# Module 4.7: Backpropagation: Computing Gradients w.r.t. Parameters

**Quantities of interest (roadmap for the remaining part):**

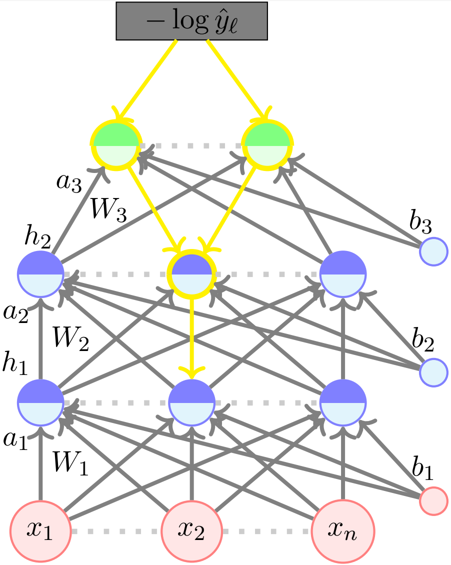
- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{111}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{111}}}_{\text{and now talk to the weights}}$$

- Our focus is on *Cross entropy loss* and *Softmax* output.

Recall that,

$$\mathbf{a_k} = \mathbf{b_k} + W_k \mathbf{h_{k-1}}$$



Recall that,

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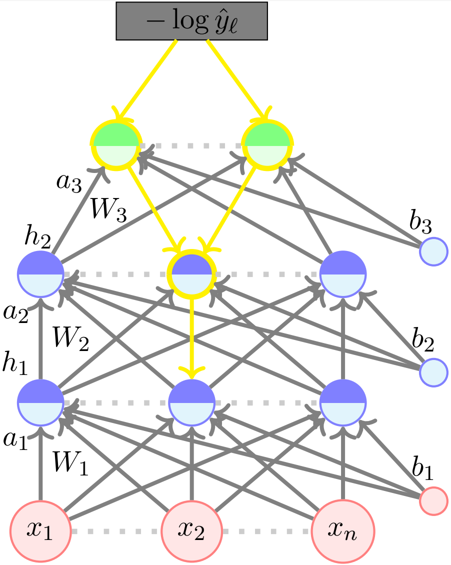
$$\frac{\partial a_{ki}}{\partial W_{kij}} = h_{k-1,j}$$



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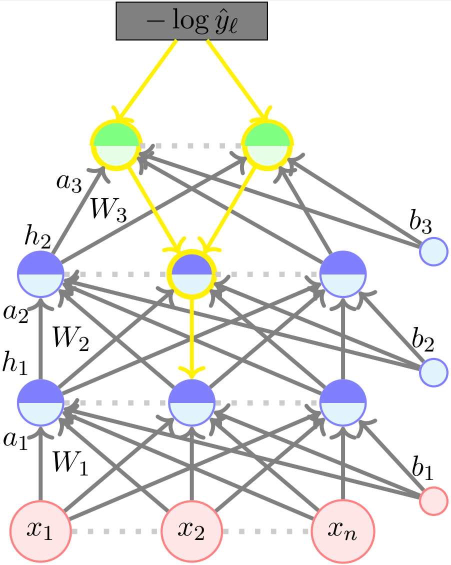
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{kij}}$$



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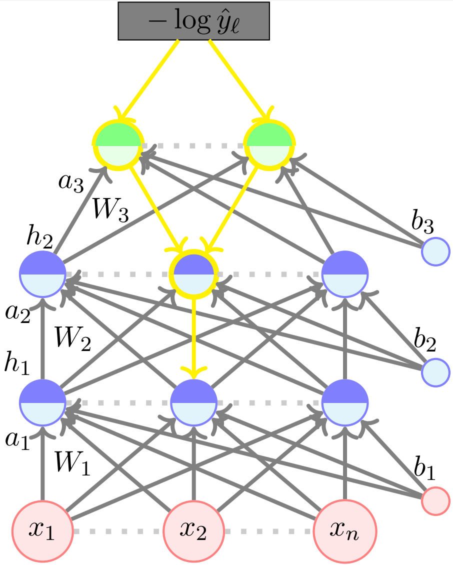

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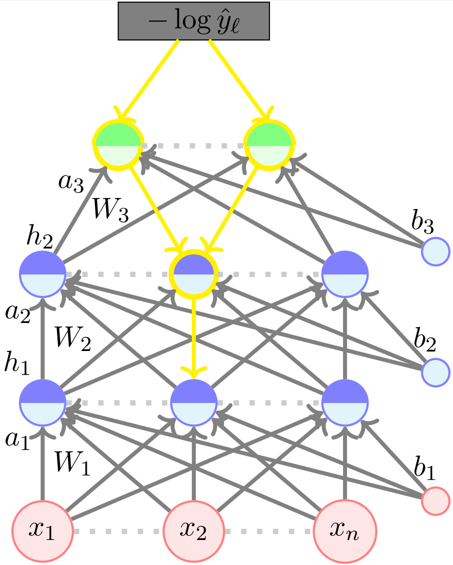

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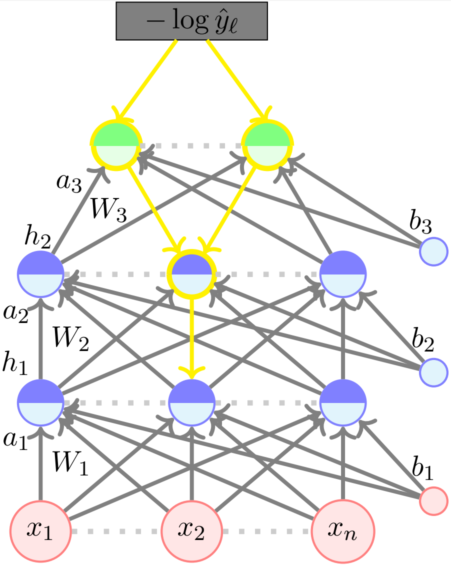
\mathbf{a_k} &= \mathbf{b_k} + W_k \mathbf{h_{k-1}} \\
\frac{\partial a_{ki}}{\partial W_{kij}} &= h_{k-1,j} \\
\frac{\partial \mathscr{L}(\theta)}{\partial W_{kij}} &= \frac{\partial \mathscr{L}(\theta)}{\partial a_{ki}} \frac{\partial a_{ki}}{\partial W_{k,i,j}} \\
&= \frac{\partial \mathscr{L}(\theta)}{\partial a_{ki}} h_{k-1,j}
\end{aligned}$$

$$\nabla_{W_K} \mathscr{L}(\theta) =$$



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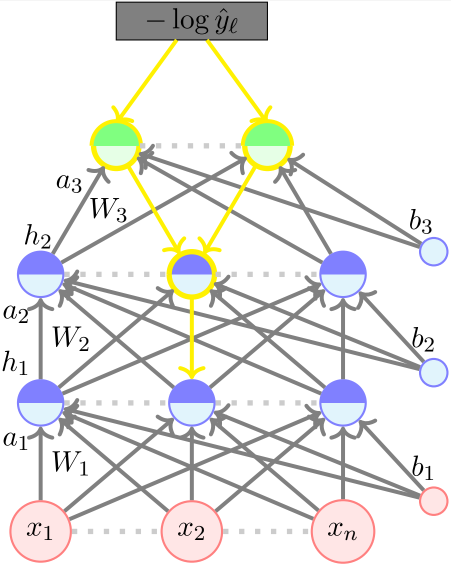
$$\nabla_{W_K} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{\partial W_{k00}} & \frac{\partial \mathscr{L}(\theta)}{\partial W_{k01}} & \cdots & \cdots & \frac{\partial \mathscr{L}(\theta)}{\partial W_{k0n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \frac{\partial \mathscr{L}(\theta)}{\partial W_{k,n-1,n-1}} \end{bmatrix}$$



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Lets take a simple example of a $W_k \in \mathbb{R}^{3\times 3}$ and see what each entry looks like

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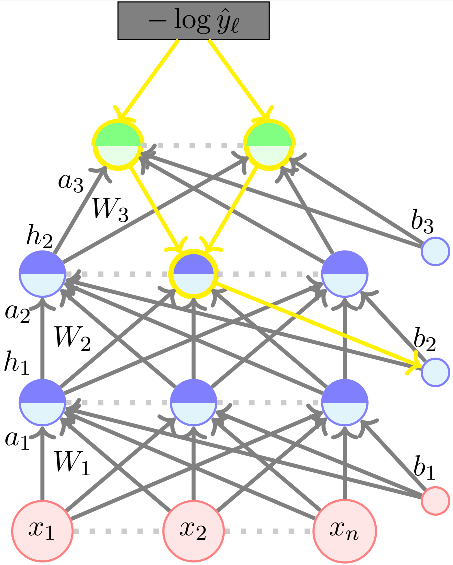
Finally, coming to the biases



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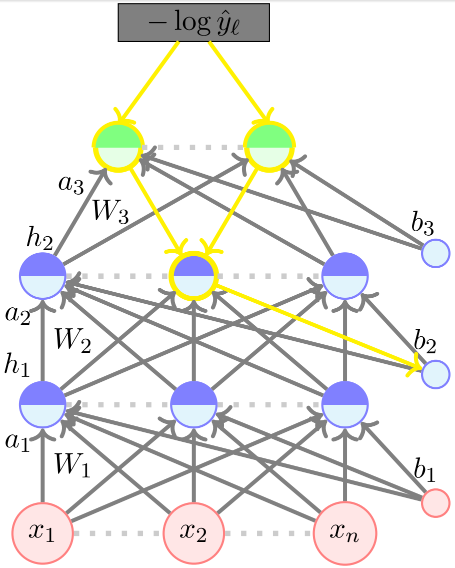
$$a_{ki} = b_{ki} + \sum_j W_{kij} h_{k-1,j}$$



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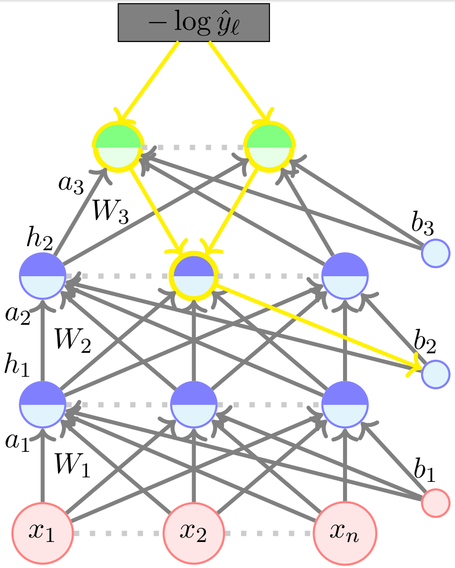

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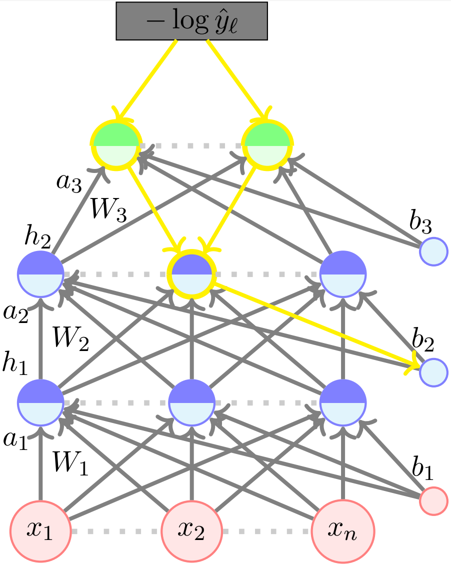

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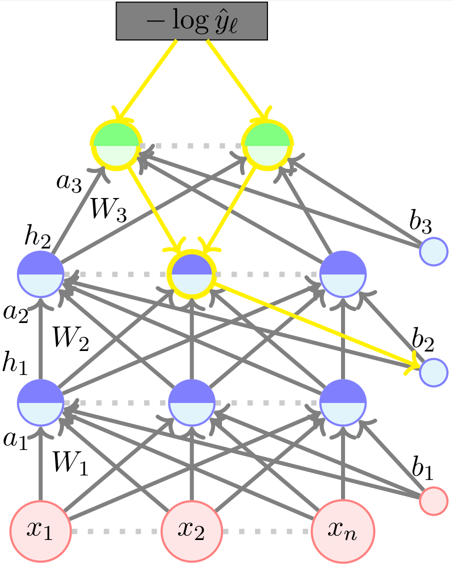
We can now write the gradient w.r.t. the vector $b_k$



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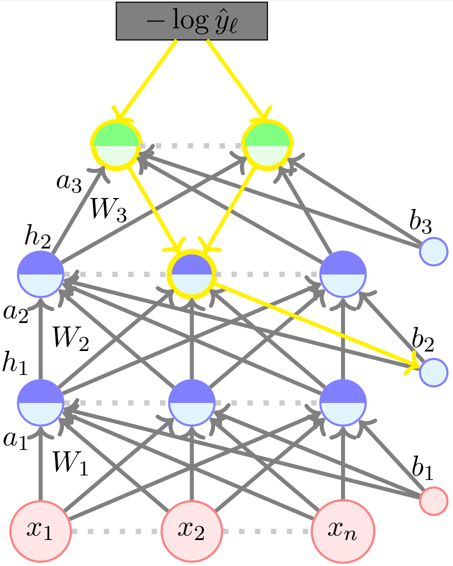
$$\nabla_{\mathbf{b_k}} \mathscr{L}(\theta) = \begin{bmatrix} \frac{\partial \mathscr{L}(\theta)}{a_{k0}} \\ \frac{\partial \mathscr{L}(\theta)}{a_{k1}} \\ \vdots \\ \frac{\partial \mathscr{L}(\theta)}{a_{kn}} \end{bmatrix}$$



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