Module 4.3: Output Functions and Loss Functions

We need to answer two questions

- How to choose the loss function $\mathscr{L}(\theta)$?
- How to compute $\nabla\theta$ which is composed of:
  $\nabla W_1, \nabla W_2, ..., \nabla W_{L-1} \in \mathbb{R}^{n\times n}, \nabla W_L \in \mathbb{R}^{n\times k}$
  $\nabla b_1, \nabla b_2, ..., \nabla b_{L-1} \in \mathbb{R}^{n}$ and $\nabla b_L \in \mathbb{R}^{k}$?

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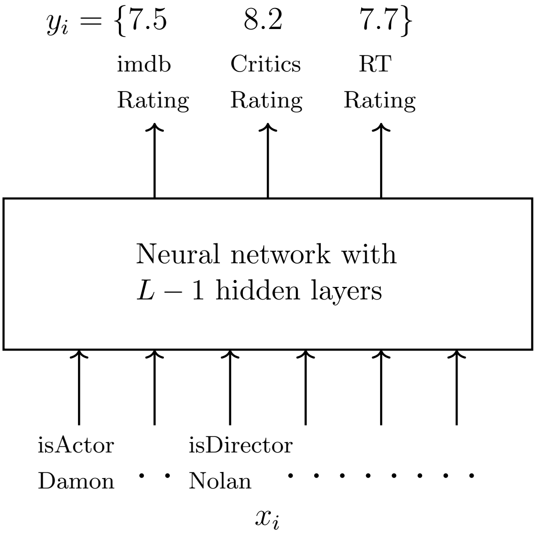

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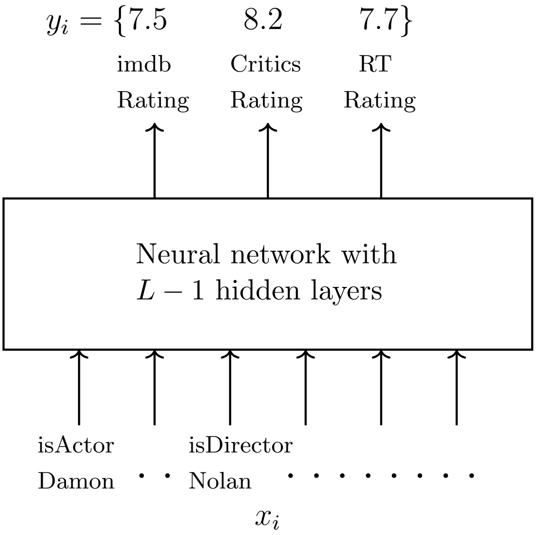

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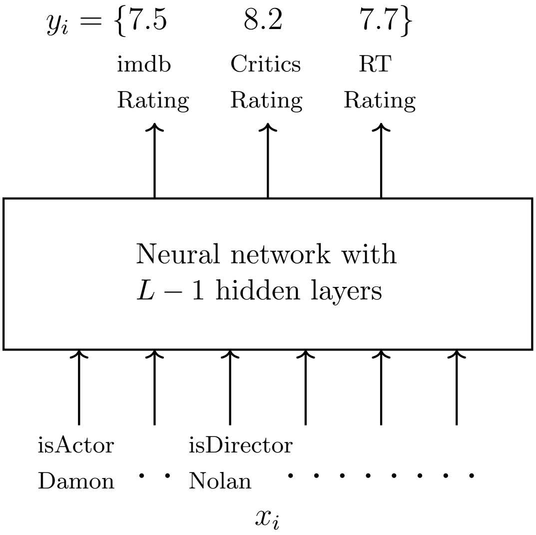

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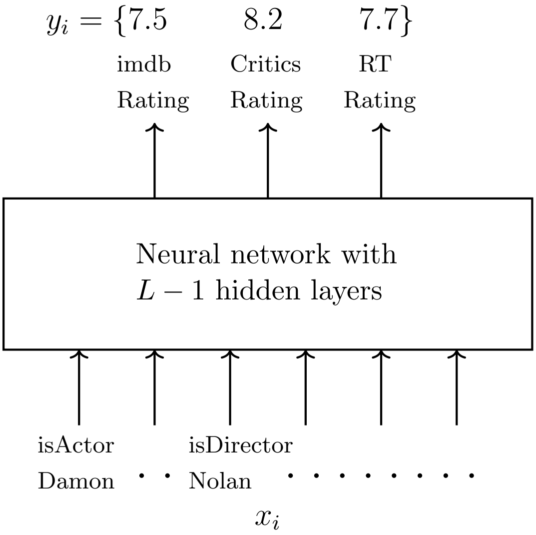

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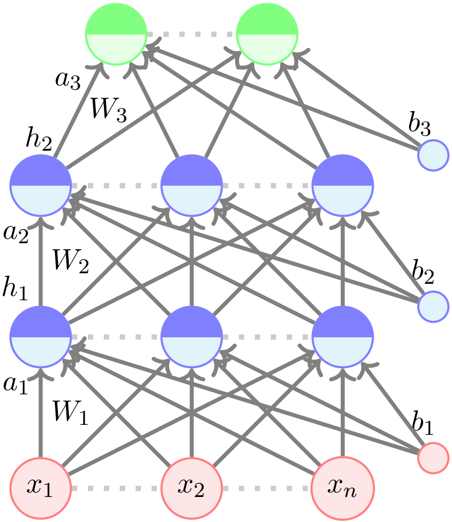
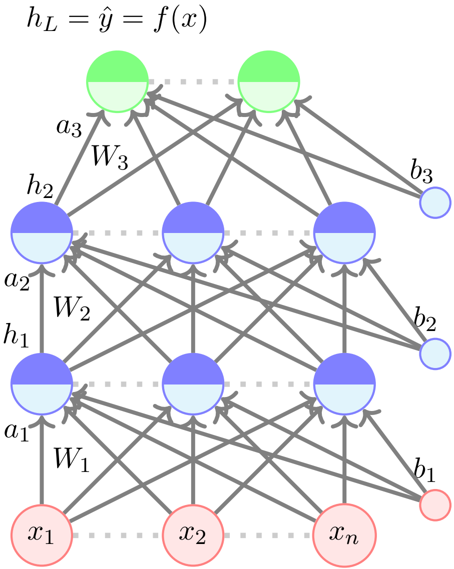
- Here $y_i \in \mathbb{R}^3$
- The loss function should capture how much $\hat{y}_i$ deviates from y_i
- If $y_i \in \mathbb{R}^n$ then the squared error loss can capture this deviation

$$\mathscr{L}(\theta) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{3}(\hat{y}_{ij} - y_{ij})^2$$

$h_L = \hat{y} = f(x)$



- A related question: What should the output function 'O' be if $y_i \in \mathbb{R}$?



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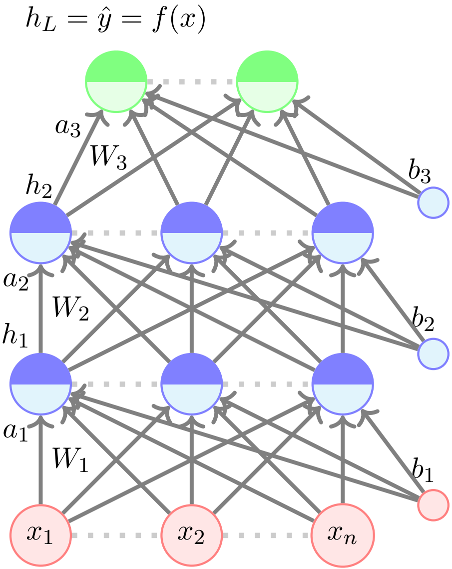

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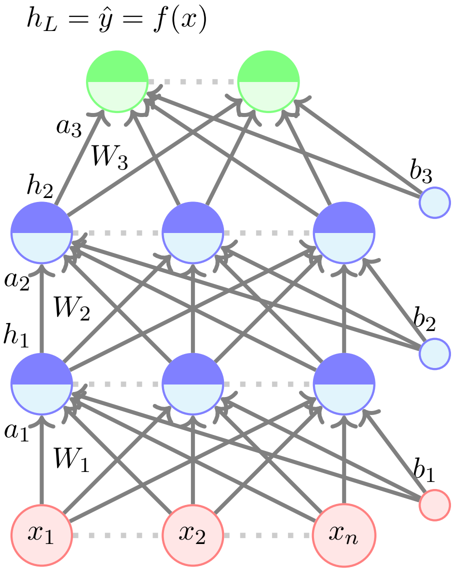

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- No, because it restricts $\hat{y}_i$ to a value between 0 & 1 but we want $\hat{y}_i \in \mathbb{R}$
- So, in such cases it makes sense to have 'O' as linear function

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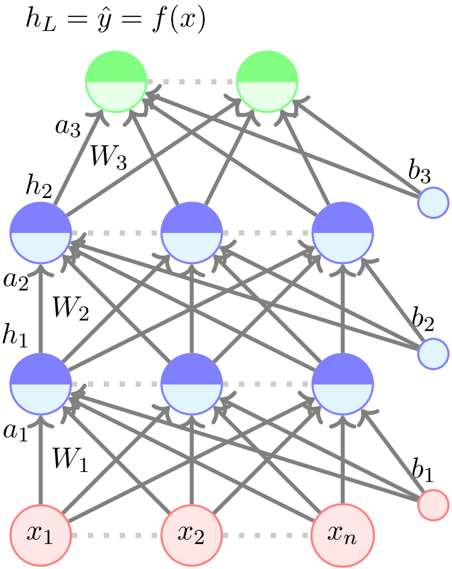

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- $\hat{y}_i = f(x_i)$ is no longer bounded between 0 and 1

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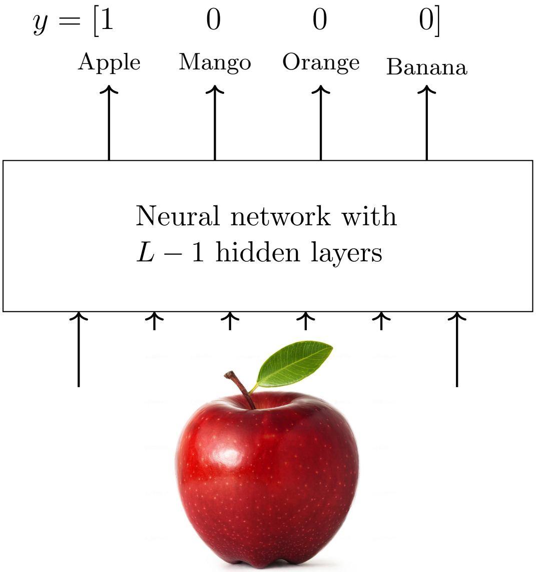

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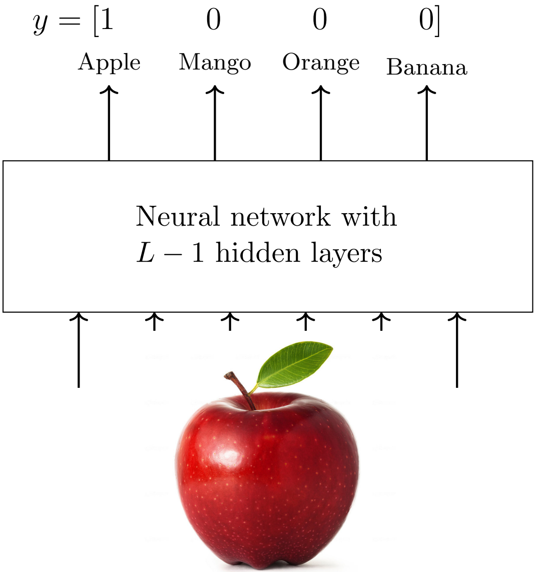

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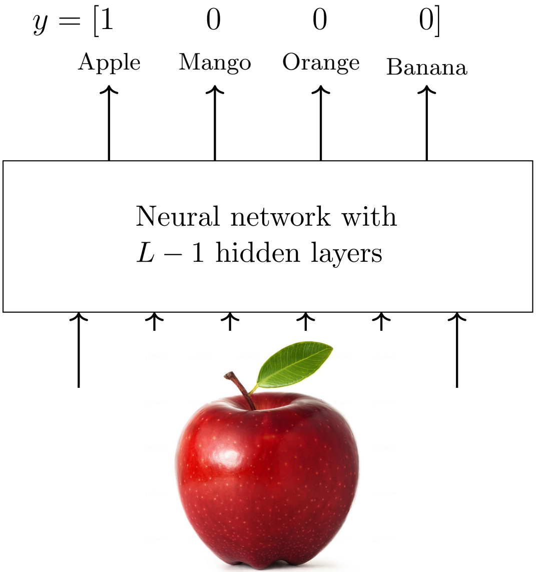

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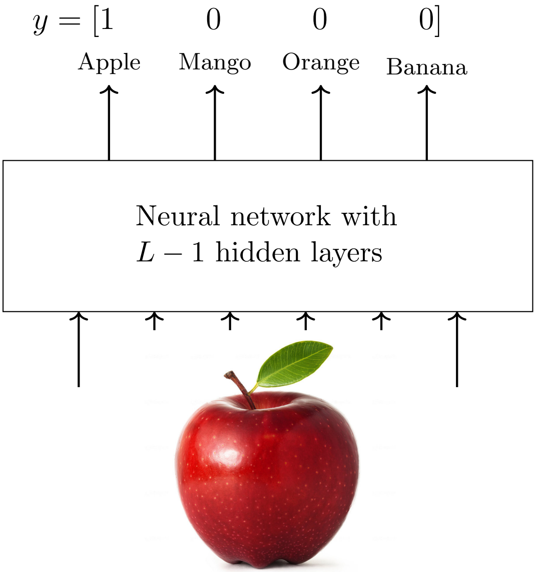

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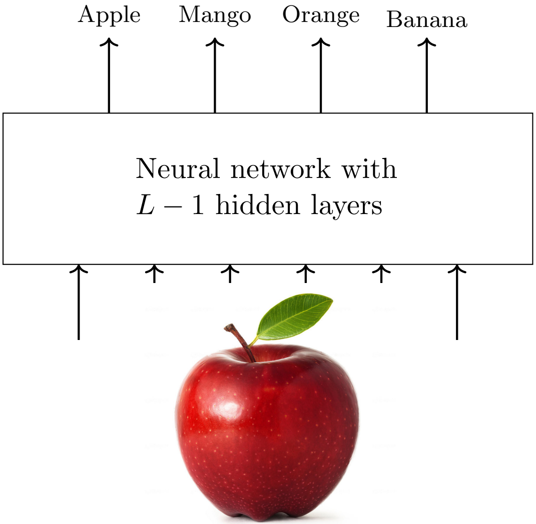
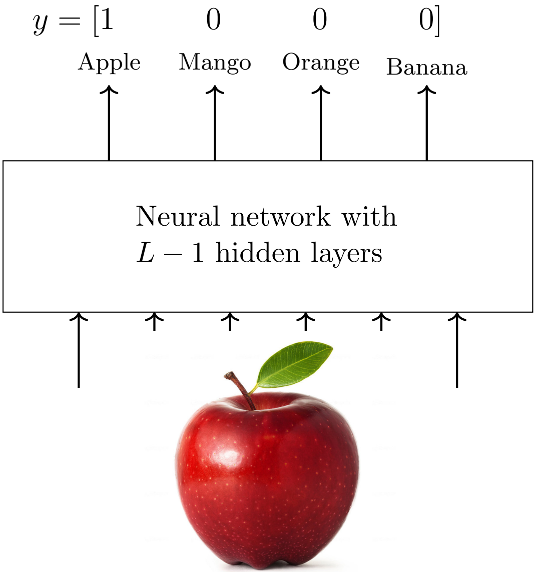
- Here again we could use the squared error loss to capture the deviation
- But can you think of a better function?

$$y = [1 \quad 0 \quad 0 \quad 0]$$



- Notice that y is a probability distribution



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- Therefore we should also ensure that $\hat{y}$ is a probability distribution

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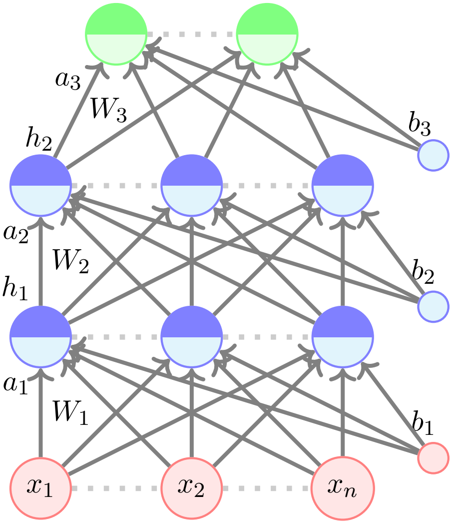

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- What choice of the output activation 'O' will ensure this ?

$$a_L = W_L h_{L-1} + b_L$$

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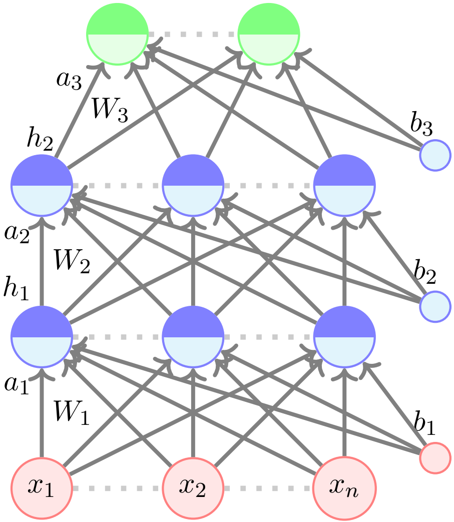

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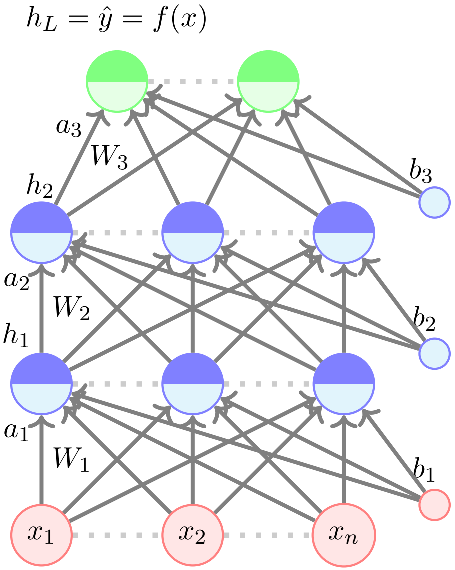
$$\hat{y}_j = O(a_L)_j = \frac{e^{a_{L,j}}}{\sum_{i=1}^{k} e^{a_{L,i}}}$$

$O(a_L)_j$ is the j^{th} element of $\hat{y}$ and $a_{L,j}$ is the j^{th} element of the vector a_L.



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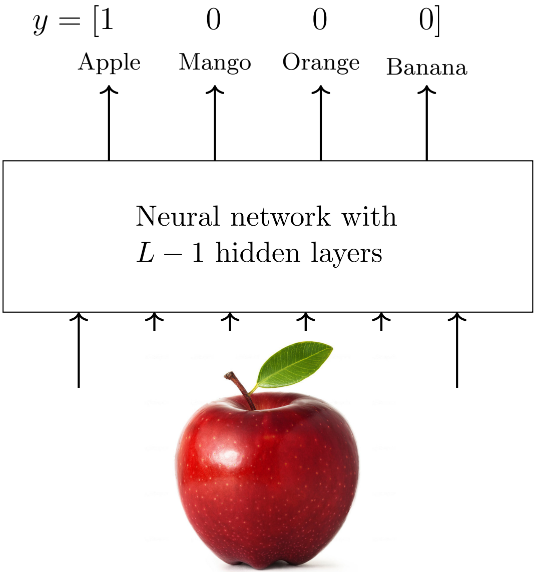
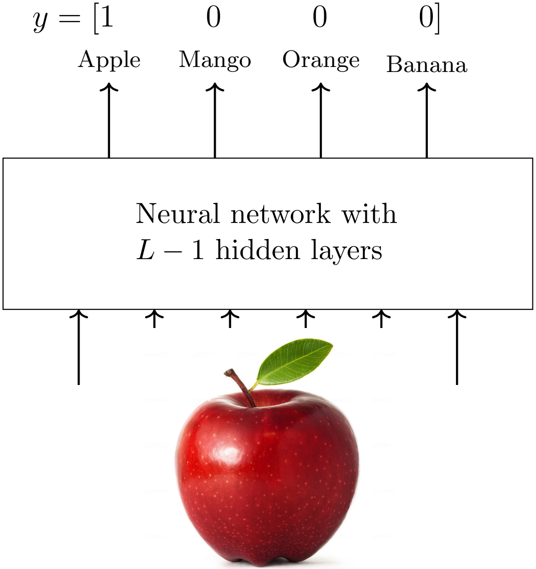
- This function is called the *softmax* function



- Now that we have ensured that both y & $\hat{y}$ are probability distributions can you think of a function which captures the difference between them?



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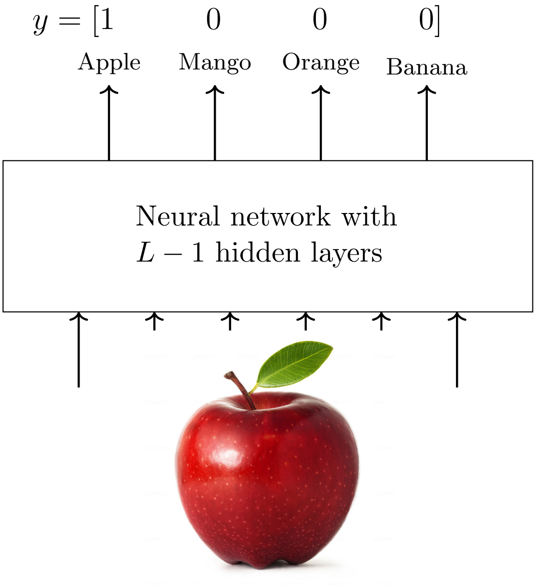
- Cross-entropy

$$\mathscr{L}(\theta) = -\sum_{c=1}^{k} y_c \log \hat{y}_c$$



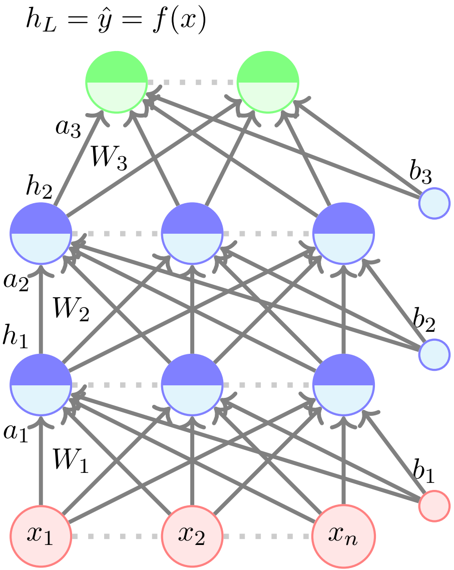
- Now that we have ensured that both y & $\hat{y}$ are probability distributions can you think of a function which captures the difference between them?
- Cross-entropy

$$\mathscr{L}(\theta) = -\sum_{c=1}^{k} y_c \log \hat{y}_c$$

- Notice that

$$\begin{aligned} y_c &= 1 \quad \text{if } c = \ell \text{ (the true class label)} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

$$\therefore \quad \mathscr{L}(\theta) = -\log \hat{y}_\ell$$



- So, for classification problem (where you have to choose 1 of K classes), we use the following objective function

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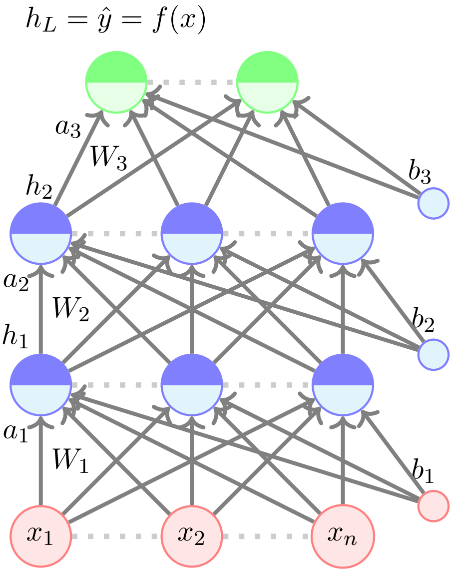

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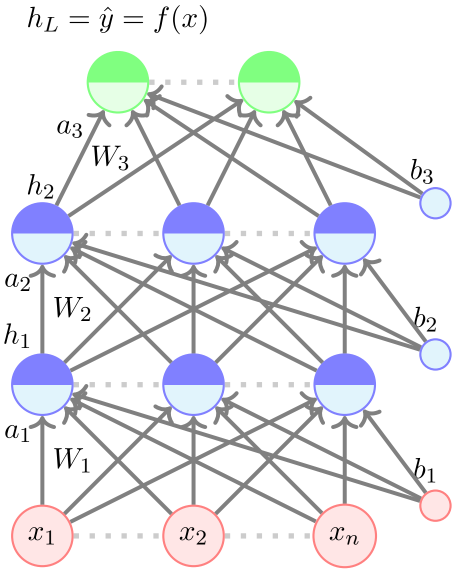
  Is $\hat{y}_\ell$ a function of $\theta = [W_1, W_2, .., W_L, b_1, b_2, .., b_L]$?



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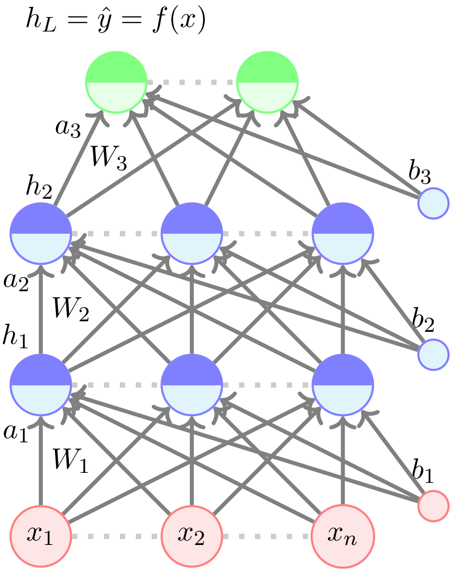

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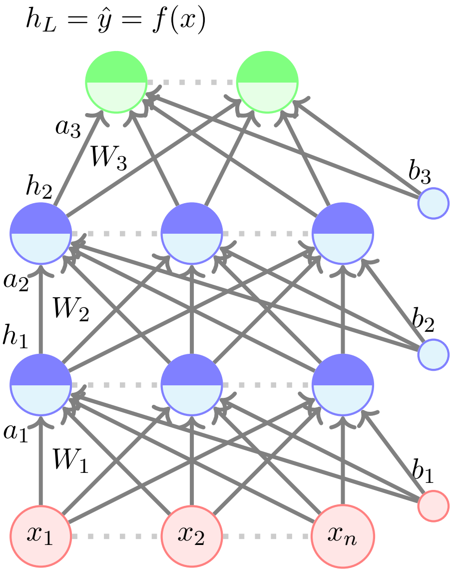

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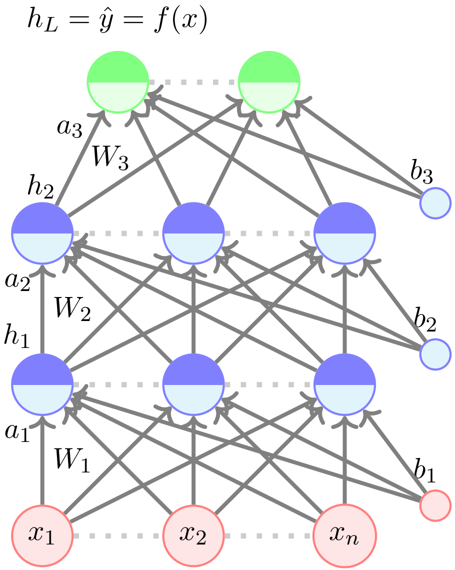

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- $\log \hat{y}_\ell$ is called the *log-likelihood* of the data.

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Output Activation		
Loss Function		

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- For the rest of this lecture we will focus on the case where the output activation is a softmax function and the loss function is cross entropy