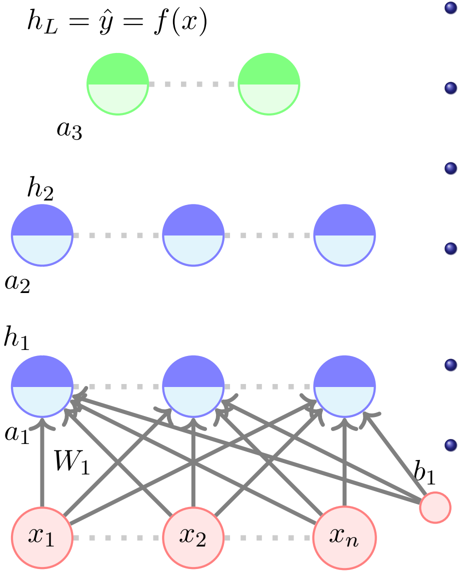

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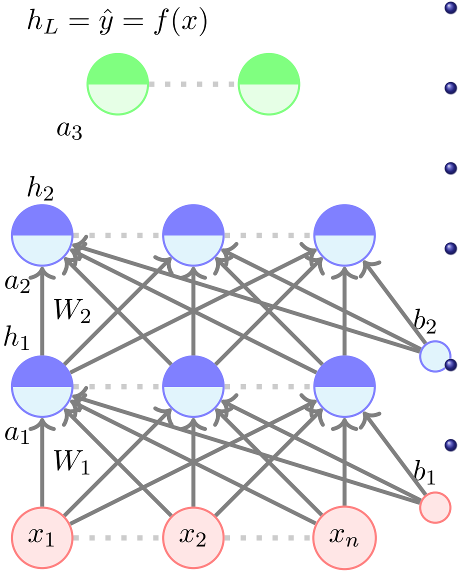

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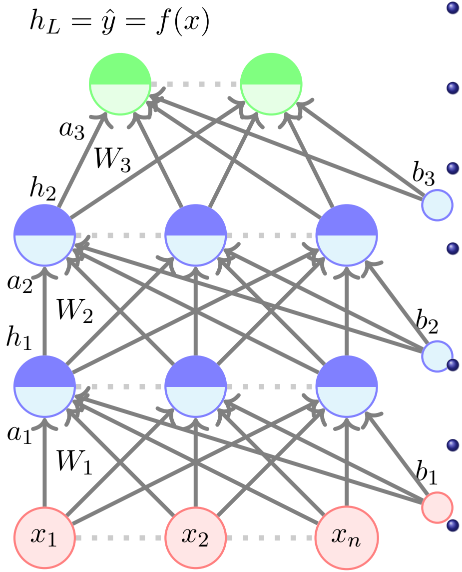

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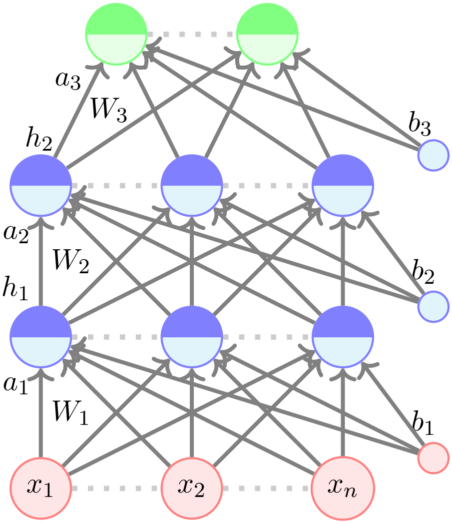
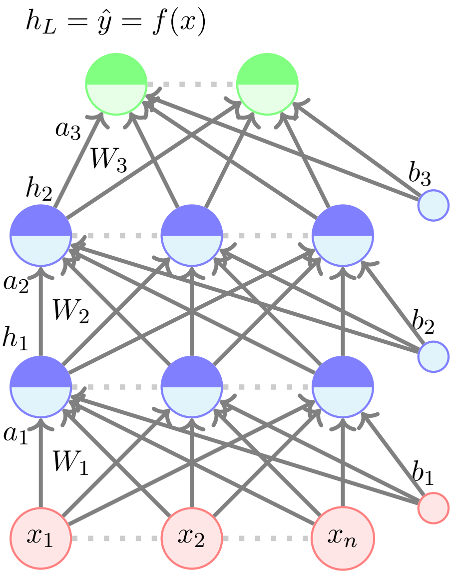
$h_L = \hat{y} = f(x)$



- The pre-activation at layer i is given by

$$a_i(x) = b_i + W_i h_{i-1}(x)$$



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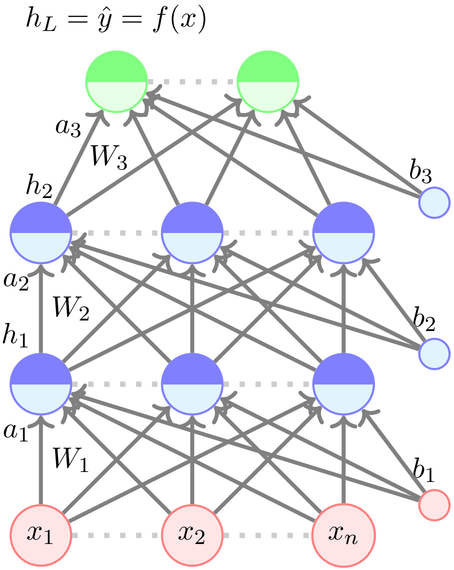

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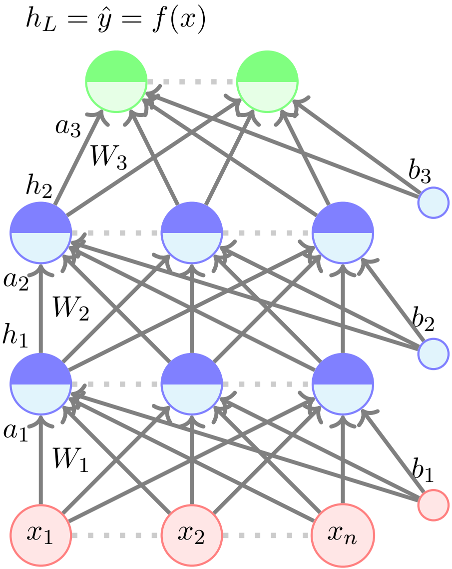

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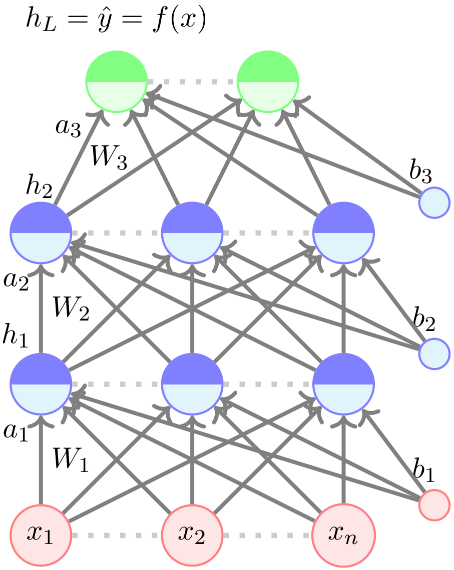

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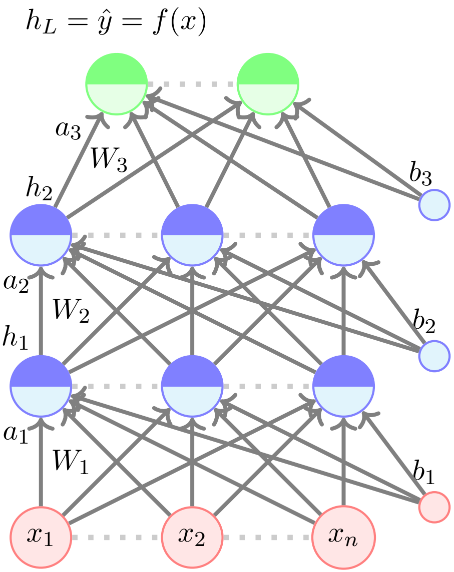

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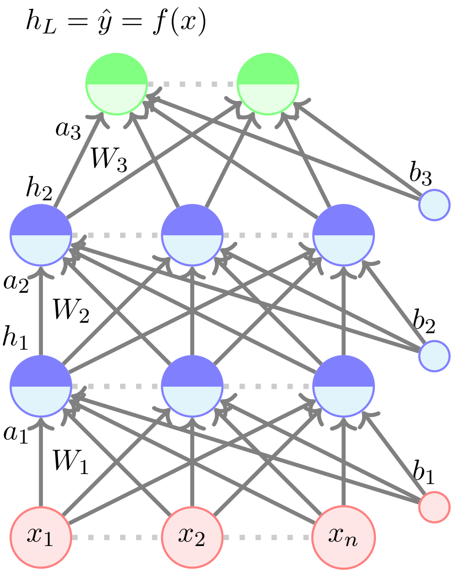
- To simplify notation we will refer to $a_i(x)$ as a_i and $h_i(x)$ as h_i



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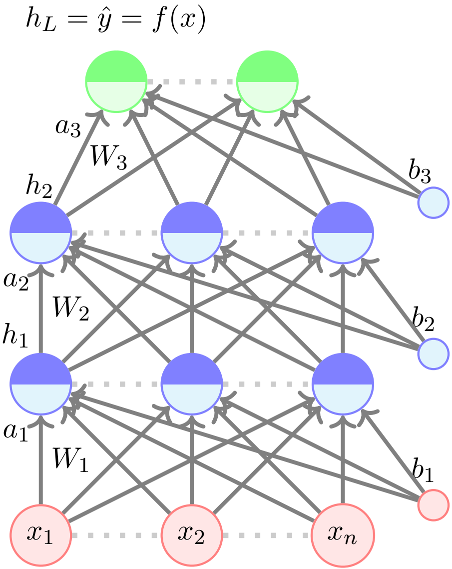
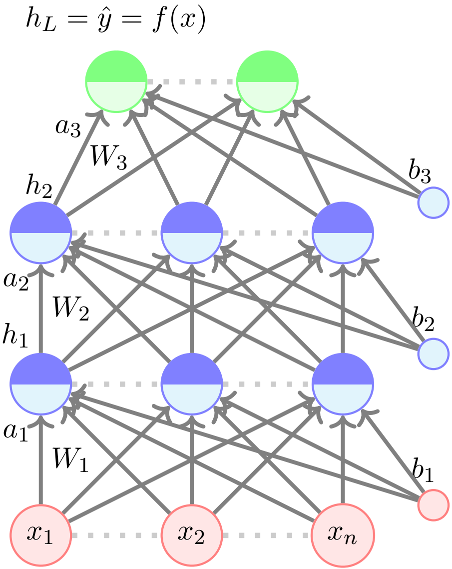


- **Data:** $\{x_i, y_i\}_{i=1}^{N}$



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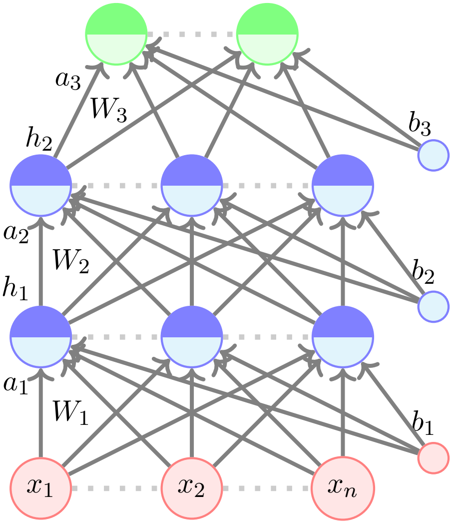

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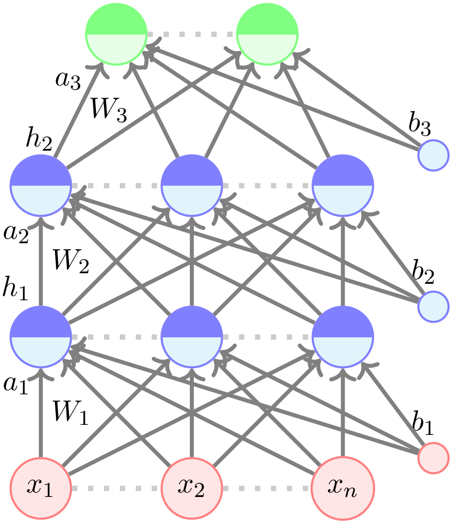

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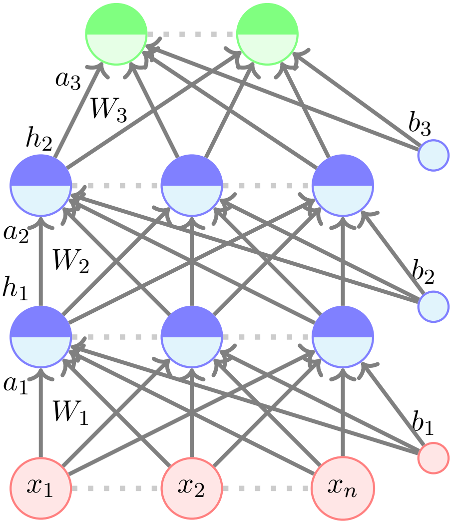

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- **Algorithm:** Backpropagation
- **Objective/Loss/Error function:** Say,

$$min \ \frac{1}{N} \sum_{i=1}^{N} (\hat{y}_i - y_i)^2$$

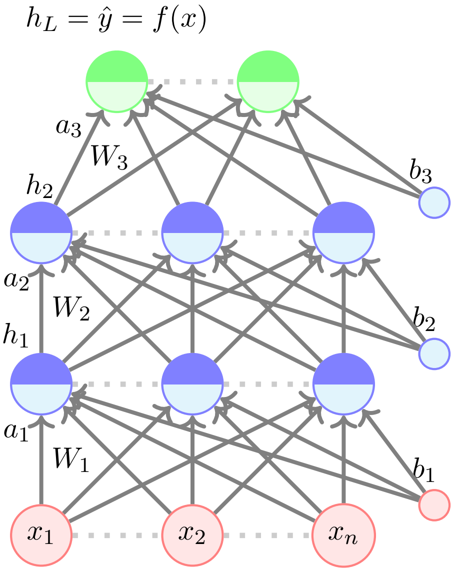
$$In \ general, \ min \ \mathscr{L}(\theta)$$

where $\mathscr{L}(\theta)$ is some function of the parameters

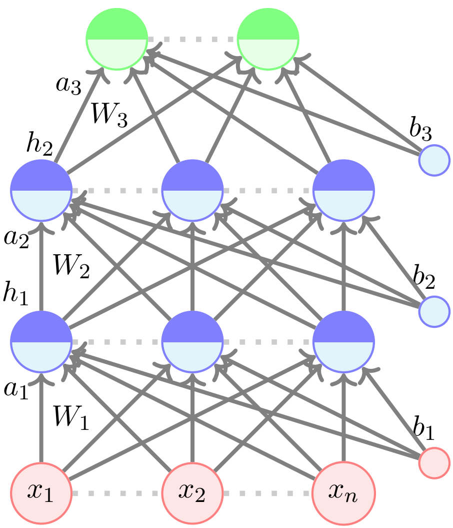
Module 4.2: Learning Parameters of Feedforward Neural Networks (Intuition)

The story so far...

- We have introduced feedforward neural networks
- We are now interested in finding an algorithm for learning the parameters of this model



- Recall our gradient descent algorithm



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Algorithm: gradient_descent()

$t \leftarrow 0;$
$max\_iterations \leftarrow 1000;$
$Initialize \quad w_0, b_0;$
while $t{+}{+} < max\_iterations$ **do**
  $w_{t+1} \leftarrow w_t - \eta \nabla w_t;$
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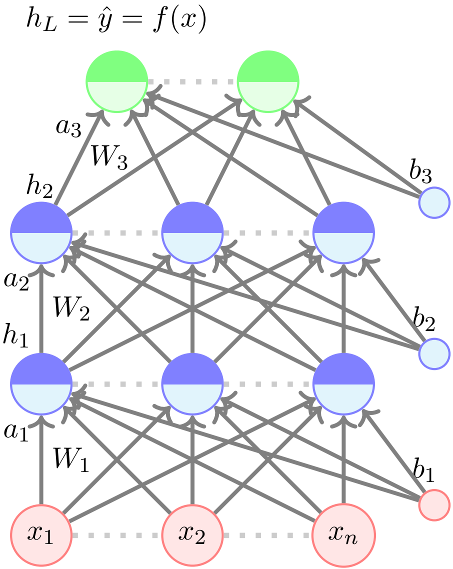

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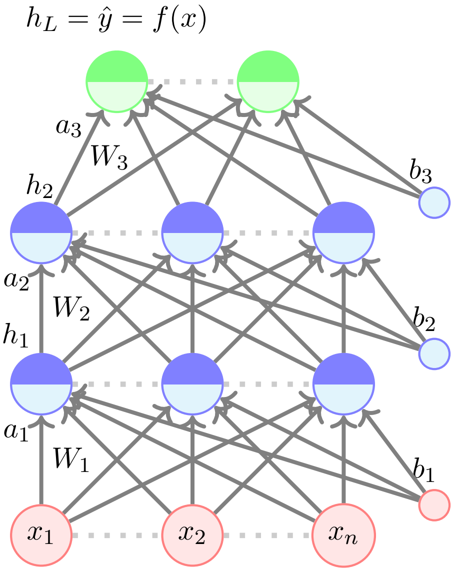

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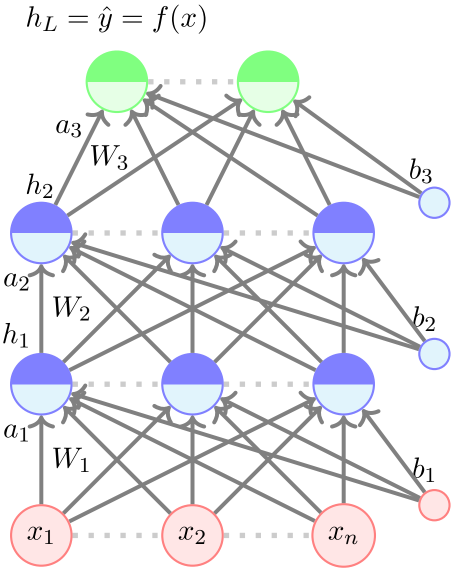

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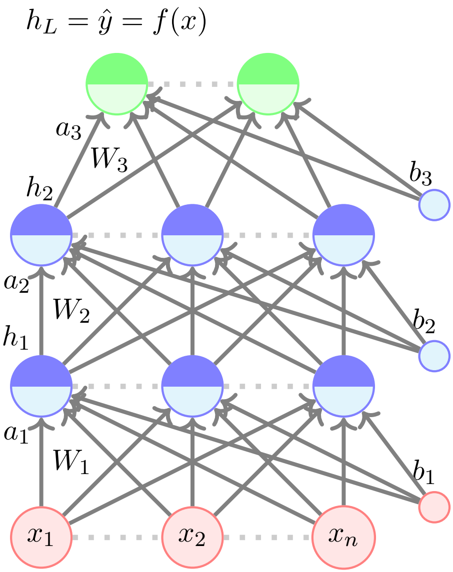

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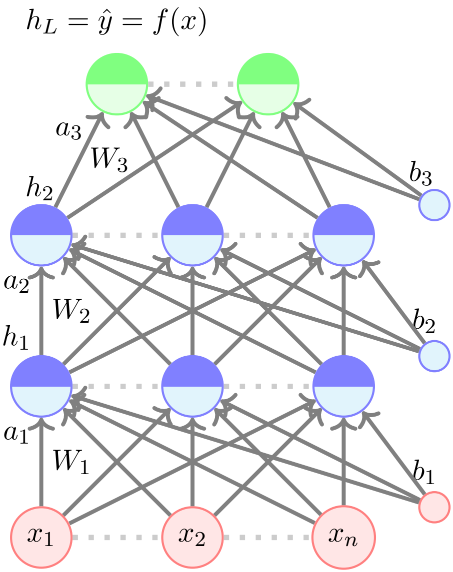

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- We can still use the same algorithm for learning the parameters of our model

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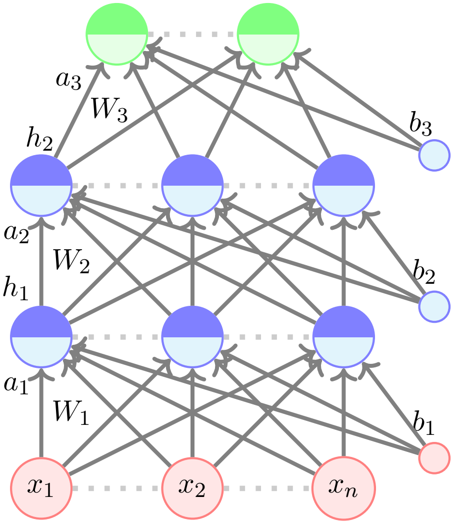

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\vdots & \vdots & \vdots \\
\frac{\partial \mathscr{L}(\theta)}{\partial W_{1n1}} & \dots & \frac{\partial \mathscr{L}(\theta)}{\partial W_{1nn}}
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- ... and similar entries for partial derivatives w.r.t. the elements of $b_1, b_2, ..., b_L$
- $\nabla\theta$ is thus composed of
  $\nabla W_1, \nabla W_2, ... \nabla W_L \in \mathbb{R}^{n\times n}, \nabla W_L \in \mathbb{R}^{n\times k}$,
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Module 4.3: Output Functions and Loss Functions

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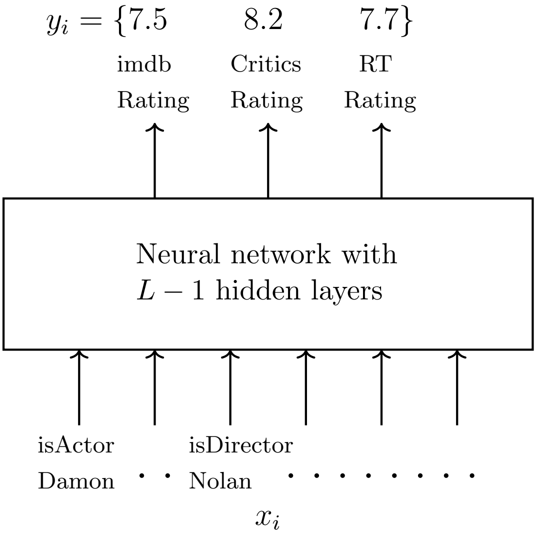

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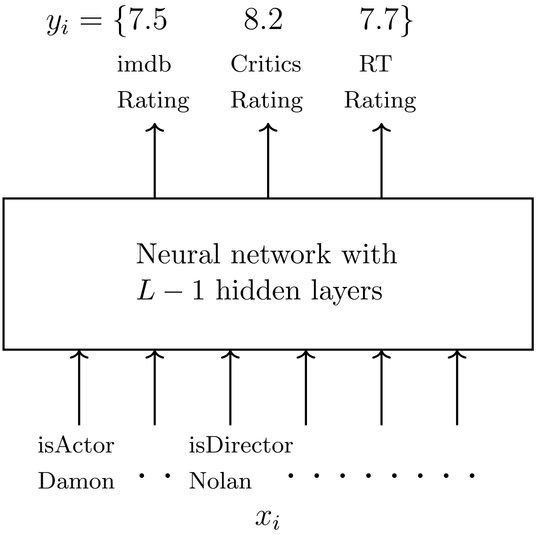

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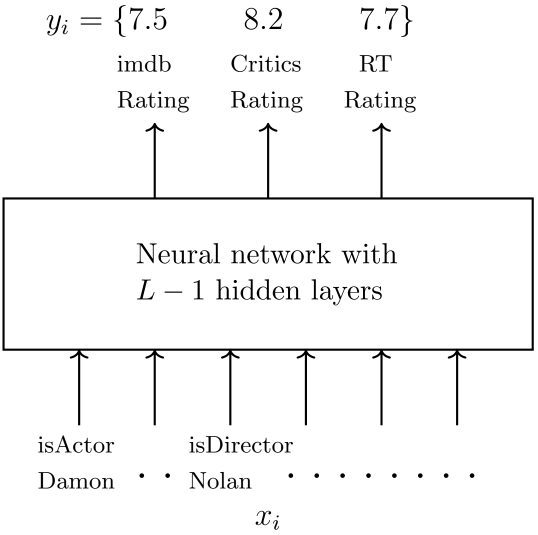

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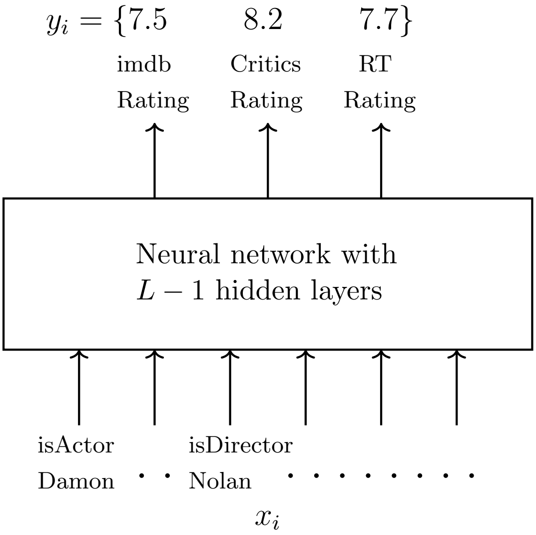

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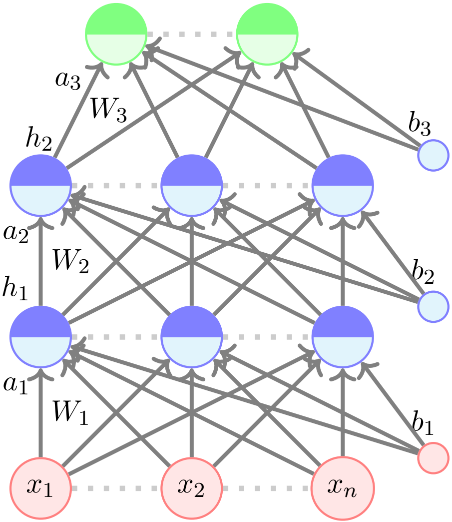
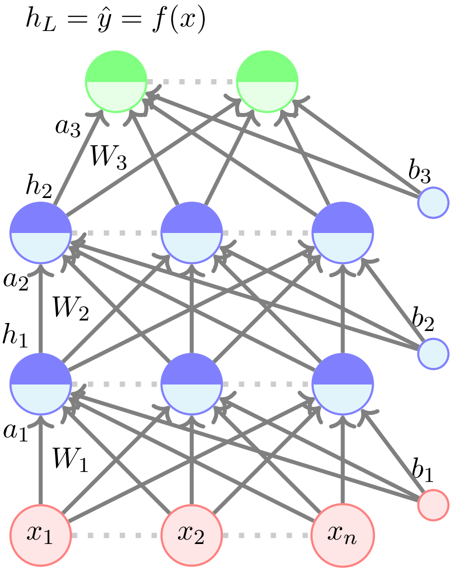
- Consider our movie example again but this time we are interested in predicting ratings
- Here $y_i \in \mathbb{R}^3$
- The loss function should capture how much $\hat{y}_i$ deviates from y_i
- If $y_i \in \mathbb{R}^n$ then the squared error loss can capture this deviation

$$\mathscr{L}(\theta) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{3}(\hat{y}_{ij} - y_{ij})^2$$

$h_L = \hat{y} = f(x)$



- A related question: What should the output function 'O' be if $y_i \in \mathbb{R}$?



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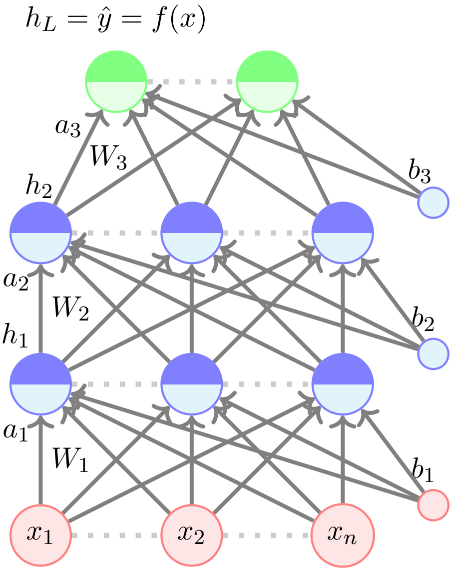

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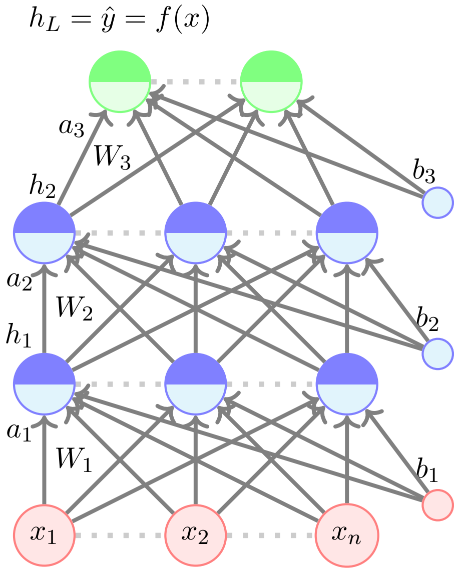

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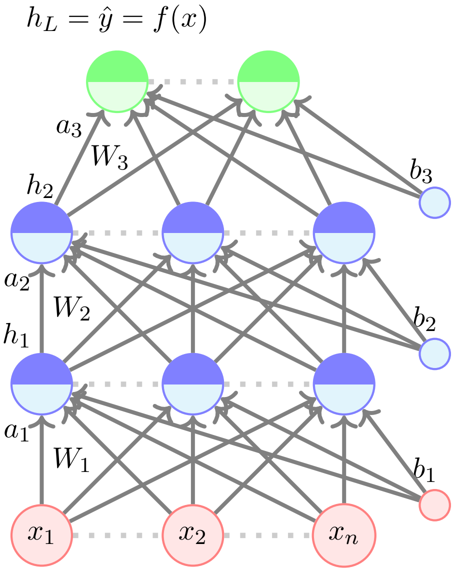
- So, in such cases it makes sense to have 'O' as identity function

$$f(x) = h_L = O(a_L)$$
$$= a_L = W_L h_{L-1} + b_L$$



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- $\hat{y}_i = f(x_i)$ is no longer bounded between 0 and 1

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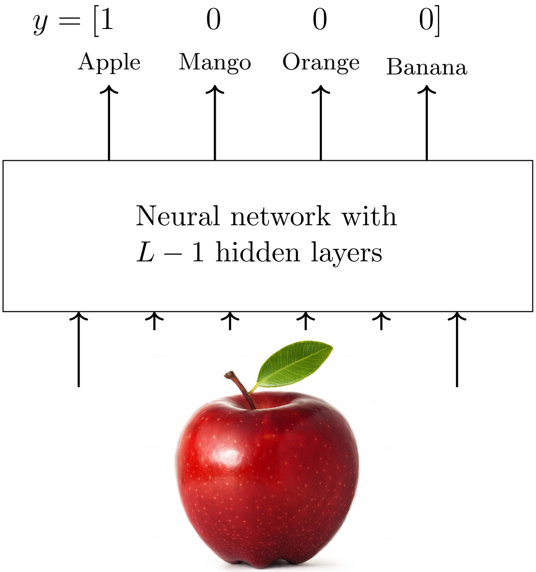
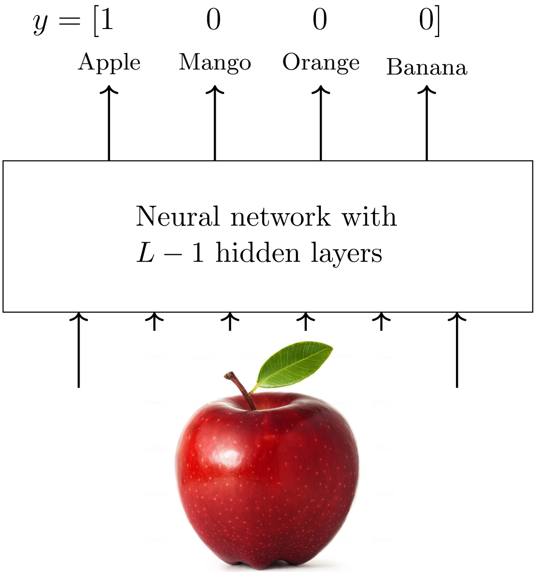


- Now let us consider another problem for which a different loss function would be appropriate



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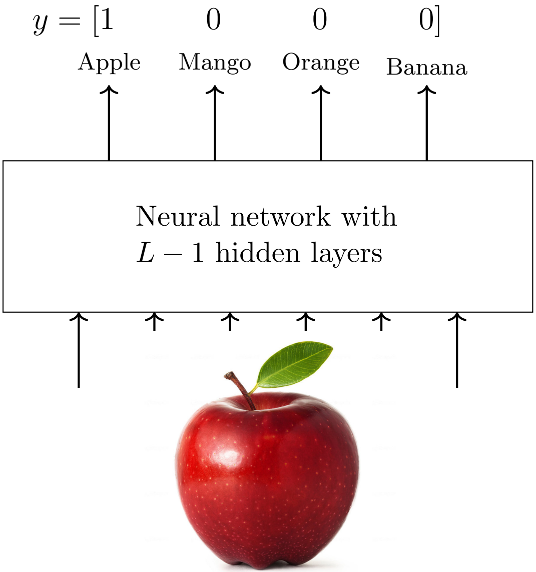

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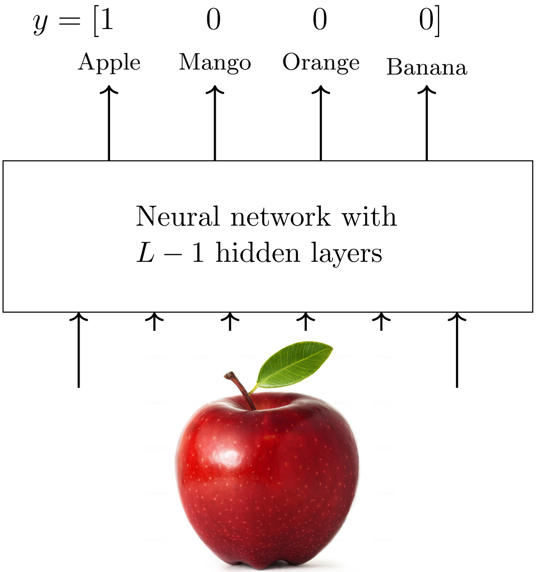

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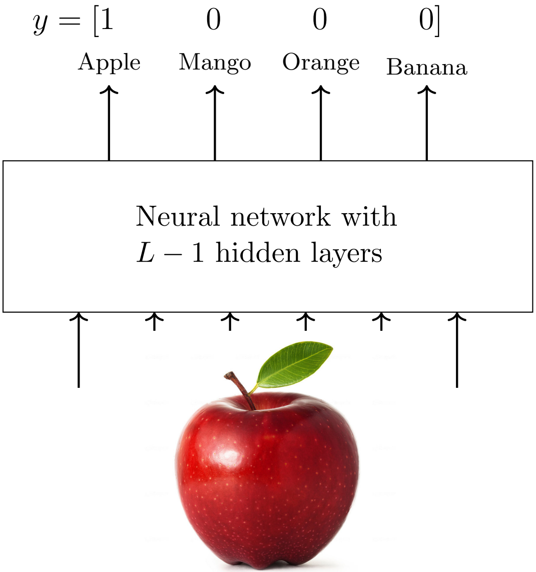
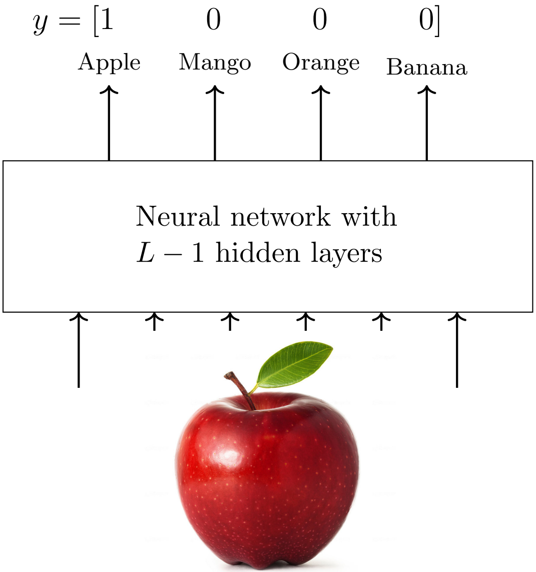
- Suppose we want to classify an image into 1 of k classes
- Here again we could use the squared error loss to capture the deviation
- But can you think of a better function?



- Notice that y is a probability distribution



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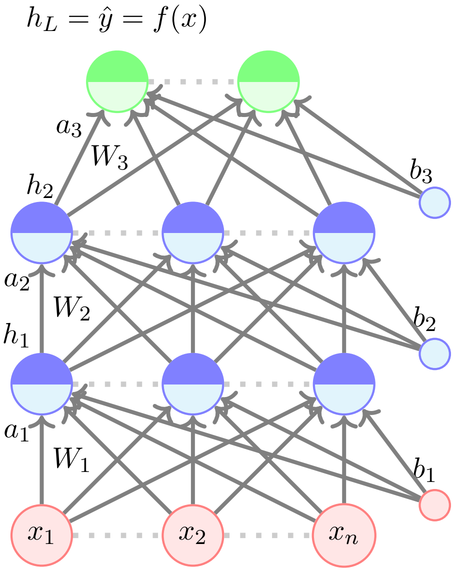

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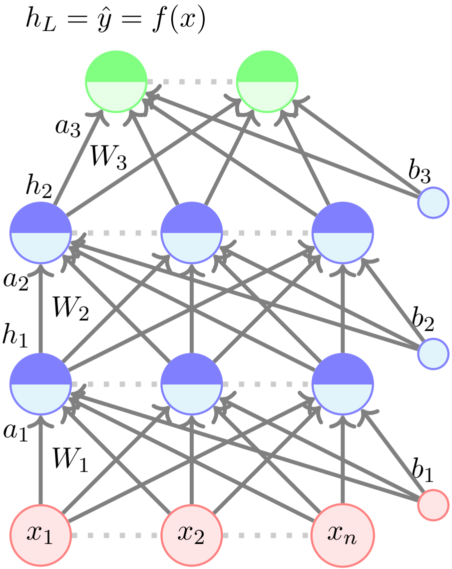
- What choice of the output activation 'O' will ensure this ?

$$a_L = W_L h_{L-1} + b_L$$



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$O(a_L)_j$ is the j^{th} element of $\hat{y}$
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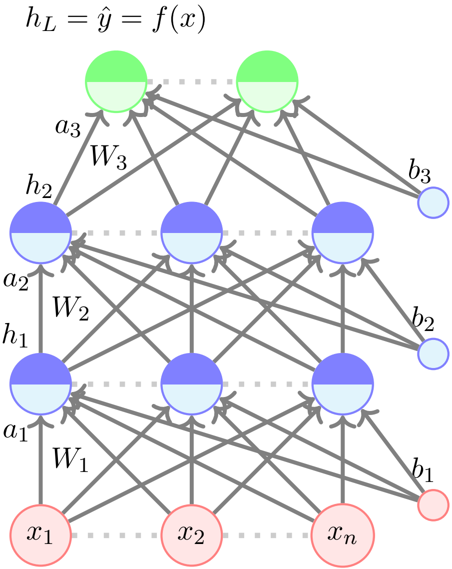

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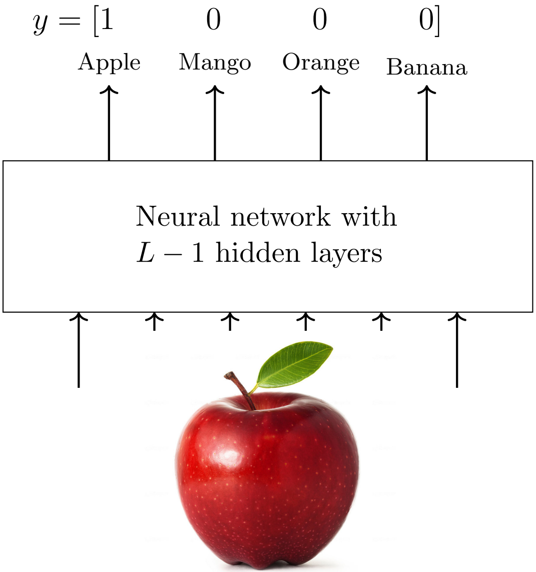
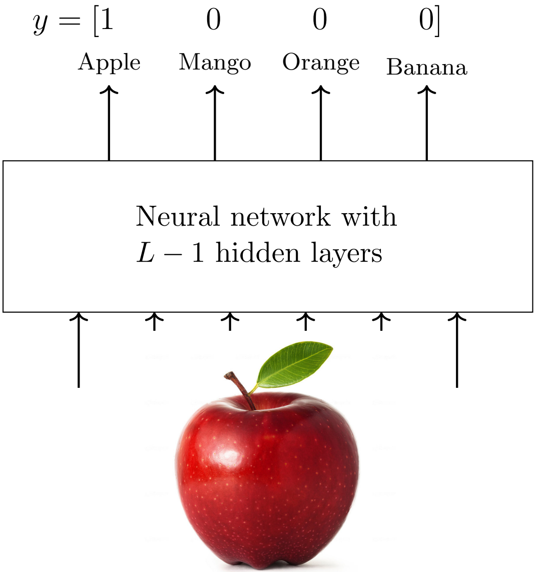
- This function is called the *softmax* function



- Now that we have ensured that both y & $\hat{y}$ are probability distributions can you think of a function which captures the difference between them?



- Now that we have ensured that both y & $\hat{y}$ are probability distributions can you think of a function which captures the difference between them?
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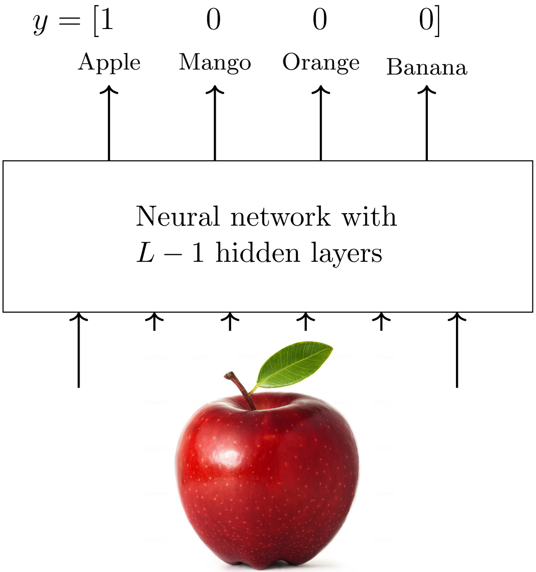
$$\mathscr{L}(\theta) = -\sum_{c=1}^{k} y_c \log \hat{y}_c$$



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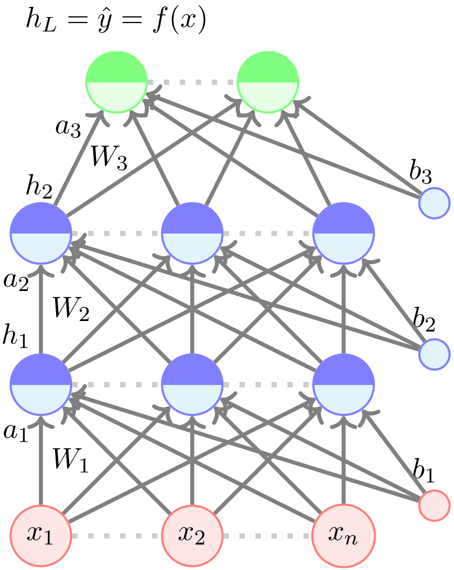
$$\mathscr{L}(\theta) = -\sum_{c=1}^{k} y_c \log \hat{y}_c$$

- Notice that

$$\begin{aligned} y_c &= 1 \quad \text{if } c = \ell \text{ (the true class label)} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

$$\mathscr{L}(\theta) = -\log \hat{y}_\ell$$



- So, for classification problem (where you have to choose 1 of K classes), we use the following objective function

$$\underset{\theta}{\text{minimize}} \quad \mathscr{L}_\theta = -\log \hat{y}_\ell$$

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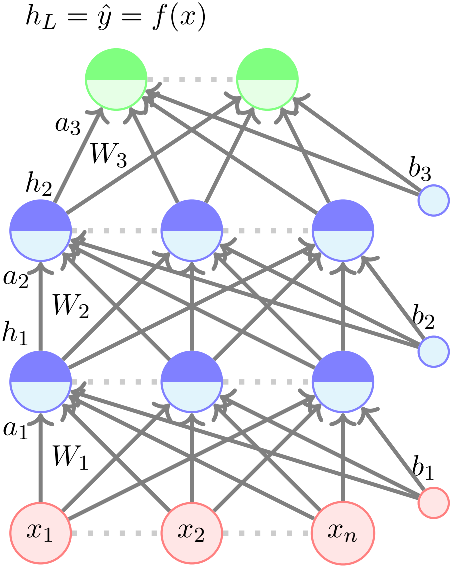

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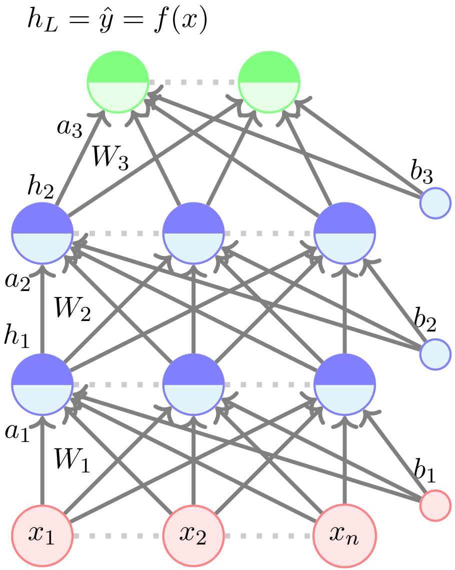

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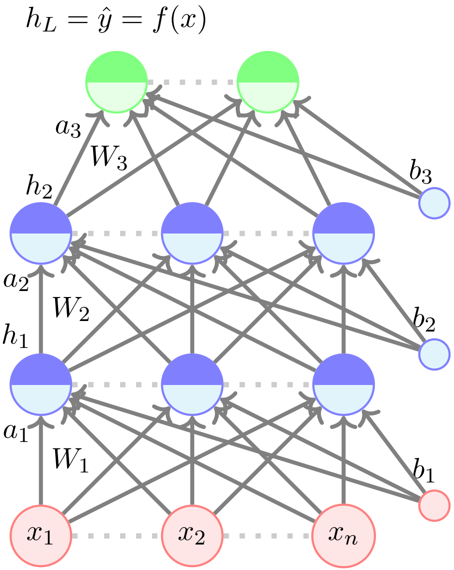

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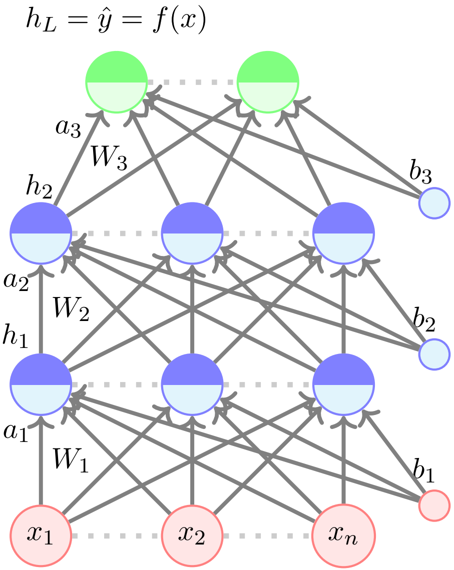

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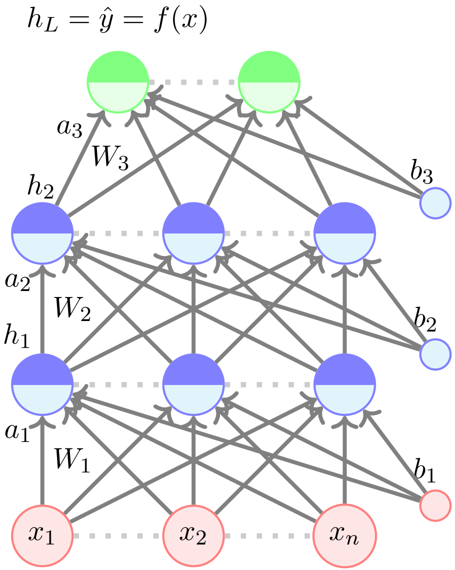
- It is the probability that x belongs to the ℓ^{th} class (bring it as close to 1).



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- It is the probability that x belongs to the ℓ^{th} class (bring it as close to 1).
- $\log \hat{y}_\ell$ is called the *log-likelihood* of the data.

	Outputs	
	Real Values	Probabilities
Output Activation		
Loss Function		

	Outputs	
	Real Values	Probabilities
Output Activation	Linear	
Loss Function		

	Outputs	
	Real Values	Probabilities
Output Activation	Linear	Softmax
Loss Function		

	Outputs	
	Real Values	Probabilities
Output Activation	Linear	Softmax
Loss Function	Squared Error	

	Outputs	
	Real Values	Probabilities
Output Activation	Linear	Softmax
Loss Function	Squared Error	Cross Entropy

	Outputs	
	Real Values	Probabilities
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- Of course, there could be other loss functions depending on the problem at hand but the two loss functions that we just saw are encountered very often

	Outputs	
	Real Values	Probabilities
Output Activation	Linear	Softmax
Loss Function	Squared Error	Cross Entropy

- Of course, there could be other loss functions depending on the problem at hand but the two loss functions that we just saw are encountered very often
- For the rest of this lecture we will focus on the case where the output activation is a softmax function and the loss function is cross entropy

Module 4.4: Backpropagation (Intuition)

We need to answer two questions

- How to choose the loss function $\mathscr{L}(\theta)$?
- How to compute $\nabla\theta$ which is composed of $\nabla W_1, \nabla W_2, ..., \nabla W_{L-1} \in \mathbb{R}^{n \times n}, \nabla W_L \in \mathbb{R}^{n \times k}$, $\nabla b_1, \nabla b_2, ..., \nabla b_{L-1} \in \mathbb{R}^{n}$ and $\nabla b_L \in \mathbb{R}^{k}$?

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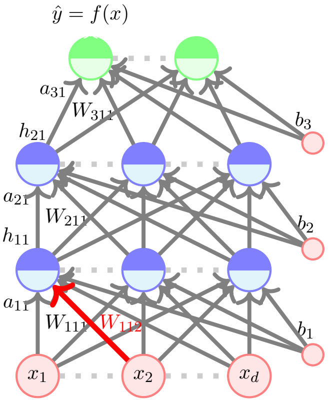
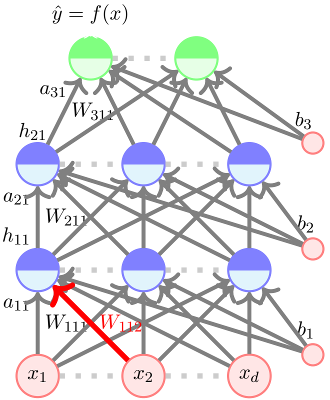
- Let us focus on this one weight.



Algorithm: gradient descent()

$t \leftarrow 0;$
$max\_iterations \leftarrow 1000;$
$Initialize \quad \theta_0;$
while $t{+}{+} < max\_iterations$ **do**
  $\theta_{t+1} \leftarrow \theta_t - \eta \nabla \theta_t;$
end

- Let us focus on this one weight.
- To learn this weight using SGD we need a formula for $\frac{\partial \mathscr{L}(\theta)}{\partial W_{112}}$.



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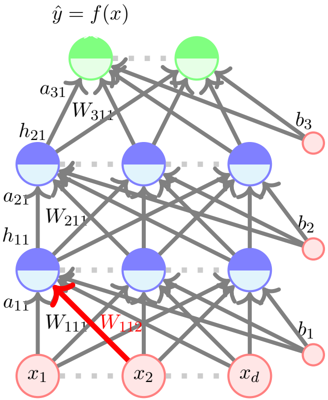
end

- Let us focus on this one weight.
- To learn this weight using SGD we need a formula for $\frac{\partial \mathscr{L}(\theta)}{\partial W_{112}}$.
- We will see how to calculate this.



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$t \leftarrow 0$;
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$Initialize \quad \theta_0$;
while $t{+}{+} < max\_iterations$ **do**
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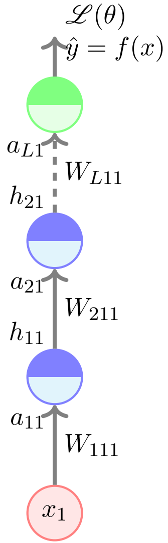
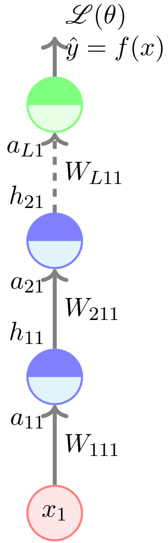
- First let us take the simple case when we have a deep but thin network.



- First let us take the simple case when we have a deep but thin network.
- In this case it is easy to find the derivative by chain rule.



- First let us take the simple case when we have a deep but thin network.
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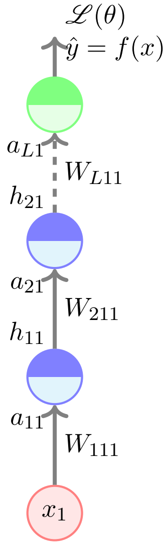
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{111}} = \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_{L11}} \frac{\partial a_{L11}}{\partial h_{21}} \frac{\partial h_{21}}{\partial a_{21}} \frac{\partial a_{21}}{\partial h_{11}} \frac{\partial h_{11}}{\partial a_{11}} \frac{\partial a_{11}}{\partial W_{111}}$$



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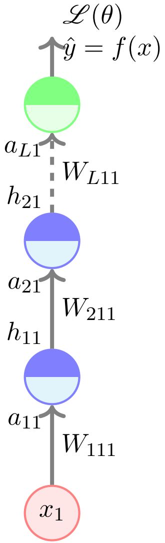
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{111}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{11}} \frac{\partial h_{11}}{\partial W_{111}} \qquad \text{(just compressing the chain rule)}$$



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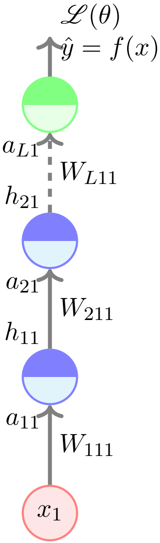
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{211}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{21}} \frac{\partial h_{21}}{\partial W_{211}}$$



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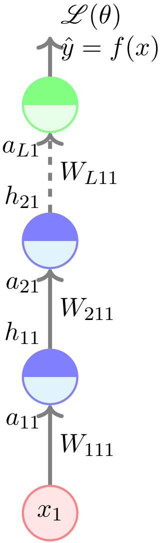
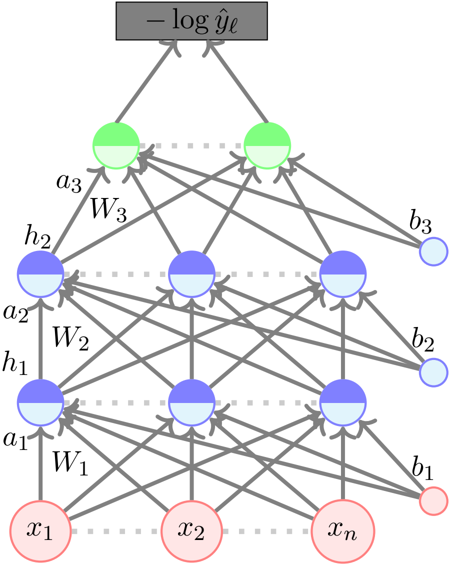
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{211}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{21}} \frac{\partial h_{21}}{\partial W_{211}}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{L11}} = \frac{\partial \mathscr{L}(\theta)}{\partial a_{L1}} \frac{\partial a_{L1}}{\partial W_{L11}}$$

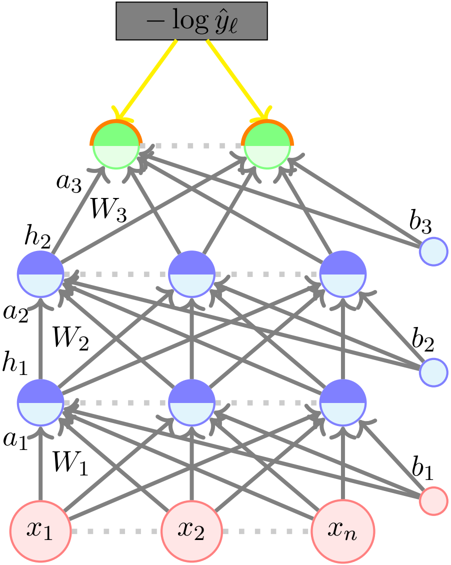


Let us see an intuitive explanation of backpropagation before we get into the mathematical details

- We get a certain loss at the output and we try to figure out who is responsible for this loss

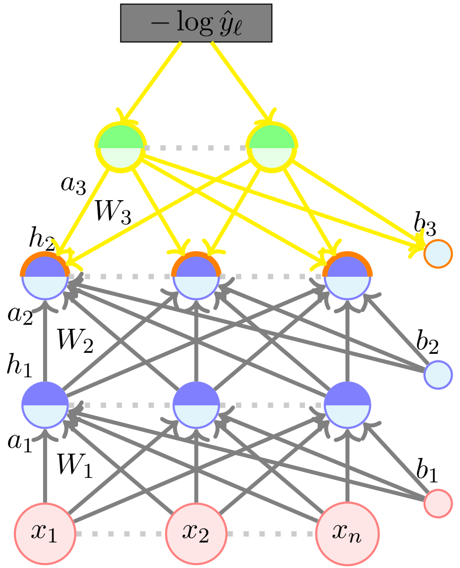


- We get a certain loss at the output and we try to figure out who is responsible for this loss
- So, we talk to the output layer and say "Hey! You are not producing the desired output, better take responsibility".

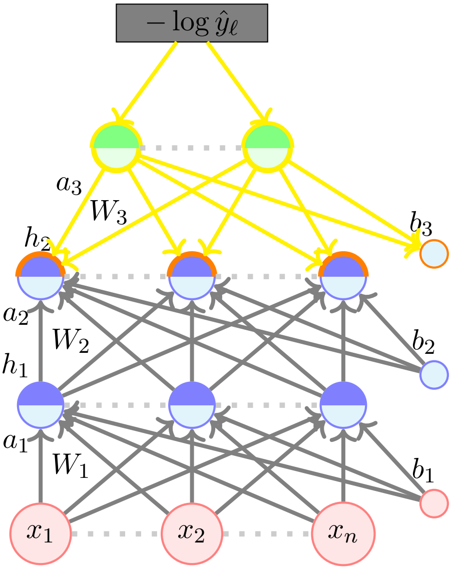


- We get a certain loss at the output and we try to figure out who is responsible for this loss
- So, we talk to the output layer and say "Hey! You are not producing the desired output, better take responsibility".
- The output layer says "Well, I take responsibility for my part but please understand that I am only as the good as the hidden layer and weights below me". After all ...

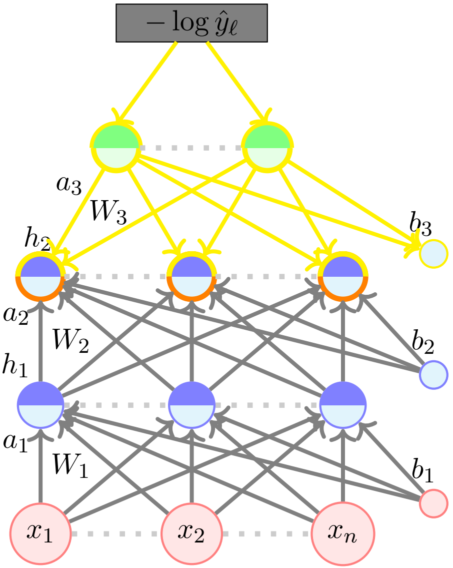
$$f(x) = \hat{y} = O(W_L h_{L-1} + b_L)$$



- So, we talk to W_L, b_L and h_L and ask them "What is wrong with you?"



- So, we talk to W_L, b_L and h_L and ask them "What is wrong with you?"
- W_L and b_L take full responsibility but h_L says "Well, please understand that I am only as good as the pre-activation layer"



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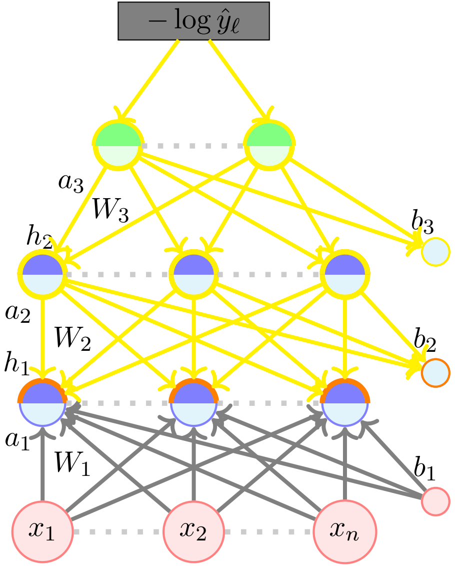
- The pre-activation layer in turn says that I am only as good as the hidden layer and weights below me.



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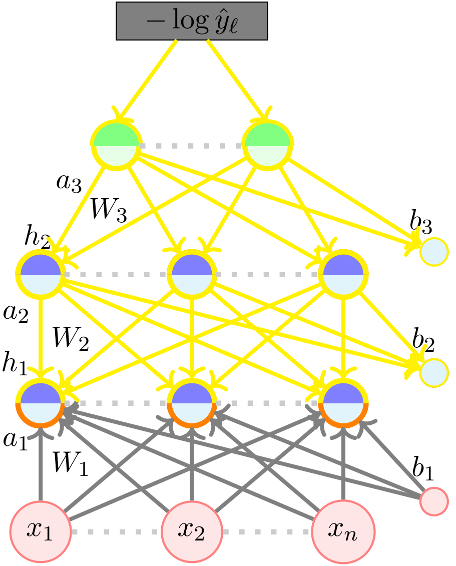
- We continue in this manner and realize that the responsibility lies with all the weights and biases (i.e. all the parameters of the model)



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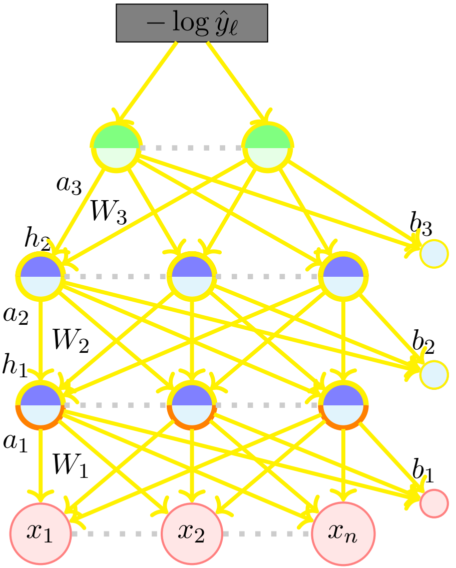

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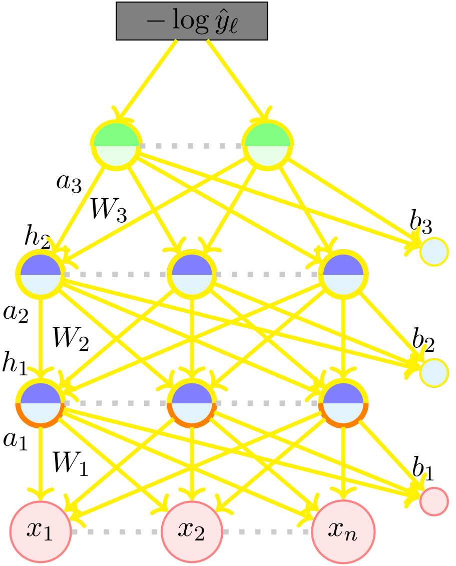
$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial f(x)} \frac{\partial f(x)}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\text{and now talk to the weights}}$$



$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the}\\\text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\substack{\text{Talk to the}\\\text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the}\\\text{previous hidden}\\\text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the}\\\text{previous}\\\text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now}\\\text{talk to}\\\text{the}\\\text{weights}}}$$

Quantities of interest (roadmap for the remaining part):

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the}\\\text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\substack{\text{Talk to the}\\\text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the}\\\text{previous hidden}\\\text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the}\\\text{previous}\\\text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now}\\\text{talk to}\\\text{the}\\\text{weights}}}$$

Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the}\\ \text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\substack{\text{Talk to the}\\ \text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the}\\ \text{previous hidden}\\ \text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the}\\ \text{previous}\\ \text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now}\\ \text{talk to}\\ \text{the}\\ \text{weights}}}$$

Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the}\\ \text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\substack{\text{Talk to the}\\ \text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the}\\ \text{previous hidden}\\ \text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the}\\ \text{previous}\\ \text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now}\\ \text{talk to}\\ \text{the}\\ \text{weights}}}$$

Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\text{and now talk to the weights}}$$

Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the} \\ \text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\substack{\text{Talk to the} \\ \text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the} \\ \text{previous hidden} \\ \text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the} \\ \text{previous} \\ \text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now} \\ \text{talk to} \\ \text{the} \\ \text{weights}}}$$

- Our focus is on *Cross entropy loss* and *Softmax* output.

Module 4.5: Backpropagation: Computing Gradients w.r.t. the Output Units

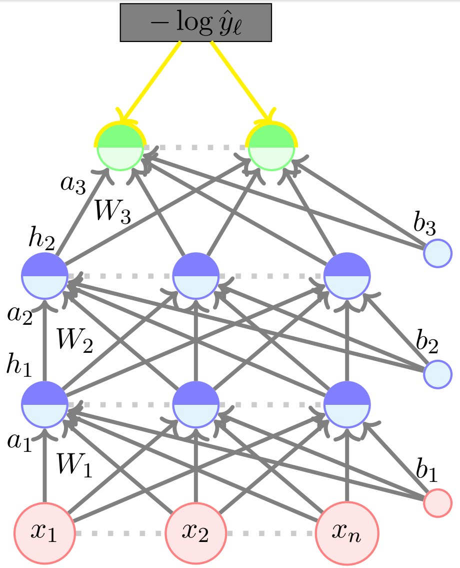
Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the} \\ \text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\substack{\text{Talk to the} \\ \text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the} \\ \text{previous hidden} \\ \text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the} \\ \text{previous} \\ \text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now} \\ \text{talk to} \\ \text{the} \\ \text{weights}}}$$

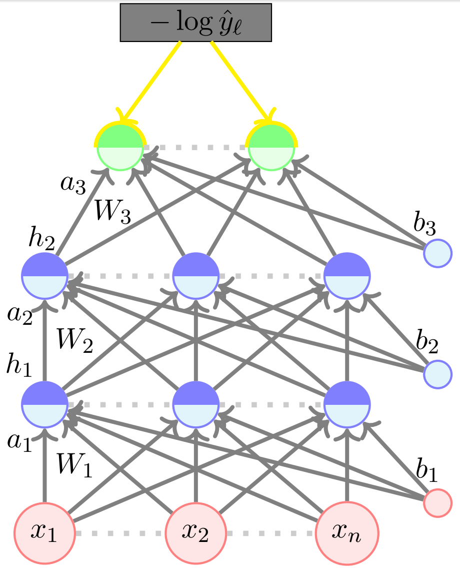
- Our focus is on *Cross entropy loss* and *Softmax* output.

Let us first consider the partial derivative w.r.t. i-th output



Let us first consider the partial derivative w.r.t. i-th output

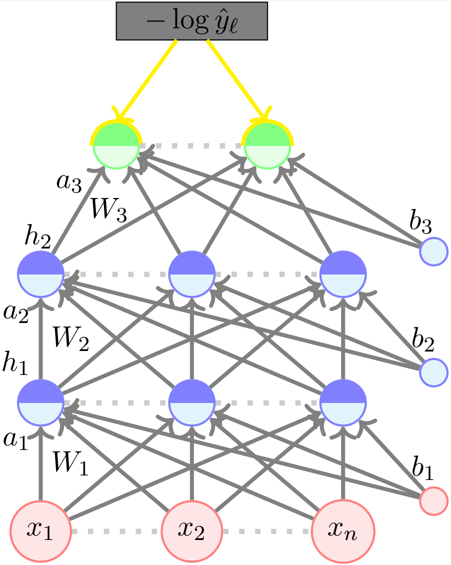
$$\mathscr{L}(\theta) = -\log \hat{y}_\ell \quad (\ell = \text{true class label})$$



Let us first consider the partial derivative w.r.t. i-th output

$$\mathscr{L}(\theta) = -\log \hat{y}_\ell \quad (\ell = \text{true class label})$$

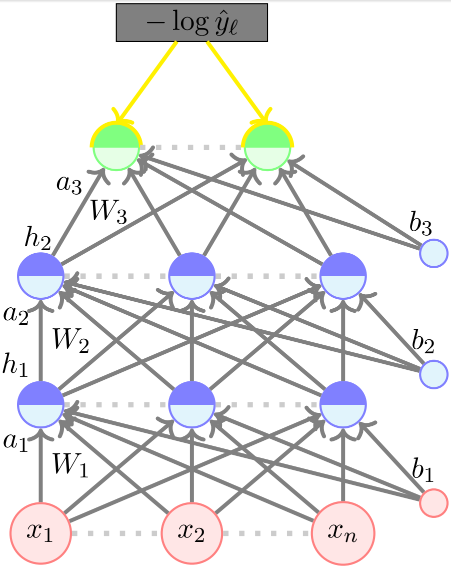
$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) =$$



Let us first consider the partial derivative w.r.t. i-th output

$$\mathscr{L}(\theta) = -\log \hat{y}_\ell \quad (\ell = \text{true class label})$$

$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = \frac{\partial}{\partial \hat{y}_i}(-\log \hat{y}_\ell)$$

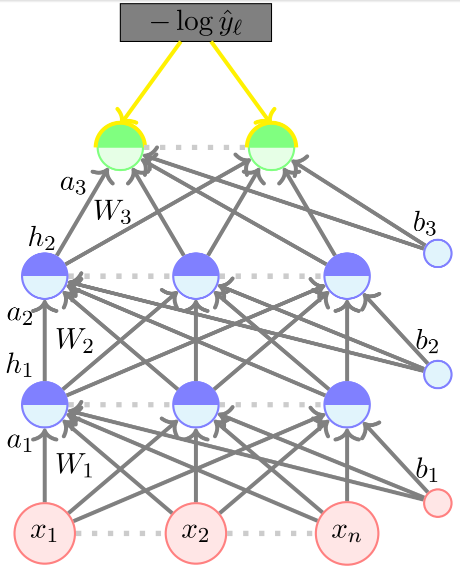


Let us first consider the partial derivative w.r.t. i-th output

$$\mathscr{L}(\theta) = -\log \hat{y}_\ell \quad (\ell = \text{true class label})$$

$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = \frac{\partial}{\partial \hat{y}_i}(-\log \hat{y}_\ell)$$

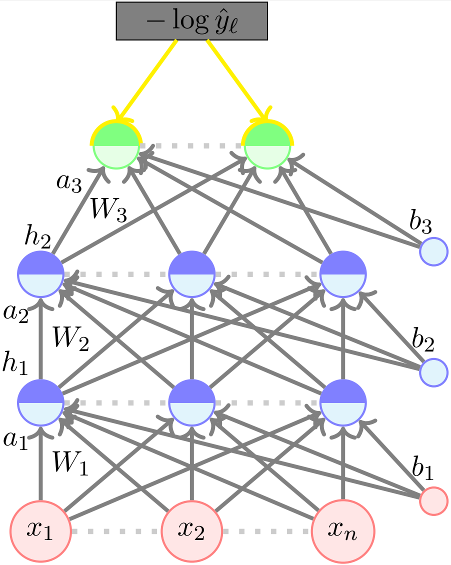
$$= -\frac{1}{\hat{y}_\ell} \quad \text{if } i = \ell$$



Let us first consider the partial derivative w.r.t. i-th output

$$\mathscr{L}(\theta) = -\log \hat{y}_\ell \quad (\ell = \text{true class label})$$

$$\begin{aligned}\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) &= \frac{\partial}{\partial \hat{y}_i}(-\log \hat{y}_\ell) \\ &= -\frac{1}{\hat{y}_\ell} \quad \text{if } i = \ell \\ &= \quad 0 \qquad \textit{otherwise}\end{aligned}$$



Let us first consider the partial derivative w.r.t. i-th output

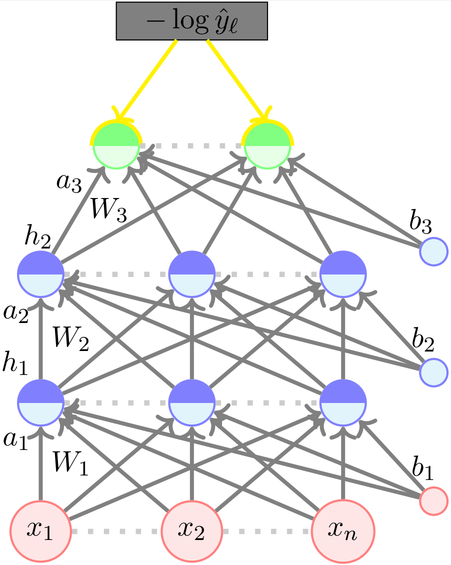
$$\mathscr{L}(\theta) = -\log \hat{y}_\ell \quad (\ell = \text{true class label})$$

$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = \frac{\partial}{\partial \hat{y}_i}(-\log \hat{y}_\ell)$$

$$= -\frac{1}{\hat{y}_\ell} \quad \text{if } i = \ell$$

$$= 0 \qquad otherwise$$

More compactly,



Let us first consider the partial derivative w.r.t. i-th output

$$\mathscr{L}(\theta) = -\log \hat{y}_\ell \quad (\ell = \text{true class label})$$

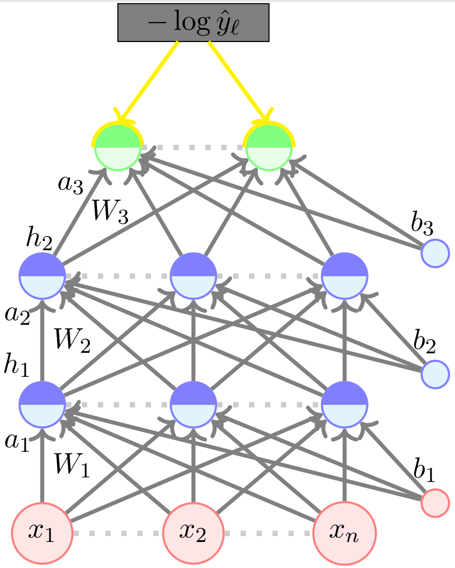
$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = \frac{\partial}{\partial \hat{y}_i}(-\log \hat{y}_\ell)$$

$$= -\frac{1}{\hat{y}_\ell} \quad \text{if } i = \ell$$

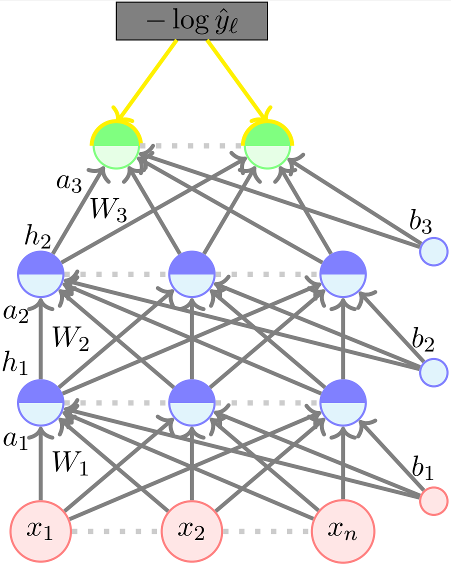
$$= \quad 0 \qquad otherwise$$

More compactly,

$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = -\frac{\mathbb{1}_{(i=\ell)}}{\hat{y}_\ell}$$

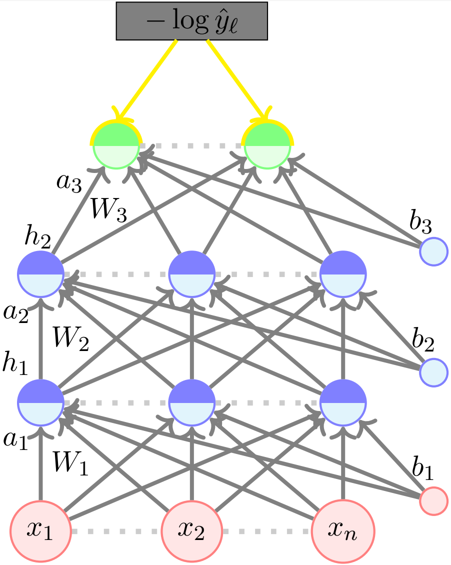


$$\frac{\partial}{\partial \hat{y}_i}\left(\mathscr{L}(\theta)\right) = -\frac{\mathbb{1}_{(\ell=i)}}{\hat{y}_\ell}$$



$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = -\frac{\mathbb{1}_{(\ell=i)}}{\hat{y}_\ell}$$

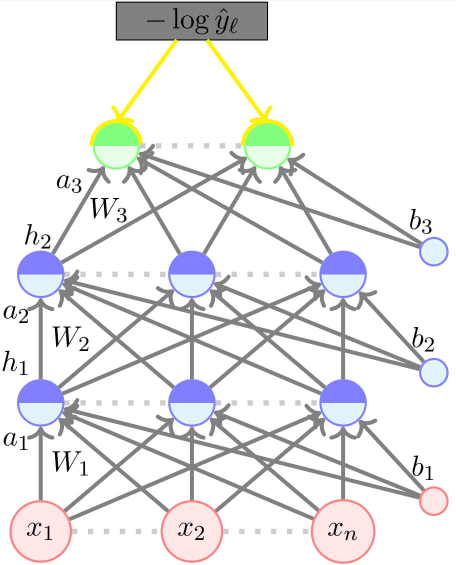
We can now talk about the gradient w.r.t. the vector $\hat{y}$



$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = -\frac{\mathbb{1}_{(\ell=i)}}{\hat{y}_\ell}$$

We can now talk about the gradient w.r.t. the vector $\hat{y}$

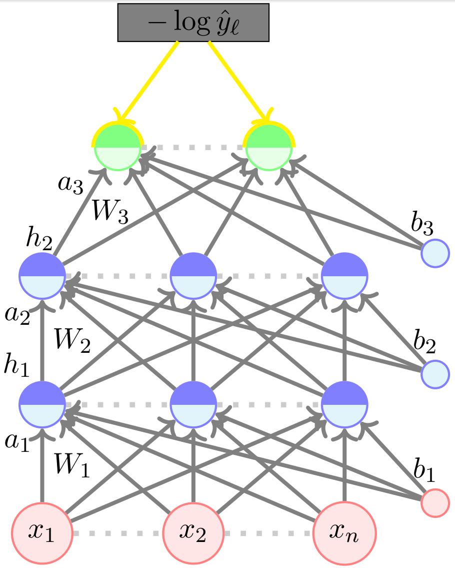
$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) \quad = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = -\frac{\mathbb{1}_{(\ell=i)}}{\hat{y}_\ell}$$

We can now talk about the gradient w.r.t. the vector $\hat{y}$

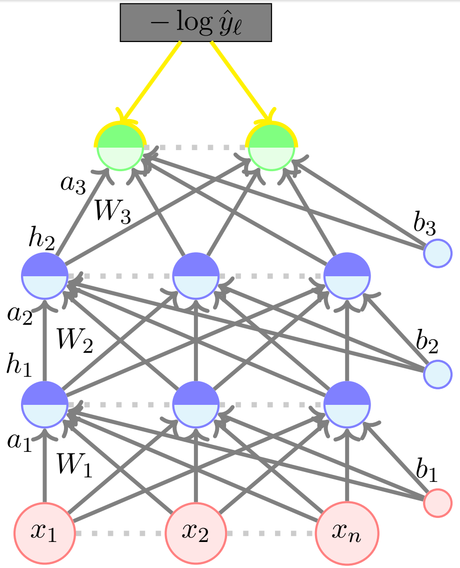
$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \\ \\ \end{bmatrix}$$



$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = -\frac{\mathbb{1}_{(\ell=i)}}{\hat{y}_\ell}$$

We can now talk about the gradient w.r.t. the vector $\hat{y}$

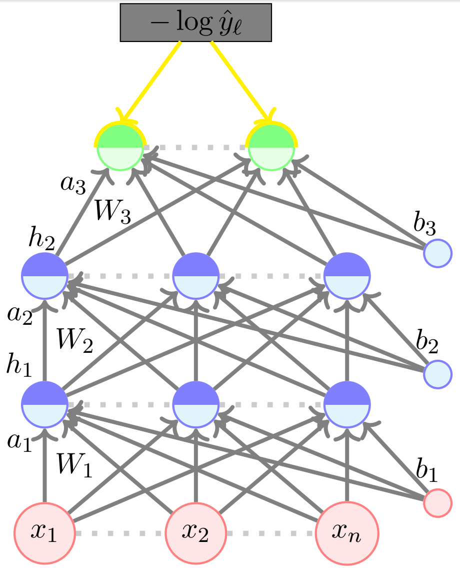
$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \\ \end{bmatrix}$$



$$\frac{\partial}{\partial \hat{y}_i}(\mathscr{L}(\theta)) = -\frac{\mathbb{1}_{(\ell=i)}}{\hat{y}_\ell}$$

We can now talk about the gradient w.r.t. the vector $\hat{y}$

$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_k} \end{bmatrix}$$



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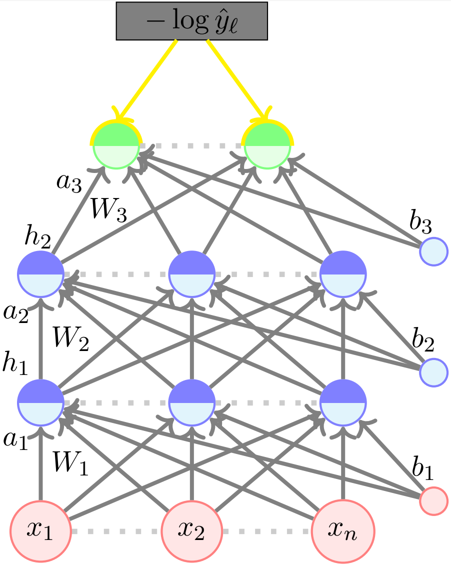

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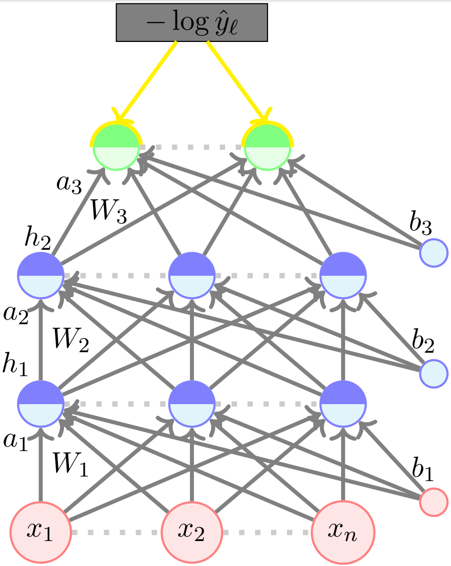

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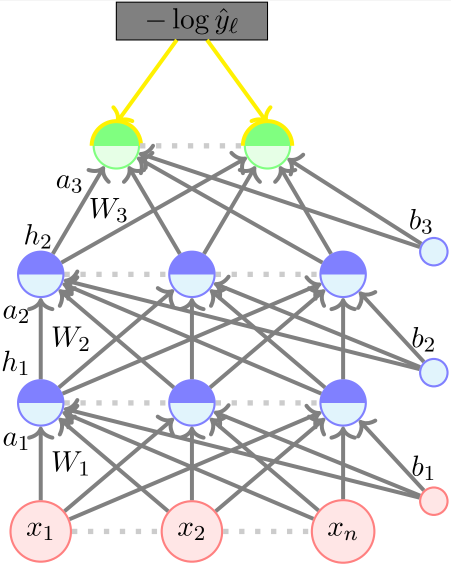
$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_k} \end{bmatrix} = -\frac{1}{\hat{y}_\ell}\begin{bmatrix} \mathbb{1}_{\ell=1} \\ \\ \\ \end{bmatrix}$$



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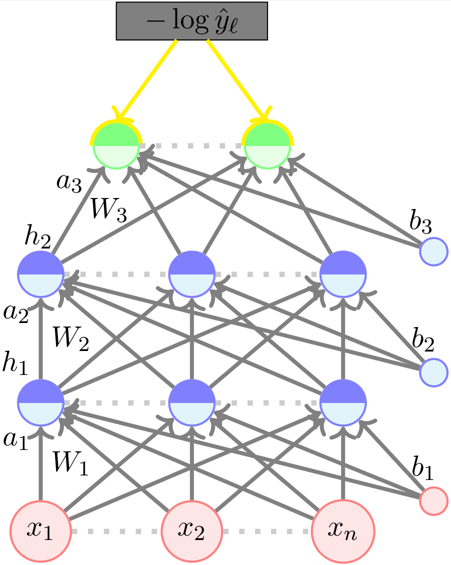
$$\nabla_{\hat{\mathbf{y}}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_k} \end{bmatrix} = -\frac{1}{\hat{y}_\ell} \begin{bmatrix} \mathbb{1}_{\ell=1} \\ \mathbb{1}_{\ell=2} \\ \\ \\ \end{bmatrix}$$



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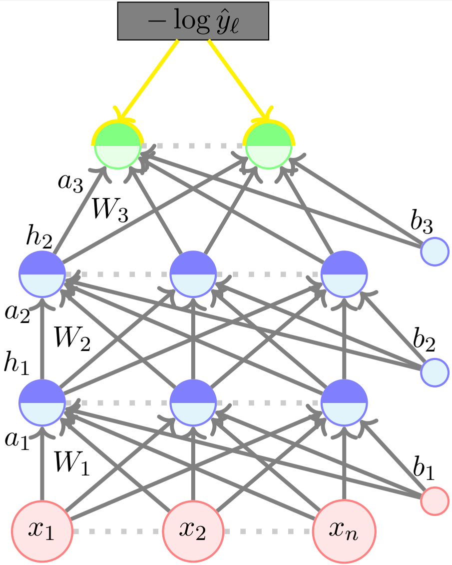
$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_k} \end{bmatrix} = -\frac{1}{\hat{y}_\ell}\begin{bmatrix} \mathbb{1}_{\ell=1} \\ \mathbb{1}_{\ell=2} \\ \vdots \\ \\ \end{bmatrix}$$



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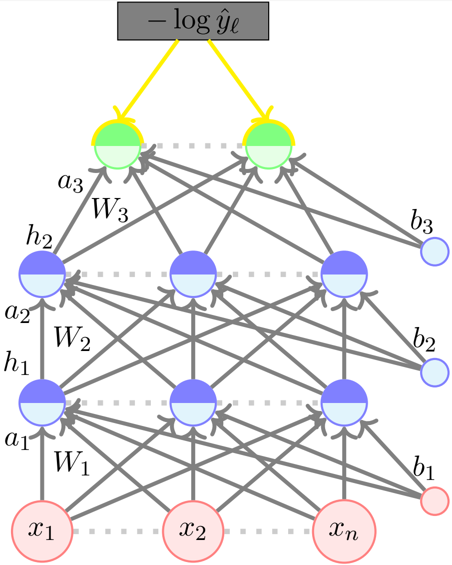
$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_k} \end{bmatrix} = -\frac{1}{\hat{y}_\ell}\begin{bmatrix} \mathbb{1}_{\ell=1} \\ \mathbb{1}_{\ell=2} \\ \vdots \\ \mathbb{1}_{\ell=k} \end{bmatrix}$$



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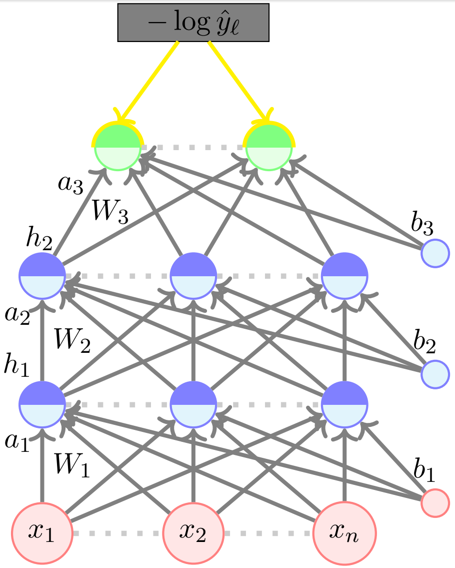
$$\nabla_{\mathbf{\hat{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_k} \end{bmatrix} = -\frac{1}{\hat{y}_\ell}\begin{bmatrix} \mathbb{1}_{\ell=1} \\ \mathbb{1}_{\ell=2} \\ \vdots \\ \mathbb{1}_{\ell=k} \end{bmatrix}$$

$$= \frac{1}{e(\ell)}\hat{y}_\ell$$



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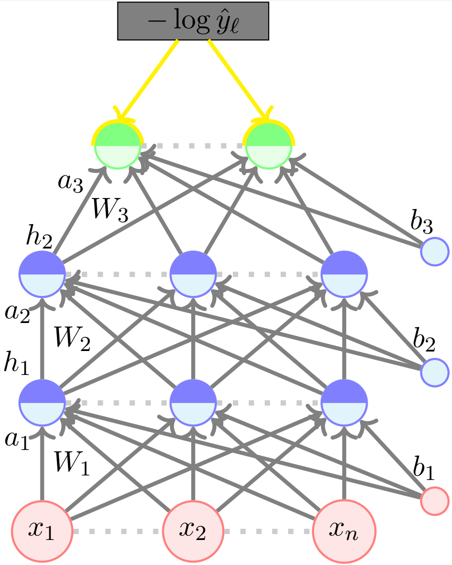
We can now talk about the gradient w.r.t. the vector $\hat{y}$

$$\nabla_{\hat{\mathbf{y}}}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}_k} \end{bmatrix} = -\frac{1}{\hat{y}_\ell}\begin{bmatrix} \mathbb{1}_{\ell=1} \\ \mathbb{1}_{\ell=2} \\ \vdots \\ \mathbb{1}_{\ell=k} \end{bmatrix}$$

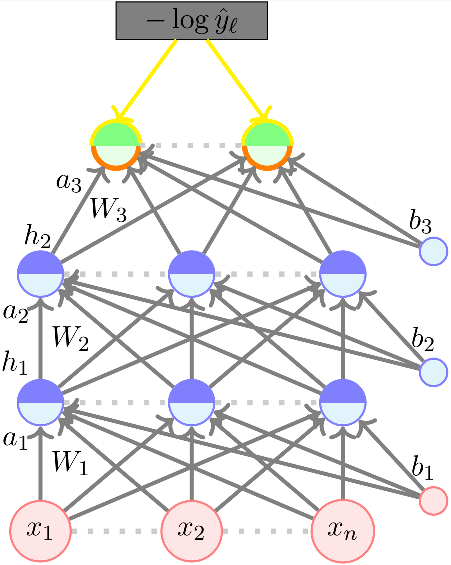
$$= \frac{1}{e(\ell)}\hat{y}_\ell$$

where $e(\ell)$ is a k-dimensional vector whose ℓ-th element is 1 and all other elements are 0.



What we are actually interested in is

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{Li}} = \frac{\partial(-\log \hat{y}_\ell)}{\partial a_{Li}}$$



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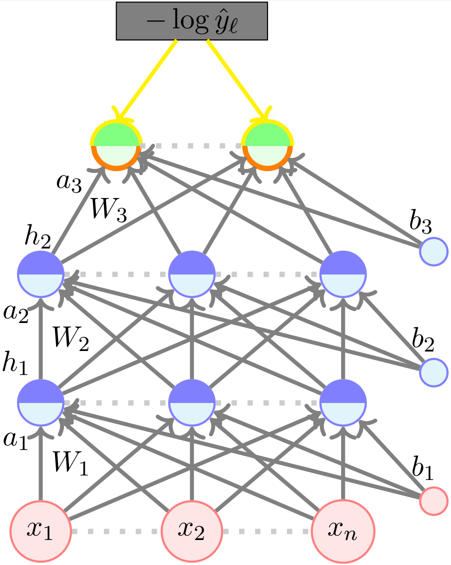
$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{Li}} = \frac{\partial(-\log \hat{y}_\ell)}{\partial a_{Li}}$$
$$= \frac{\partial(-\log \hat{y}_\ell)}{\partial \hat{y}_\ell} \frac{\partial \hat{y}_\ell}{\partial a_{Li}}$$



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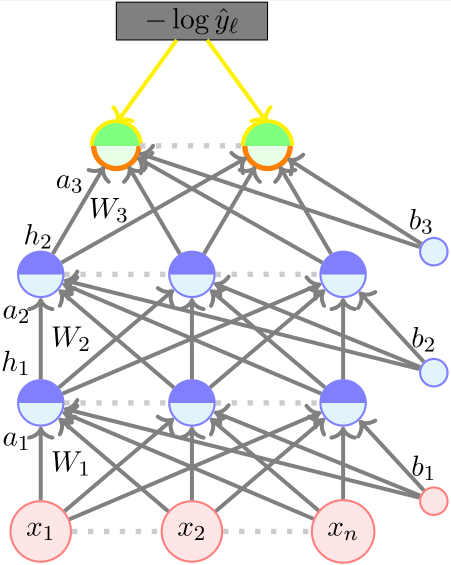
$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{Li}} = \frac{\partial(-\log \hat{y}_\ell)}{\partial a_{Li}} = \frac{\partial(-\log \hat{y}_\ell)}{\partial \hat{y}_\ell} \frac{\partial \hat{y}_\ell}{\partial a_{Li}}$$

Does $\hat{y}_\ell$ depend on a_{Li} ? Indeed, it does.



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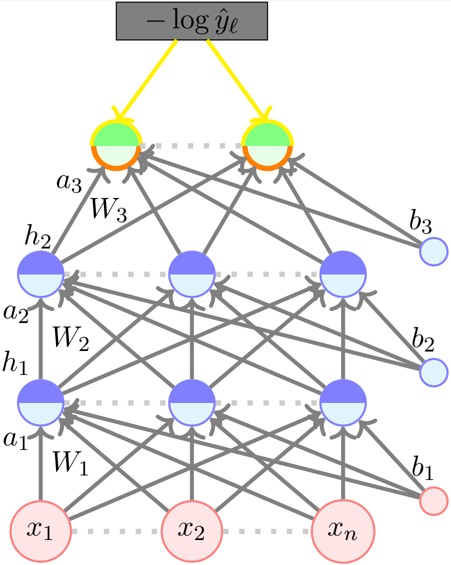
$$= \frac{\partial(-\log \hat{y}_\ell)}{\partial \hat{y}_\ell} \frac{\partial \hat{y}_\ell}{\partial a_{Li}}$$

Does $\hat{y}_\ell$ depend on a_{Li} ? Indeed, it does.

$$\hat{y}_\ell = \frac{exp(a_{L\ell})}{\sum_i exp(a_{Li})}$$



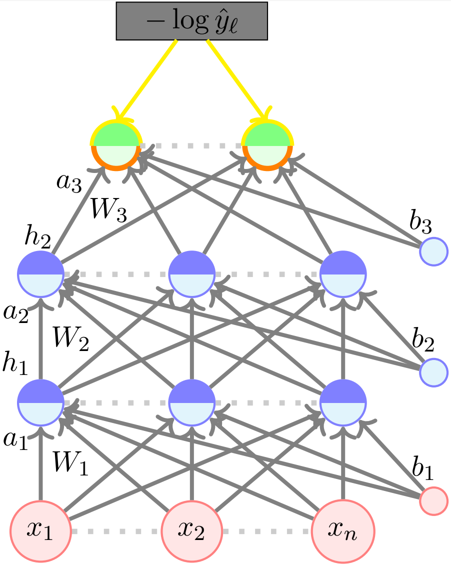
What we are actually interested in is

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{Li}} = \frac{\partial(-\log \hat{y}_\ell)}{\partial a_{Li}}$$
$$= \frac{\partial(-\log \hat{y}_\ell)}{\partial \hat{y}_\ell} \frac{\partial \hat{y}_\ell}{\partial a_{Li}}$$

Does $\hat{y}_\ell$ depend on a_{Li} ? Indeed, it does.

$$\hat{y}_\ell = \frac{exp(a_{L\ell})}{\sum_i exp(a_{Li})}$$

Having established this, we will now derive the full expression on the next slide



$$\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell =$$

$$\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell = \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell$$

$$
\begin{aligned}
\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell \\
&= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} softmax(\mathbf{a}_L)_\ell
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell \\
&= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} softmax(\mathbf{a}_L)_\ell \\
&= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_\ell}
\end{aligned}
$$

$$\begin{aligned}\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell \\ &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} softmax(\mathbf{a}_L)_\ell \\ &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_\ell}\end{aligned}$$

$$\frac{\partial \frac{g(x)}{h(x)}}{\partial x} = \frac{\partial g(x)}{\partial x} \frac{1}{h(x)} - \frac{g(x)}{h(x)^2} \frac{\partial h(x)}{\partial x}$$

$$\begin{aligned}\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell \\ &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} softmax(\mathbf{a}_L)_\ell \\ &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_\ell} \\ &= \frac{-1}{\hat{y}_\ell} \left(\frac{\frac{\partial}{\partial a_{Li}} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell \left(\frac{\partial}{\partial a_{Li}} \sum_{i'} \exp(\mathbf{a}_L)_{i'} \right)}{(\sum_{i'} (\exp(\mathbf{a}_L)_{i'})^2} \right)\end{aligned}$$

$$\frac{\partial \frac{g(x)}{h(x)}}{\partial x} = \frac{\partial g(x)}{\partial x} \frac{1}{h(x)} - \frac{g(x)}{h(x)^2} \frac{\partial h(x)}{\partial x}$$

$$
\begin{aligned}
\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell \\
&= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} softmax(\mathbf{a}_L)_\ell \\
&= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_\ell} \\
&= \frac{-1}{\hat{y}_\ell} \left(\frac{\frac{\partial}{\partial a_{Li}} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell \left(\frac{\partial}{\partial a_{Li}} \sum_{i'} \exp(\mathbf{a}_L)_{i'} \right)}{(\sum_{i'} (\exp(\mathbf{a}_L)_{i'})^2} \right) \\
&= \frac{-1}{\hat{y}_\ell} \left(\frac{\mathbb{1}_{(\ell=i)} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \frac{\exp(\mathbf{a}_L)_i}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \right)
\end{aligned}
$$

$$
\frac{\partial \frac{g(x)}{h(x)}}{\partial x} = \frac{\partial g(x)}{\partial x} \frac{1}{h(x)} - \frac{g(x)}{h(x)^2} \frac{\partial h(x)}{\partial x}
$$

$$\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell = \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell$$

$$= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} softmax(\mathbf{a}_L)_\ell$$

$$= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_\ell}$$

$$= \frac{-1}{\hat{y}_\ell} \left(\frac{\frac{\partial}{\partial a_{Li}} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell \left(\frac{\partial}{\partial a_{Li}} \sum_{i'} \exp(\mathbf{a}_L)_{i'} \right)}{(\sum_{i'} (\exp(\mathbf{a}_L)_{i'})^2} \right)$$

$$= \frac{-1}{\hat{y}_\ell} \left(\frac{\mathbb{1}_{(\ell=i)} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \frac{\exp(\mathbf{a}_L)_i}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \right)$$

$$= \frac{-1}{\hat{y}_\ell} \left(\mathbb{1}_{(\ell=i)} softmax(\mathbf{a}_L)_\ell - softmax(\mathbf{a}_L)_\ell softmax(\mathbf{a}_L)_i \right)$$

$$\frac{\partial \frac{g(x)}{h(x)}}{\partial x} = \frac{\partial g(x)}{\partial x} \frac{1}{h(x)} - \frac{g(x)}{h(x)^2} \frac{\partial h(x)}{\partial x}$$

$$
\begin{aligned}
\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell &= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell \\
&= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} softmax(\mathbf{a}_L)_\ell \\
&= \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_\ell} \\
&= \frac{-1}{\hat{y}_\ell} \left(\frac{\frac{\partial}{\partial a_{Li}} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell \left(\frac{\partial}{\partial a_{Li}} \sum_{i'} \exp(\mathbf{a}_L)_{i'} \right)}{(\sum_{i'} (\exp(\mathbf{a}_L)_{i'})^2} \right) \\
&= \frac{-1}{\hat{y}_\ell} \left(\frac{\mathbb{1}_{(\ell=i)} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \frac{\exp(\mathbf{a}_L)_i}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \right) \\
&= \frac{-1}{\hat{y}_\ell} \left(\mathbb{1}_{(\ell=i)} softmax(\mathbf{a}_L)_\ell - softmax(\mathbf{a}_L)_\ell softmax(\mathbf{a}_L)_i \right) \\
&= \frac{-1}{\hat{y}_\ell} \left(\mathbb{1}_{(\ell=i)} f(\mathbf{x})_\ell - f(\mathbf{x})_\ell f(\mathbf{x})_i \right)
\end{aligned}
$$

$$
\frac{\partial \frac{g(x)}{h(x)}}{\partial x} = \frac{\partial g(x)}{\partial x} \frac{1}{h(x)} - \frac{g(x)}{h(x)^2} \frac{\partial h(x)}{\partial x}
$$

$$\frac{\partial}{\partial a_{Li}} - \log \hat{y}_\ell = \frac{-1}{\hat{y}_\ell} \frac{\partial}{\partial a_{Li}} \hat{y}_\ell$$

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$$= \frac{-1}{\hat{y}_\ell} \left(\frac{\mathbb{1}_{(\ell=i)} \exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} - \frac{\exp(\mathbf{a}_L)_\ell}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \frac{\exp(\mathbf{a}_L)_i}{\sum_{i'} \exp(\mathbf{a}_L)_{i'}} \right)$$

$$= \frac{-1}{\hat{y}_\ell} \left(\mathbb{1}_{(\ell=i)} softmax(\mathbf{a}_L)_\ell - softmax(\mathbf{a}_L)_\ell softmax(\mathbf{a}_L)_i \right)$$

$$= \frac{-1}{\hat{y}_\ell} \left(\mathbb{1}_{(\ell=i)} f(\mathbf{x})_\ell - f(\mathbf{x})_\ell f(\mathbf{x})_i \right)$$

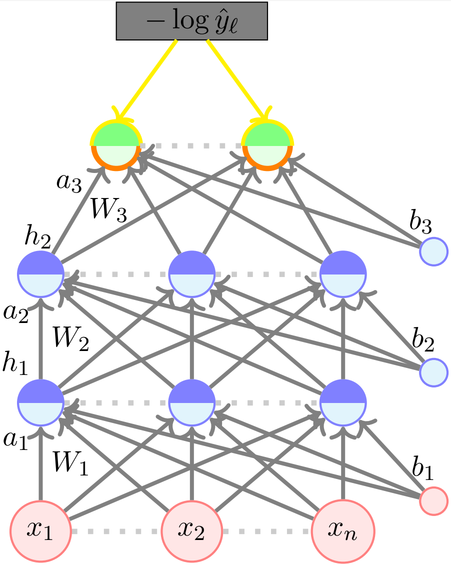
$$= -\left(\mathbb{1}_{(\ell=i)} - f(\mathbf{x})_i \right)$$

$$\frac{\partial \frac{g(x)}{h(x)}}{\partial x} = \frac{\partial g(x)}{\partial x} \frac{1}{h(x)} - \frac{g(x)}{h(x)^2} \frac{\partial h(x)}{\partial x}$$

So far we have derived the partial derivative w.r.t. the i-th element of $\mathbf{a}_L$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{L,i}} = -(\mathbb{1}_{\ell=i} - \hat{y}_\ell)$$

We can now write the gradient w.r.t. the vector $\mathbf{a}_L$



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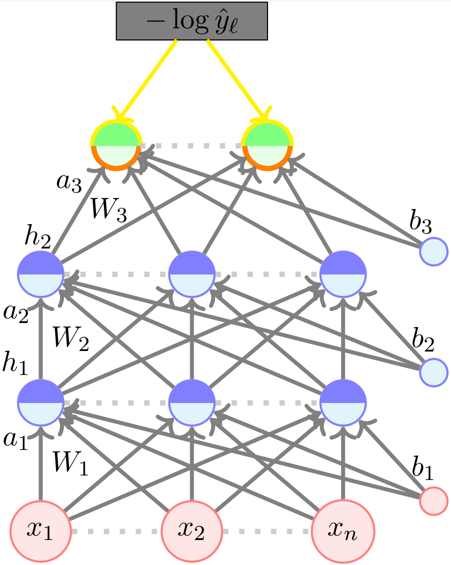
We can now write the gradient w.r.t. the vector $\mathbf{a}_L$

$$\nabla_{\mathbf{a_L}}$$



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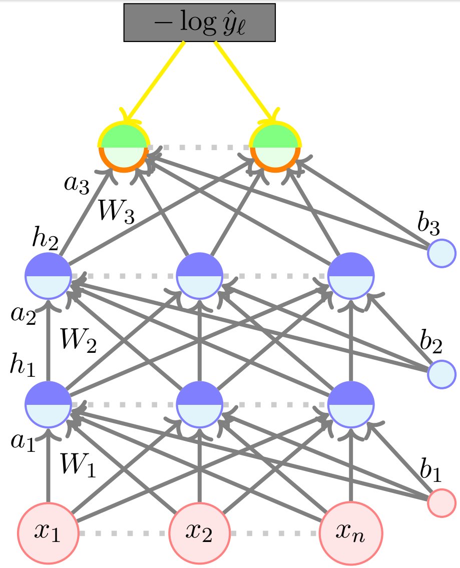
$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{L,i}} = -(\mathbb{1}_{\ell=i} - \hat{y}_\ell)$$

We can now write the gradient w.r.t. the vector $\mathbf{a}_L$

$$\nabla_{\mathbf{a_L}} = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_1} \\ \\ \\ \\ \end{bmatrix}$$



So far we have derived the partial derivative w.r.t. the i-th element of $\mathbf{a}_L$

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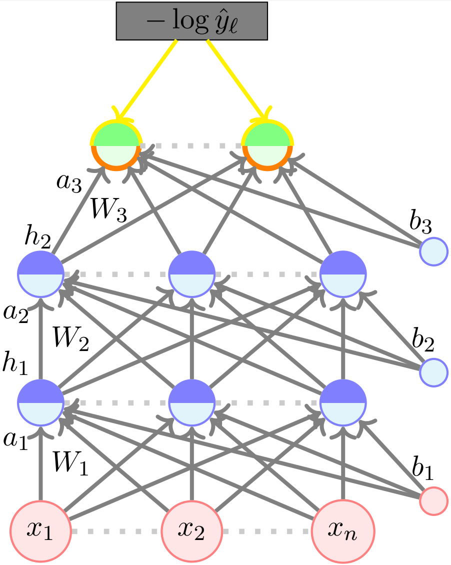
$$\nabla_{\mathbf{a_L}} = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_1} \\ \vdots \\ \\ \end{bmatrix}$$



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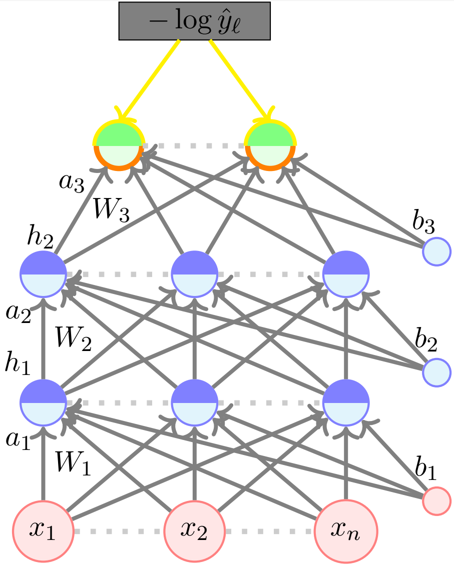
$$\nabla_{\mathbf{a_L}} = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_1} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_k} \end{bmatrix}$$



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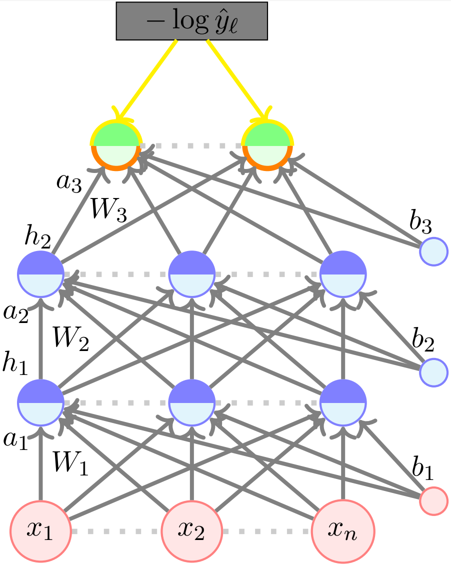

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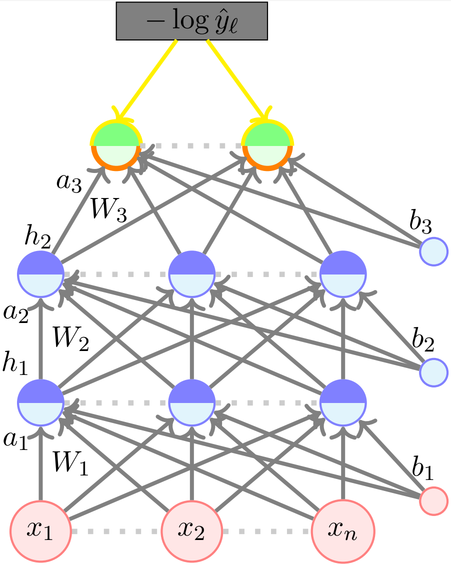

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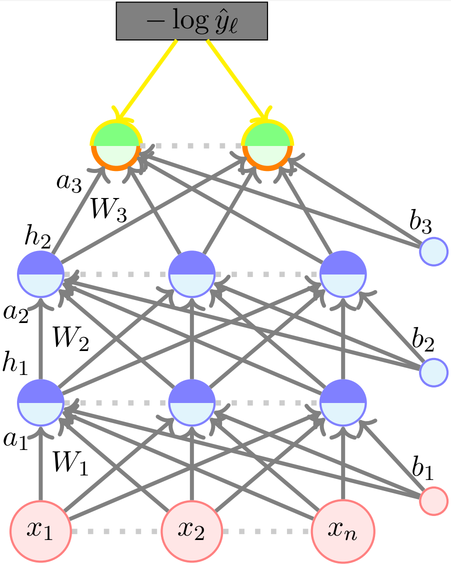

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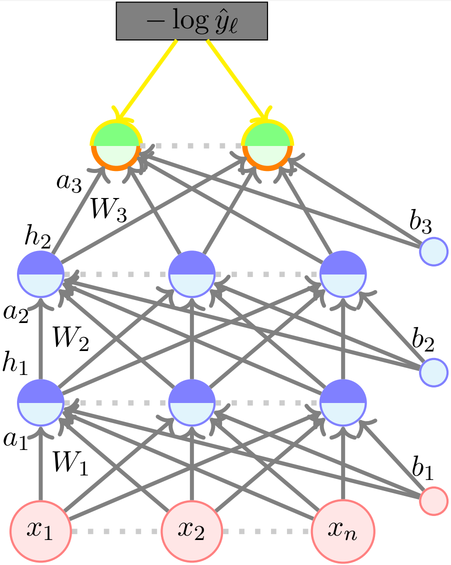

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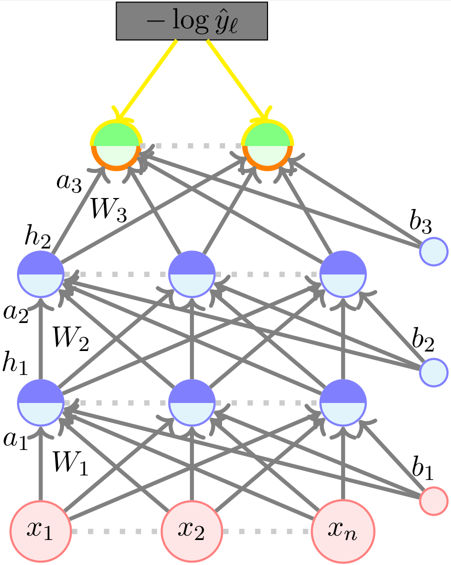

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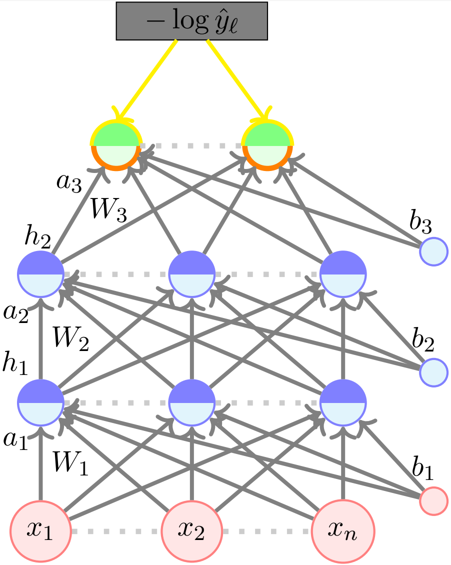
$$= -(\mathbf{e}(\ell) - \mathbf{f}(x))$$



Module 4.6: Backpropagation: Computing Gradients w.r.t. Hidden Units

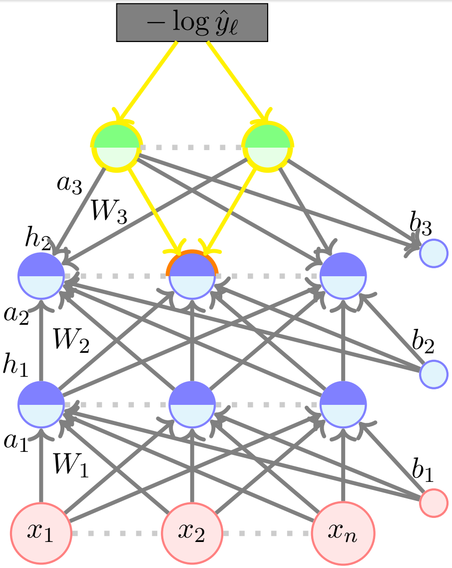
Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\text{and now talk to the weights}}$$

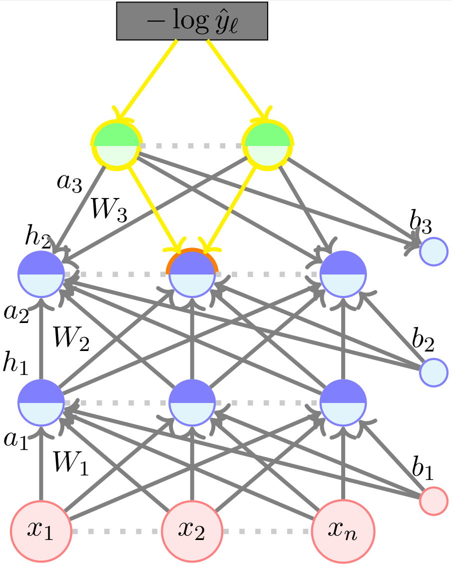
- Our focus is on *Cross entropy loss* and *Softmax* output.

Chain rule along multiple paths: If a function $p(z)$ can be written as a function of intermediate results $q_i(z)$ then we have :



Chain rule along multiple paths: If a function $p(z)$ can be written as a function of intermediate results $q_i(z)$ then we have :

$$\frac{\partial p(z)}{\partial z} = \sum_m \frac{\partial p(z)}{\partial q_m(z)} \frac{\partial q_m(z)}{\partial z}$$



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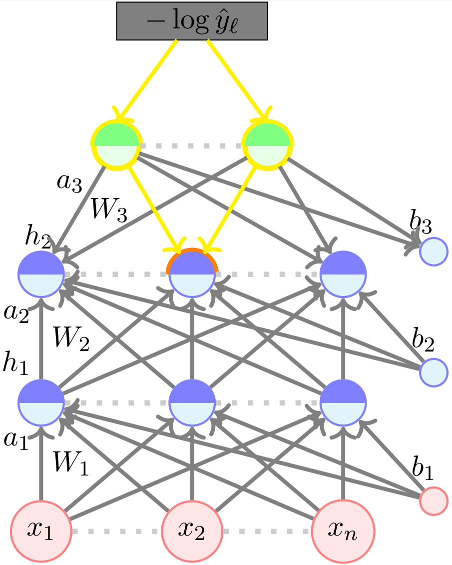
In our case:

- $p(z)$ is the loss function $\mathscr{L}(\theta)$



Chain rule along multiple paths: If a function $p(z)$ can be written as a function of intermediate results $q_i(z)$ then we have :

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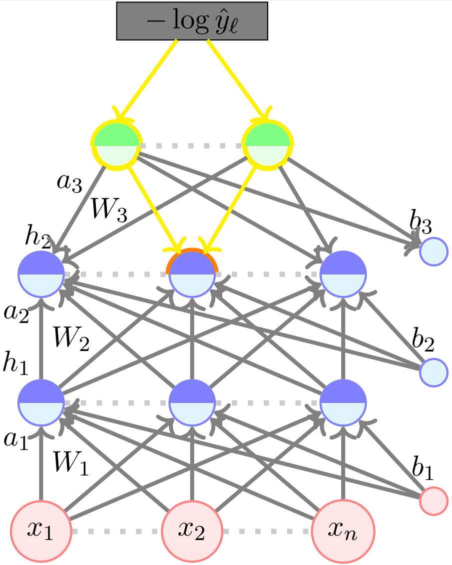
In our case:

- $p(z)$ is the loss function $\mathscr{L}(\theta)$
- $z = h_{ij}$

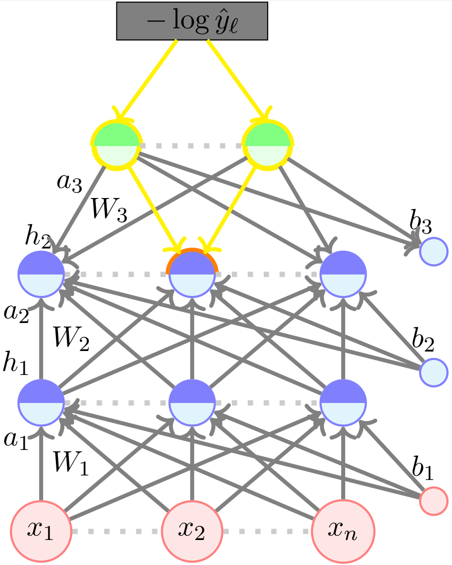


Chain rule along multiple paths: If a function $p(z)$ can be written as a function of intermediate results $q_i(z)$ then we have :

$$\frac{\partial p(z)}{\partial z} = \sum_m \frac{\partial p(z)}{\partial q_m(z)} \frac{\partial q_m(z)}{\partial z}$$

In our case:

- $p(z)$ is the loss function $\mathscr{L}(\theta)$
- $z = h_{ij}$
- $q_m(z) = a_{Lm}$



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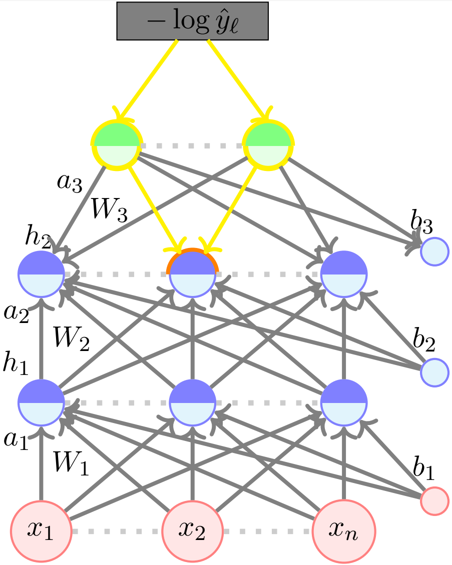
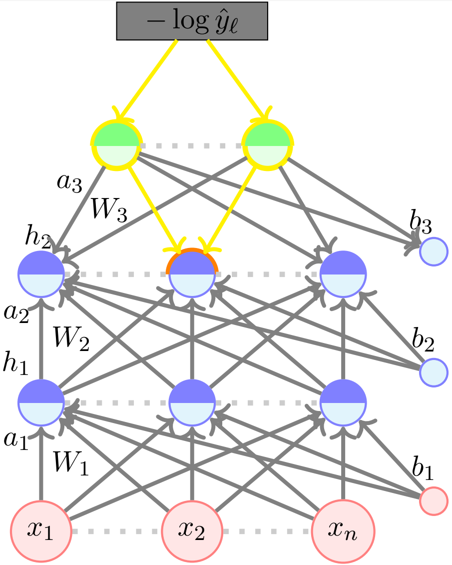
$$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}}$$



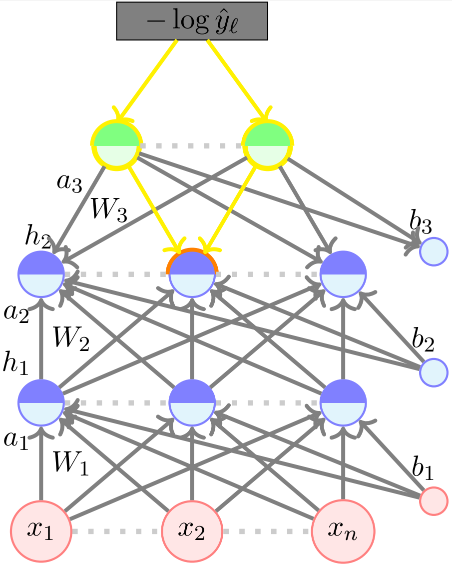
$$a_{i+1} = W_{i+1} h_{ij} + b_{i+1}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} \frac{\partial a_{i+1,m}}{\partial h_{ij}}$$



$$a_{i+1} = W_{i+1} h_{ij} + b_{i+1}$$

$$
\begin{aligned}
\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} &= \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} \frac{\partial a_{i+1,m}}{\partial h_{ij}} \\
&= \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} W_{i+1,m,j}
\end{aligned}
$$



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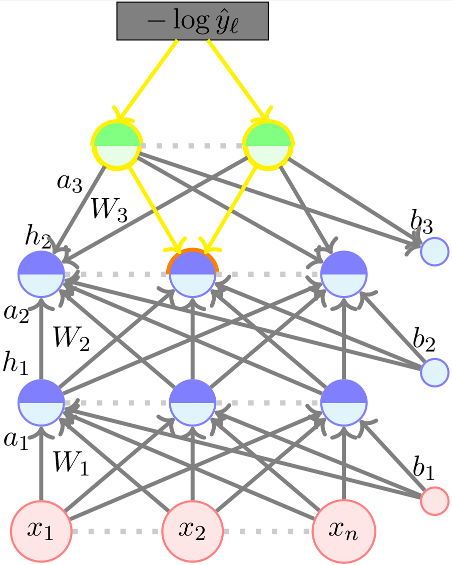
$$= \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} W_{i+1,m,j}$$

Now consider these two vectors,



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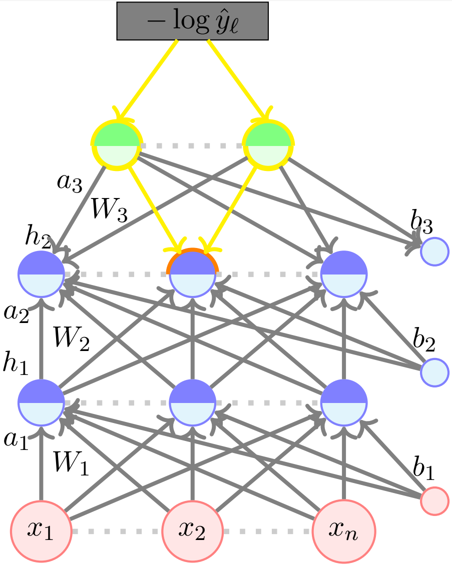
$$= \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} W_{i+1,m,j}$$

Now consider these two vectors,

$$\nabla_{a_{i+1}} \mathscr{L}(\theta) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} ; W_{i+1,\cdot,j} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



$$a_{i+1} = W_{i+1} h_{ij} + b_{i+1}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} \frac{\partial a_{i+1,m}}{\partial h_{ij}}$$

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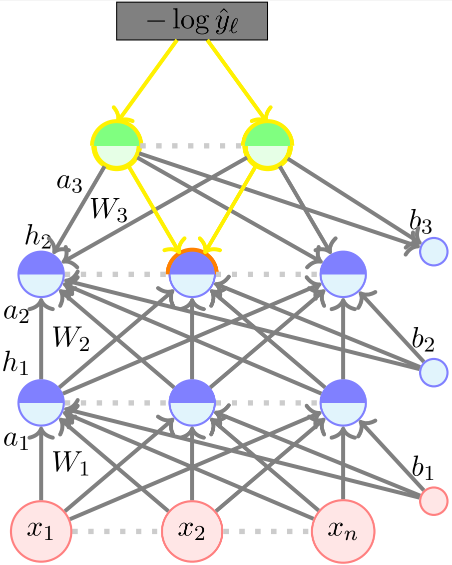

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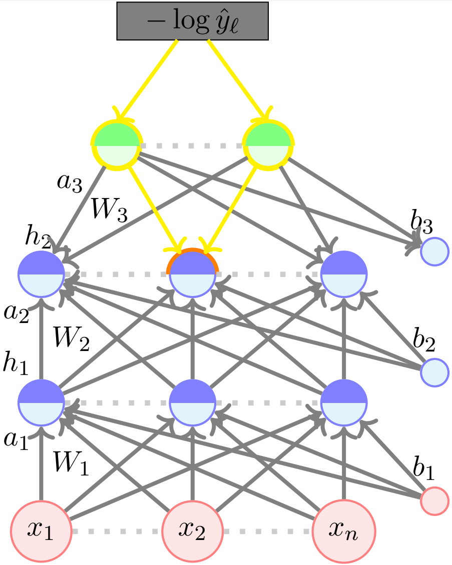

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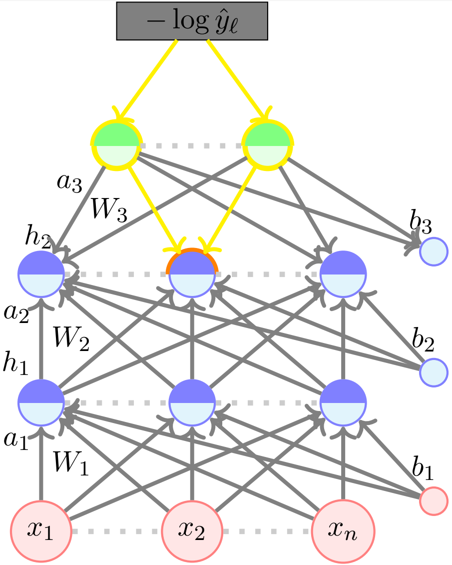

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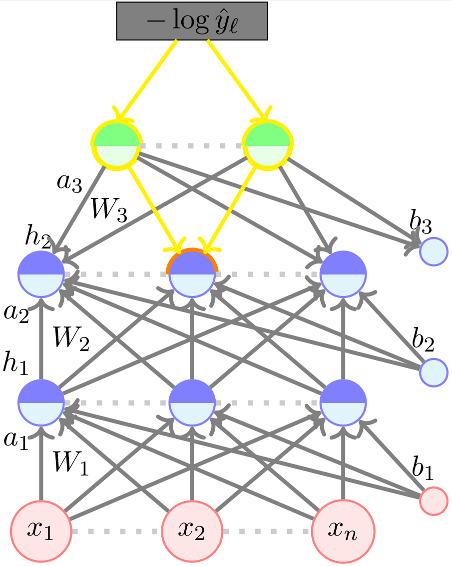

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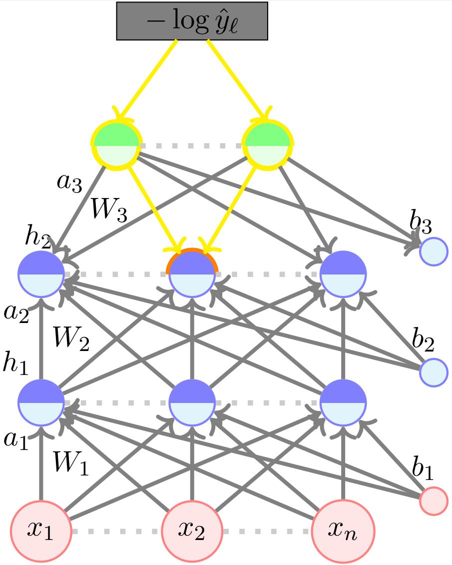

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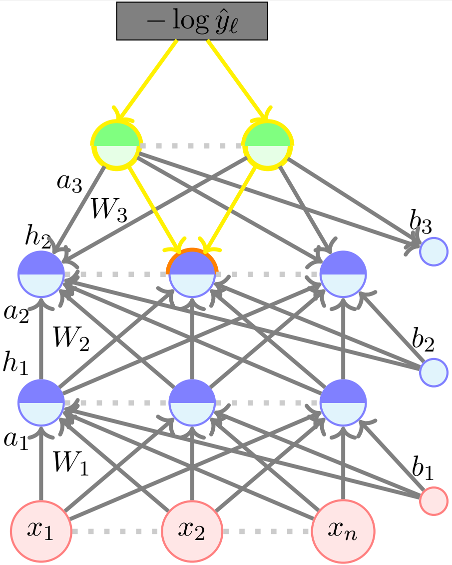

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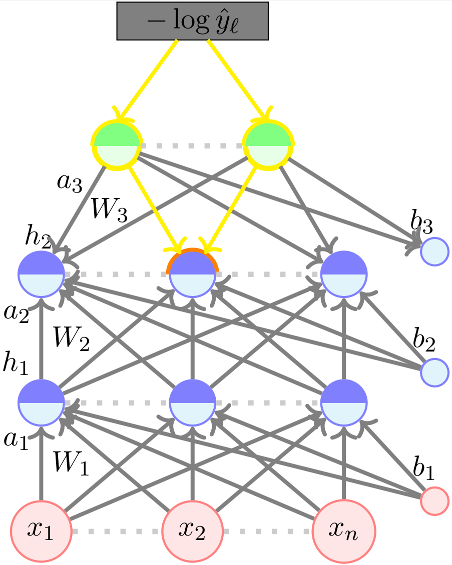
$W_{i+1,\cdot,j}$ is the j-th column of W_{i+1};



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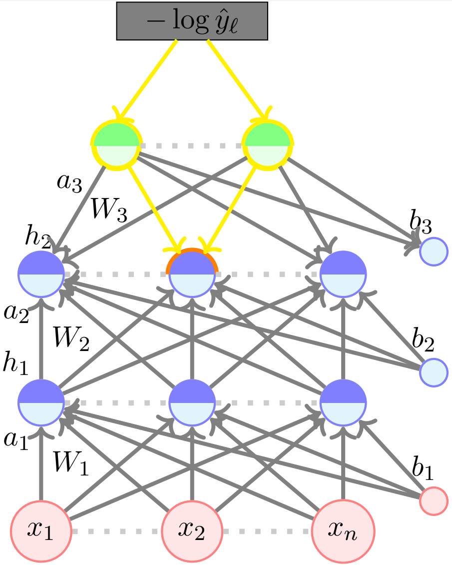

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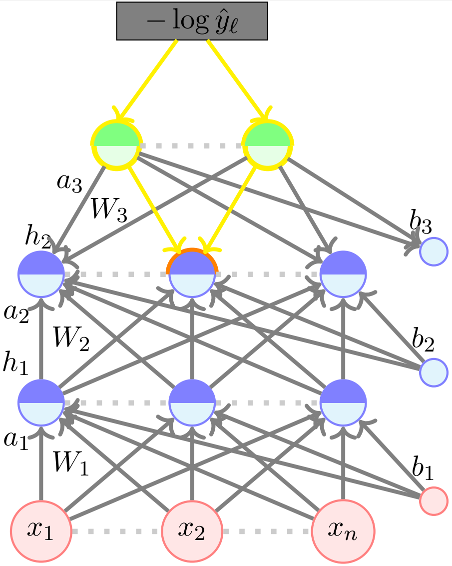
$W_{i+1,\cdot,j}$ is the j-th column of W_{i+1}; see that,

$$(W_{i+1,\cdot,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) =$$



$$a_{i+1} = W_{i+1} h_{ij} + b_{i+1}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} \frac{\partial a_{i+1,m}}{\partial h_{ij}}$$

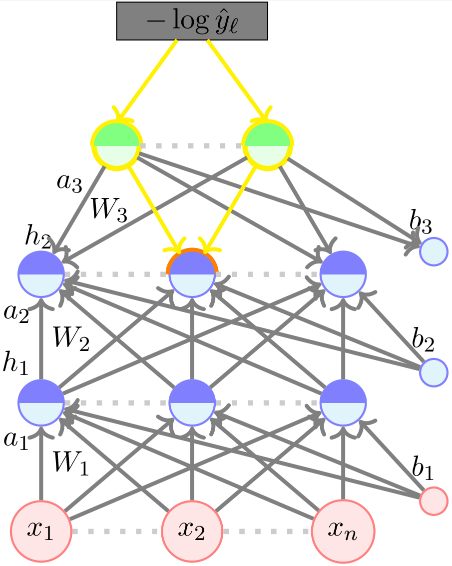
$$= \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} W_{i+1,m,j}$$

Now consider these two vectors,

$$\nabla_{a_{i+1}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,k}} \end{bmatrix}; W_{i+1,\cdot,j} = \begin{bmatrix} W_{i+1,1,j} \\ \vdots \\ W_{i+1,k,j} \end{bmatrix}$$

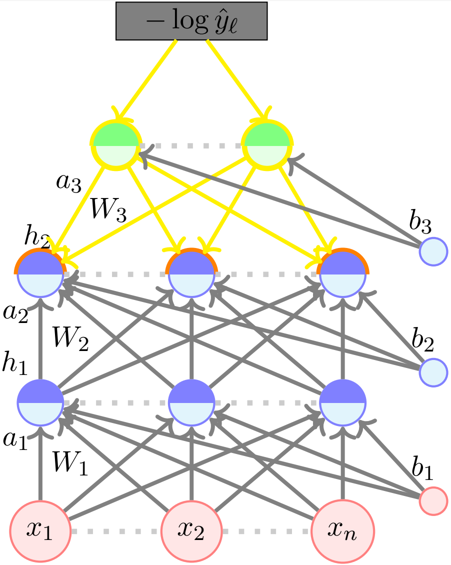
$W_{i+1,\cdot,j}$ is the j-th column of W_{i+1}; see that,

$$(W_{i+1,\cdot,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) = \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} W_{i+1,m,j}$$



$$a_{i+1} = W_{i+1} h_{ij} + b_{i+1}$$

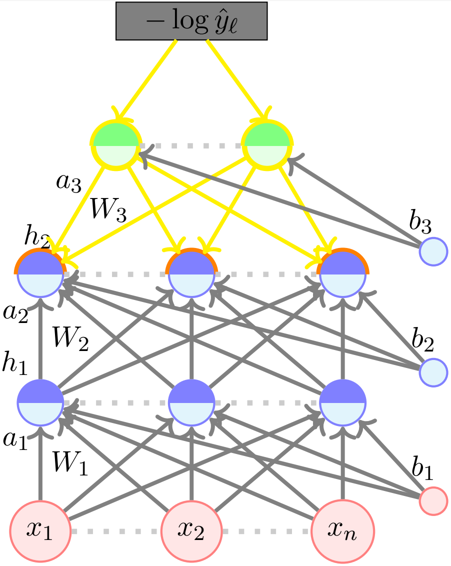
$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

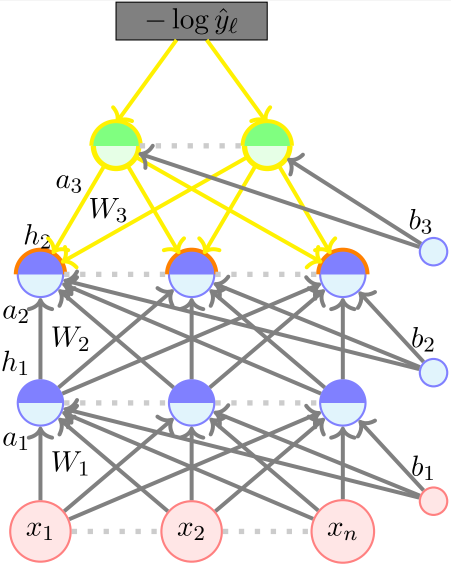
$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta)$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

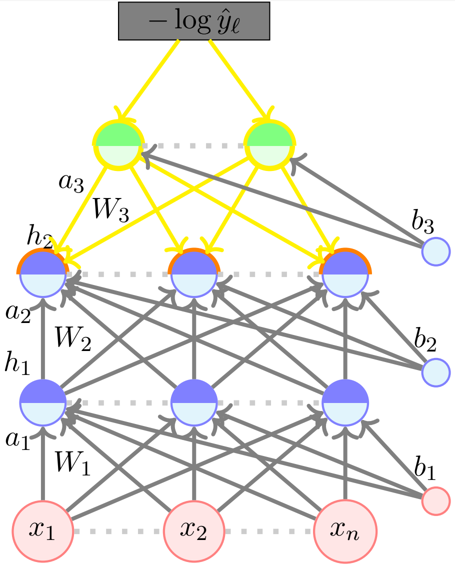
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



$$\textit{We have}, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

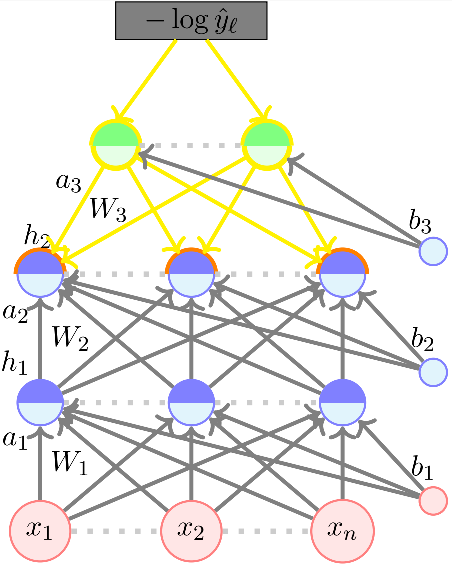
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

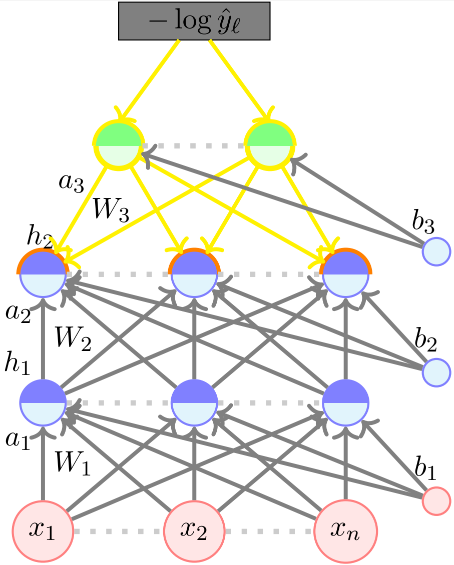
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \\ \\ \\ \end{bmatrix}$$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

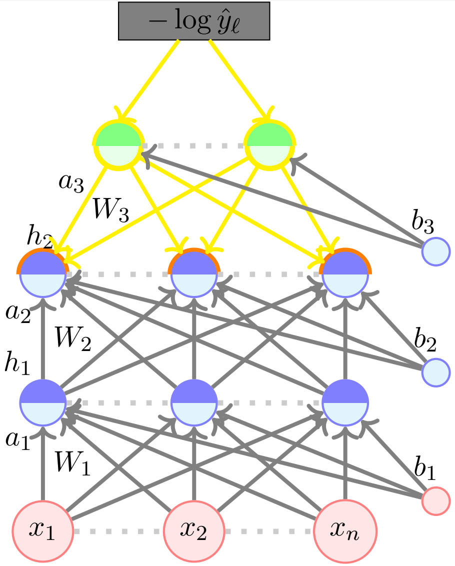
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \\ \\ \\ \end{bmatrix}$$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

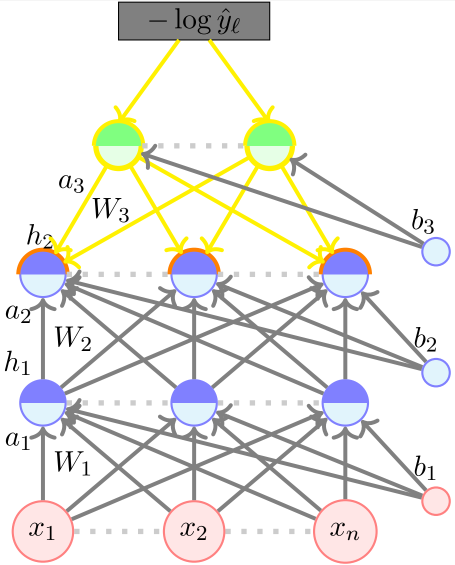
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \\ \\ \end{bmatrix}$$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

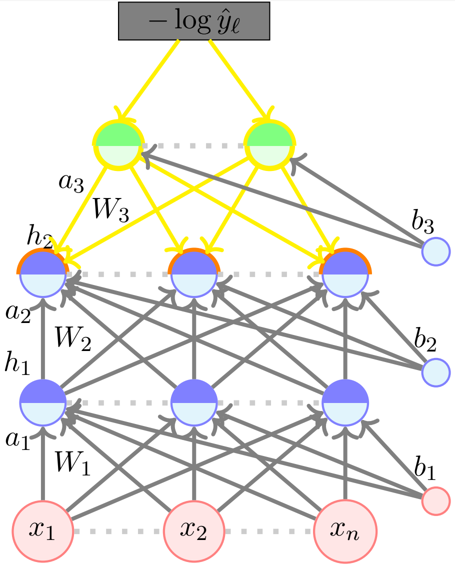
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \\ \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ \\ \end{bmatrix}$$



$$\textit{We have,} \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

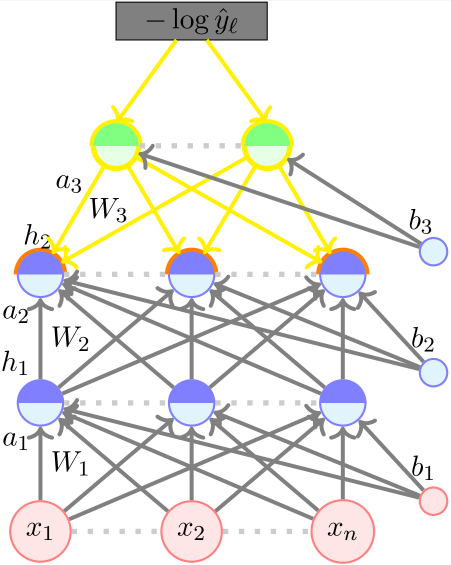
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ \\ \end{bmatrix}$$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ (W_{i+1,\cdot,n})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \end{bmatrix}$$

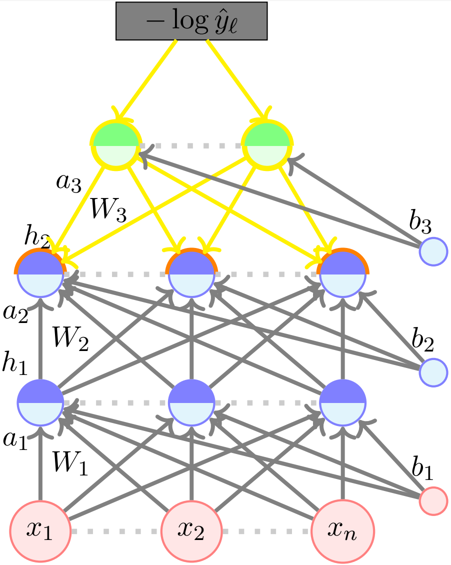


$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ (W_{i+1,\cdot,n})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \end{bmatrix}$$

$$= (W_{i+1})^T (\nabla_{a_{i+1}} \mathscr{L}(\theta))$$



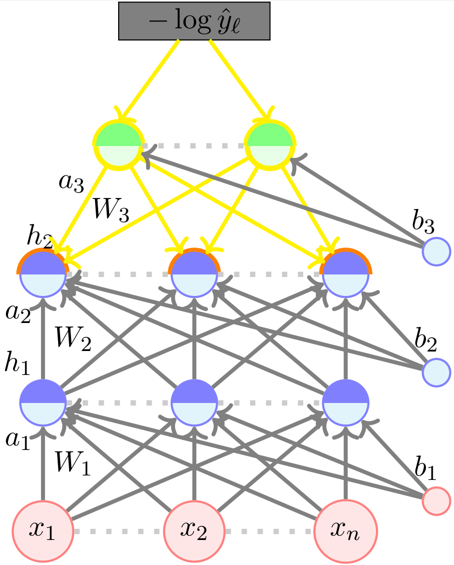
$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ (W_{i+1,\cdot,n})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \end{bmatrix}$$

$$= (W_{i+1})^T (\nabla_{a_{i+1}} \mathscr{L}(\theta))$$

- We are almost done except that we do not know how to calculate $\nabla_{a_{i+1}} \mathscr{L}(\theta)$ for $i < L - 1$



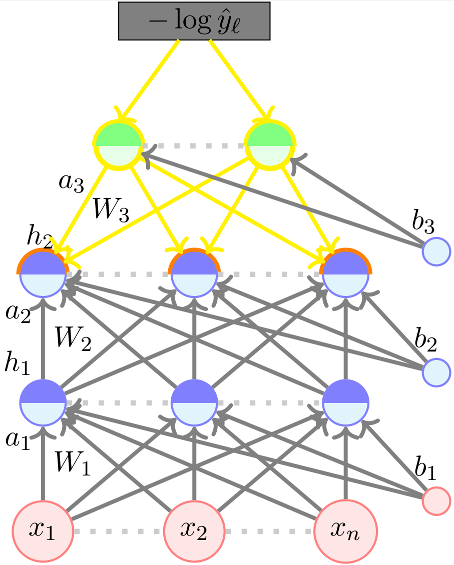
$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

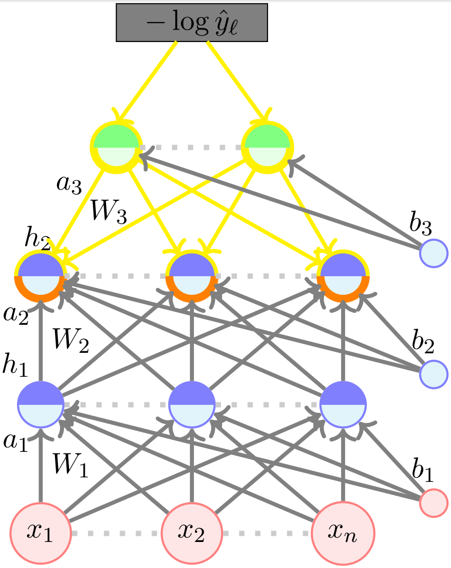
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ (W_{i+1,\cdot,n})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \end{bmatrix}$$

$$= (W_{i+1})^T (\nabla_{a_{i+1}} \mathscr{L}(\theta))$$

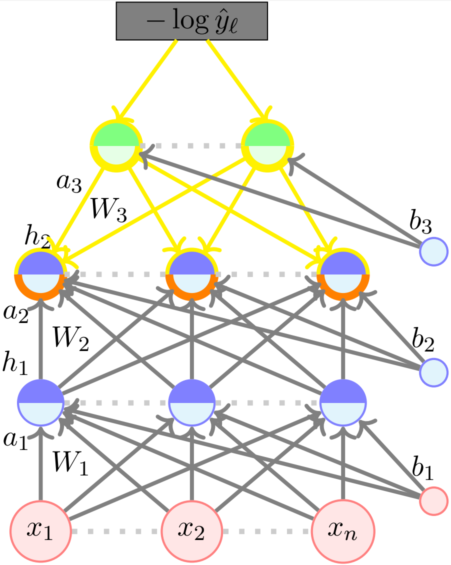
- We are almost done except that we do not know how to calculate $\nabla_{a_{i+1}} \mathscr{L}(\theta)$ for $i < L - 1$
- We will see how to compute that



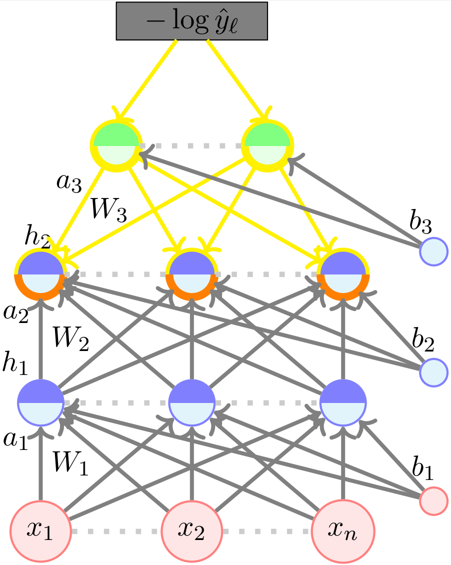
$\nabla_{a_i}\mathscr{L}(\theta)$



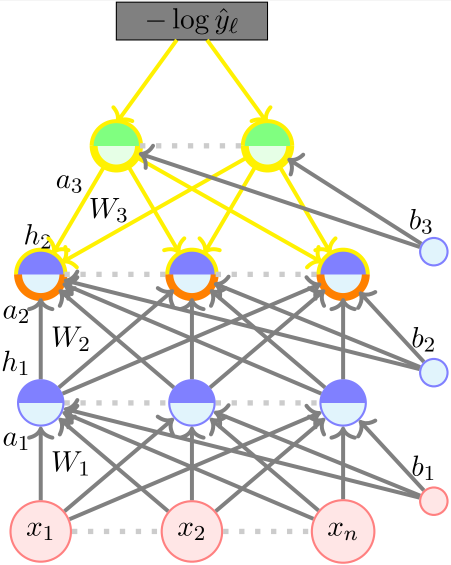
$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



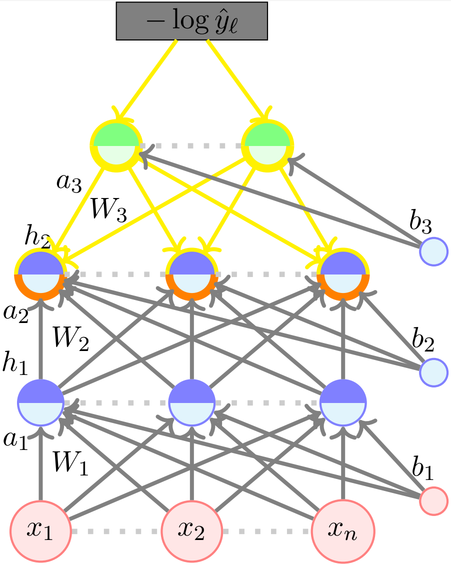
$$\nabla_{a_i} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \\ \\ \end{bmatrix}$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \\ \end{bmatrix}$$

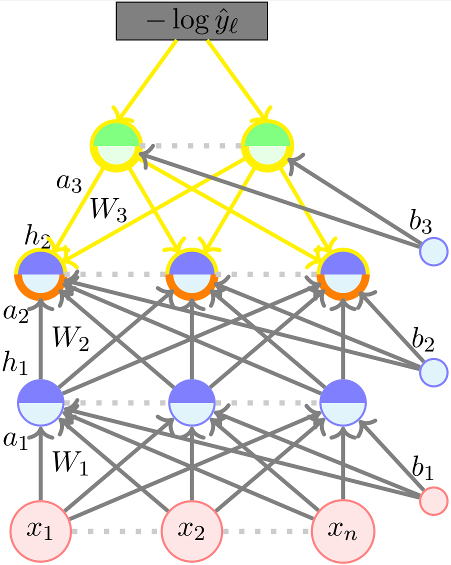


$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$



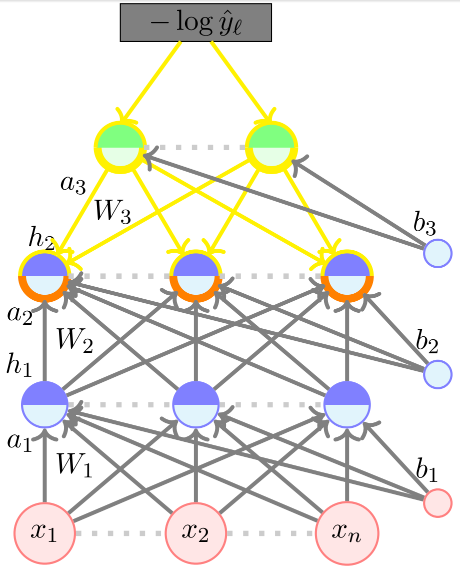
$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}}$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

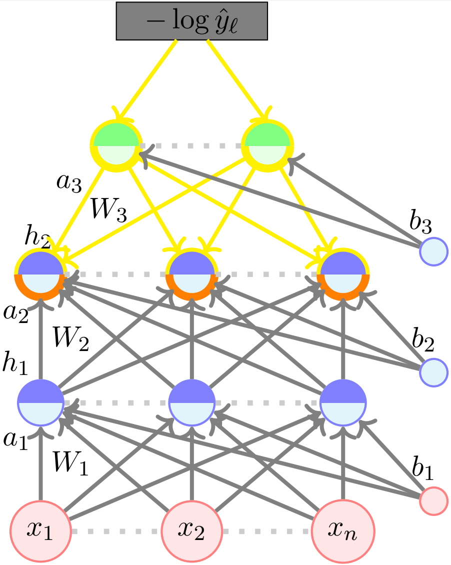
$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

$$= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g'(a_{ij}) \quad [\because h_{ij} = g(a_{ij})]$$

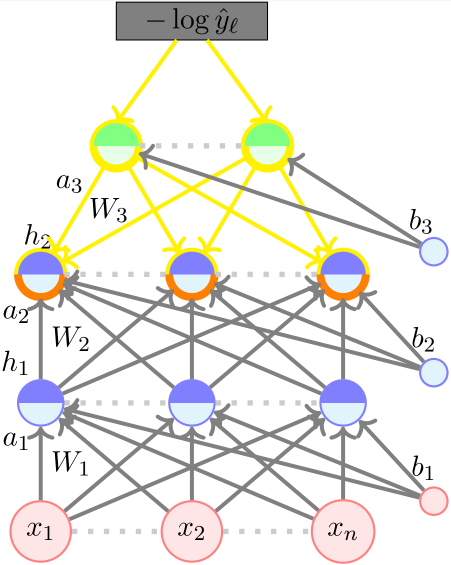


$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

$$= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g'(a_{ij}) \quad [\because h_{ij} = g(a_{ij})]$$

$$\nabla_{a_i}\mathscr{L}(\theta)$$

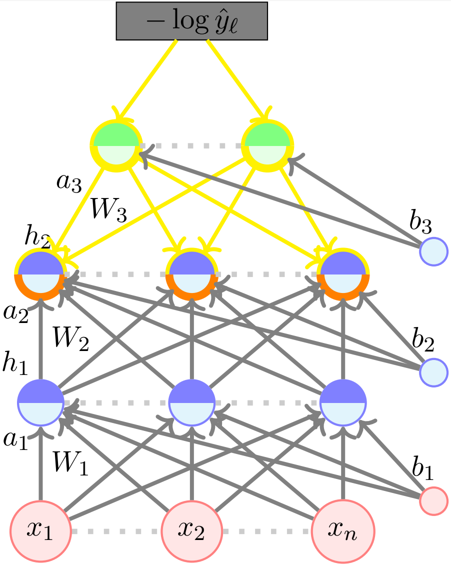


$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

$$= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g'(a_{ij}) \quad [\because h_{ij} = g(a_{ij})]$$

$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

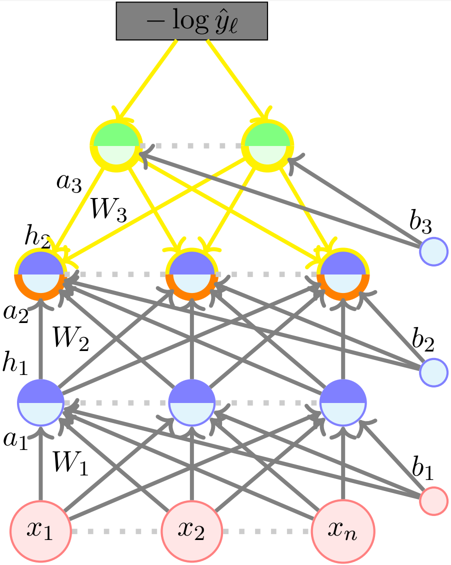


$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

$$= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g'(a_{ij}) \quad [\because h_{ij} = g(a_{ij})]$$

$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} g'(a_{i1}) \\ \\ \\ \end{bmatrix}$$

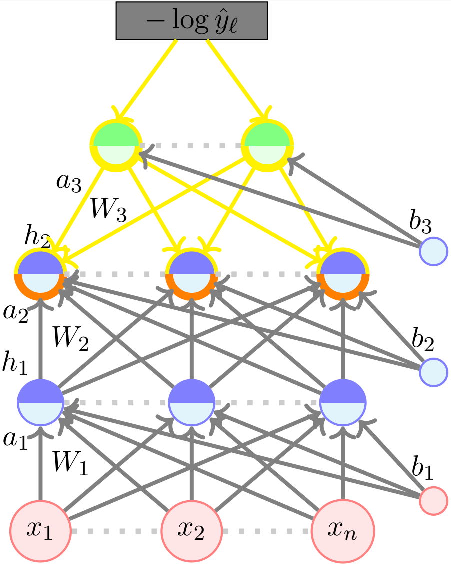


$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

$$= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g'(a_{ij}) \quad [\because h_{ij} = g(a_{ij})]$$

$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} g'(a_{i1}) \\ \vdots \\ \\ \end{bmatrix}$$

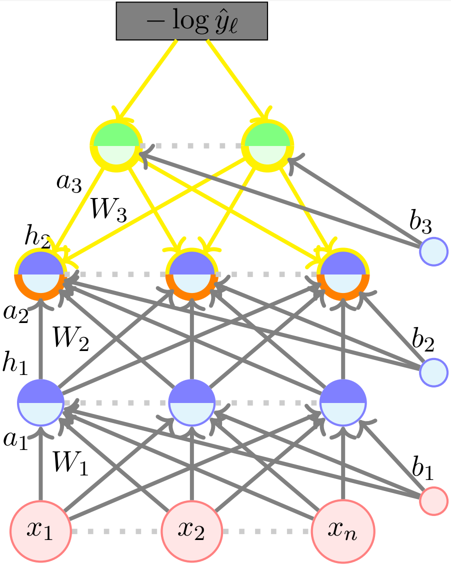


$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

$$= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g^{'}(a_{ij}) \quad [\because h_{ij} = g(a_{ij})]$$

$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} g^{'}(a_{i1}) \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} g^{'}(a_{in}) \end{bmatrix}$$



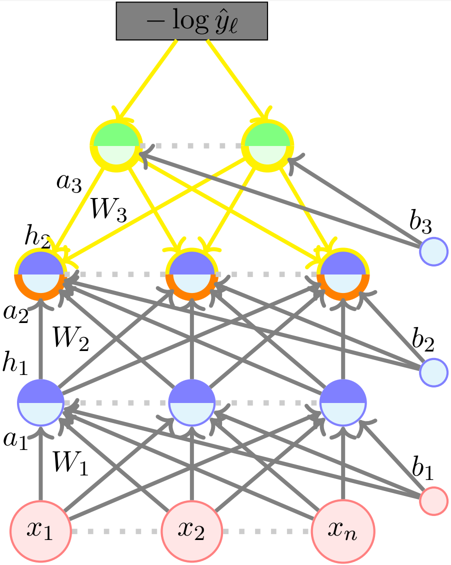
$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

$$= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g^{'}(a_{ij}) \quad [\because h_{ij} = g(a_{ij})]$$

$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} g^{'}(a_{i1}) \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} g^{'}(a_{in}) \end{bmatrix}$$

$$= \nabla_{h_i}\mathscr{L}(\theta) \odot [\ldots, g^{'}(a_{ik}), \ldots]$$



Module 4.7: Backpropagation: Computing Gradients w.r.t. Parameters

Quantities of interest (roadmap for the remaining part):

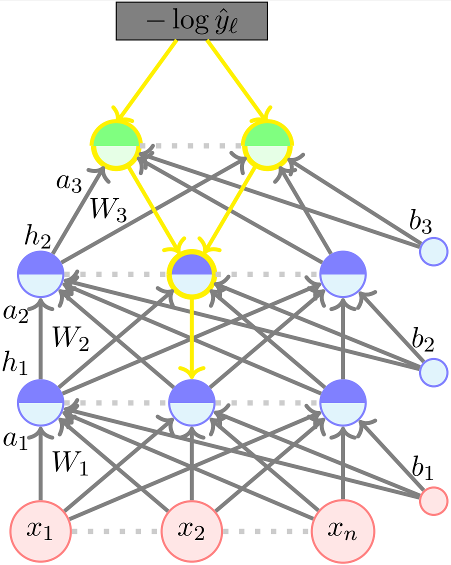
- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\text{and now talk to the weights}}$$

- Our focus is on *Cross entropy loss* and *Softmax* output.

Recall that,

$$a_k = b_k + W_k h_{k-1}$$



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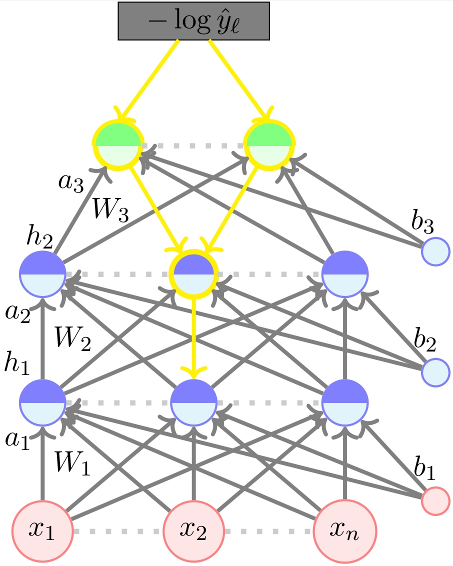

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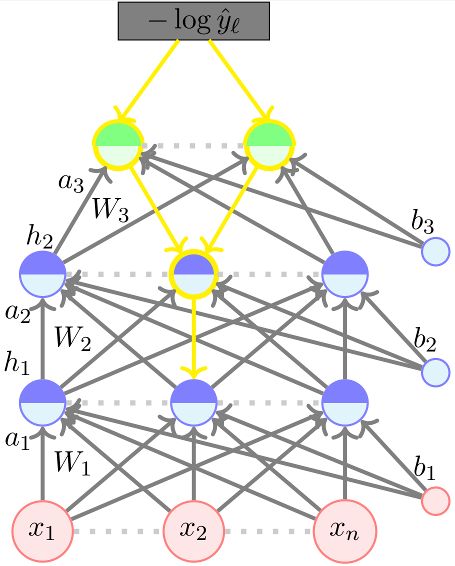

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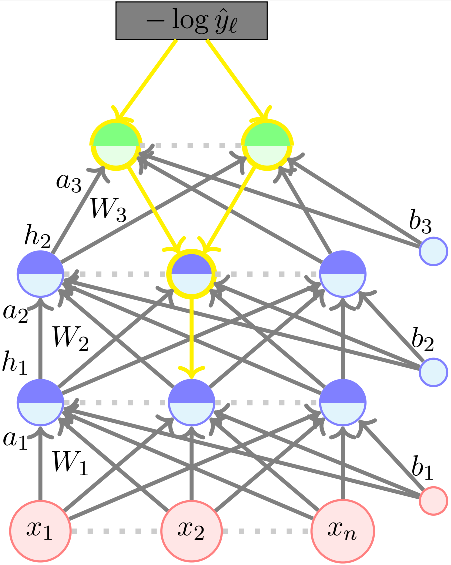
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{kij}}$$



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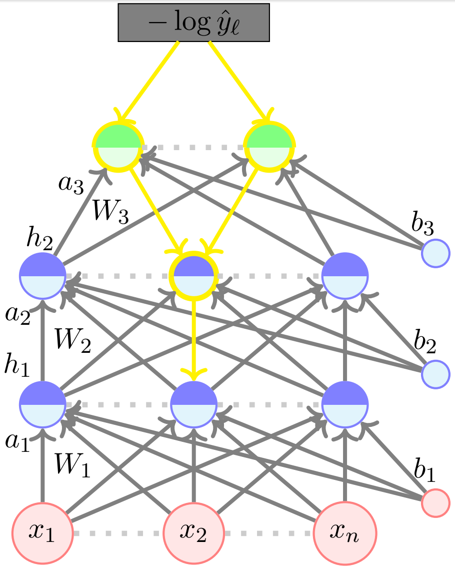

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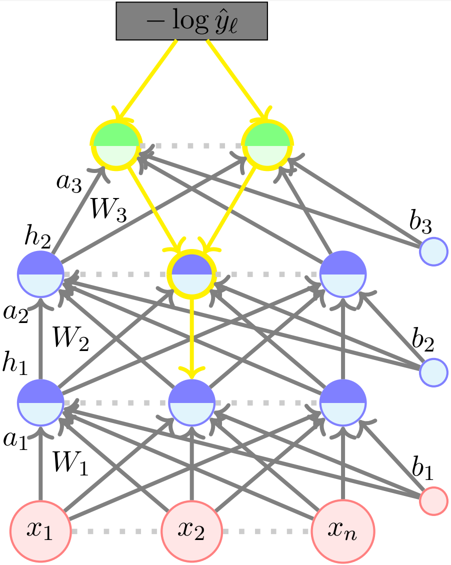

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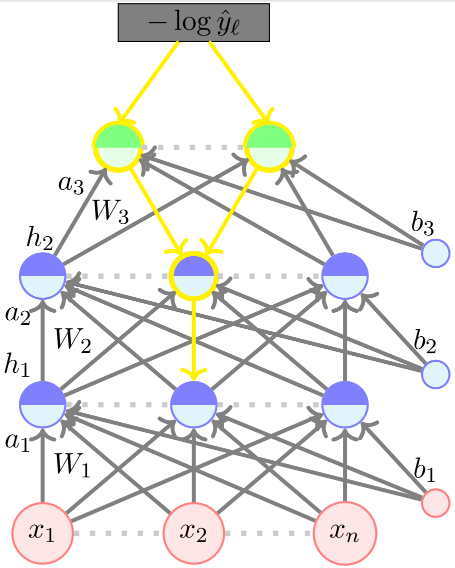
$$\nabla_{W_K} \mathscr{L}(\theta) =$$



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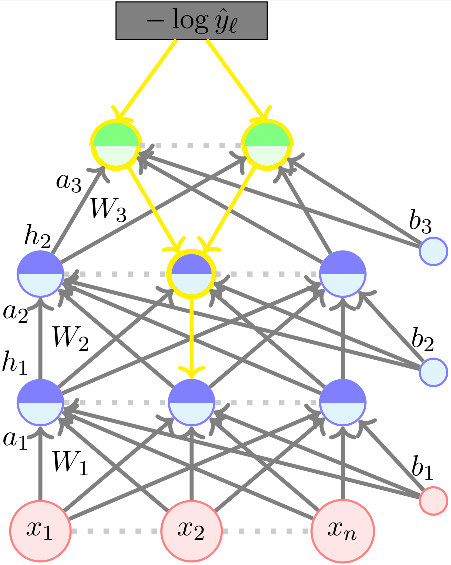
$$\nabla_{W_K} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial W_{k00}} & \frac{\partial \mathscr{L}(\theta)}{\partial W_{k01}} & \cdots & \cdots & \frac{\partial \mathscr{L}(\theta)}{\partial W_{k0n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \frac{\partial \mathscr{L}(\theta)}{\partial W_{k,n-1,n-1}} \end{bmatrix}$$



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Lets take a simple example of a $W_k \in \mathbb{R}^{3\times 3}$ and see what each entry looks like

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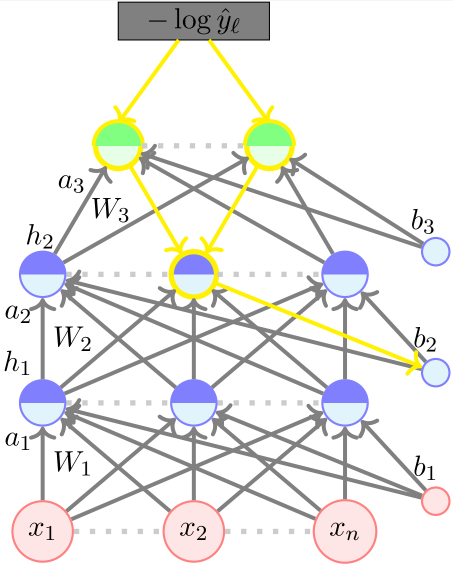
Finally, coming to the biases



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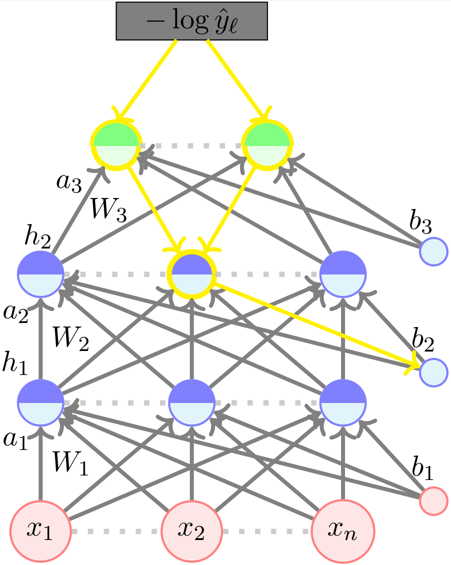
$$a_{ki} = b_{ki} + W_{kij} h_{k-1,j}$$



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$$a_{ki} = b_{ki} + W_{kij}h_{k-1,j}$$
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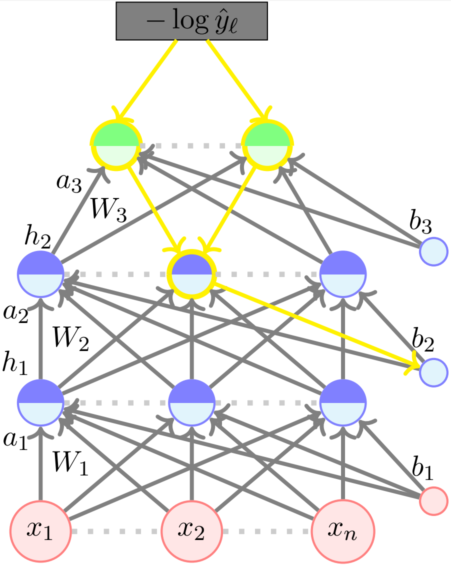

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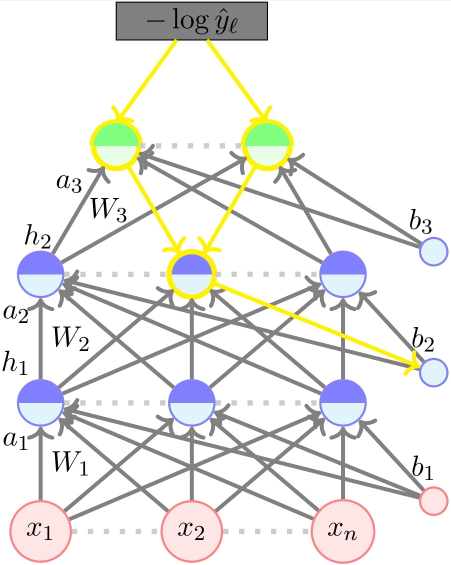

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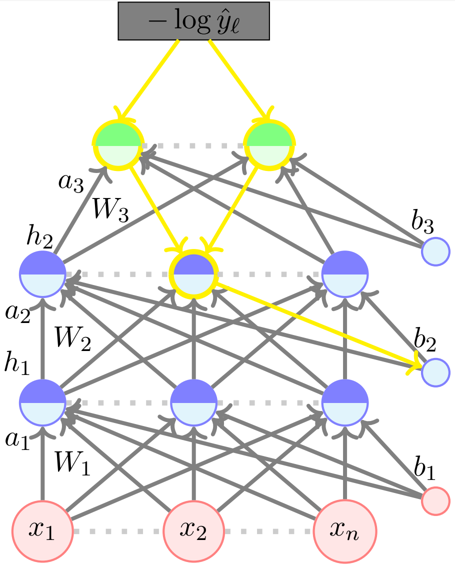
$$
\begin{aligned}
a_{ki} &= b_{ki} + W_{kij} h_{k-1,j} \\
\frac{\partial \mathscr{L}(\theta)}{\partial b_{ki}} &= \frac{\partial \mathscr{L}(\theta)}{\partial a_{ki}} \frac{\partial a_{ki}}{\partial b_{ki}} \\
&= \frac{\partial \mathscr{L}(\theta)}{\partial a_{ki}}
\end{aligned}
$$

We can now write the gradient w.r.t. the vector b_k



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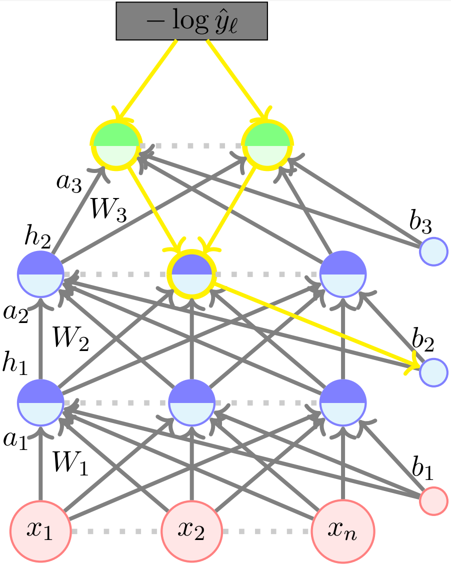
$$\nabla_{b_k} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{a_{k_0}} \\ \frac{\partial \mathscr{L}(\theta)}{a_{k_1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{a_{k_n}} \end{bmatrix}$$



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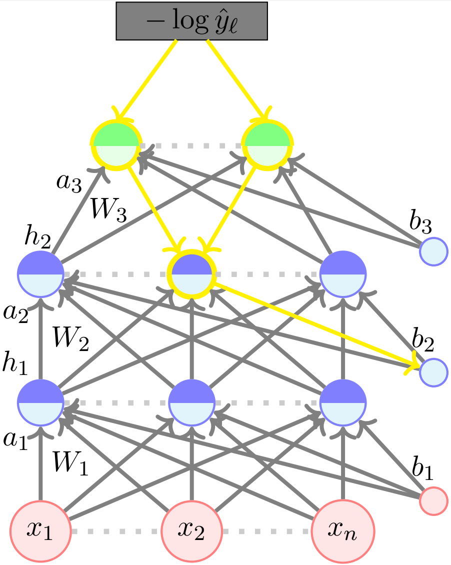

Module 4.8: Backpropagation: Pseudo code

Finally, we have all the pieces of the puzzle

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$$\nabla_{a_L}\mathscr{L}(\theta) \quad \text{(gradient w.r.t. output layer)}$$

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Finally, we have all the pieces of the puzzle

$$\nabla_{a_L} \mathscr{L}(\theta) \quad \text{(gradient w.r.t. output layer)}$$

$$\nabla_{h_k} \mathscr{L}(\theta), \nabla_{a_k} \mathscr{L}(\theta) \quad \text{(gradient w.r.t. hidden layers } 0 < k < L\text{)}$$

$$\nabla_{W_k} \mathscr{L}(\theta), \nabla_{b_k} \mathscr{L}(\theta) \quad \text{(gradient w.r.t. weights and biases)}$$

Finally, we have all the pieces of the puzzle

$$\nabla_{a_L} \mathscr{L}(\theta) \quad \text{(gradient w.r.t. output layer)}$$

$$\nabla_{h_k} \mathscr{L}(\theta), \nabla_{a_k} \mathscr{L}(\theta) \quad \text{(gradient w.r.t. hidden layers } 0 < k < L\text{)}$$

$$\nabla_{W_k} \mathscr{L}(\theta), \nabla_{b_k} \mathscr{L}(\theta) \quad \text{(gradient w.r.t. weights and biases)}$$

We can now write the full learning algorithm

Algorithm: gradient_descent()

$t \leftarrow 0;$

$max\_iterations \leftarrow 1000;$

$Initialize \quad \theta_0 = [W_1^0, ..., W_L^0, b_1^0, ..., b_L^0];$

Algorithm: gradient_descent()

$t \leftarrow 0;$

$max\_iterations \leftarrow 1000;$

$Initialize \quad \theta_0 = [W_1^0, ..., W_L^0, b_1^0, ..., b_L^0];$

while t++ $< max\_iterations$ **do**

end

Algorithm: gradient_descent()

$t \leftarrow 0;$

$max\_iterations \leftarrow 1000;$

$Initialize \quad \theta_0 = [W_1^0, ..., W_L^0, b_1^0, ..., b_L^0];$

while $t{+}{+} < max\_iterations$ **do**

$\quad h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y} = forward\_propagation(\theta_t);$

end

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$h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y} = forward\_propagation(\theta_t);$

$\nabla\theta_t = backward\_propagation(h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y});$

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$\quad h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y} = forward\_propagation(\theta_t);$

$\quad \nabla\theta_t = backward\_propagation(h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y});$

$\quad \theta_{t+1} \leftarrow \theta_t - \eta\nabla\theta_t;$

end

Algorithm: forward_propagation(θ)

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for $k = 1$ *to* $L - 1$ **do**

end

Algorithm: forward_propagation(θ)

for $k = 1$ *to* $L - 1$ **do**

$a_k = b_k + W_k h_{k-1};$

end

Algorithm: forward_propagation(θ)

for $k = 1$ *to* $L - 1$ **do**

$a_k = b_k + W_k h_{k-1};$

$h_k = g(a_k);$

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Algorithm: forward_propagation(θ)

for $k = 1$ *to* $L - 1$ **do**

$a_k = b_k + W_k h_{k-1}$;

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$a_L = b_L + W_L h_{L-1}$;

Algorithm: forward_propagation(θ)

for $k = 1$ *to* $L - 1$ **do**

$a_k = b_k + W_k h_{k-1};$

$h_k = g(a_k);$

end

$a_L = b_L + W_L h_{L-1};$

$\hat{y} = O(a_L);$

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

$\nabla_{a_L} \mathscr{L}(\theta) = -(e(y) - f(x))$;

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

$\nabla_{a_L} \mathscr{L}(\theta) = -(e(y) - f(x))$;

for $k = L$ *to* 1 **do**

end

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

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  // Compute gradients w.r.t. parameters ;

end

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

$\nabla_{a_L} \mathscr{L}(\theta) = -(e(y) - f(x))$;

for $k = L$ *to* 1 **do**

// Compute gradients w.r.t. parameters ;

$\nabla_{W_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta) h_{k-1}^T$;

end

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

$\nabla_{a_L} \mathscr{L}(\theta) = -(e(y) - f(x))$;

for $k = L$ *to* 1 **do**

  // Compute gradients w.r.t. parameters ;

  $\nabla_{W_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta) h_{k-1}^T$;

  $\nabla_{b_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta)$;

end

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

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for $k = L$ *to* 1 **do**

// Compute gradients w.r.t. parameters ;

$\nabla_{W_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta) h_{k-1}^T$;

$\nabla_{b_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta)$;

// Compute gradients w.r.t. layer below ;

end

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

$\nabla_{a_L} \mathscr{L}(\theta) = -(e(y) - f(x))$;

for $k = L$ *to* 1 **do**

// Compute gradients w.r.t. parameters ;

$\nabla_{W_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta) h_{k-1}^T$;

$\nabla_{b_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta)$;

// Compute gradients w.r.t. layer below ;

$\nabla_{h_{k-1}} \mathscr{L}(\theta) = W_k^T (\nabla_{a_k} \mathscr{L}(\theta))$;

end

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

$\nabla_{a_L} \mathscr{L}(\theta) = -(e(y) - f(x))$;

for $k = L$ *to* 1 **do**

// Compute gradients w.r.t. parameters ;

$\nabla_{W_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta) h_{k-1}^T$;

$\nabla_{b_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta)$;

// Compute gradients w.r.t. layer below ;

$\nabla_{h_{k-1}} \mathscr{L}(\theta) = W_k^T (\nabla_{a_k} \mathscr{L}(\theta))$;

// Compute gradients w.r.t. layer below (pre-activation);

end

Just do a forward propagation and compute all h_i's, a_i's and $f(x)$

Algorithm: back_propagation($h_1, h_2, ..., h_{L-1}, a_1, a_2, ..., a_L, \hat{y}$)

//Compute output gradient ;

$\nabla_{a_L} \mathscr{L}(\theta) = -(e(y) - f(x))$;

for $k = L$ *to* 1 **do**

// Compute gradients w.r.t. parameters ;

$\nabla_{W_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta) h_{k-1}^T$;

$\nabla_{b_k} \mathscr{L}(\theta) = \nabla_{a_k} \mathscr{L}(\theta)$;

// Compute gradients w.r.t. layer below ;

$\nabla_{h_{k-1}} \mathscr{L}(\theta) = W_k^T (\nabla_{a_k} \mathscr{L}(\theta))$;

// Compute gradients w.r.t. layer below (pre-activation);

$\nabla_{a_{k-1}} \mathscr{L}(\theta) = \nabla_{h_{k-1}} \mathscr{L}(\theta) \odot [\ldots, g'(a_{k-1,j}), \ldots]$;

end

Module 4.9: Derivative of the activation function

Now, the only thing we need to figure out is how to compute g'

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$
\begin{aligned}
g(z) &= \sigma(z) \\
&= \frac{1}{1+e^{-z}}
\end{aligned}
$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$\begin{aligned} g(z) &= \sigma(z) \\ &= \frac{1}{1+e^{-z}} \end{aligned}$$

$$g'(z) = (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z})$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$
\begin{aligned}
g(z) &= \sigma(z) \\
&= \frac{1}{1+e^{-z}} \\
g'(z) &= (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z}) \\
&= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z})
\end{aligned}
$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$
\begin{aligned}
g(z) &= \sigma(z) \\
&= \frac{1}{1+e^{-z}} \\
g'(z) &= (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z}) \\
&= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z}) \\
&= \frac{1}{1+e^{-z}}\left(\frac{1+e^{-z}-1}{1+e^{-z}}\right)
\end{aligned}
$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$
\begin{aligned}
g(z) &= \sigma(z) \\
&= \frac{1}{1+e^{-z}} \\
g'(z) &= (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z}) \\
&= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z}) \\
&= \frac{1}{1+e^{-z}}\left(\frac{1+e^{-z}-1}{1+e^{-z}}\right) \\
&= g(z)(1-g(z))
\end{aligned}
$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$
\begin{aligned}
g(z) &= \sigma(z) \\
&= \frac{1}{1+e^{-z}} \\
g'(z) &= (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z}) \\
&= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z}) \\
&= \frac{1}{1+e^{-z}}\left(\frac{1+e^{-z}-1}{1+e^{-z}}\right) \\
&= g(z)(1-g(z))
\end{aligned}
$$

tanh

$$
\begin{aligned}
g(z) &= \tanh(z) \\
&= \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}
\end{aligned}
$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$
\begin{aligned}
g(z) &= \sigma(z) \\
&= \frac{1}{1+e^{-z}} \\
g'(z) &= (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z}) \\
&= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z}) \\
&= \frac{1}{1+e^{-z}}\left(\frac{1+e^{-z}-1}{1+e^{-z}}\right) \\
&= g(z)(1-g(z))
\end{aligned}
$$

tanh

$$
\begin{aligned}
g(z) &= \tanh(z) \\
&= \frac{e^z - e^{-z}}{e^z + e^{-z}} \\
g'(z) &= \frac{\left(\begin{array}{l} (e^z + e^{-z})\frac{d}{dz}(e^z - e^{-z}) \\ \qquad - (e^z - e^{-z})\frac{d}{dz}(e^z + e^{-z}) \end{array} \right)}{(e^z + e^{-z})^2}
\end{aligned}
$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$
\begin{aligned}
g(z) &= \sigma(z) \\
&= \frac{1}{1+e^{-z}} \\
g'(z) &= (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z}) \\
&= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z}) \\
&= \frac{1}{1+e^{-z}}\left(\frac{1+e^{-z}-1}{1+e^{-z}}\right) \\
&= g(z)(1-g(z))
\end{aligned}
$$

tanh

$$
\begin{aligned}
g(z) &= \tanh(z) \\
&= \frac{e^z - e^{-z}}{e^z + e^{-z}} \\
g'(z) &= \frac{\begin{pmatrix} (e^z+e^{-z})\frac{d}{dz}(e^z-e^{-z}) \\ -(e^z-e^{-z})\frac{d}{dz}(e^z+e^{-z}) \end{pmatrix}}{(e^z+e^{-z})^2} \\
&= \frac{(e^z+e^{-z})^2-(e^z-e^{-z})^2}{(e^z+e^{-z})^2}
\end{aligned}
$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$g(z) = \sigma(z)$$
$$= \frac{1}{1+e^{-z}}$$
$$g'(z) = (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z})$$
$$= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z})$$
$$= \frac{1}{1+e^{-z}}\left(\frac{1+e^{-z}-1}{1+e^{-z}}\right)$$
$$= g(z)(1-g(z))$$

tanh

$$g(z) = \tanh(z)$$
$$= \frac{e^z - e^{-z}}{e^z + e^{-z}}$$
$$g'(z) = \frac{\left(\begin{array}{l}(e^z + e^{-z})\frac{d}{dz}(e^z - e^{-z}) \\ \quad - (e^z - e^{-z})\frac{d}{dz}(e^z + e^{-z})\end{array}\right)}{(e^z + e^{-z})^2}$$
$$= \frac{(e^z + e^{-z})^2 - (e^z - e^{-z})^2}{(e^z + e^{-z})^2}$$
$$= 1 - \frac{(e^z - e^{-z})^2}{(e^z + e^{-z})^2}$$

Now, the only thing we need to figure out is how to compute g'

Logistic function

$$\begin{aligned} g(z) &= \sigma(z) \\ &= \frac{1}{1+e^{-z}} \\ g'(z) &= (-1)\frac{1}{(1+e^{-z})^2}\frac{d}{dz}(1+e^{-z}) \\ &= (-1)\frac{1}{(1+e^{-z})^2}(-e^{-z}) \\ &= \frac{1}{1+e^{-z}}\left(\frac{1+e^{-z}-1}{1+e^{-z}}\right) \\ &= g(z)(1-g(z)) \end{aligned}$$

tanh

$$\begin{aligned} g(z) &= \tanh(z) \\ &= \frac{e^z - e^{-z}}{e^z + e^{-z}} \\ g'(z) &= \frac{\left(\begin{array}{c} (e^z+e^{-z})\frac{d}{dz}(e^z-e^{-z}) \\ -(e^z-e^{-z})\frac{d}{dz}(e^z+e^{-z}) \end{array} \right)}{(e^z+e^{-z})^2} \\ &= \frac{(e^z+e^{-z})^2-(e^z-e^{-z})^2}{(e^z+e^{-z})^2} \\ &= 1-\frac{(e^z-e^{-z})^2}{(e^z+e^{-z})^2} \\ &= 1-(g(z))^2 \end{aligned}$$