CS7015 (Deep Learning) : Lecture 9

Regularization: Bias Variance Tradeoff, L2 regularization, Early stopping, Dataset augmentation, Parameter sharing and tying, Injecting noise at input, Ensemble methods, Dropout

February 10, 2017

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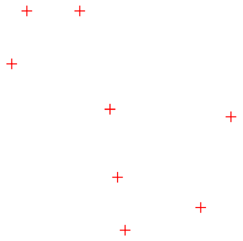
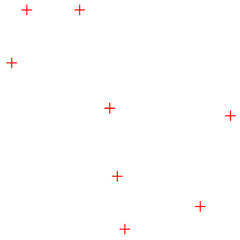
Department of Computer Science and Engineering
Indian Institute of Technology Madras

We will begin with a quick overview of bias, variance and the trade-off between them.



The points were drawn from a sinusoidal function (the true $f(x)$)

- Let us consider the problem of fitting a curve through a given set of points



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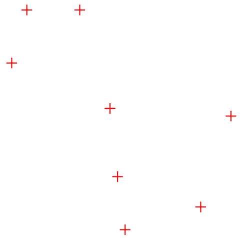
- We consider two models :



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- Let us consider the problem of fitting a curve through a given set of points
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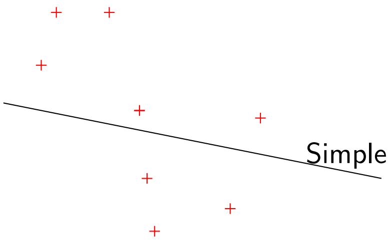
$$\underset{(degree:1)}{Simple} \quad y = \hat{f}(x) = w_1 x + w_0$$



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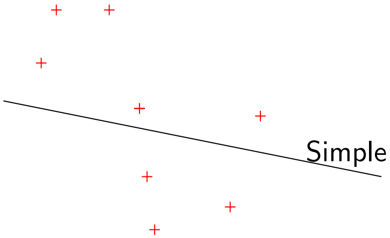

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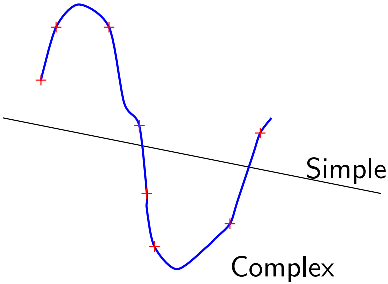
$$\begin{array}{cl} \textit{Simple} \\ (\textit{degree}:1) & y = \hat{f}(x) = w_1 x + w_0 \\ \textit{Complex} \\ (\textit{degree}:25) & y = \hat{f}(x) = \sum_{i=1}^{25} w_i x^i + w_0 \end{array}$$



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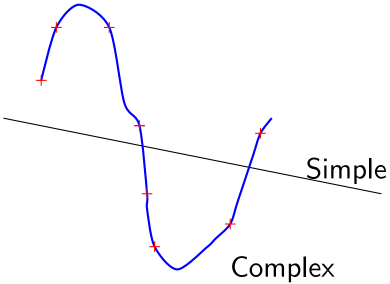

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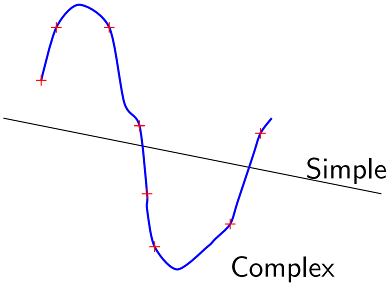

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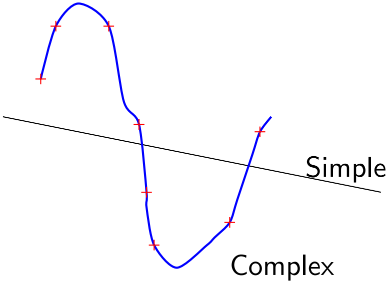
- Note that in both cases we are making an assumption about how y is related to x. We have no idea about the true relation $f(x)$
- The training data consists of 100 points



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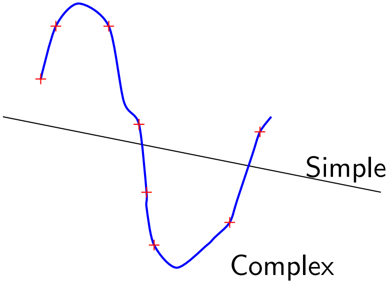
- We sample 25 points from the training data and train a simple and a complex model



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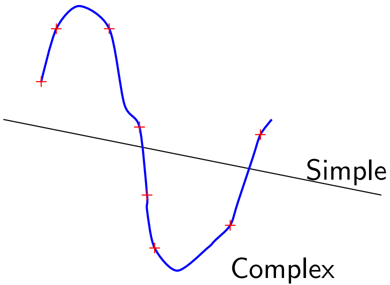
- We repeat the process 'k' times to train multiple models (each model sees a different sample of the training data)



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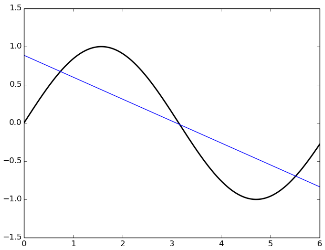
- We make a few observations from these plots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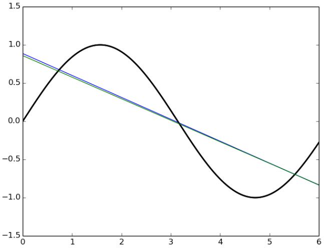
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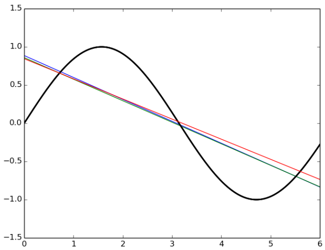
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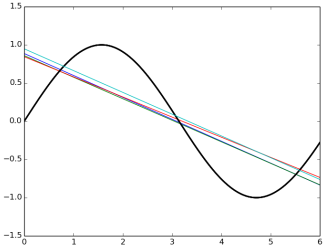

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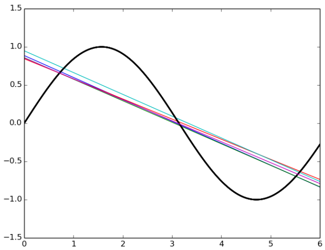

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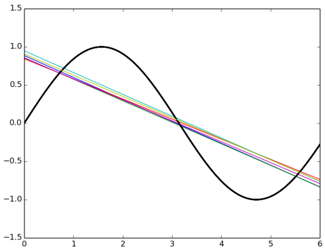

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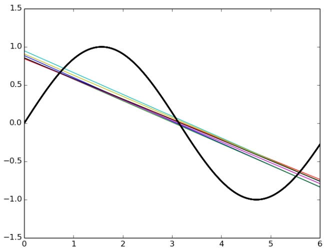
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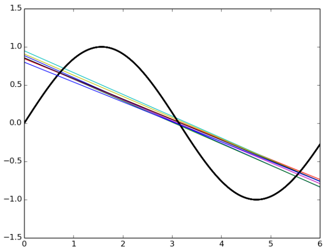
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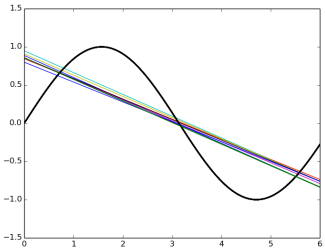
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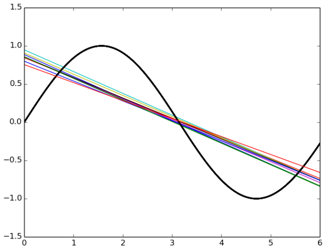

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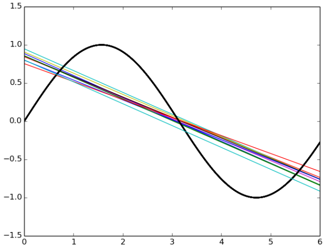
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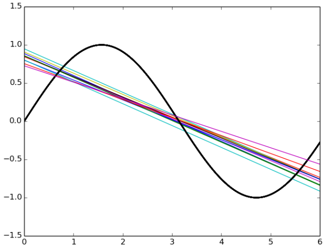
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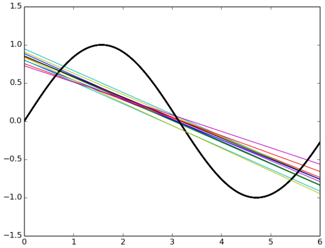

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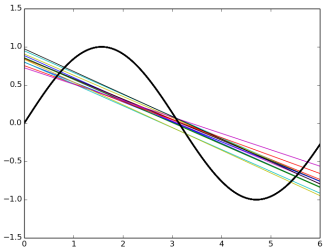
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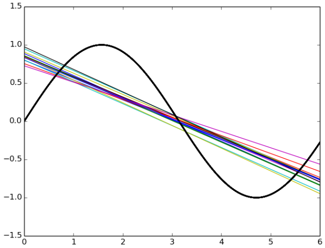
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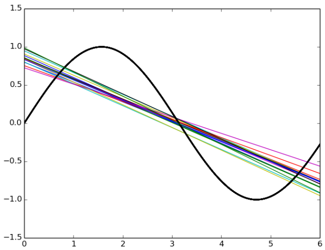
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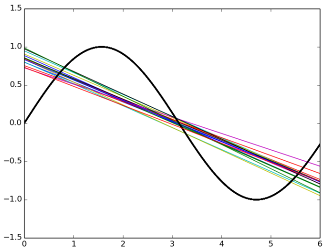
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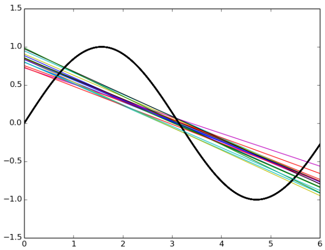
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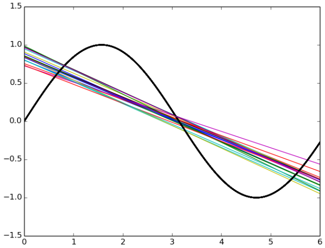
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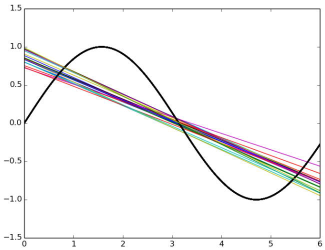

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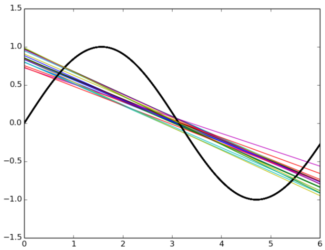

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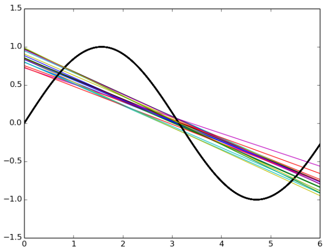

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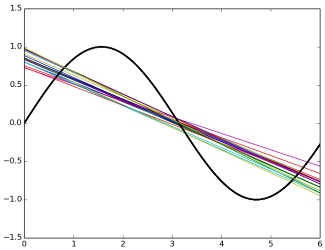

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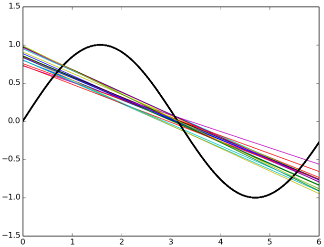

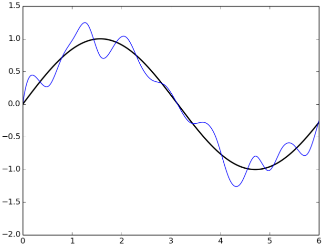

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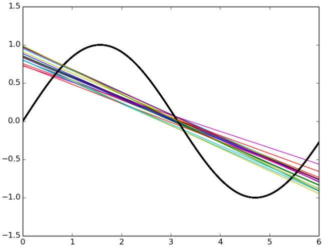

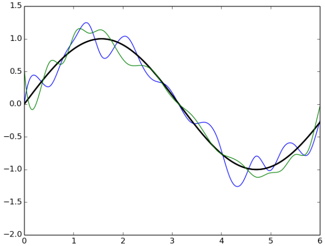

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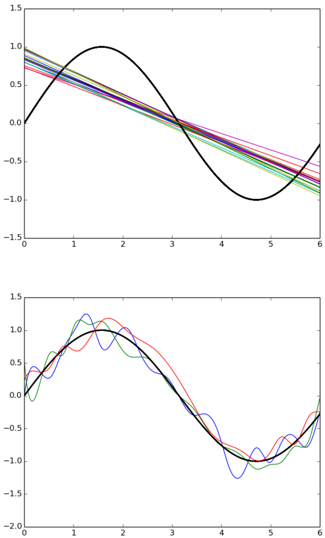

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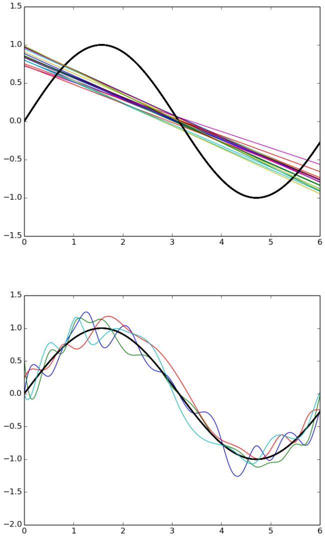

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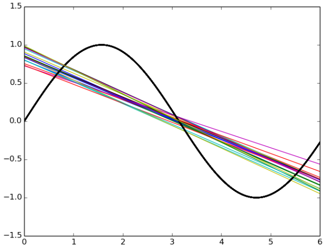

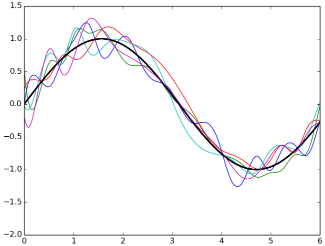

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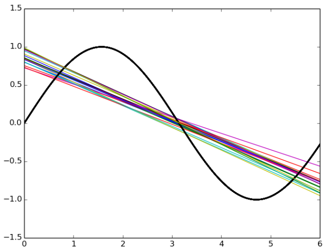

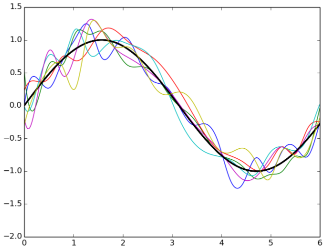

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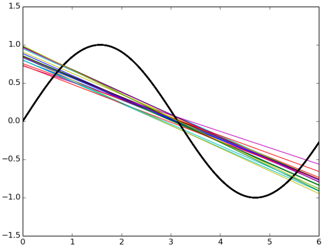

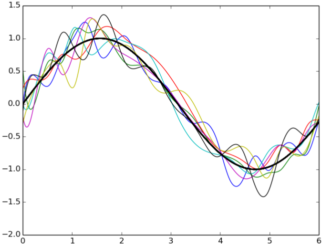

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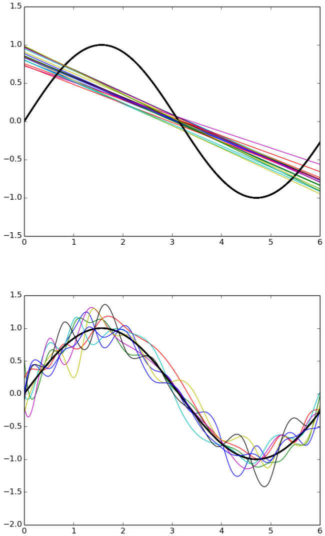

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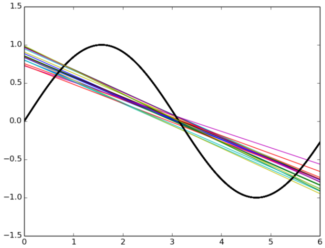

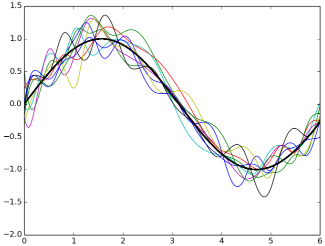

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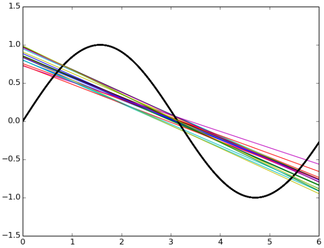

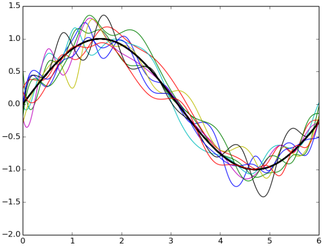

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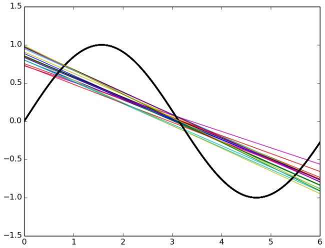

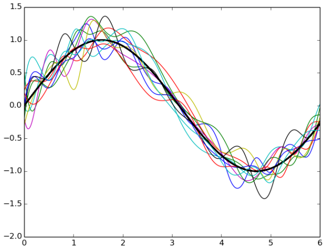

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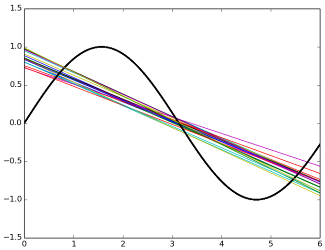

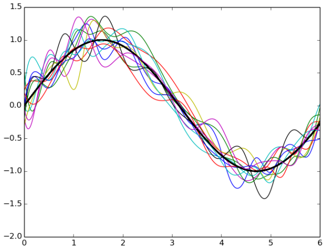

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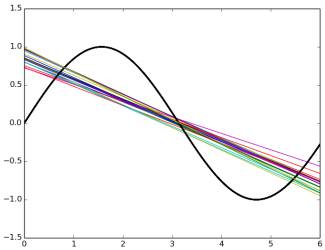

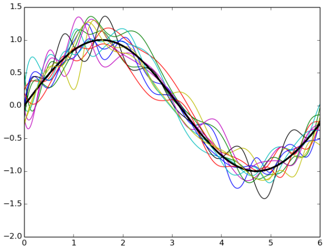

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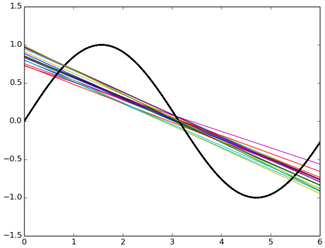

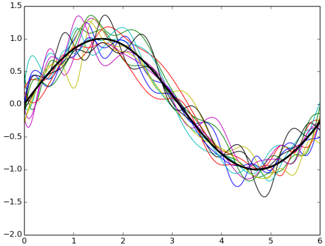

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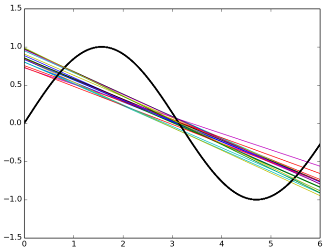

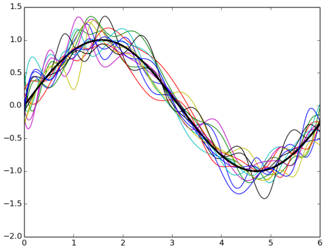

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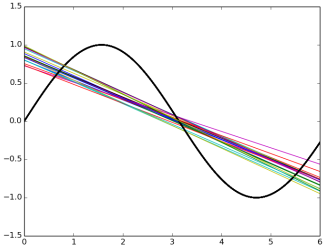

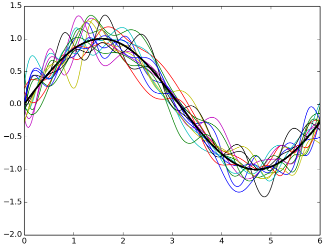

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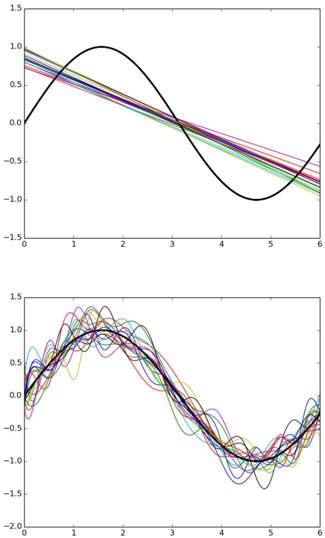

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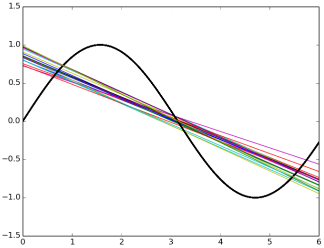

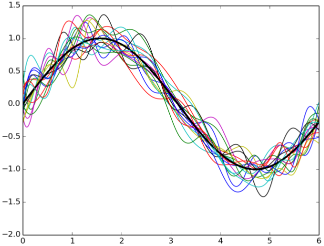

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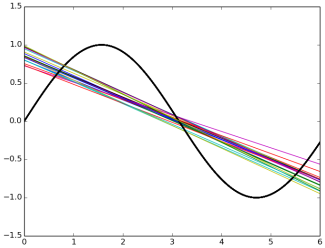

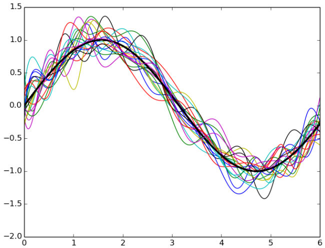

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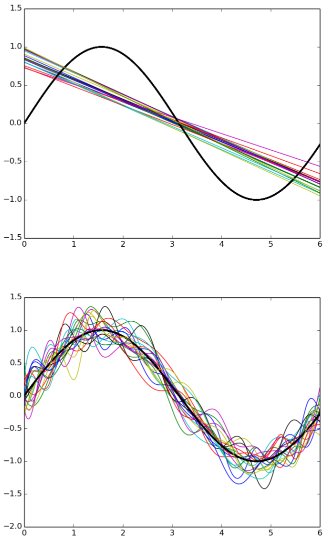

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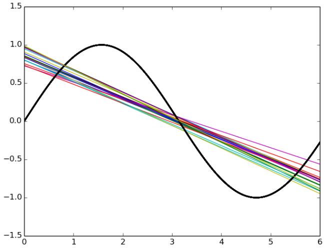
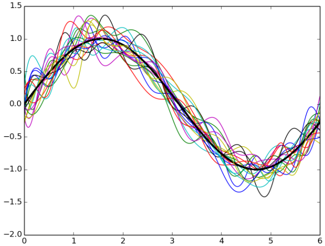
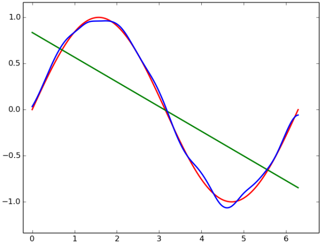




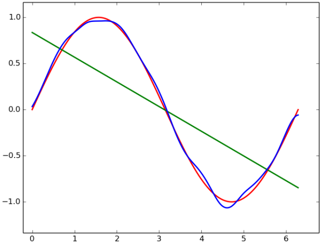
- Simple models trained on different samples of the data do not differ much from each other
- However they are very far from the true sinusoidal curve (under fitting)
- On the other hand, complex models trained on different samples of the data are very different from each other (high variance)



Green Line: Average value of $\hat{f}(x)$ for the simple model
Blue Curve: Average value of $\hat{f}(x)$ for the complex model
Red Curve: True model ($f(x)$)

- Let $f(x)$ be the true model (sinusoidal in this case) and $\hat{f}(x)$ be our estimate of the model (simple or complex, in this case) then,

$$\text{Bias}\ (\hat{f}(x)) = E[\hat{f}(x)] - f(x)$$



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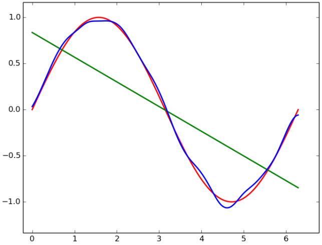

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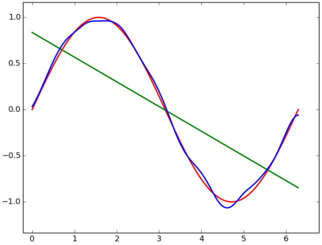

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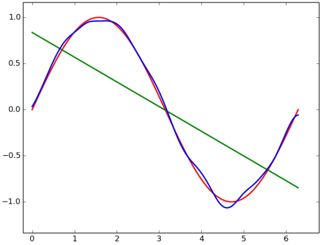
- Mathematically, this means that the simple model has a high bias



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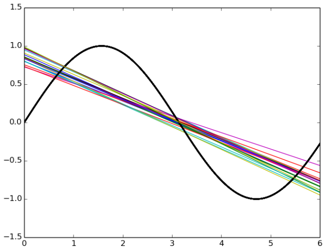
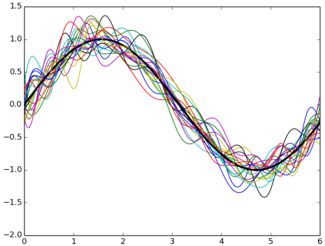
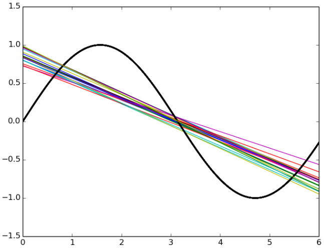


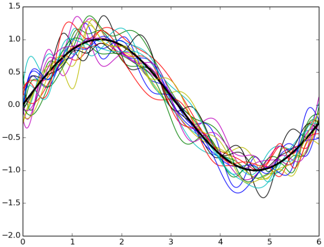


- We now define,

$$\text{Variance } (\hat{f}(x)) = E[(\hat{f}(x) - E[\hat{f}(x)])^2]$$

(Standard definition from statistics)





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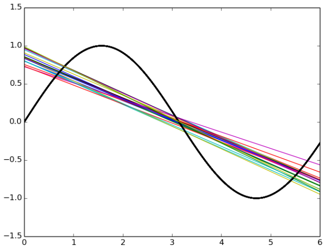

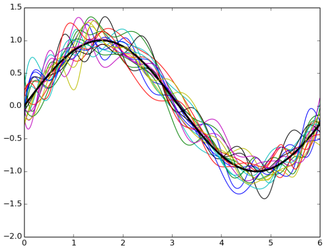

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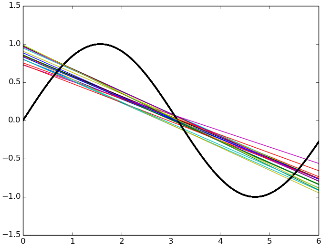
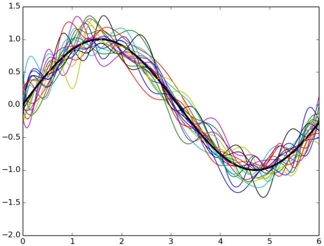
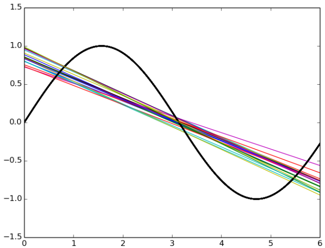
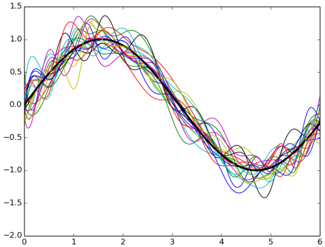
- It is clear that the simple model has a low variance whereas the complex model has a high variance



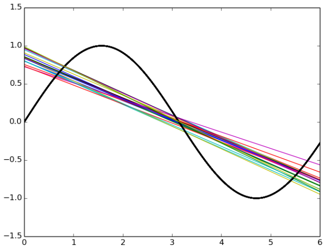


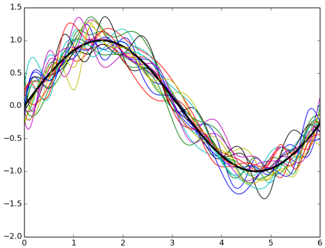
- In summary (informally)





- In summary (informally)
- Simple model: high bias, low variance





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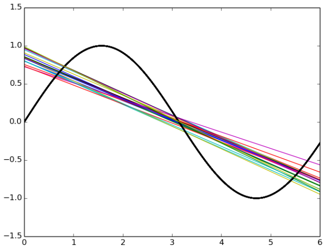

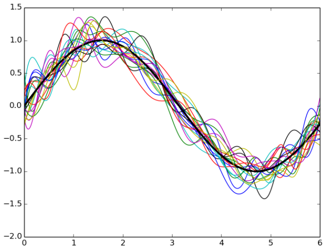

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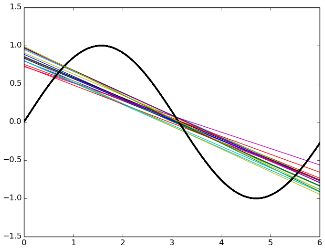

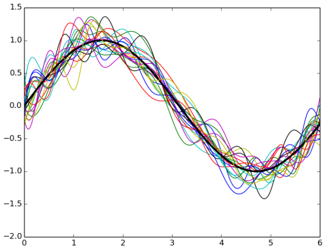

- In summary (informally)
- Simple model: high bias, low variance
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- Both bias and variance contribute to the mean square error. Let us see how,

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- See proof here

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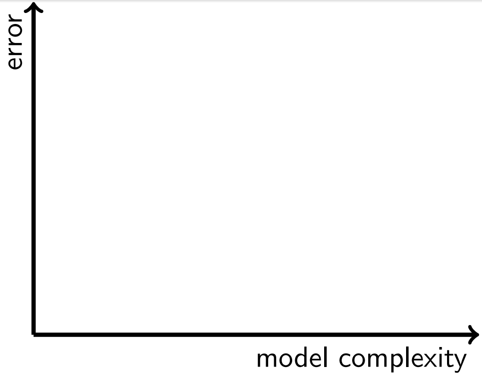

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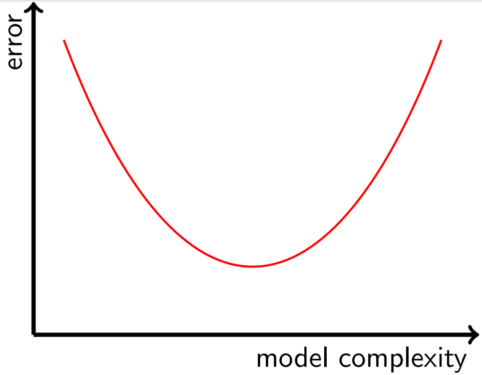

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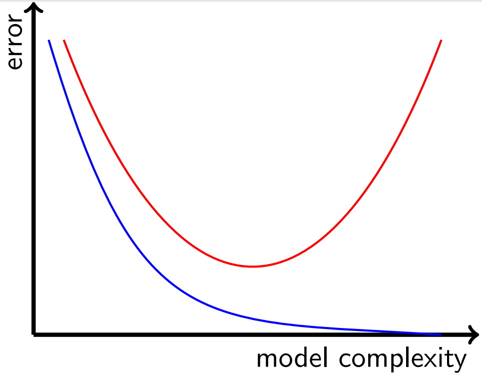

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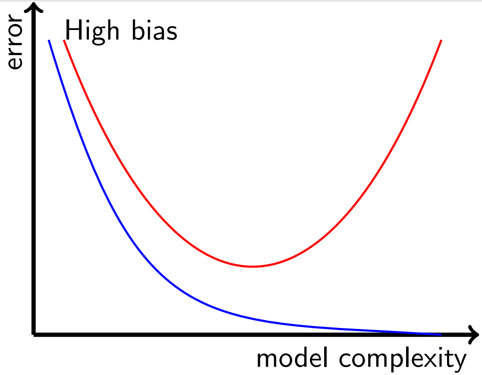

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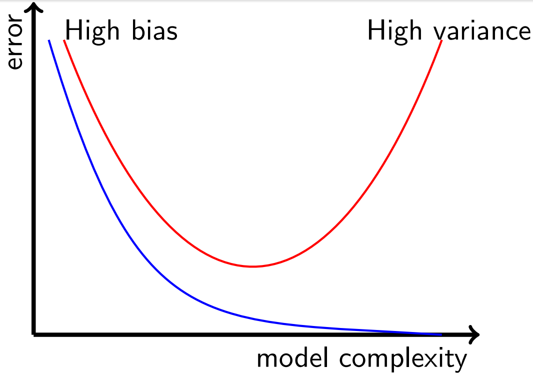

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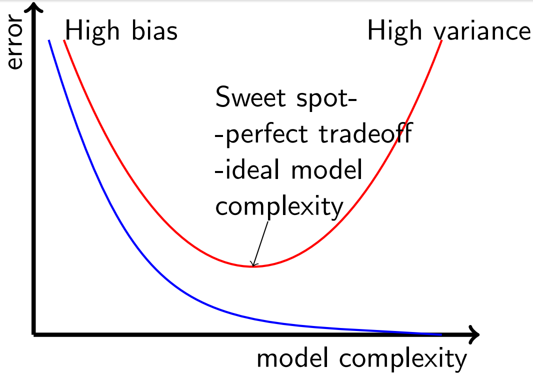

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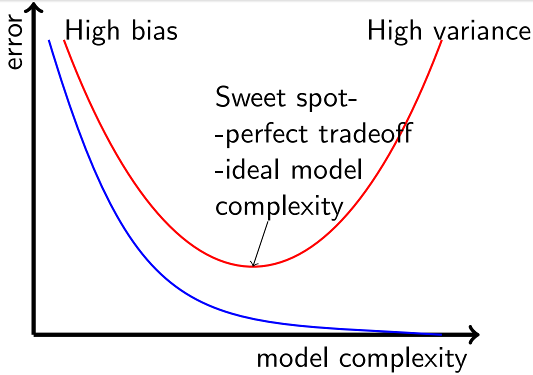

$$E[(y - \hat{f}(x))^2] = Bias^2 + Variance + \sigma^2 \text{ (irreducible error)}$$

- The parameters of $\hat{f}(x)$ (all w_i's) are trained using a training set $\{(x_i, y_i)\}_{i=1}^n$
- However, at test time we are interested in evaluating the model on a validation (unseen) set which was not used for training
- This gives rise to the following two entities of interest:
  $train_{err}$ (say, mean square error)
  $test_{err}$ (say, mean square error)
- Typically these errors exhibit the trend shown in the adjacent figure

Intuitions developed so far

- Let there be n training points and m test (validation) points

$$train_{err} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{f}(x_i))^2$$

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- As the model complexity increases $train_{err}$ becomes overly optimistic and gives us a wrong picture of how close $\hat{f}$ is to f
- The validation error gives the real picture of how close $\hat{f}$ is to f
- We will concretize this intuition mathematically now and eventually show how to account for the optimism in the training error

- Let D=$\{x_i, y_i\}_{i=1}^{m+n}$, then for any point (x,y) we have,

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- We will see how to estimate this empirically using the observation y_i & prediction $\hat{y}_i$

$$
\begin{aligned}
&E[(\hat{y}_i - y_i)^2] \\
&\quad = E[(\hat{f}(x_i) - f(x_i) - \varepsilon_i)^2] \quad (y_i = f(x_i) + \varepsilon_i)
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$$\therefore \; E[(\hat{f}(x_i) - f(x_i))^2] = E[(\hat{y}_i - y_i)^2] - E[\varepsilon_i^2] + 2E[\,\varepsilon_i(\hat{f}(x_i) - f(x_i))\,]$$

We will take a small detour to understand how to empirically estimate an Expectation and then return to our derivation

- Suppose we have observed the goals scored(z) in k matches as z_1=2, z_2=1, z_3=0, ... z_k=2

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$$E[(\hat{y}_i - y_i)^2] = \frac{1}{m}\sum_{i=1}^{m}(\hat{y}_i - y_i)^2$$

... returning back to our derivation

$$E[(\hat{f}(x_i) - f(x_i))^2] \; = \; E[(\hat{y}_i - y_i)^2] \; - \; E[\varepsilon_i^2] \; + \; 2E[\; \varepsilon_i(\hat{f}(x_i) - f(x_i)) \;]$$

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- We can empirically evaluate R.H.S using training observations or test observations

Case 1: Using test observations

$$E[(\hat{f}(x_i) - f(x_i))^2] \;=\; E[(\hat{y}_i - y_i)^2] \;-\; E[\varepsilon_i^2] \;+\; 2E[\;\varepsilon_i(\hat{f}(x_i) - f(x_i))\;]$$

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$\therefore\ \ \text{true error} = \text{empirical test error} + \text{small constant}$

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  $\therefore\ E[\varepsilon_i \cdot (\hat{f}(x_i) - f(x_i))] = E[\varepsilon_i] \cdot E[\hat{f}(x_i) - f(x_i))] = 0 \cdot E[\hat{f}(x_i) - f(x_i))] = 0$
  $\therefore$ true error = empirical test error + small constant

- Hence, we should always use a validation set(independent of the training set) to estimate the error

Case 2: Using training observations

$$\underbrace{E[(\hat{f}(x_i) - f(x_i))^2]}_{true\ error}$$

$$= \underbrace{\frac{1}{m}\sum_{i=1}^{n+m}(\hat{y}_i - y_i)^2}_{empirical\ estimation\ of\ error} - \underbrace{\frac{1}{m}\sum_{i=n+1}^{n+m}\varepsilon_i^2}_{small\ constant} + \underbrace{2E[\ \varepsilon_i(\hat{f}(x_i) - f(x_i))\ ]}_{= covariance\ (\varepsilon_i, \hat{f}(x_i) - f(x_i))}$$

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Now, $\varepsilon \not\perp \hat{f}(x)$ because ε was used for estimating the parameters of $\hat{f}(x)$

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Now, $\varepsilon \not\perp \hat{f}(\mathrm{x})$ because ε was used for estimating the parameters of $\hat{f}(\mathrm{x})$

$$\therefore\ E[\varepsilon_i\ .\ (\hat{f}(x_i) - f(x_i))] \neq E[\varepsilon_i]\ .\ E[\hat{f}(x_i) - f(x_i))] \neq\ 0$$

Hence, the empirical train error is smaller than the true error and does not give a true picture of the error

Case 2: Using training observations

$$\underbrace{E[(\hat{f}(x_i) - f(x_i))^2]}_{true\ error} = \underbrace{\frac{1}{m}\sum_{i=1}^{n+m}(\hat{y}_i - y_i)^2}_{empirical\ estimation\ of\ error} - \underbrace{\frac{1}{m}\sum_{i=n+1}^{n+m}\varepsilon_i^2}_{small\ constant} + \underbrace{2E[\ \varepsilon_i(\hat{f}(x_i) - f(x_i))\ ]}_{=\ covariance\ (\varepsilon_i, \hat{f}(x_i) - f(x_i))}$$

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But how is this related to model complexity? Let us see

Using Stein's Lemma (and some trickery) we can show that

$$\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i(\hat{f}(x_i) - f(x_i)) = \frac{\sigma^2}{n}\sum_{i=1}^{n}\frac{\partial \hat{f}(x_i)}{\partial y_i}$$

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- Can you link this to model complexity?
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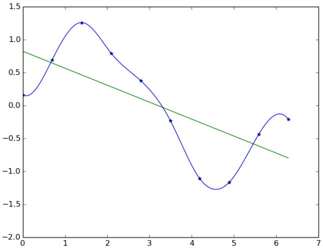
- Hence, we can say that
  true error $=$ empirical train error $+$ small constant $+ \Omega$(model complexity)

- Let us verify that indeed a complex model is more sensitive to minor changes in the data



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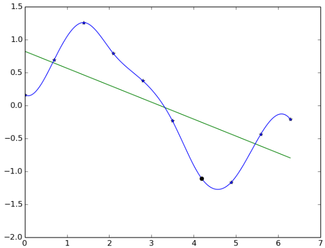

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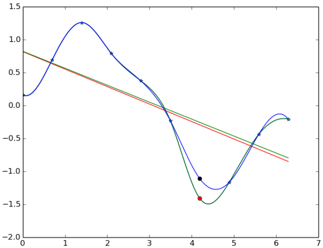

- Let us verify that indeed a complex model is more sensitive to minor changes in the data
- We have fitted a simple and complex model for some given data
- We now change one of these data points
- The simple model does not change much as compared to the complex model

- Hence while training, instead of minimizing the training error $\mathscr{L}_{train}(\theta)$ we should minimize

$$\min_{w.r.t\ \theta} \mathscr{L}_{train}(\theta) + \Omega(\theta) = \mathscr{L}(\theta)$$

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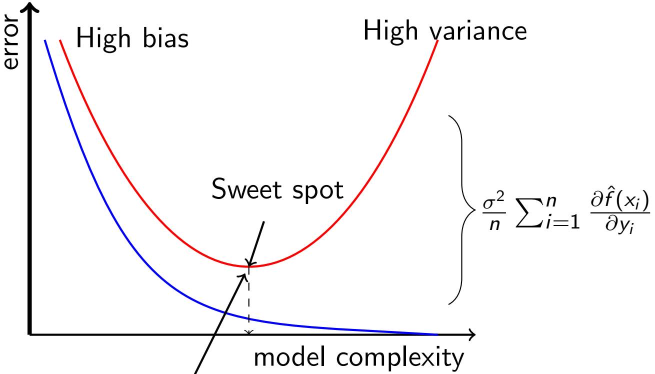
- This is the basis for all regularization methods
- We can show that L_1 regularization, L_2 regularization, early stopping and injecting noise in input are all instances of this form of regularization.



$\Omega(\theta)$ should ensure that model has reasonable complexity

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- Hence we need some form of regularization.

Different forms of regularization

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- Let us see the geometric interpretation of this

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- Let us analyse the case when $\alpha \neq 0$

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where $D = (\Lambda + \alpha\mathbb{I})^{-1}\Lambda$, is a diagonal matrix which we will see in more detail soon

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- if $\lambda_i >> \alpha$ then $\frac{\lambda_i}{\lambda_i+\alpha} = 1$
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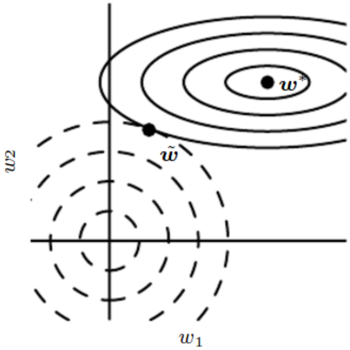
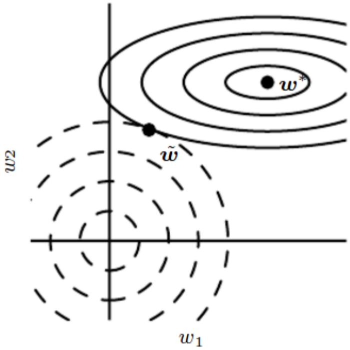
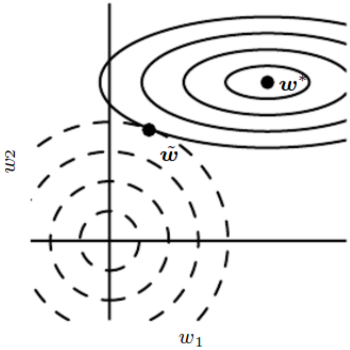
- Thus only significant directions (larger eigen values) will be retained.

$$\text{Effective parameters} = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \alpha} < n$$

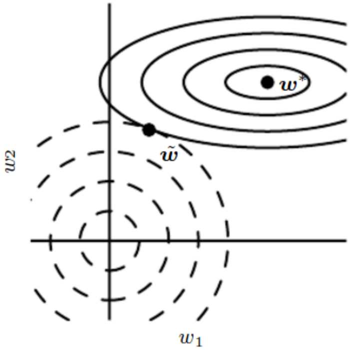

w^*
$\tilde{w}$
w_2
w_1



- The weight vector(w^*) is getting rotated to ($\tilde{w}$)



- The weight vector(w^*) is getting rotated to ($\tilde{w}$)
- All of its elements are shrinking but some are shrinking more than the others



- The weight vector(w^*) is getting rotated to ($\tilde{w}$)
- All of its elements are shrinking but some are shrinking more than the others
- This ensures that only important features are given high weights

Different forms of regularization

- $L2$ regularization
- Dataset augmentation
- Parameter Sharing and tying
- Adding Noise to the inputs
- Adding Noise to the outputs
- Early stopping
- Ensemble methods
- Dropout

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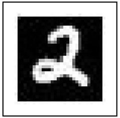
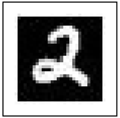


label $= 2$



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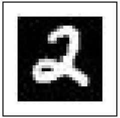
[given training data]



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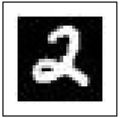

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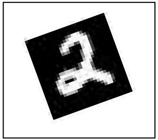
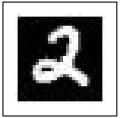


rotated by 20°



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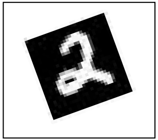

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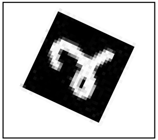
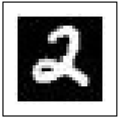


rotated by 65°



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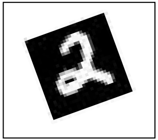

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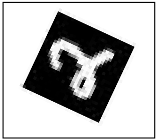

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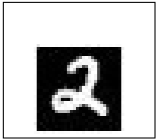
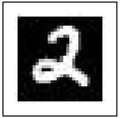


shifted vertically



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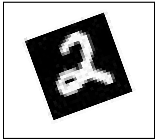

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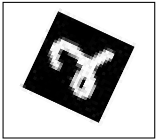

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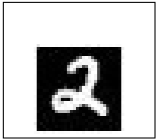

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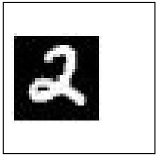

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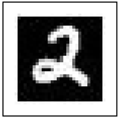

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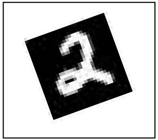

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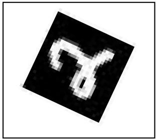

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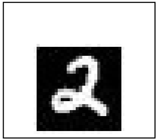

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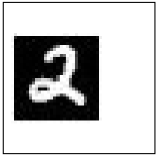

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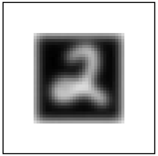
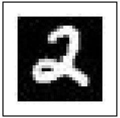


blurred



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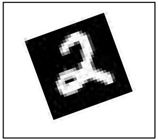

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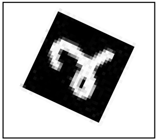

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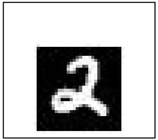

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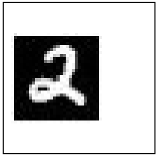

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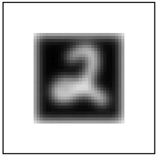

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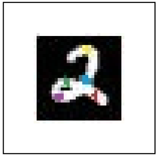
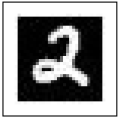


changed some pixels



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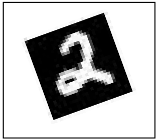
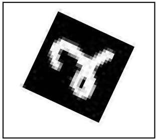
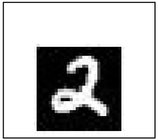


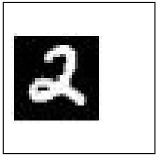
rotated by 20°



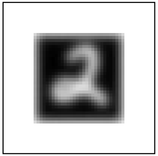
rotated by 65°



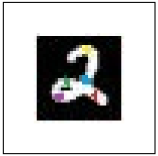
shifted vertically



shifted horizontally

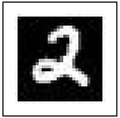


blurred



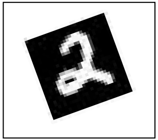
changed some pixels

label = 2

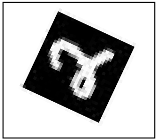


label $= 2$

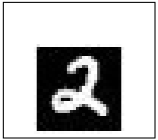
[given training data]



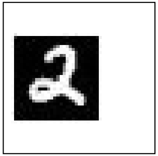
rotated by 20°



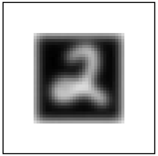
rotated by 65°



shifted vertically



shifted horizontally



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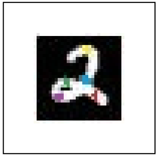

changed some pixels

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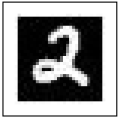
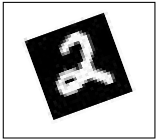
[augmented data $=$ created using some knowledge of the task]

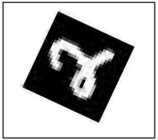


label $= 2$

[given training data]
We exploit the fact that certain transformations to the image do not change the label of the image.



rotated by 20°



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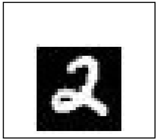

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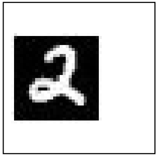
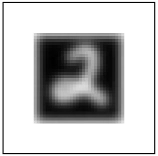


shifted horizontally



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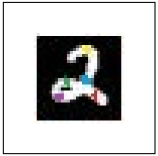

changed some pixels

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[augmented data $=$ created using some knowledge of the task]

- Typically, More data = better learning

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- Also shown to work well for speech
- For some tasks it may not be clear how to generate such data

Other forms of regularization

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- Adding Noise to the inputs
- Adding Noise to the outputs
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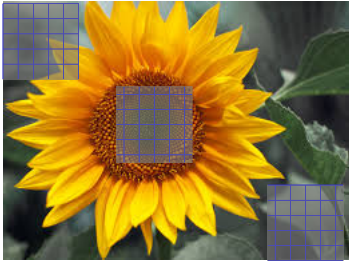
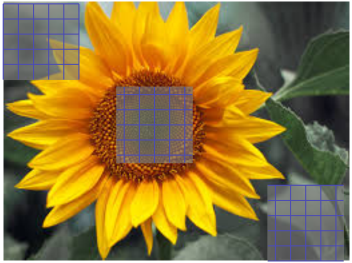
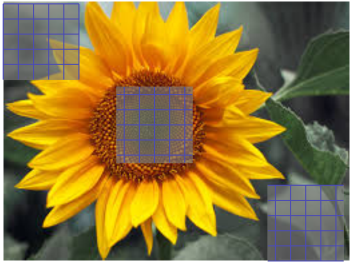


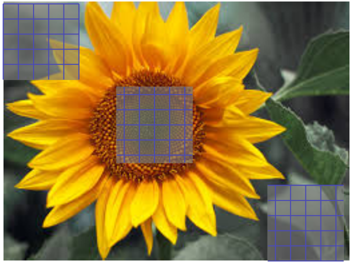


Parameter Sharing



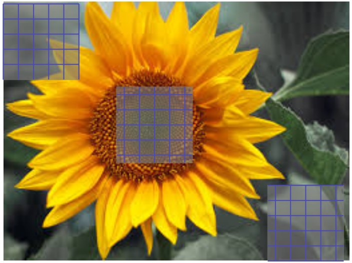
Parameter Sharing

- Used in CNNs



Parameter Sharing

- Used in CNNs
- Same filter applied at different positions of the image



Parameter Sharing

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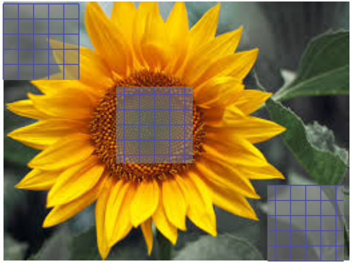
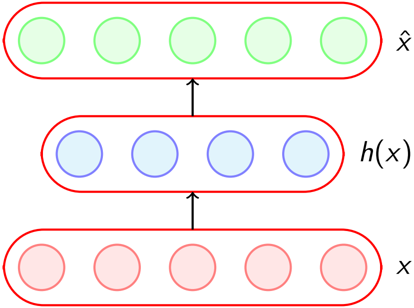
- Same filter applied at different positions of the image
- Or same weight matrix acts on different input neurons





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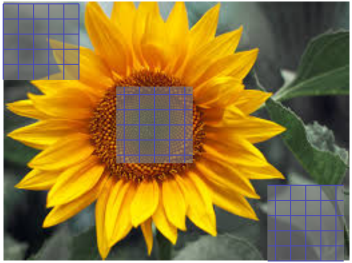

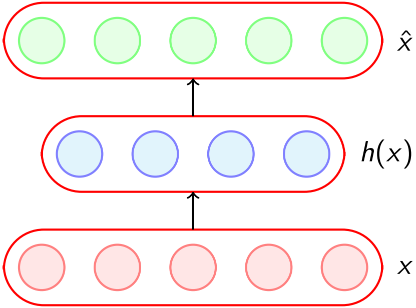

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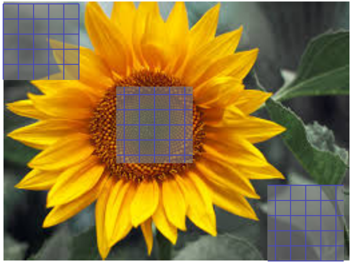
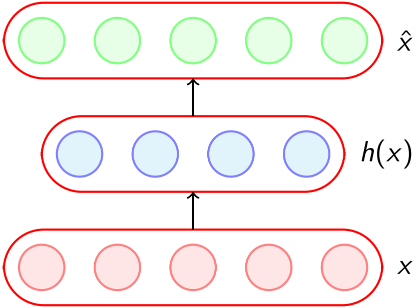
Parameter Tying





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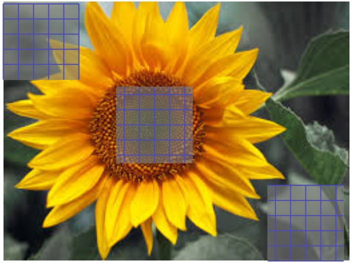
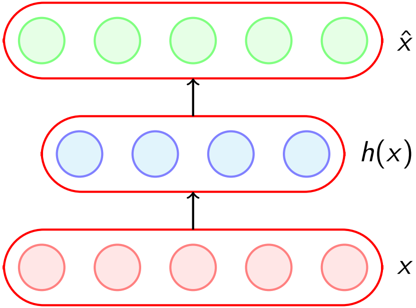
- Typically used in autoencoders





Parameter Sharing

- Used in CNNs
- Same filter applied at different positions of the image
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Parameter Tying

- Typically used in autoencoders
- The encoder and decoder weights are tied.

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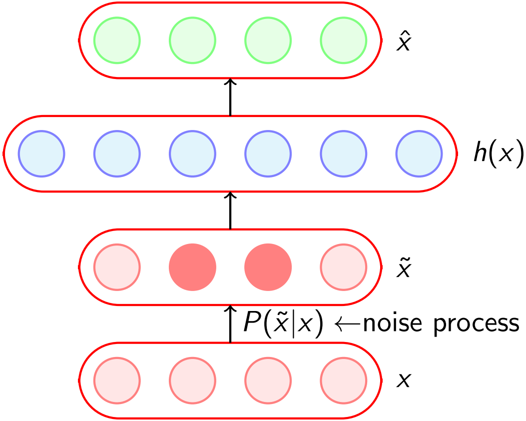
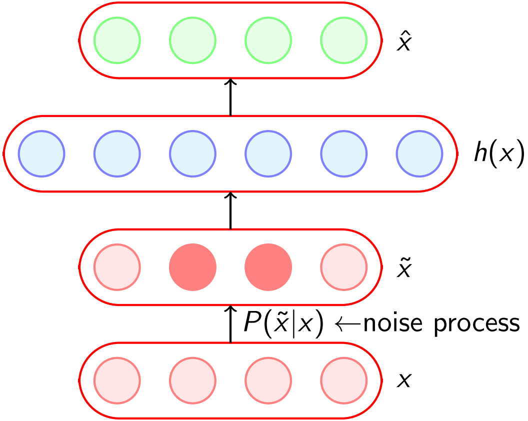
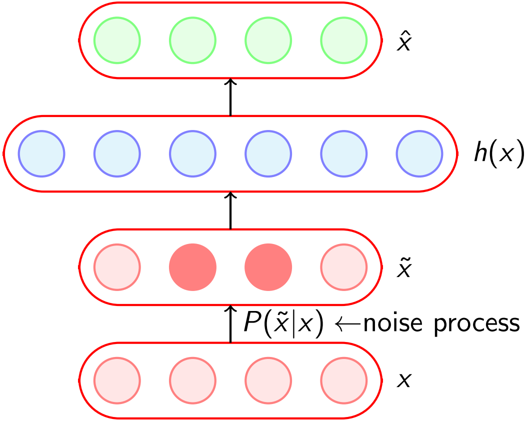


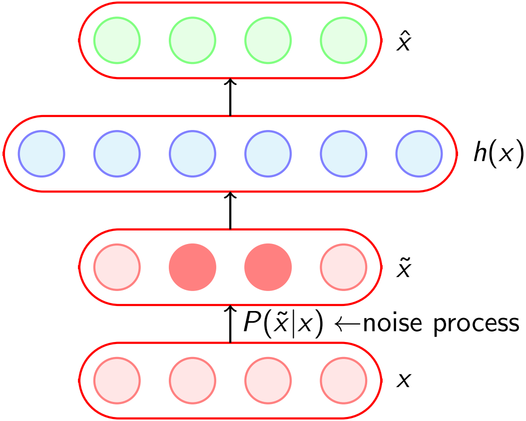
$\hat{x}$
$h(x)$
$\tilde{x}$
$P(\tilde{x}|x)$ ←noise process
x



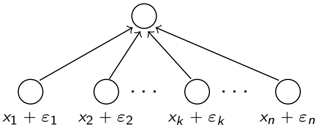
- We saw this in Autoencoder



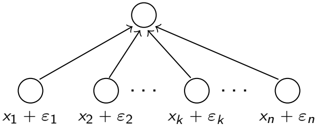
- We saw this in Autoencoder
- We can show that for a simple input output neural network, adding Gaussian noise to the input is equivalent to weight decay ($L2$ normalizaton)



- We saw this in Autoencoder
- We can show that for a simple input output neural network, adding Gaussian noise to the input is equivalent to weight decay ($L2$ normalizaton)
- Can be viewed as data augmentation

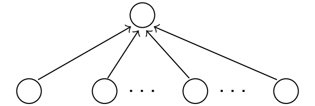


$$\varepsilon \sim \mathcal{N}(0, \sigma^2)$$



$$\varepsilon \sim \mathcal{N}(0, \sigma^2)$$

$$\widetilde{x}_i = x_i + \varepsilon_i$$



$$\varepsilon \sim \mathcal{N}(0, \sigma^2)$$

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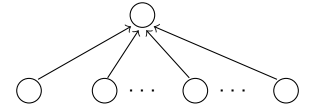
$$\widehat{y} = \sum_{i=1}^{n} w_i x_i$$



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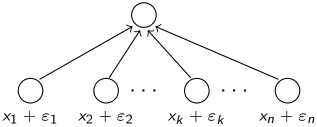

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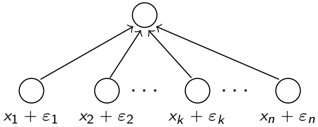

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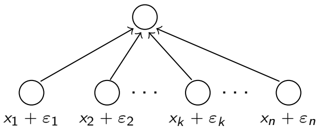
$$\begin{aligned}\widetilde{y} &= \sum_{i=1}^{n} w_i \widetilde{x}_i \\ &= \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n} w_i \varepsilon_i \\ &= \widehat{y} + \sum_{i=1}^{n} w_i \varepsilon_i\end{aligned}$$



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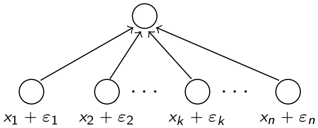
We are interested in $E[(\widetilde{y} - y)^2]$



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$$\begin{aligned}
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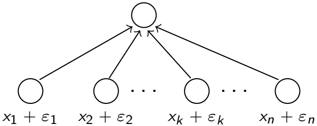
$$E\left[(\widetilde{y} - y)^2\right] = E\left[\left(\widehat{y} + \sum_{i=1}^{n} w_i \varepsilon_i - y\right)^2\right]$$



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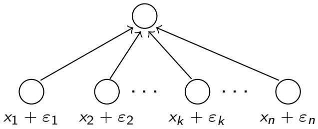

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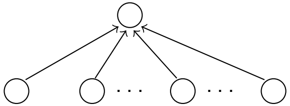
$$= E\left[(\widehat{y} - y)^2\right] + E\left[2(\widehat{y} - y)\sum_{i=1}^{n} w_i \varepsilon_i\right] + E\left[\left(\sum_{i=1}^{n} w_i \varepsilon_i\right)^2\right]$$



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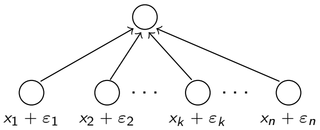
$$= E\left[(\widehat{y} - y)^2\right] + 0 + E\left[\sum_{i=1}^{n} w_i^2 \varepsilon_i^2\right]$$

($\because$ ε_i is independent of ε_j and ε_i is independent of ($\widehat{y}$-y))



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$$\widetilde{y} = \sum_{i=1}^{n} w_i \widetilde{x}_i$$

$$= \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n} w_i \varepsilon_i$$

$$= \widehat{y} + \sum_{i=1}^{n} w_i \varepsilon_i$$

We are interested in $E[(\widetilde{y} - y)^2]$

$$E\left[(\widetilde{y} - y)^2\right] = E\left[\left(\widehat{y} + \sum_{i=1}^{n} w_i \varepsilon_i - y\right)^2\right]$$

$$= E\left[\left(\left(\widehat{y} - y\right) + \left(\sum_{i=1}^{n} w_i \varepsilon_i\right)\right)^2\right]$$

$$= E\left[(\widehat{y} - y)^2\right] + E\left[2(\widehat{y} - y)\sum_{i=1}^{n} w_i \varepsilon_i\right] + E\left[\left(\sum_{i=1}^{n} w_i \varepsilon_i\right)^2\right]$$

$$= E\left[(\widehat{y} - y)^2\right] + 0 + E\left[\sum_{i=1}^{n} w_i^2 \varepsilon_i^2\right]$$

($\because$ ε_i is independent of ε_j and ε_i is independent of ($\widehat{y}$-y))

$$= (E\left[(\widehat{y} - y)^2\right] + \sigma^2 \sum_{i=1}^{n} w_i^2 \quad \text{(same as L2 norm penalty)}$$

Other forms of regularization

- $L2$ regularization
- Dataset augmentation
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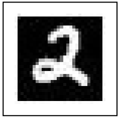
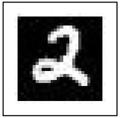


0	0	1	0	0	0	0	0	0	0	Hard targets



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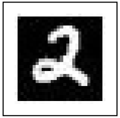
$$\text{minimize} : \sum_{i=0}^{9} p_i \log q_i$$



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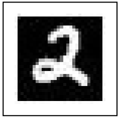
$$\text{minimize} : \sum_{i=0}^{9} p_i \log q_i$$

$$\text{true distribution} : p = \{0, 0, 1, 0, 0, 0, 0, 0, 0, 0\}$$



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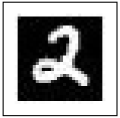

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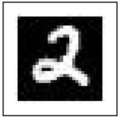
Intuition

- Do not trust the true labels, they may be noisy



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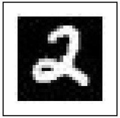
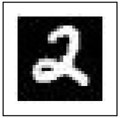
Intuition

- Do not trust the true labels, they may be noisy
- Instead, use soft targets



$\frac{\varepsilon}{9}$	$\frac{\varepsilon}{9}$	$1-\varepsilon$	$\frac{\varepsilon}{9}$	$\frac{\varepsilon}{9}$	$\frac{\varepsilon}{9}$	$\frac{\varepsilon}{9}$	$\frac{\varepsilon}{9}$	$\frac{\varepsilon}{9}$	$\frac{\varepsilon}{9}$

Soft targets



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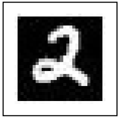
$\varepsilon =$ small positive constant



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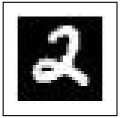

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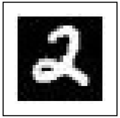

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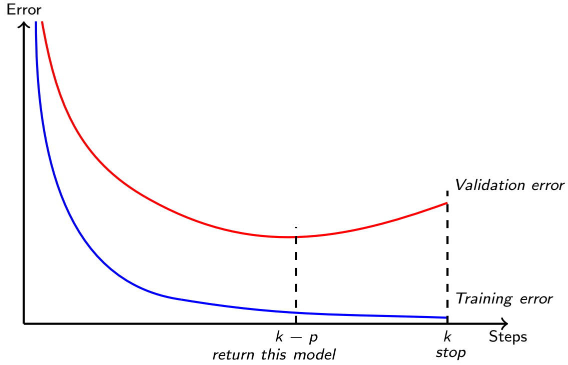
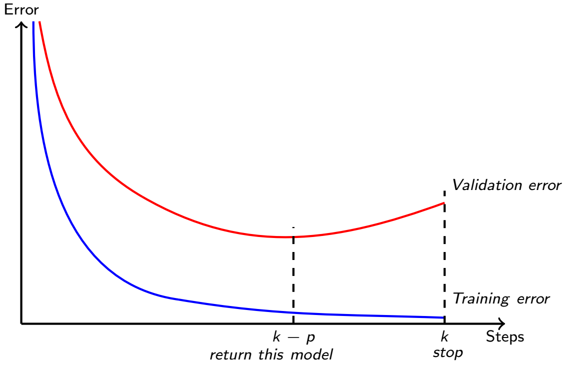


- Track the validation error



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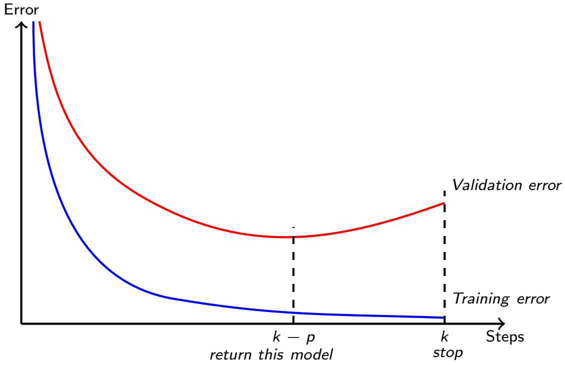

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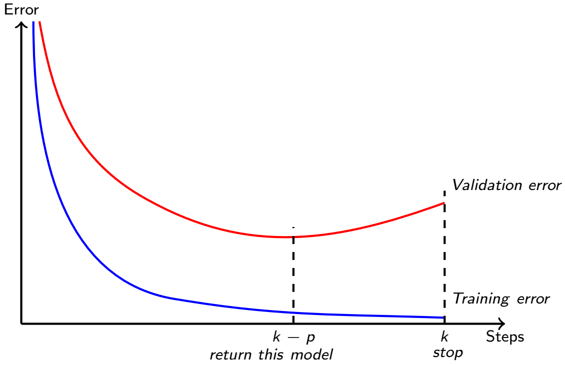

- Track the validation error
- Have a patience parameter p
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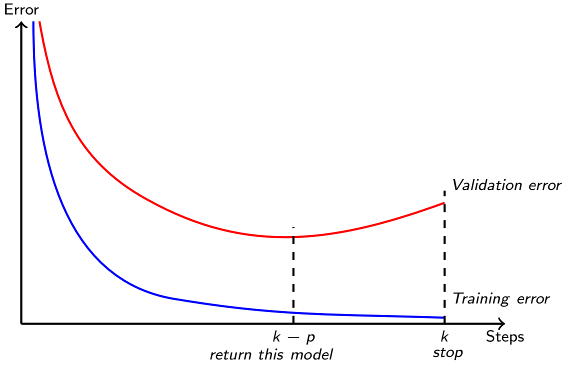
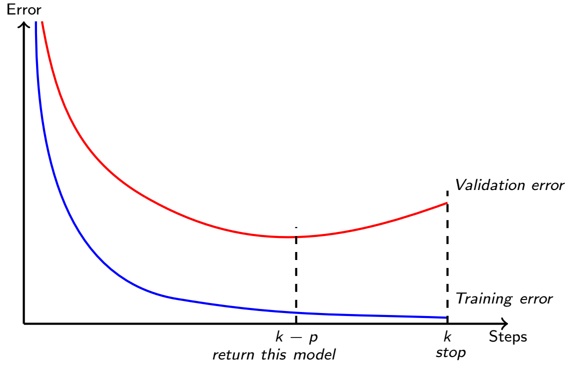
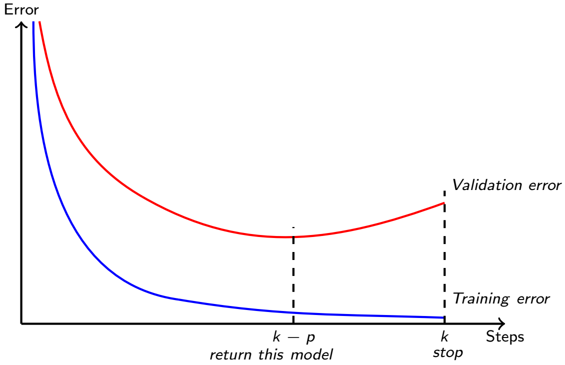
- Basically, stop the training early before it drives the training error to 0 and blows up the validation error



- Very effective and the mostly widely used form of regularization



- Very effective and the mostly widely used form of regularization
- Can be used even with other regularizers(such as L_2)



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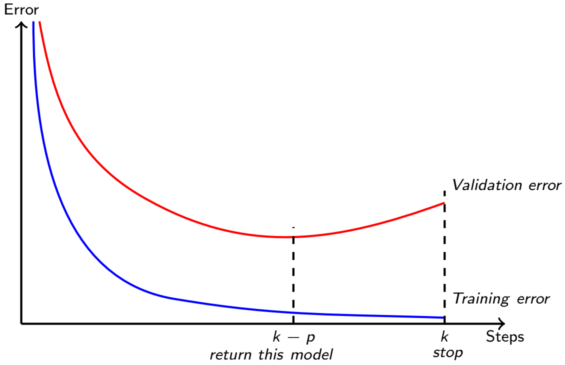

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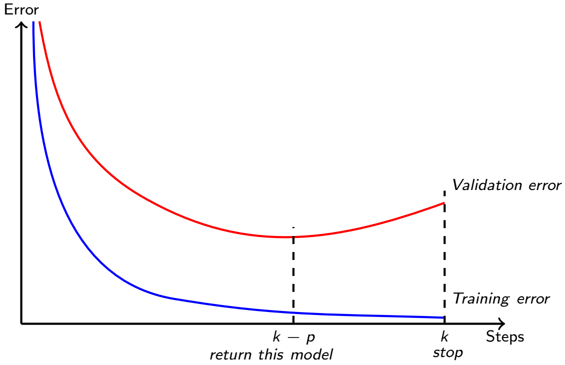
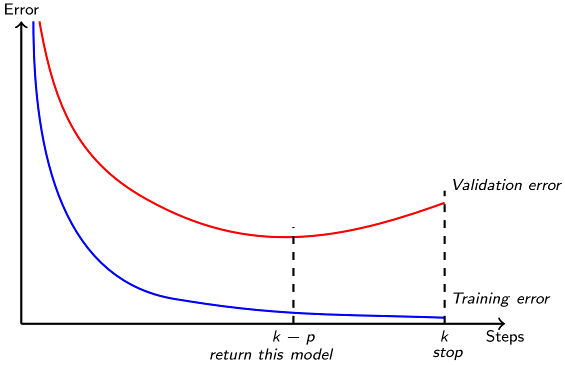
- How does it act as a regularizer ?
- We will first see an intuitive explanation and then a mathematical analysis



- Recall that the update rule in SGD is-



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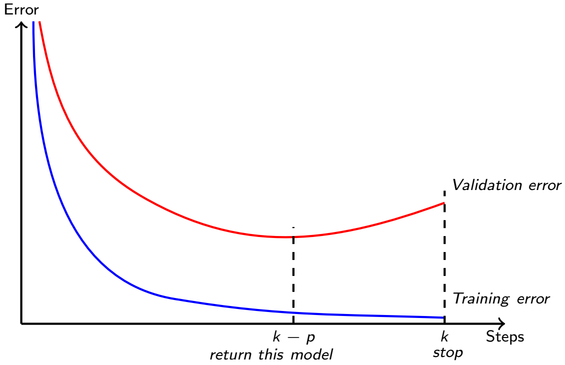

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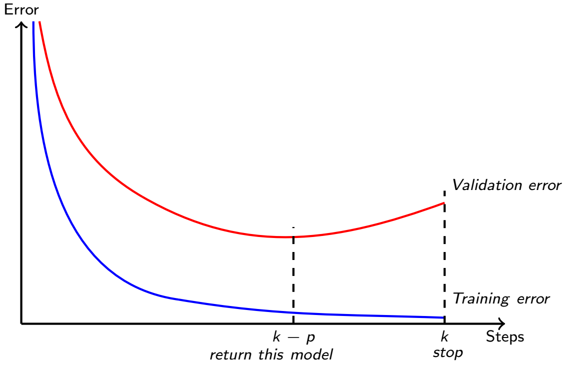

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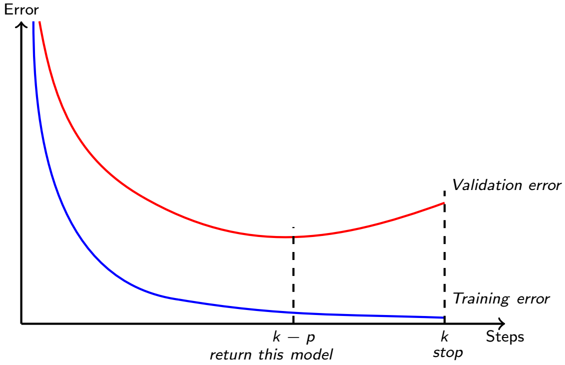

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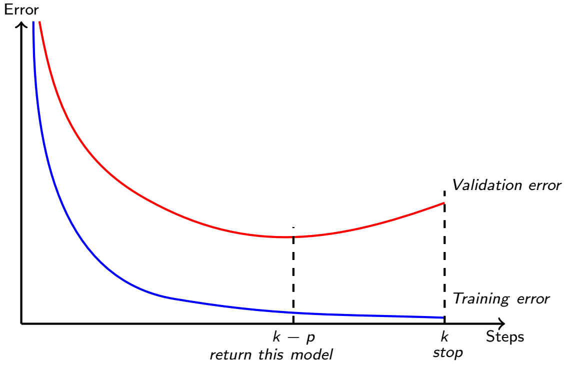

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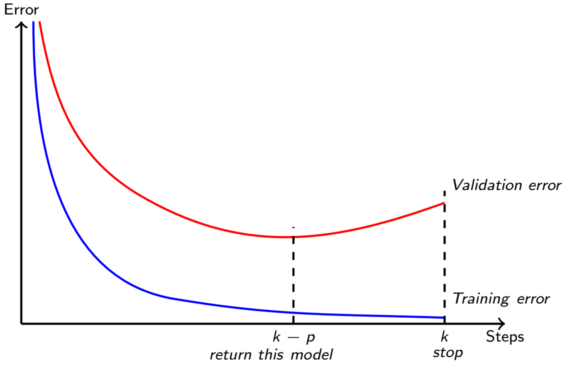

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- In other words it controls the space of exploration

We will now see a mathematical analysis of this

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- We observe that $\omega_t = \tilde{\omega}$, if we choose ε,t and α such that

$$(I - \varepsilon\Lambda)^t = (\Lambda + \alpha I)^{-1}\alpha$$

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- However if a parameter is not important ($\frac{\partial \mathscr{L}(\theta)}{\partial \omega}$ is small) then its updates will be small and the parameter will not be able to grow large in 't' steps
- Early stopping will thus effectively shrink the parameters corresponding to less important directions (same as weight decay).

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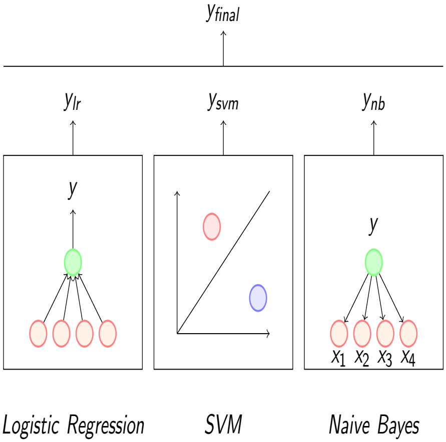
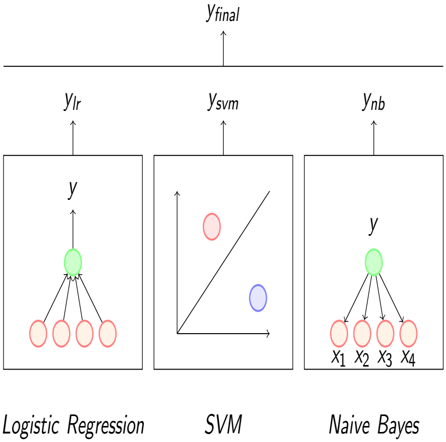


- Combine the output of different models to reduce generalization error



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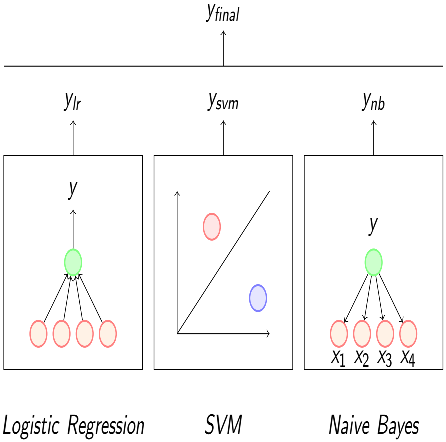

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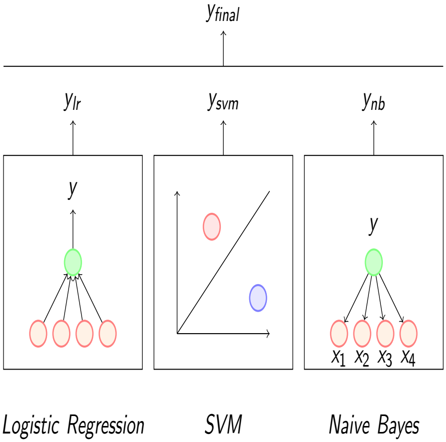
- It could be different instances of the same classifier trained with:



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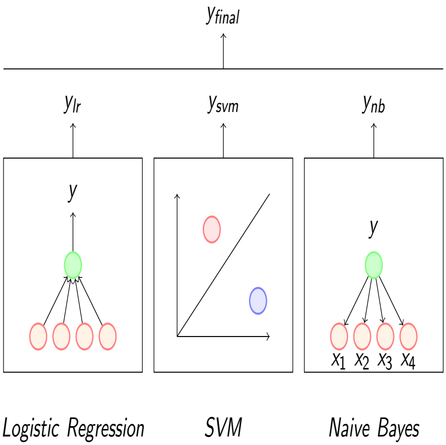

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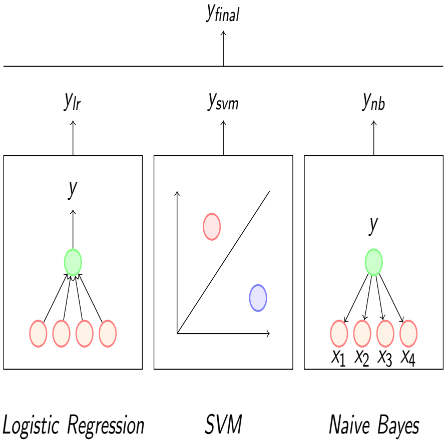

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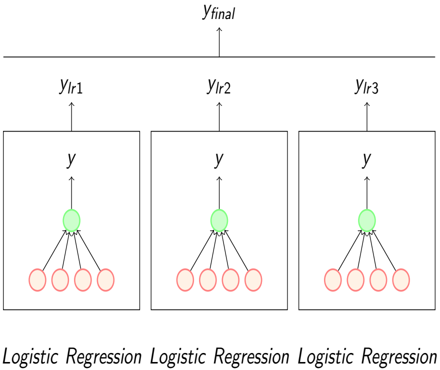
y_{final}
y_{lr1}
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y
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Logistic Regression
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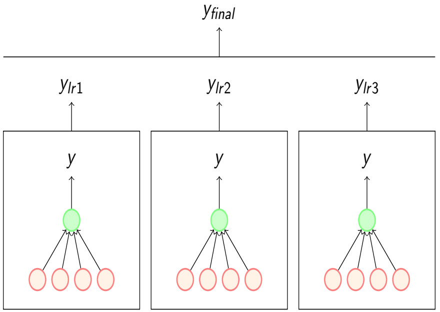

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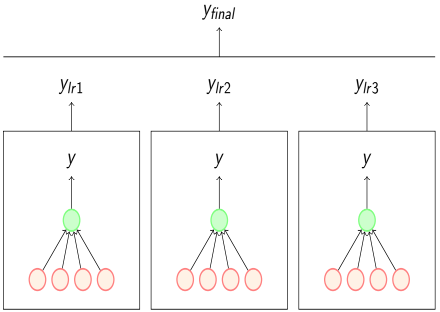

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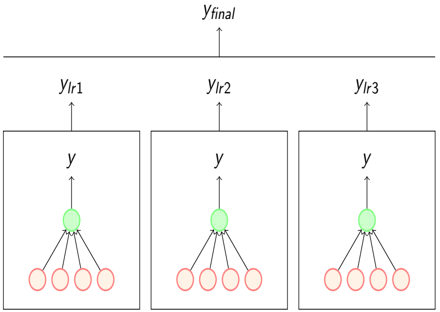

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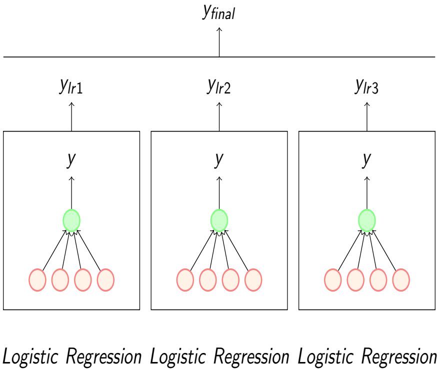

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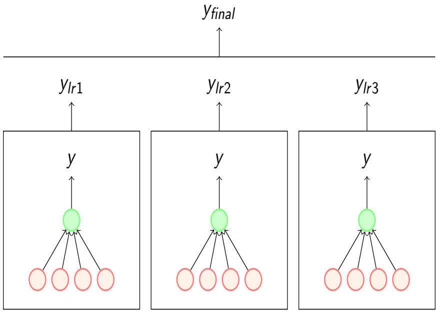

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- On average, the ensemble will perform at least as well as its individual members

Other forms of regularization

- $L2$ regularization
- Dataset augmentation
- Parameter Sharing and tying
- Adding Noise to the inputs
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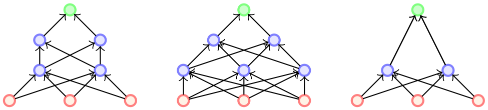

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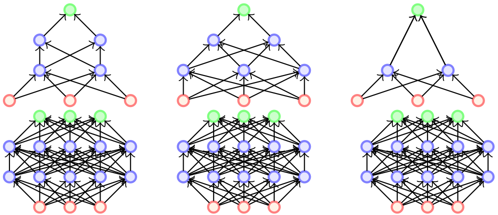

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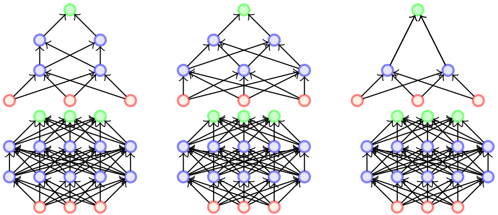

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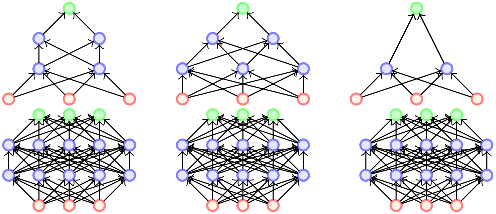
- Even if we manage to train with option 1 or option 2, combining several models at test time is infeasible in real time applications



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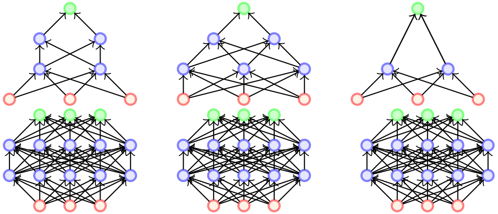

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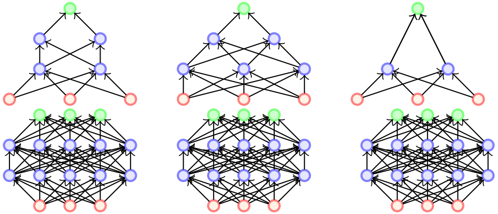

- Dropout is a technique which addresses both these issues.
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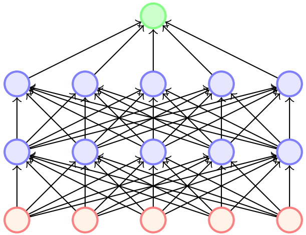
- Also gives an efficient approximate way of combining exponentially many different neural networks.



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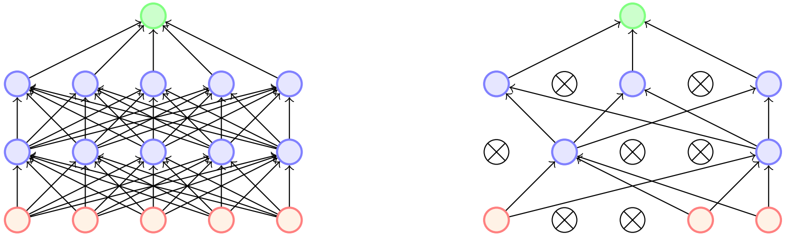

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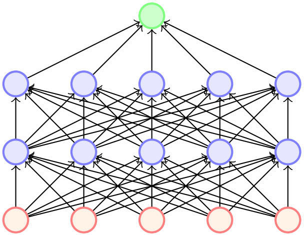

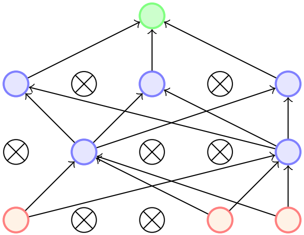

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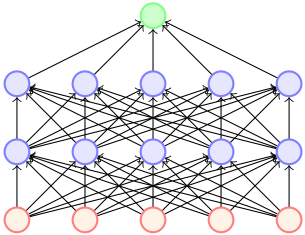
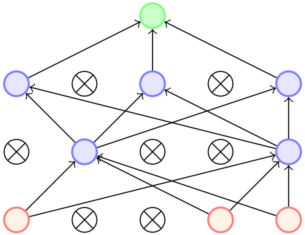
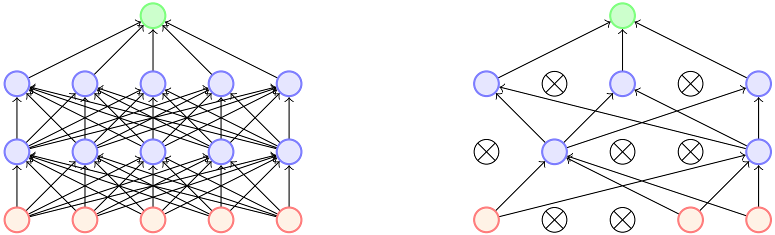
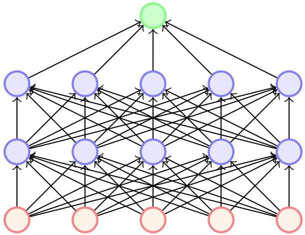
- Each node is retained with a fixed probability (typically $p = 0.5$) for hidden nodes and $p = 0.8$ for visible nodes

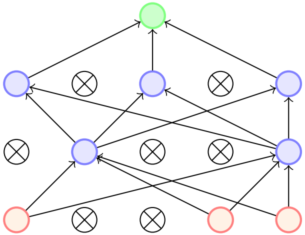






- A neural network with n nodes can be seen as a collection of 2^n possible thinned networks





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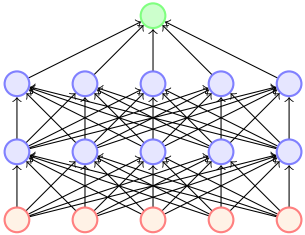

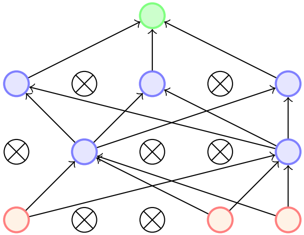

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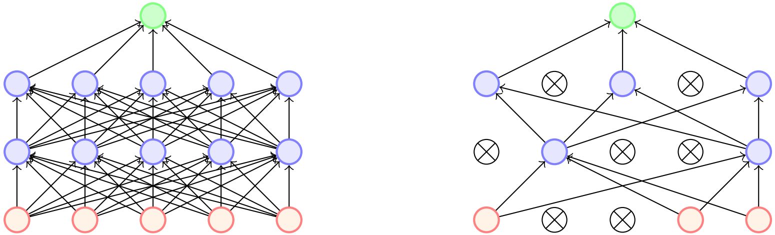

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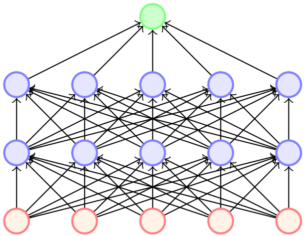
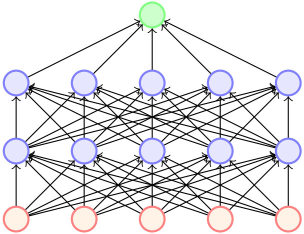
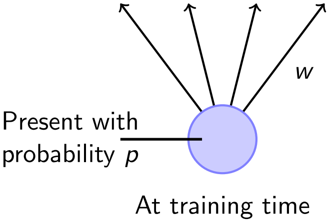
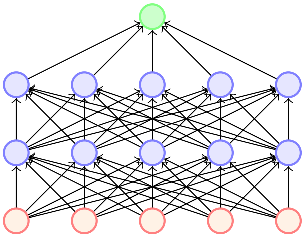
- Each thinned network gets trained rarely (or even never) but the parameter sharing ensures that no model has untrained or poorly trained parameters

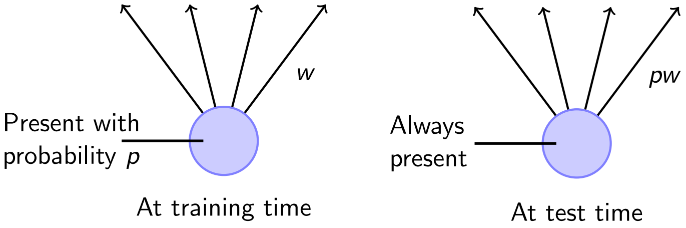






- What happens at test time?





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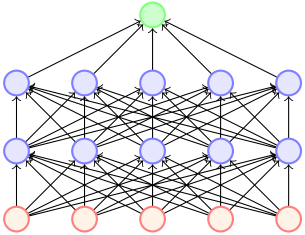

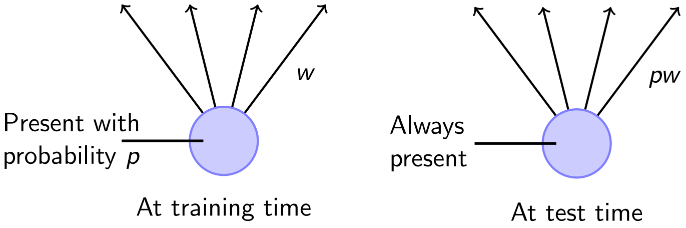

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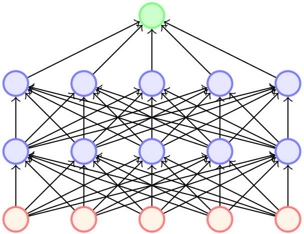
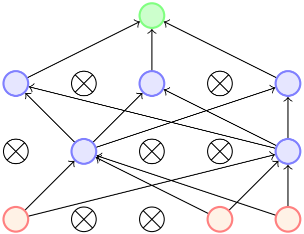
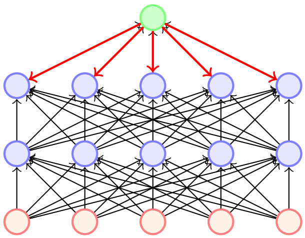
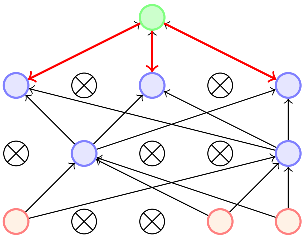
- Impossible to aggregate the outputs of 2^n thinned networks.
- Instead we use the full Neural Network and scale the output of each node by the fraction of times it was on during training.





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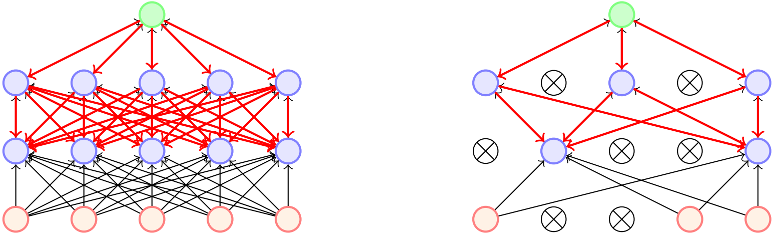
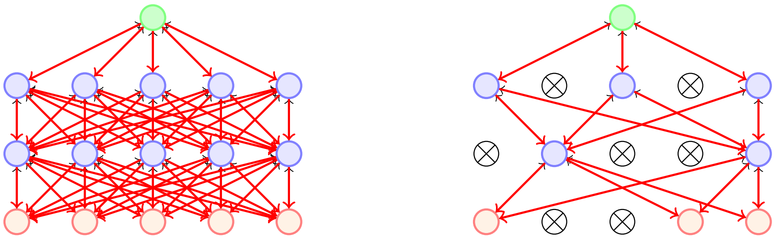


- How do you do backpropagation in such a noisy network which changes for each training instance (or batch)
- Simple: we only backpropagate over the paths which are active and only update those weights which are active in the current thinned network



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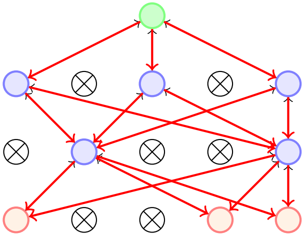

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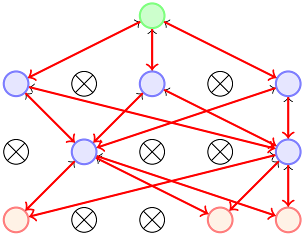

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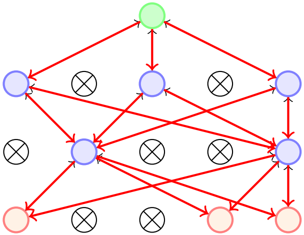

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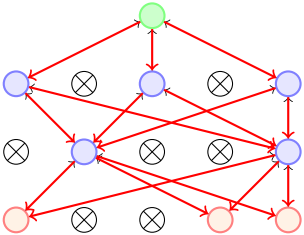

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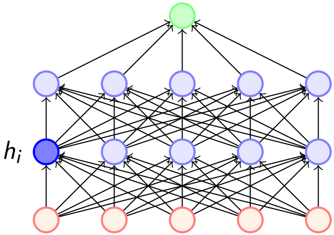
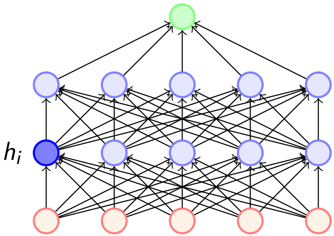
- Essentially a hidden unit cannot rely too much on other units as they may get dropped out any time
- Each hidden unit has to learn to be more robust to these random dropouts


h_i



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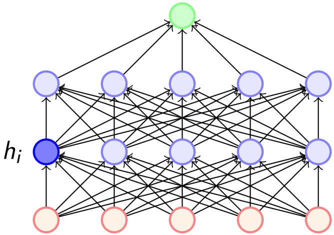

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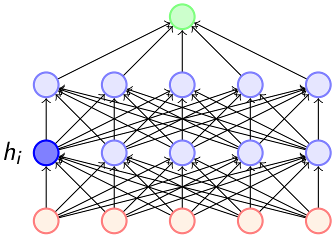

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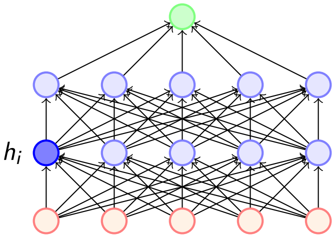

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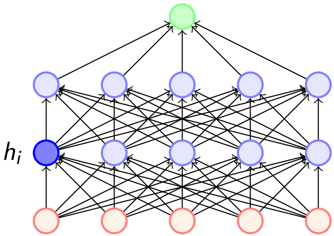

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- The model should then learn another h_i which redundantly encodes the presence of a nose
- Or the model should learn to detect the face using other features

Recap

- $L2$ regularization
- Dataset augmentation
- Parameter Sharing and tying
- Adding Noise to the inputs
- Adding Noise to the outputs
- Early stopping
- Ensemble methods
- Dropout