# Module 4.3: Output Functions and Loss Functions

### We need to answer two questions

- How to choose the loss function $\mathscr{L}(\theta)$ ?
- How to compute $\nabla\theta$ which is composed of $\nabla W_1, \nabla W_2, ..., \nabla W_{L-1} \in \mathbb{R}^{n \times n}, \nabla W_L \in \mathbb{R}^{n \times k}$,
  $\nabla b_1, \nabla b_2, ..., \nabla b_{L-1} \in \mathbb{R}^n$ and $\nabla b_L \in \mathbb{R}^k$ ?

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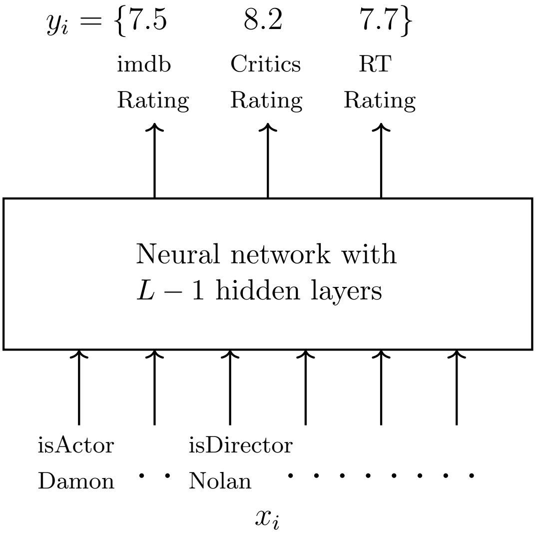


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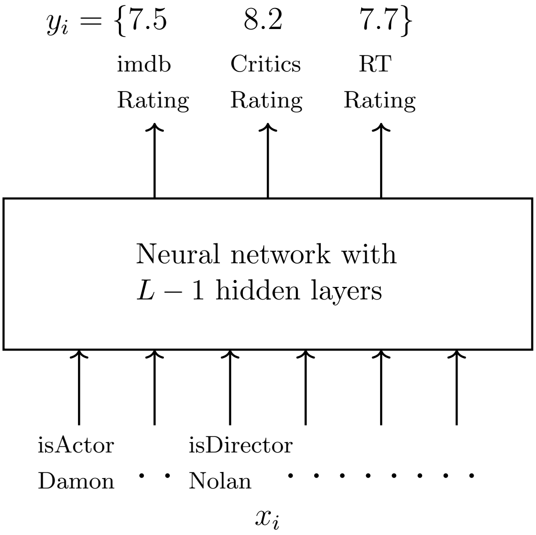


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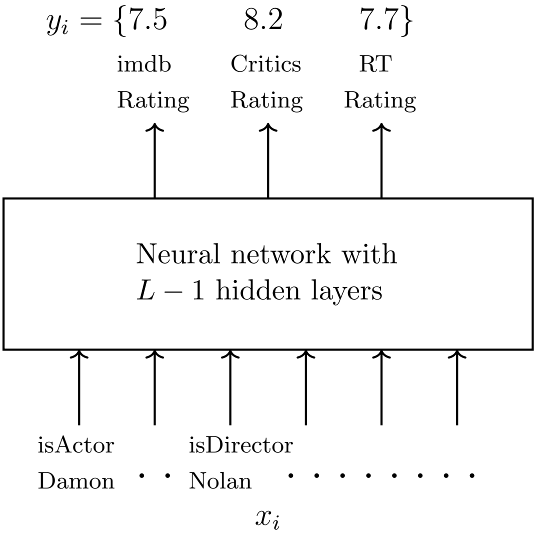


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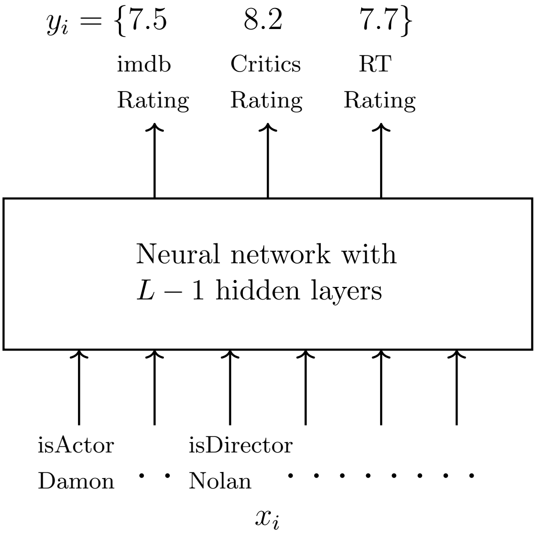


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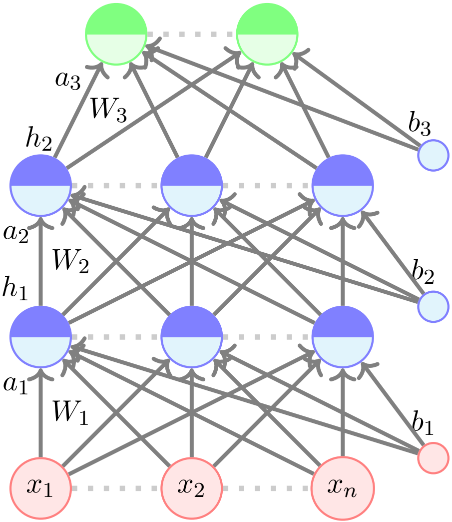
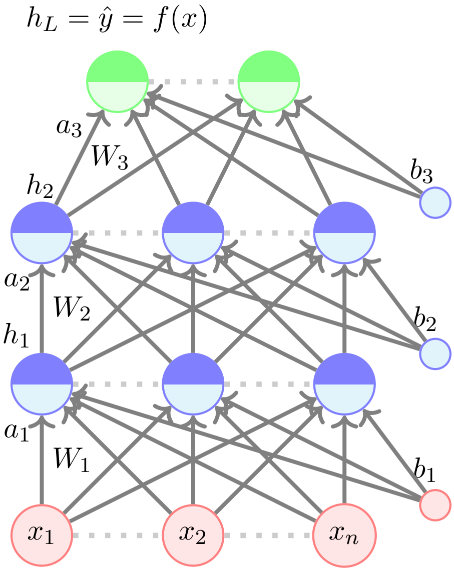
- The loss function should capture how much $\hat{y}_i$ deviates from $y_i$
- If $y_i \in \mathbb{R}^n$ then the squared error loss can capture this deviation

$$\mathscr{L}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{3} (\hat{y}_{ij} - y_{ij})^2$$

$h_L = \hat{y} = f(x)$




- A related question: What should the output function '$O$' be if $y_i \in \mathbb{R}$?

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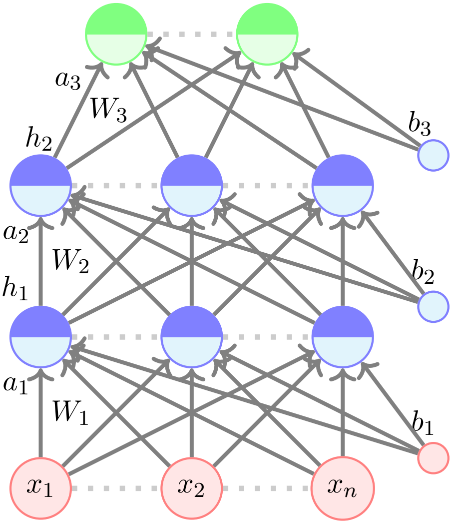


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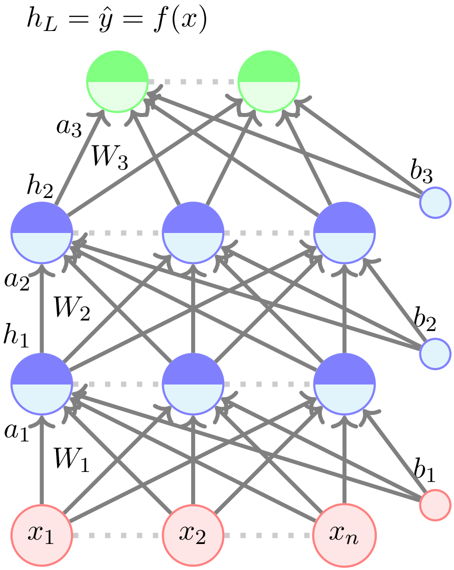


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- So, in such cases it makes sense to have '$O$' as linear function

$$
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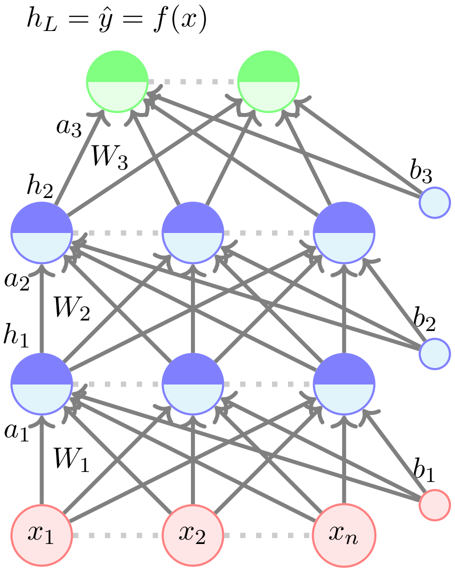


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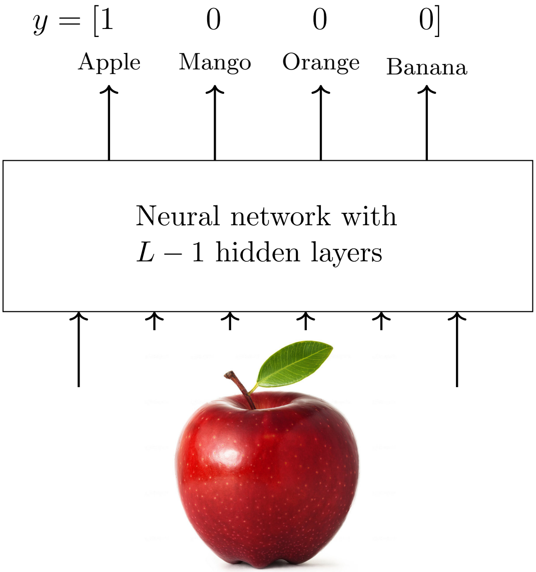
$$\begin{aligned} f(x) &= h_L = O(a_L) \\ &= W_O a_L + b_O \end{aligned}$$

- $\hat{y}_i = f(x_i)$ is no longer bounded between 0 and 1

Intentionally left blank

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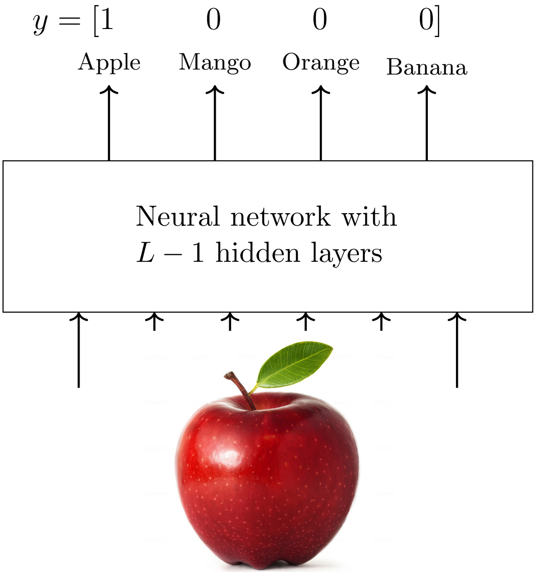


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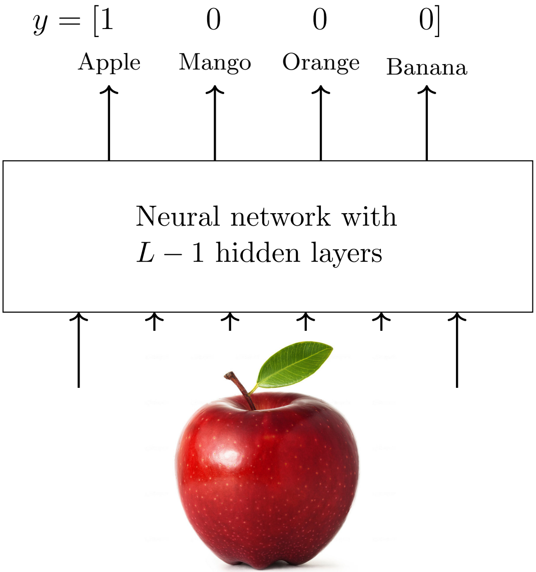


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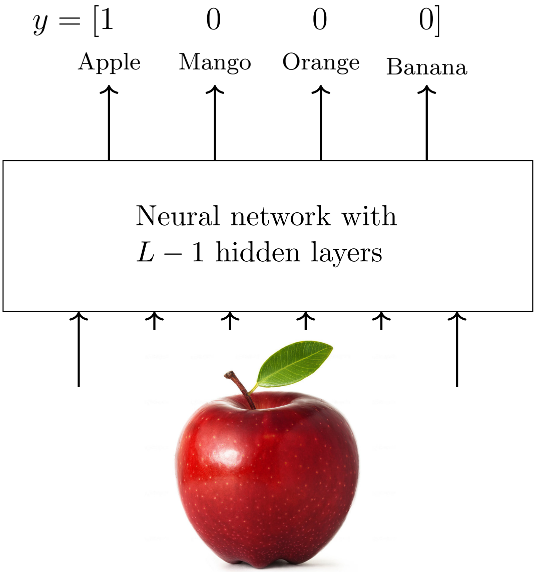


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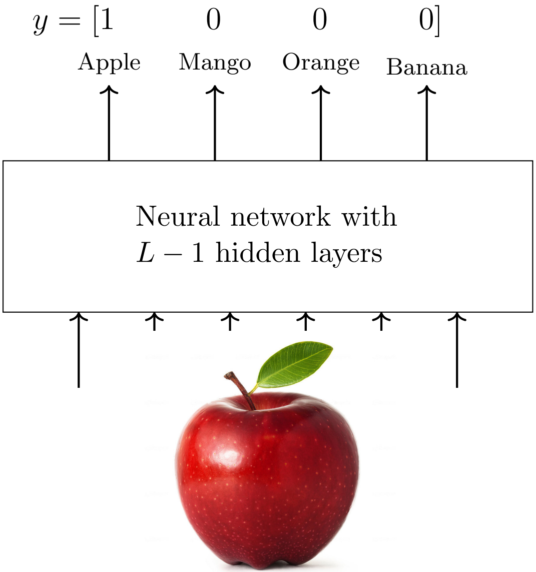
- Here again we could use the squared error loss to capture the deviation
- But can you think of a better function?

$$y = [1 \quad 0 \quad 0 \quad 0]$$

Apple Mango Orange Banana

Neural network with
$L-1$ hidden layers

- Notice that $y$ is a probability distribution

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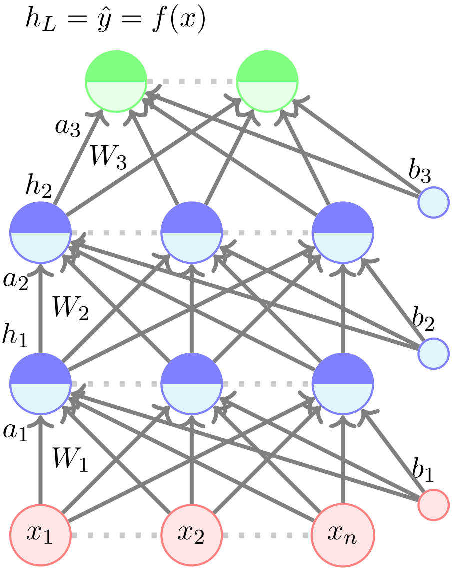


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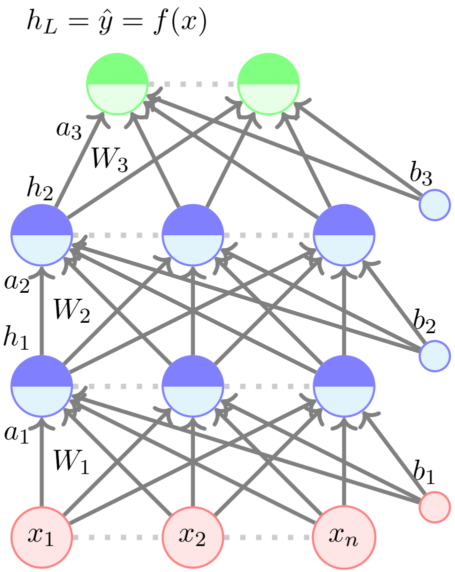
- What choice of the output activation '$O$' will ensure this ?

$$a_L = W_L h_{L-1} + b_L$$

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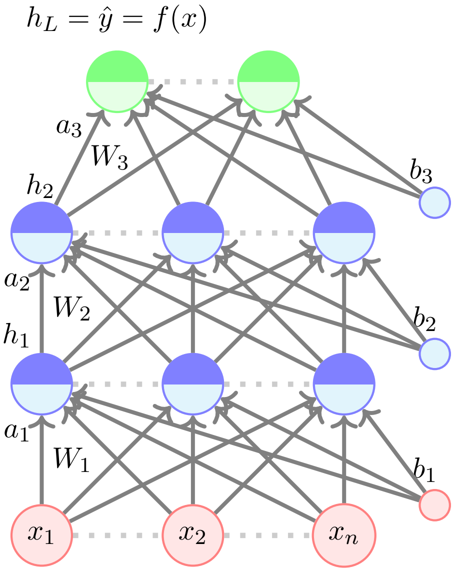
$O(a_L)_j$ is the $j^{th}$ element of $\hat{y}$ and $a_{L,j}$ is the $j^{th}$ element of the vector $a_L$.

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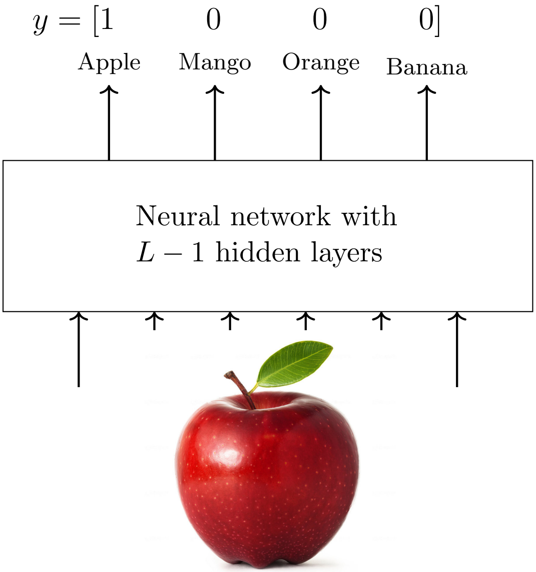
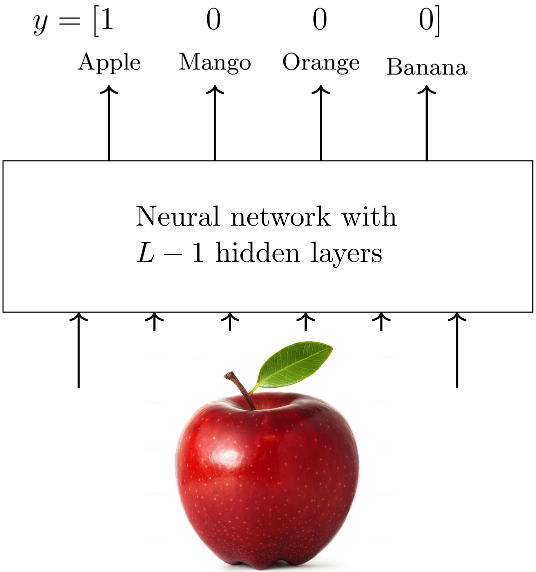
- This function is called the *softmax* function




- Now that we have ensured that both $y$ & $\hat{y}$ are probability distributions can you think of a function which captures the difference between them?

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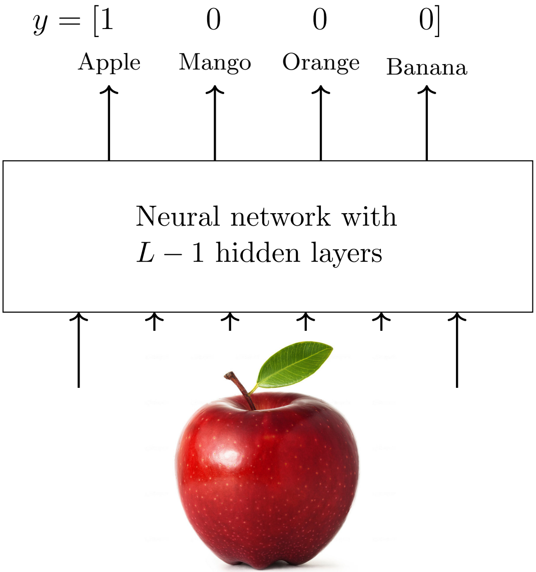


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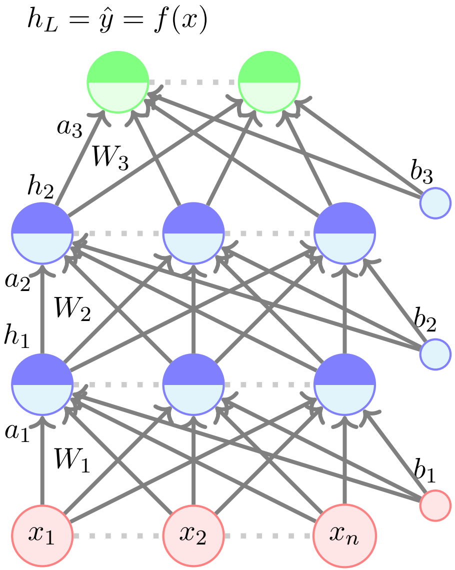
$$\mathscr{L}(\theta) = -\sum_{c=1}^{k} y_c \log \hat{y}_c$$

- Notice that

$$\begin{aligned} y_c &= 1 \quad \text{if } c = \ell \text{ (the true class label)} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

$$\therefore \quad \mathscr{L}(\theta) = -\log \hat{y}_\ell$$

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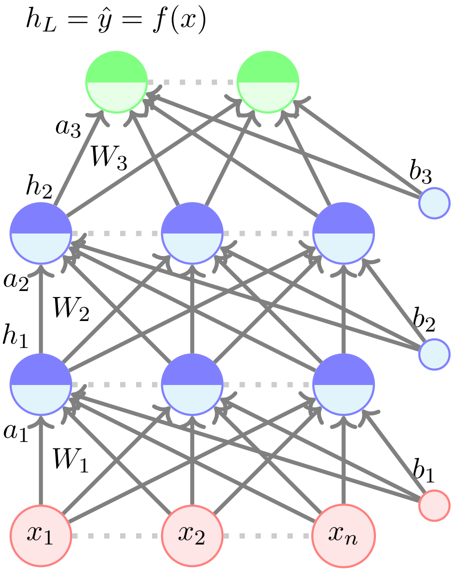


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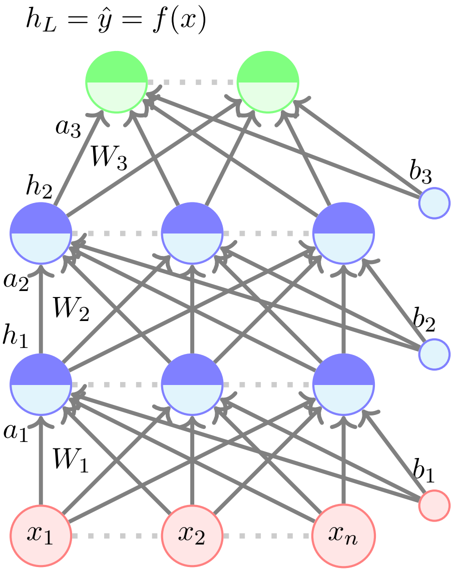


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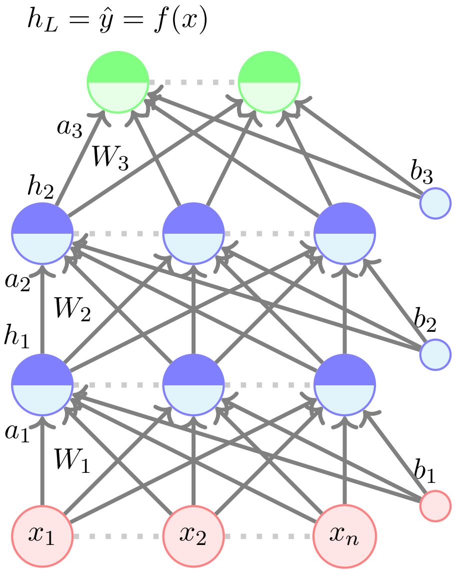


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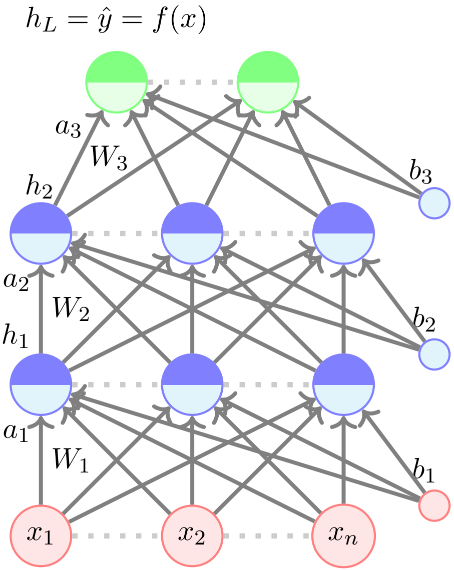


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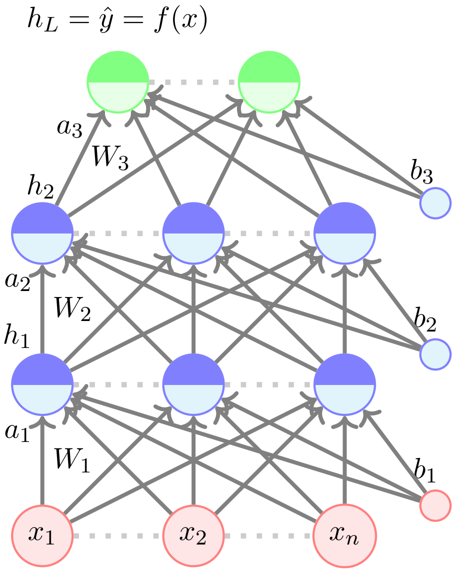


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- $\log \hat{y}_\ell$ is called the *log-likelihood* of the data.

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| Output Activation | | |
| Loss Function | | |

| | Outputs | |
|---|---|---|
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- For the rest of this lecture we will focus on the case where the output activation is a softmax function and the loss function is cross entropy