Module 4.6: Backpropagation: Computing Gradients w.r.t. Hidden Units

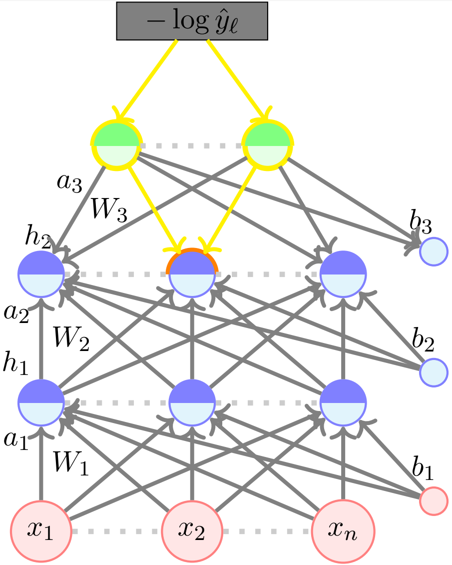
Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\text{and now talk to the weights}}$$

- Our focus is on *Cross entropy loss* and *Softmax* output.

Chain rule along multiple paths: If a function $p(z)$ can be written as a function of intermediate results $q_i(z)$ then we have :



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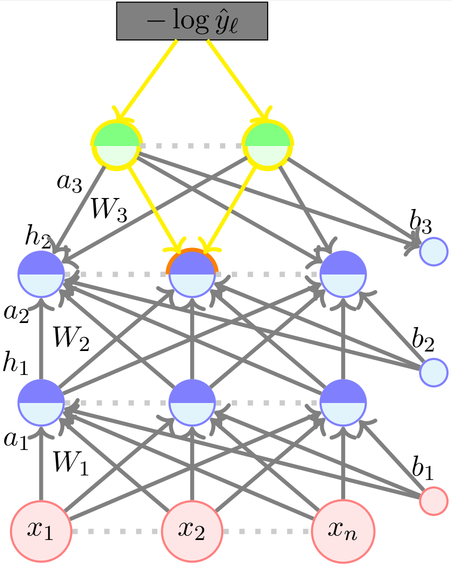

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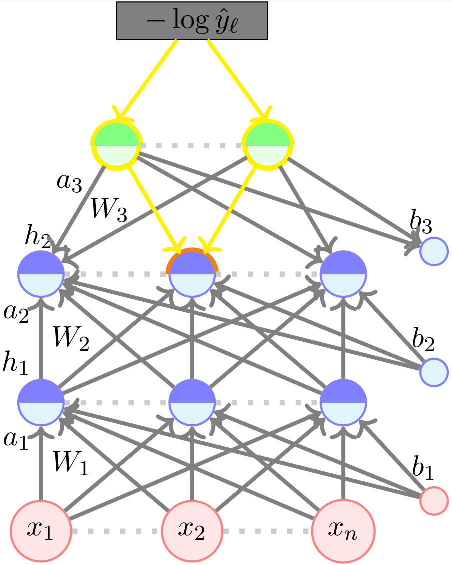
In our case:

- $p(z)$ is the loss function $\mathscr{L}(\theta)$



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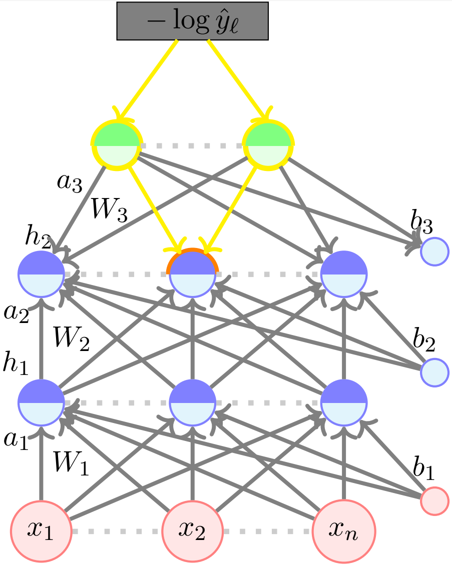
- $z = h_{ij}$



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In our case:

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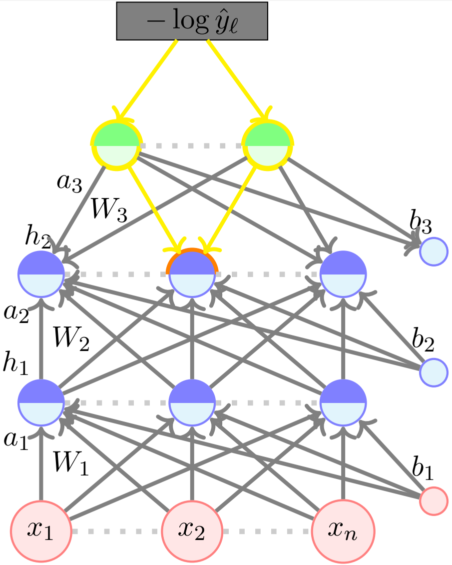
- $z = h_{ij}$
- $q_m(z) = a_{Lm}$



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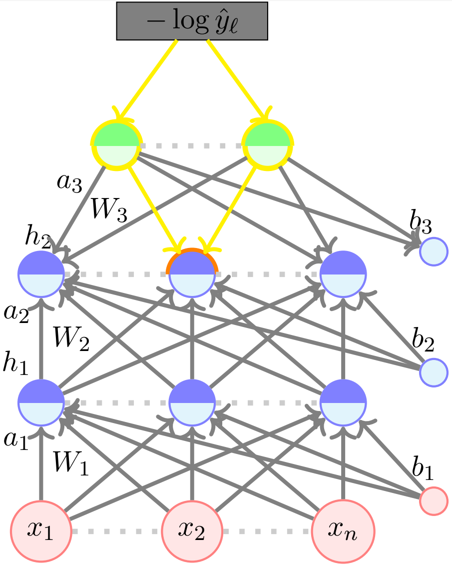
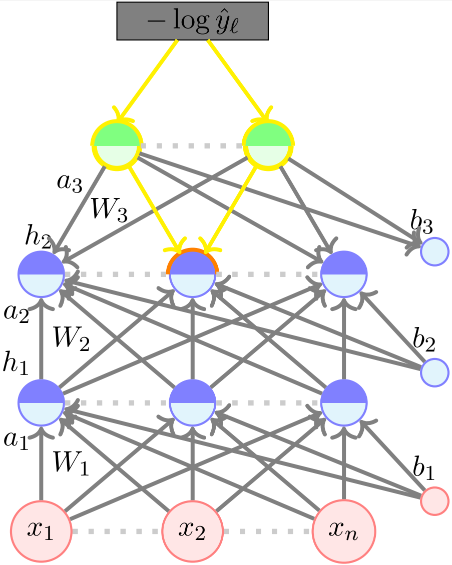
$$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}}$$



$$a_{i+1} = W_{i+1} h_{ij} + b_{i+1}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = \sum_{m=1}^{k} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i+1,m}} \frac{\partial a_{i+1,m}}{\partial h_{ij}}$$



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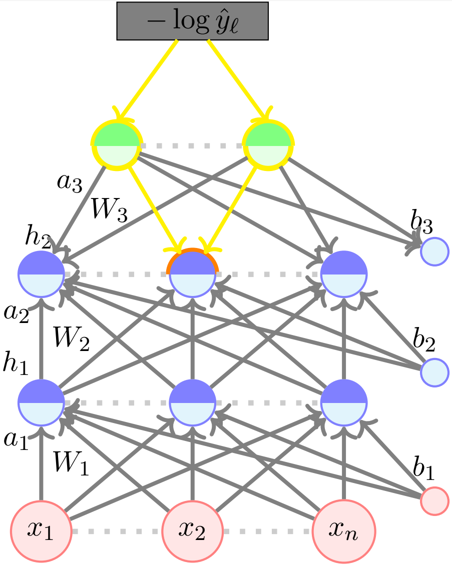

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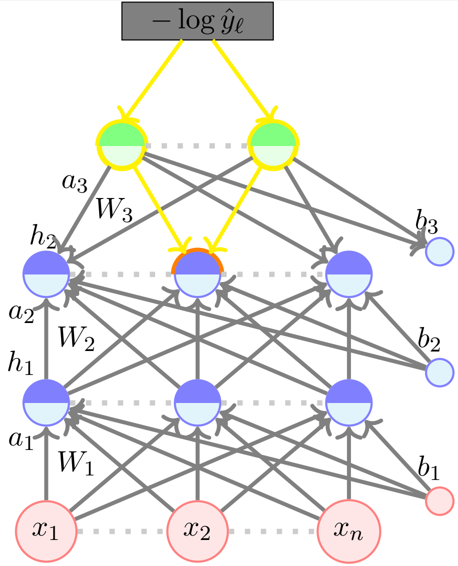

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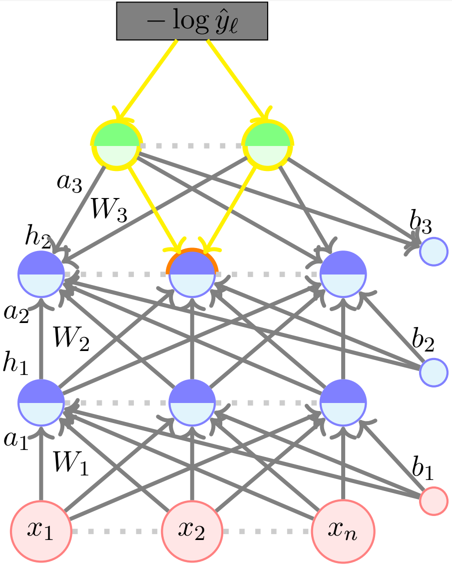

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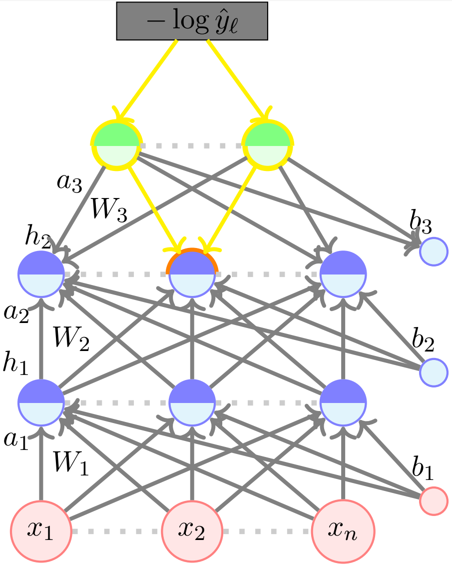

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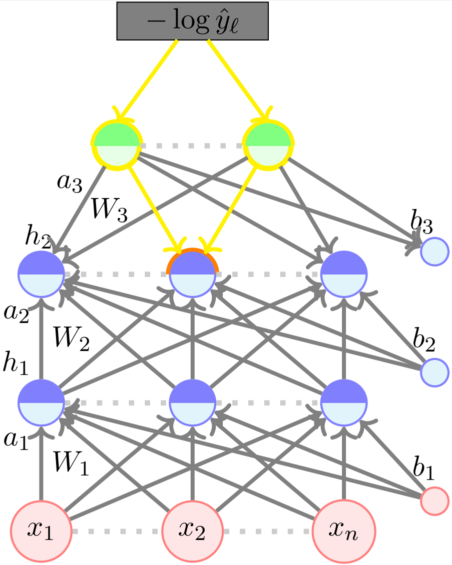

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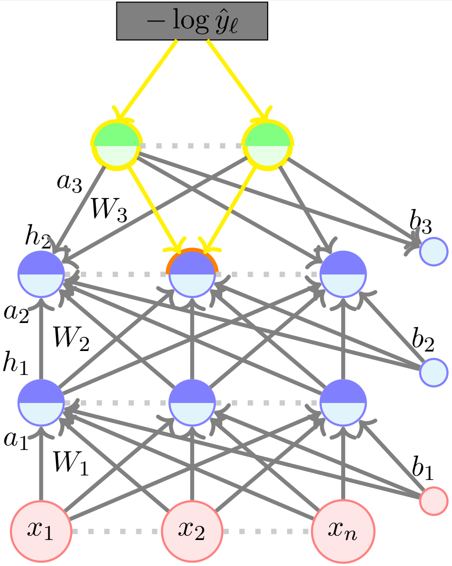

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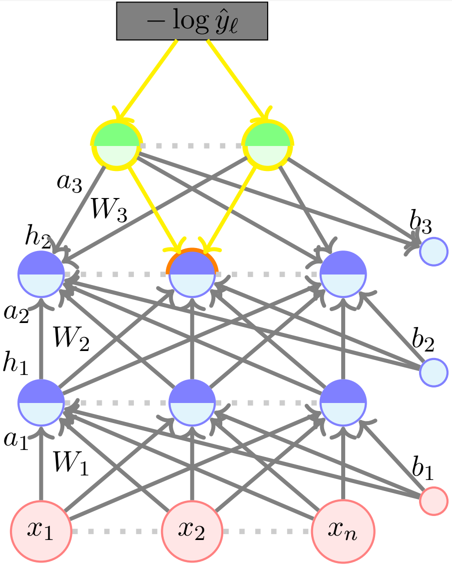

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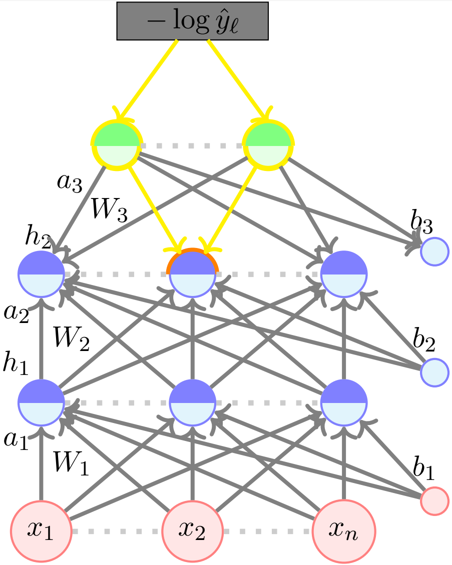

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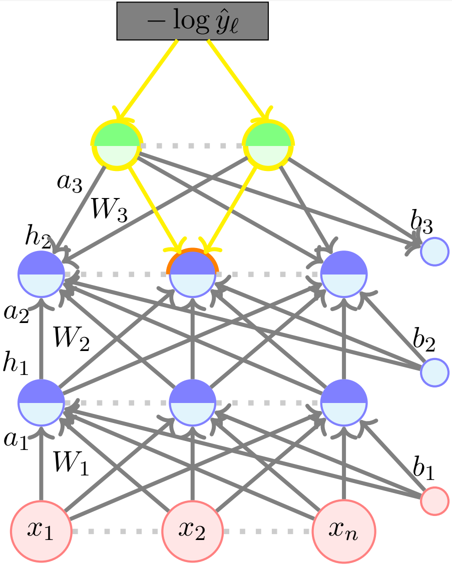

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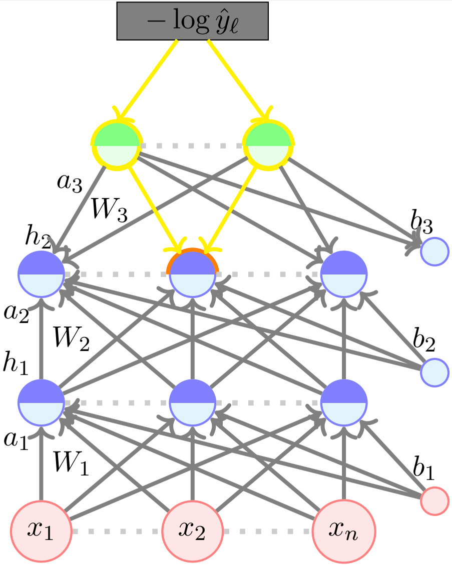

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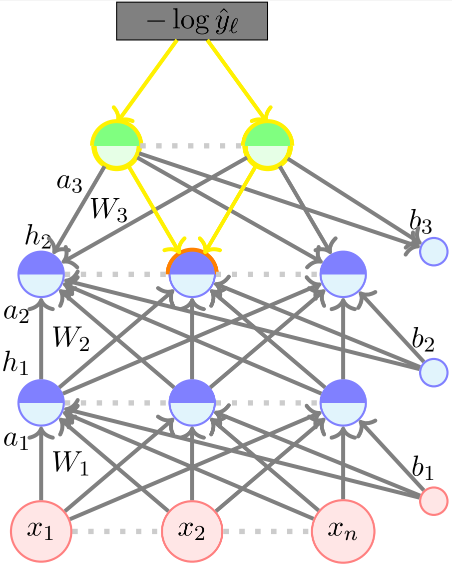

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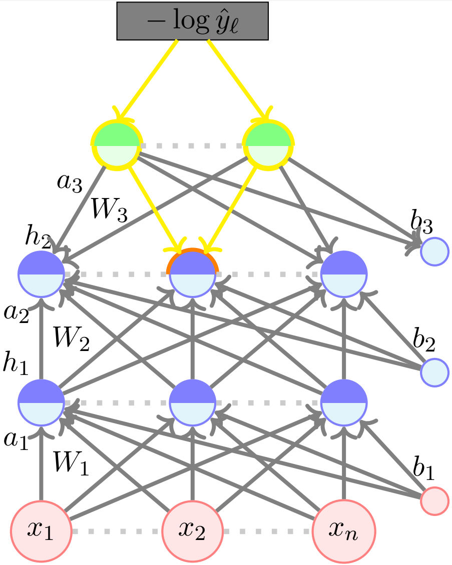
$$(W_{i+1,\,\cdot\,,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) =$$



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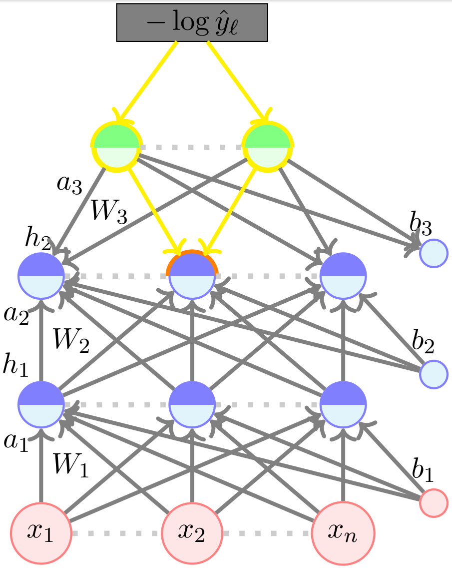

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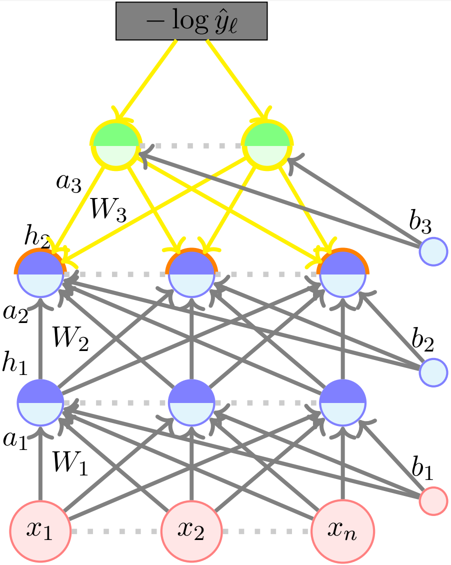

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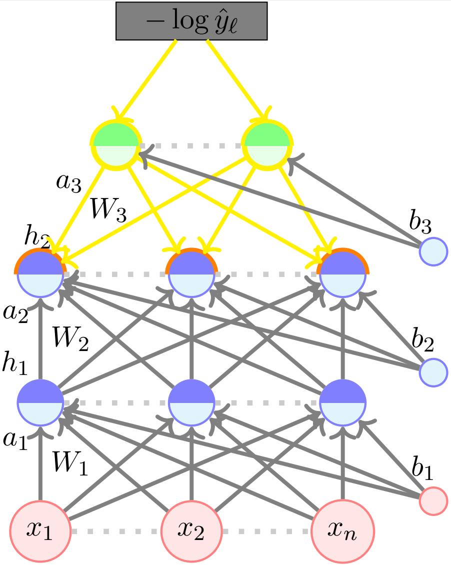
We can now write the gradient w.r.t. h_i

$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta)$$



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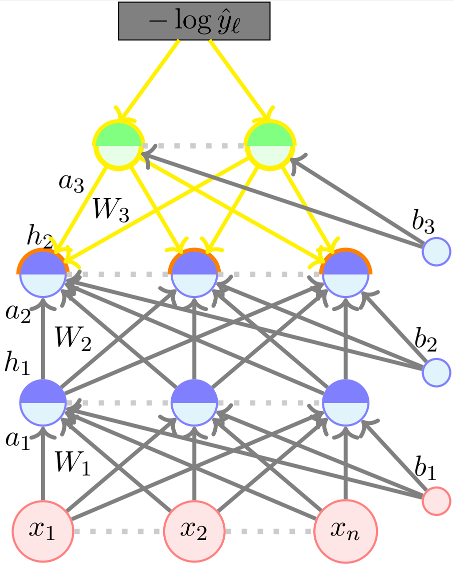

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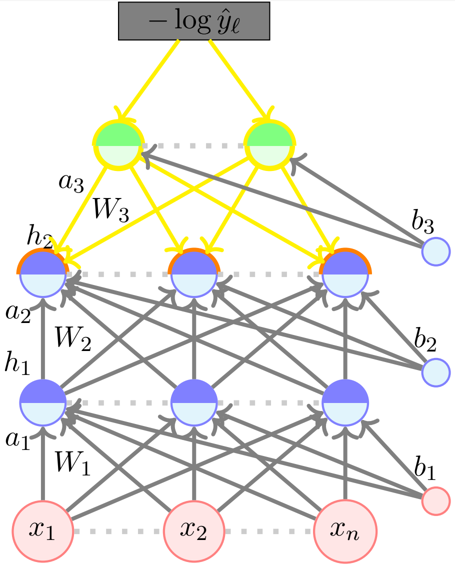

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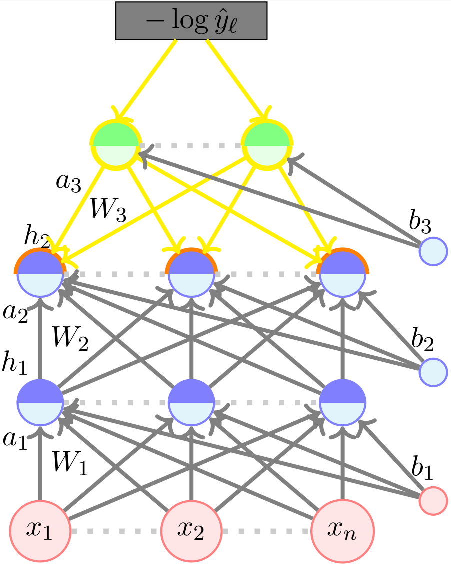

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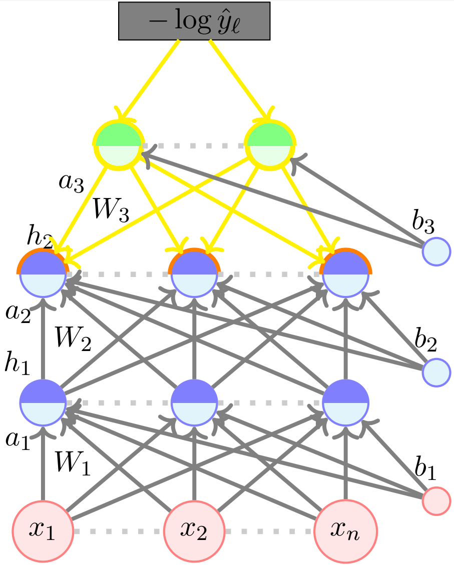

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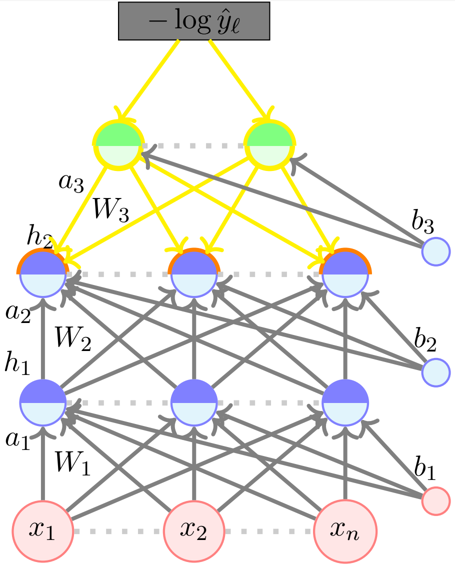

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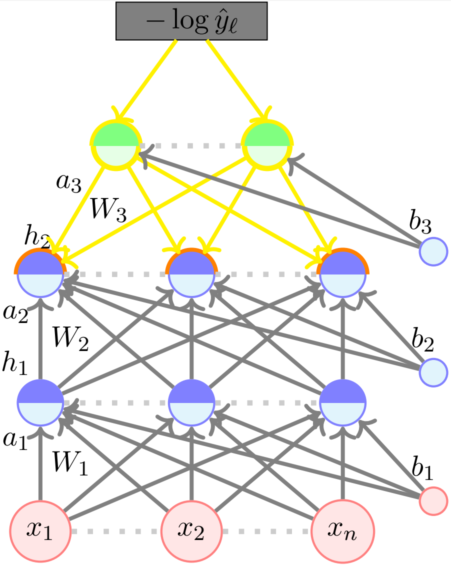
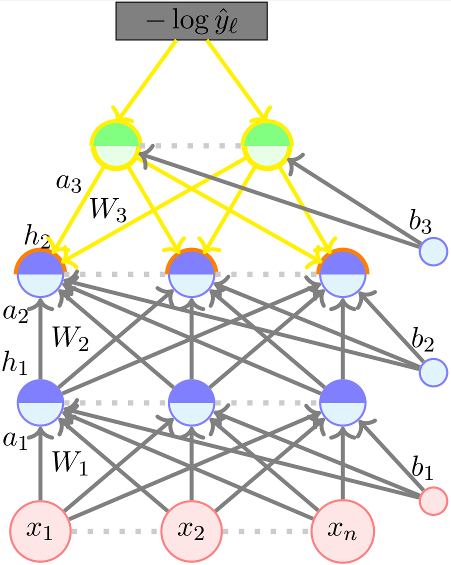


$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

We can now write the gradient w.r.t. h_i

$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ \\ \end{bmatrix}$$



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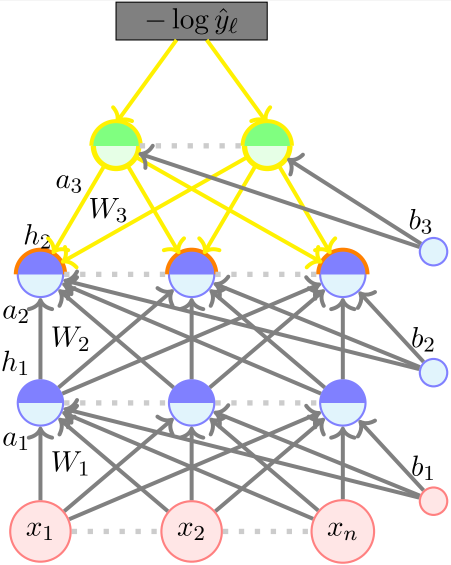

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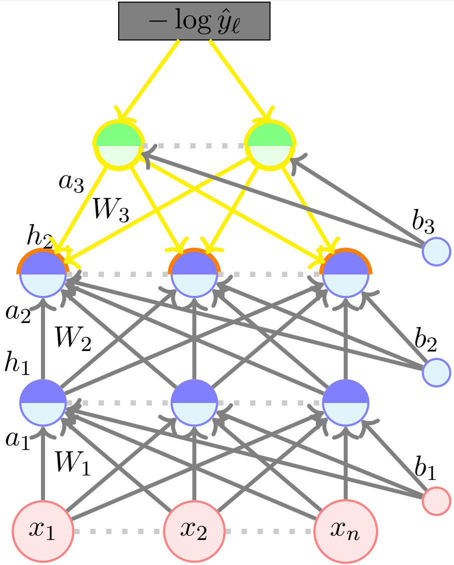
$$= (W_{i+1})^T (\nabla_{a_{i+1}} \mathscr{L}(\theta))$$



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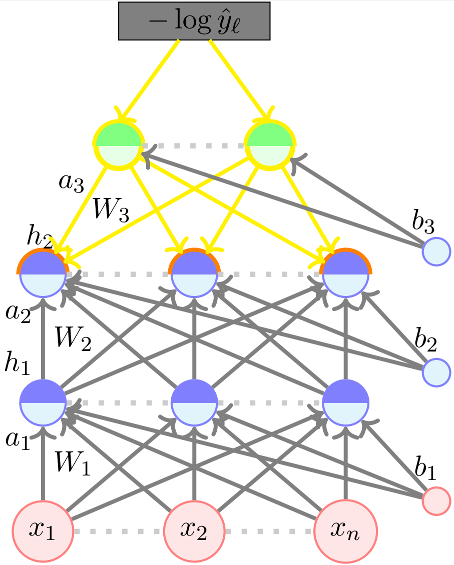
$$= (W_{i+1})^T (\nabla_{a_{i+1}} \mathscr{L}(\theta))$$

- We are almost done except that we do not know how to calculate $\nabla_{a_{i+1}} \mathscr{L}(\theta)$ for $i < L-1$



$$We\ have, \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} = (W_{i+1,.,j})^T \nabla_{a_{i+1}} \mathscr{L}(\theta)$$

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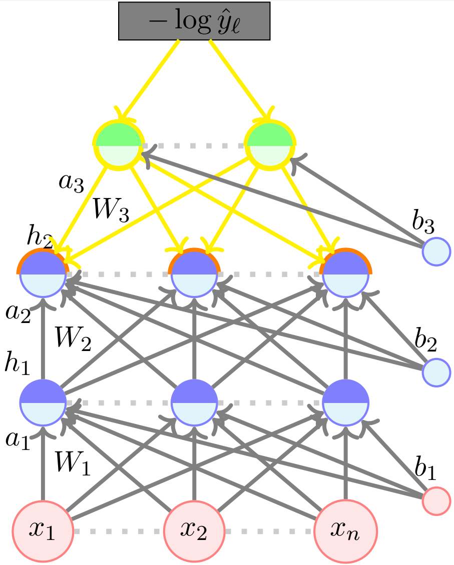
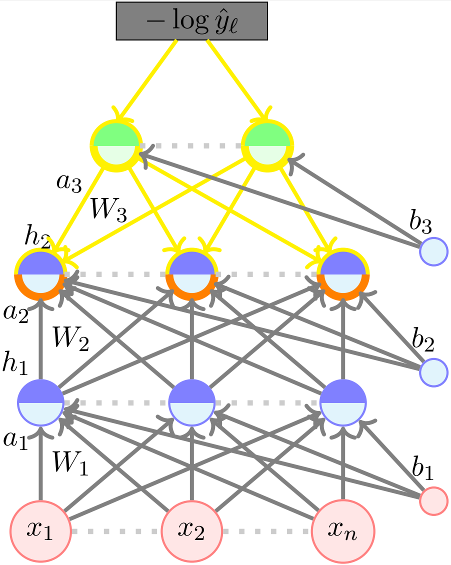
$$\nabla_{\mathbf{h_i}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{i2}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial h_{in}} \end{bmatrix} = \begin{bmatrix} (W_{i+1,\cdot,1})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ (W_{i+1,\cdot,2})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \\ \vdots \\ (W_{i+1,\cdot,n})^T \nabla_{a_{i+1}} \mathscr{L}(\theta) \end{bmatrix}$$

$$= (W_{i+1})^T (\nabla_{a_{i+1}} \mathscr{L}(\theta))$$

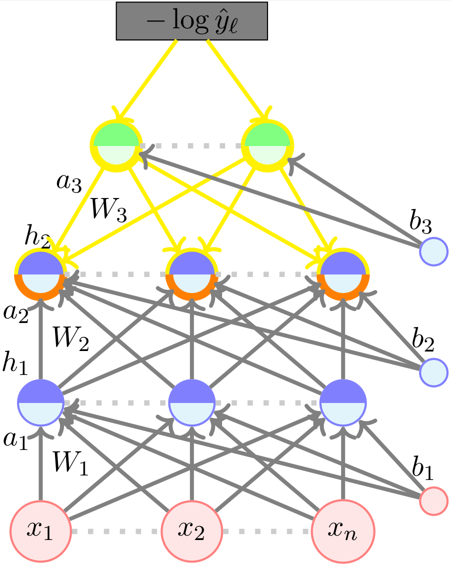
- We are almost done except that we do not know how to calculate $\nabla_{a_{i+1}} \mathscr{L}(\theta)$ for $i < L - 1$
- We will see how to compute that



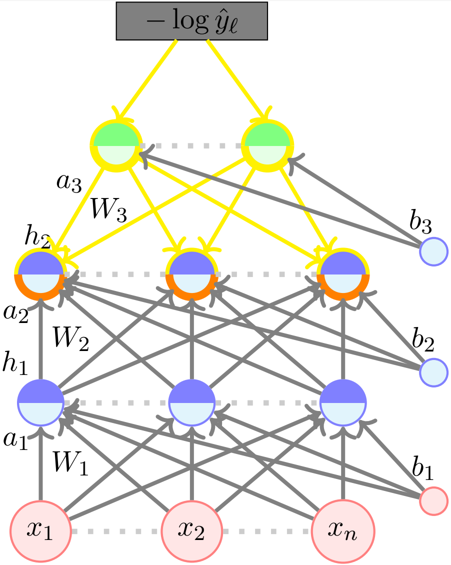
$\nabla_{a_i}\mathscr{L}(\theta)$



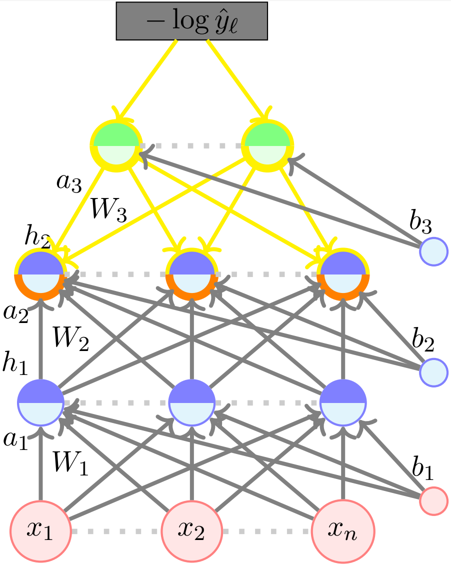
$$\nabla_{a_i} \mathscr{L}(\theta) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



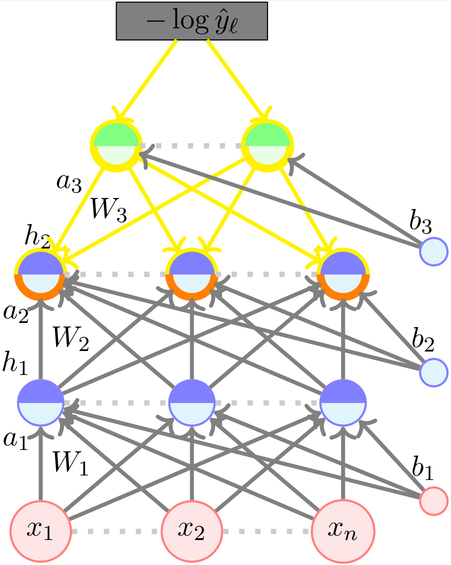
$$\nabla_{a_i} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \\ \\ \end{bmatrix}$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \\ \end{bmatrix}$$

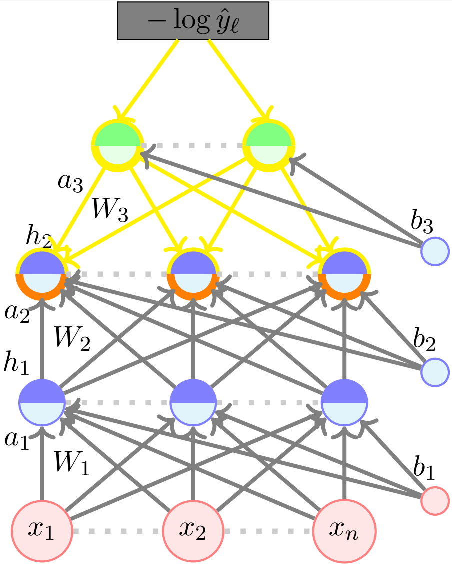


$$\nabla_{a_i} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$



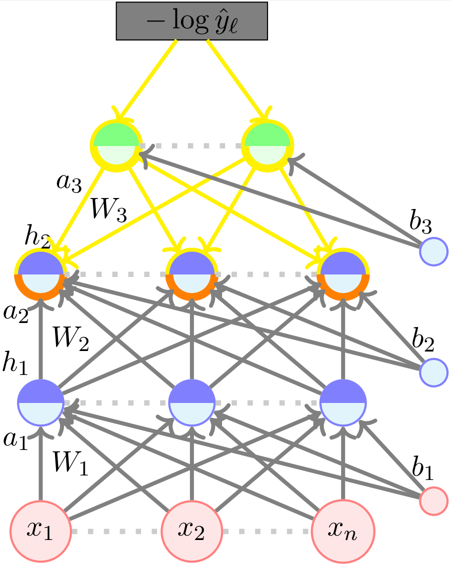
$$\nabla_{a_i} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}}$$



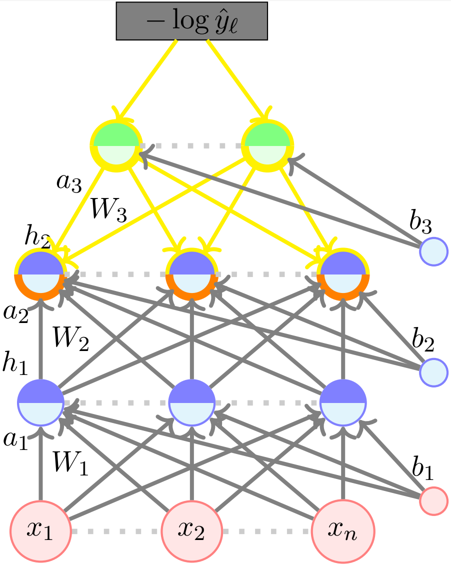
$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\begin{aligned} \frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} &= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}} \\ &= \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} g^{'}(a_{ij}) \quad [\because h_{ij} = g(a_{ij})] \end{aligned}$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

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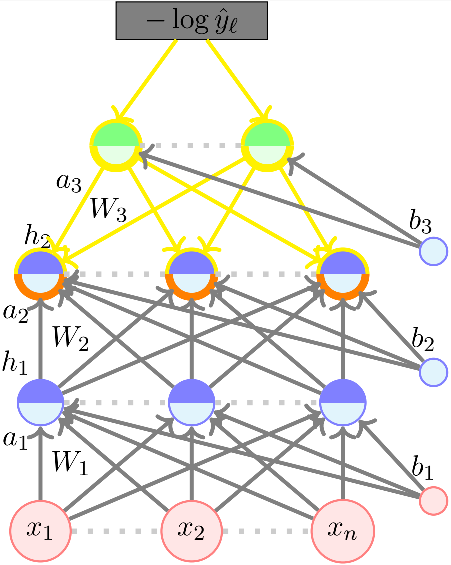
$$\nabla_{a_i}\mathscr{L}(\theta)$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

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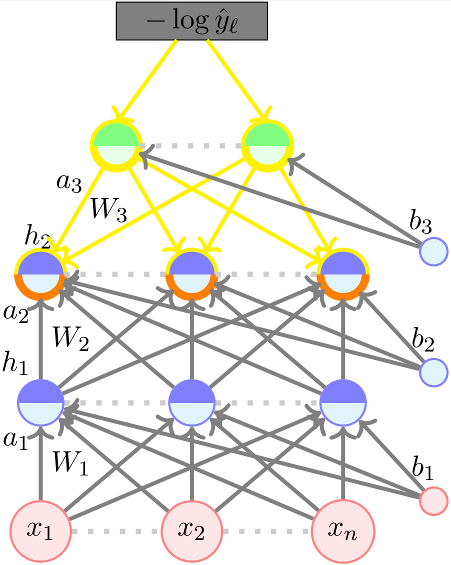
$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \quad \end{bmatrix}$$



$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial a_{i1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{\partial a_{in}} \end{bmatrix}$$

$$\frac{\partial \mathscr{L}(\theta)}{\partial a_{ij}} = \frac{\partial \mathscr{L}(\theta)}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial a_{ij}}$$

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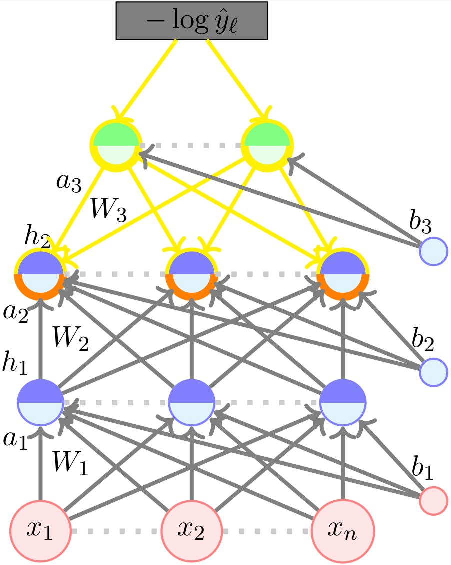
$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} g^{'}(a_{i1}) \\ \\ \\ \\ \end{bmatrix}$$



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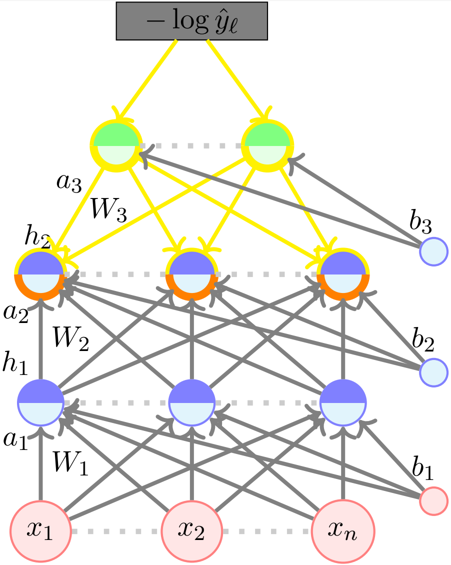
$$\nabla_{a_i}\mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial h_{i1}} g^{'}(a_{i1}) \\ \vdots \\ \\ \end{bmatrix}$$



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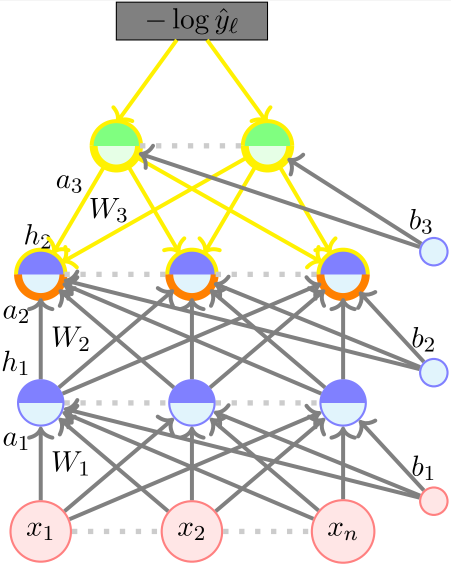

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