Module 4.4: Backpropagation (Intuition)

We need to answer two questions

- How to choose the loss function $\mathscr{L}(\theta)$?
- How to compute $\nabla\theta$ which is composed of $\nabla W_1, \nabla W_2, ..., \nabla W_{L-1} \in \mathbb{R}^{n\times n}, \nabla W_L \in \mathbb{R}^{n\times k}$, $\nabla b_1, \nabla b_2, ..., \nabla b_{L-1} \in \mathbb{R}^n$ and $\nabla b_L \in \mathbb{R}^k$?

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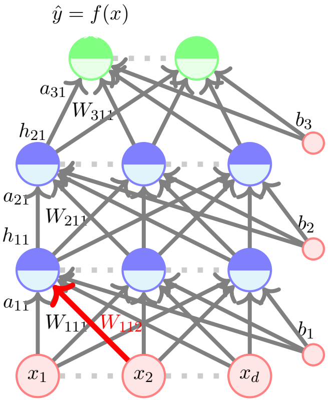
- Let us focus on this one weight.



Algorithm: gradient descent()

$t \leftarrow 0$;
$max\_iterations \leftarrow 1000$;
$Initialize \quad \theta_0$;
while $t{+}{+} < max\_iterations$ **do**
  $\theta_{t+1} \leftarrow \theta_t - \eta \nabla \theta_t$;
end

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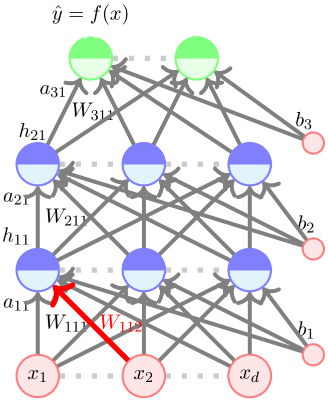
- To learn this weight using SGD we need a formula for $\frac{\partial \mathscr{L}(\theta)}{\partial W_{112}}$.



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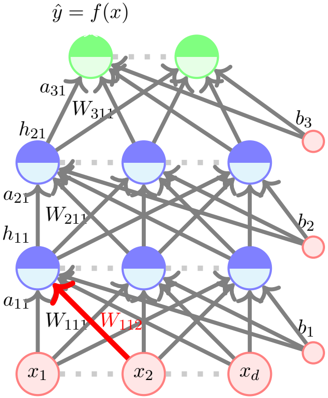
- We will see how to calculate this.



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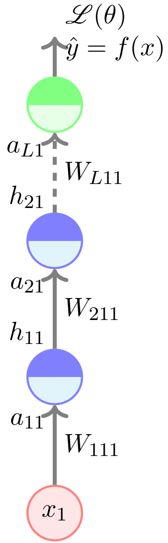
- First let us take the simple case when we have a deep but thin network.



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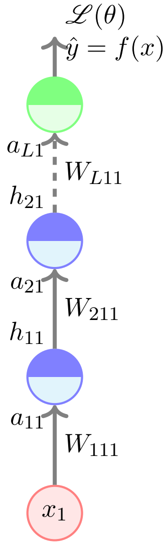
- In this case it is easy to find the derivative by chain rule.



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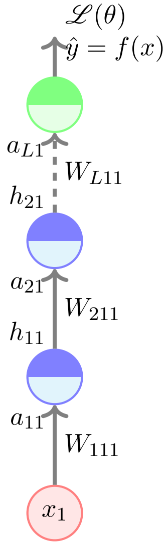
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{111}} = \frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_{L11}} \frac{\partial a_{L11}}{\partial h_{21}} \frac{\partial h_{21}}{\partial a_{21}} \frac{\partial a_{21}}{\partial h_{11}} \frac{\partial h_{11}}{\partial a_{11}} \frac{\partial a_{11}}{\partial W_{111}}$$



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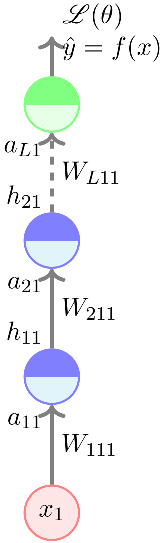

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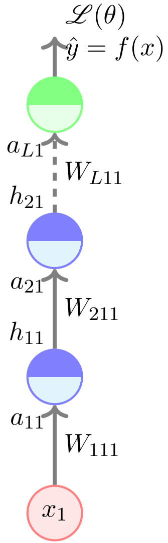

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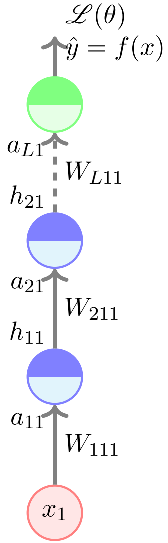
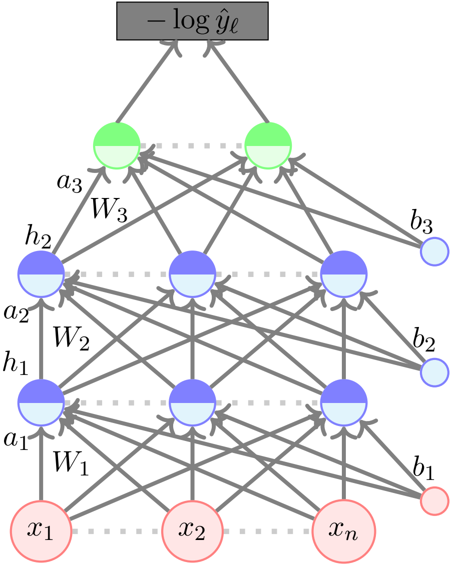


Let us see an intuitive explanation of backpropagation before we get into the mathematical details

- We get a certain loss at the output and we try to figure out who is responsible for this loss



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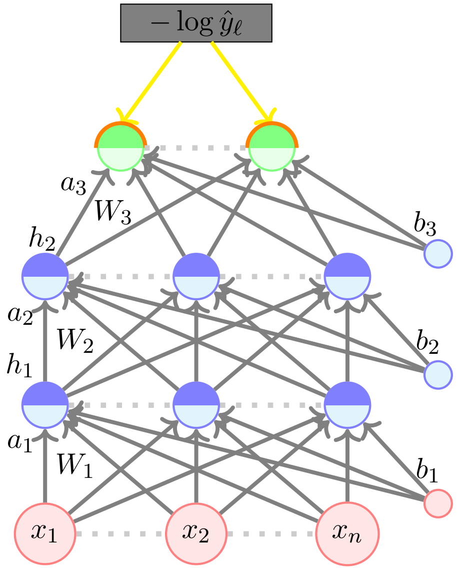
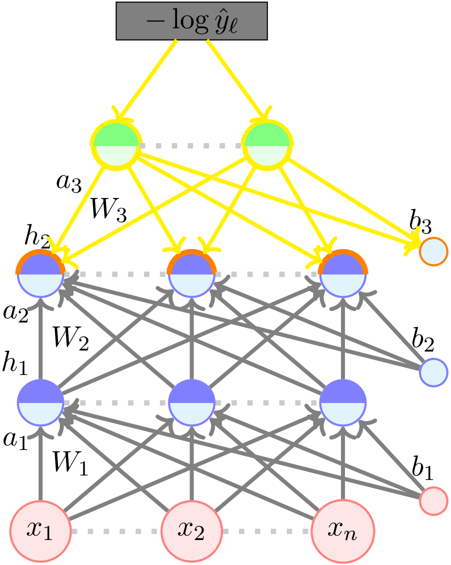
- So, we talk to the output layer and say “Hey! You are not producing the desired output, better take responsibility”.

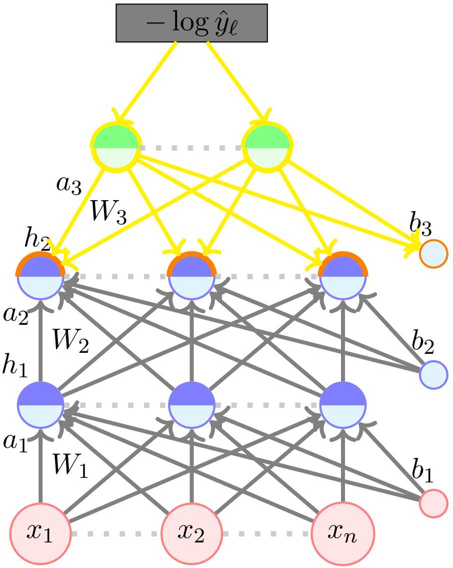


- We get a certain loss at the output and we try to figure out who is responsible for this loss
- So, we talk to the output layer and say “Hey! You are not producing the desired output, better take responsibility”.
- The output layer says “Well, I take responsibility for my part but please understand that I am only as the good as the hidden layer and weights below me”. After all ...

$$f(x) = \hat{y} = O(W_L h_{L-1} + b_L)$$



- So, we talk to W_L, b_L and h_L and ask them “What is wrong with you?”



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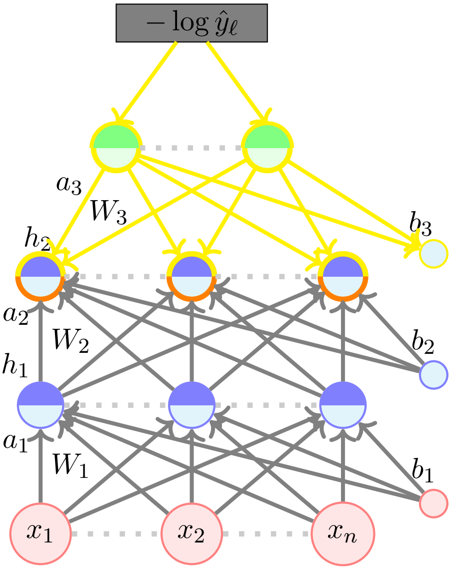
- W_L and b_L take full responsibility but h_L says "Well, please understand that I am only as good as the pre-activation layer"



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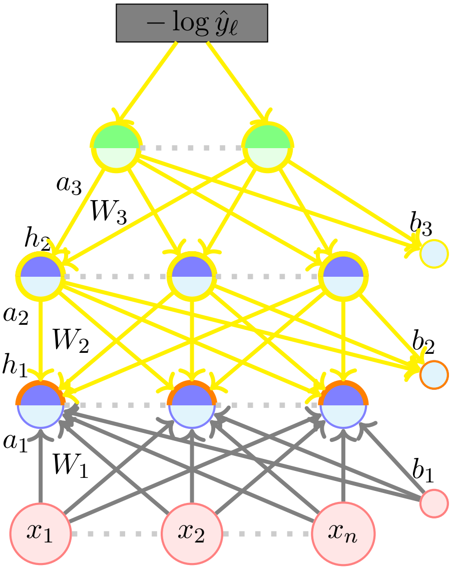

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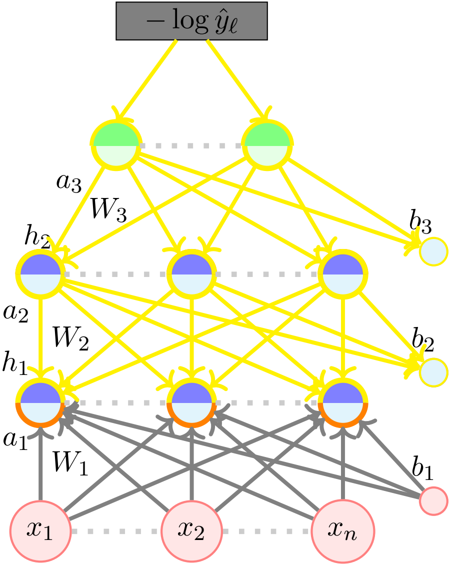
- We continue in this manner and realize that the responsibility lies with all the weights and biases (i.e. all the parameters of the model)



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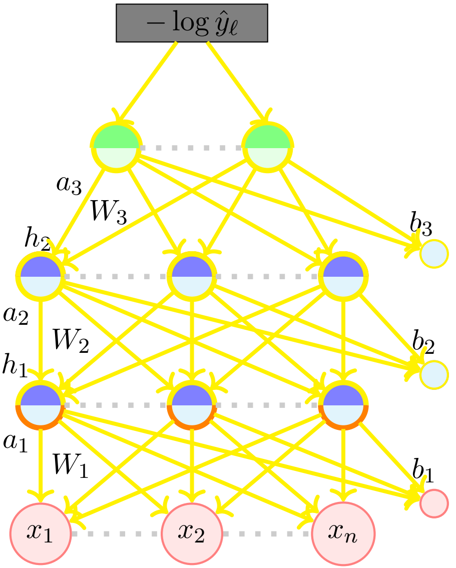

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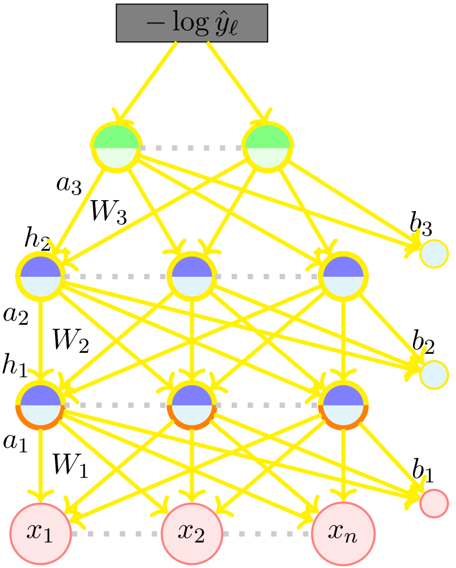
$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the}\\\text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial f(x)}\frac{\partial f(x)}{\partial a_3}}_{\substack{\text{Talk to the}\\\text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2}\frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the}\\\text{previous hidden}\\\text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1}\frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the}\\\text{previous}\\\text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now}\\\text{talk to}\\\text{the}\\\text{weights}}}$$



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Quantities of interest (roadmap for the remaining part):

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Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units

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Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units

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Quantities of interest (roadmap for the remaining part):

- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\substack{\text{Talk to the}\\ \text{weight directly}}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\substack{\text{Talk to the}\\ \text{output layer}}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\substack{\text{Talk to the}\\ \text{previous hidden}\\ \text{layer}}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\substack{\text{Talk to the}\\ \text{previous}\\ \text{hidden layer}}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\substack{\text{and now}\\ \text{talk to}\\ \text{the}\\ \text{weights}}}$$

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- Our focus is on *Cross entropy loss* and *Softmax* output.