Module 2.3: Perceptron

The story ahead ...

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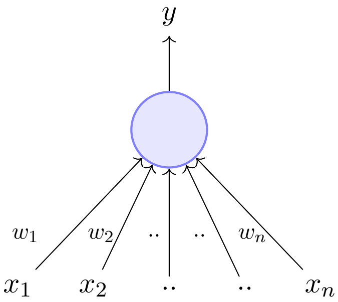
- Do we always need to hand code the threshold ?
- Are all inputs equal ? What if we want to assign more weight (importance) to some inputs ?
- What about functions which are not linearly separable ?

- Frank Rosenblatt, an American psychologist, proposed the **classical perceptron** model (1958)



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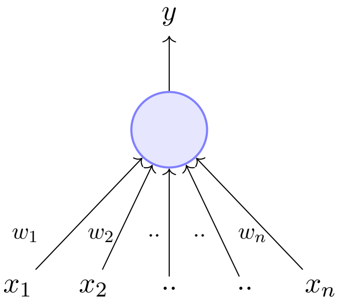

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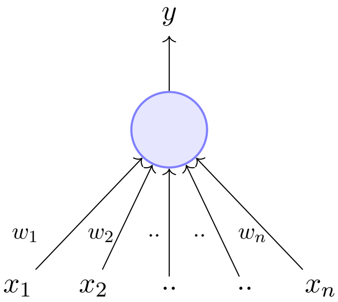
- A more general computational model than McCulloch–Pitts neurons



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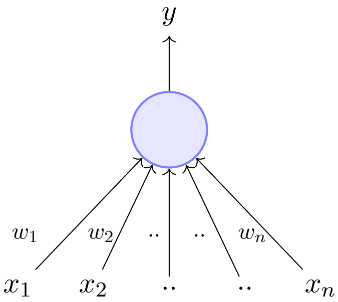

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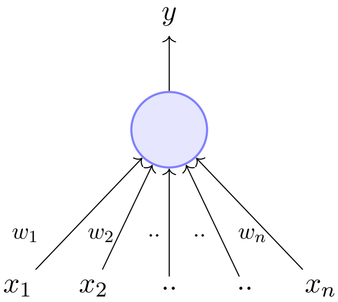

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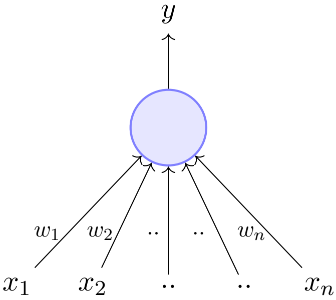
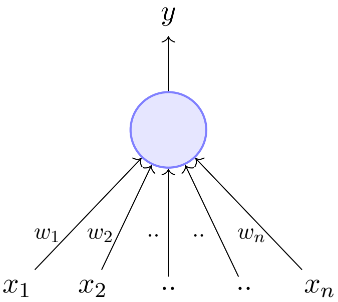
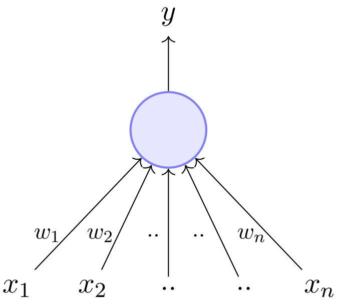
- Refined and carefully analyzed by Minsky and Papert (1969) - their model is referred to as the **perceptron** model here



y
w1
w2
..
..
wn
x1
x2
..
..
xn



$$y = 1 \quad if \sum_{i=1}^{n} w_i * x_i \geq \theta$$



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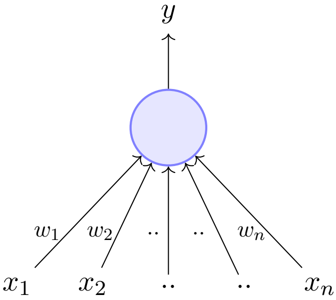

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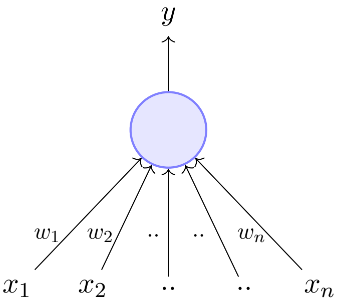

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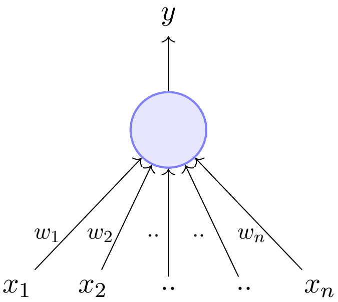

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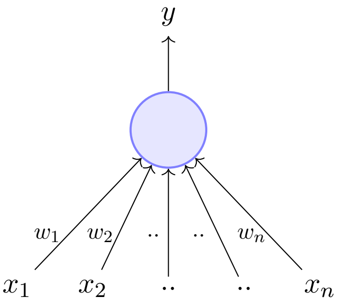

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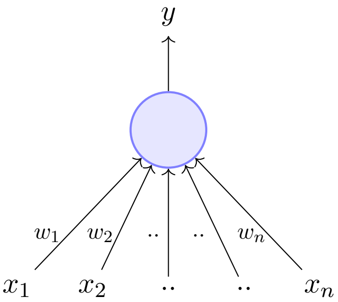

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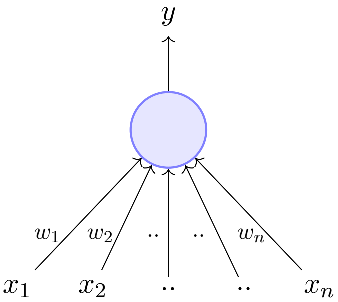

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$$where, \quad x_0 = 1 \quad and \quad w_0 = -\theta$$

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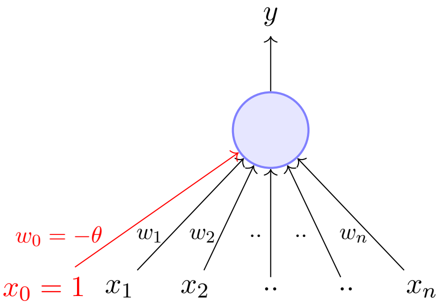

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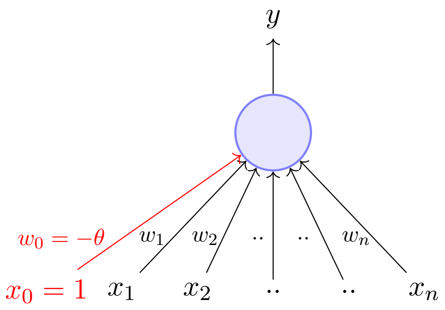

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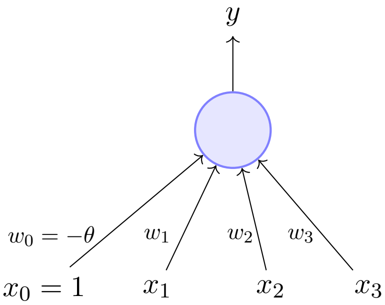
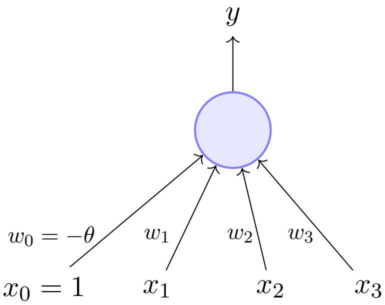
We will now try to answer the following questions:

- Why are we trying to implement boolean functions?
- Why do we need weights ?
- Why is $w_0 = -\theta$ called the bias ?



- Consider the task of predicting whether we would like a movie or not



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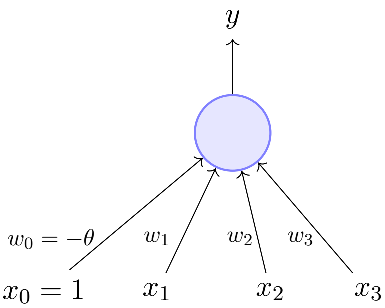

$$x_1 = isActorDamon$$
$$x_2 = isGenreThriller$$
$$x_3 = isDirectorNolan$$

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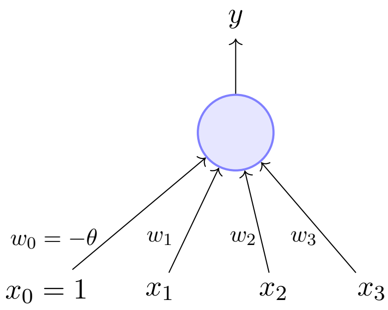

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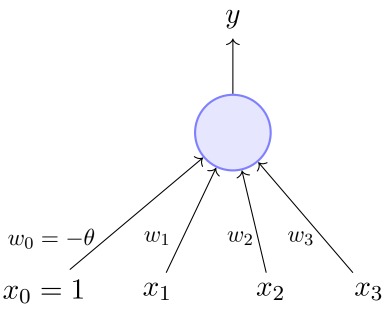
- Based on our past viewing experience (**data**), we may give a high weight to *isDirectorNolan* as compared to the other inputs
- Specifically, even if the actor is not *Matt Damon* and the genre is not *thriller* we would still want to cross the threshold θ by assigning a high weight to *isDirectorNolan*



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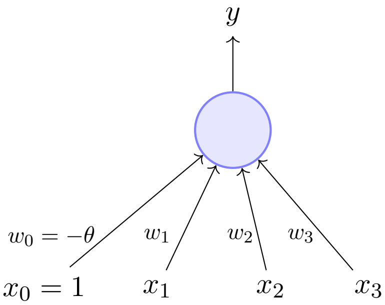

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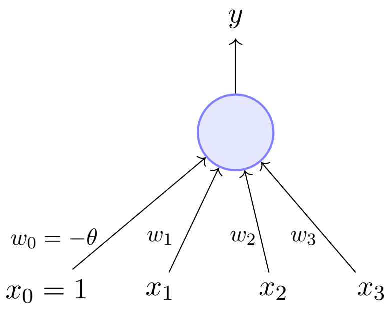

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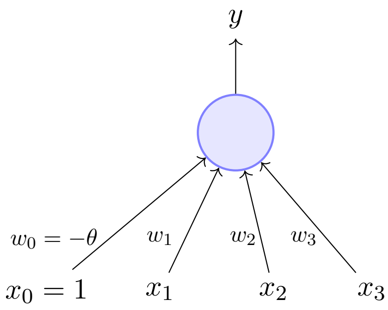

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- The weights $(w_1, w_2, ..., w_n)$ and the bias (w_0) will depend on the data (viewer history in this case)

What kind of functions can be implemented using the perceptron? Any difference from McCulloch Pitts neurons?

McCulloch Pitts Neuron
(assuming no inhibitory inputs)

$$
\begin{aligned}
y &= 1 \quad if \sum_{i=0}^{n} x_i \geq \theta \\
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Perceptron

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$$y = 1 \quad if \sum_{i=0}^{n} {\color{red} w_i} * x_i \geq \theta$$

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- Then what is the difference?

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- In other words, a single perceptron can only be used to implement linearly separable functions
- Then what is the difference? The weights (including threshold) can be learned and the inputs can be real valued
- We will first revisit some boolean functions and then see the perceptron learning algorithm (for learning weights)

x_1	x_2	OR
0	0	

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0	0	0

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^{2} w_i x_i$

x_1	x_2	OR	
0	0	0	$w_0 + \sum_{i=1}^{2} w_i x_i < 0$

x_1	x_2	OR	
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1	0	1	

x_1	x_2	OR	
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$$w_0 + w_1 \cdot 0 + w_2 \cdot 0 < 0 \implies w_0 < 0$$

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$$w_0 + w_1 \cdot 0 + w_2 \cdot 1 \geq 0 \implies w_2 > -w_0$$

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0	0	0	$w_0 + \sum_{i=1}^{2} w_i x_i < 0$
1	0	1	$w_0 + \sum_{i=1}^{2} w_i x_i \geq 0$
0	1	1	$w_0 + \sum_{i=1}^{2} w_i x_i \geq 0$
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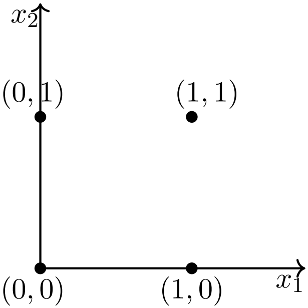

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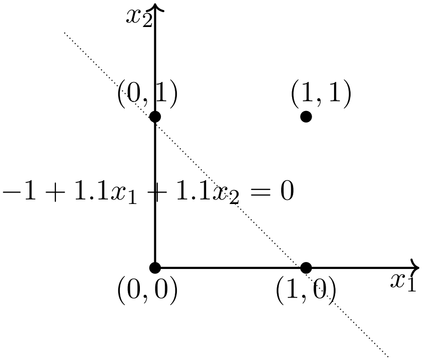

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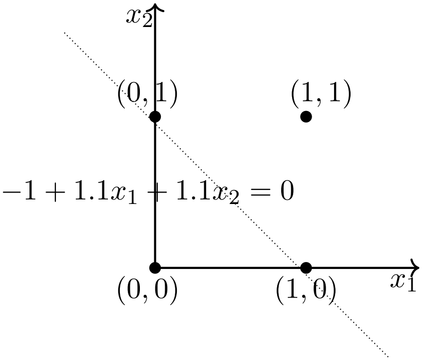

- Note that we can come up with a similar set of inequalities and find the value of θ for a McCulloch Pitts neuron also

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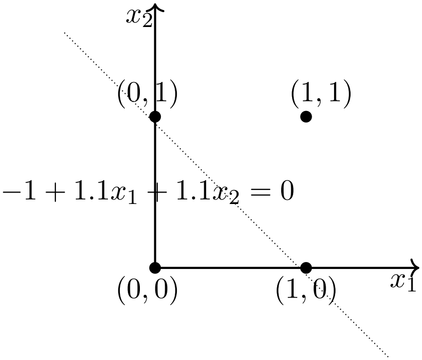

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- Note that we can come up with a similar set of inequalities and find the value of θ for a McCulloch Pitts neuron also (Try it!)