Module 4.7: Backpropagation: Computing Gradients w.r.t. Parameters

Quantities of interest (roadmap for the remaining part):

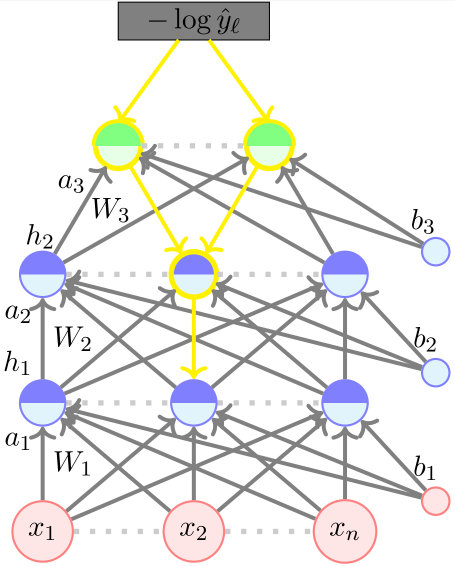
- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial W_{11}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathscr{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{11}}}_{\text{and now talk to the weights}}$$

- Our focus is on *Cross entropy loss* and *Softmax* output.

Recall that,

$$a_k = b_k + W_k h_{k-1}$$



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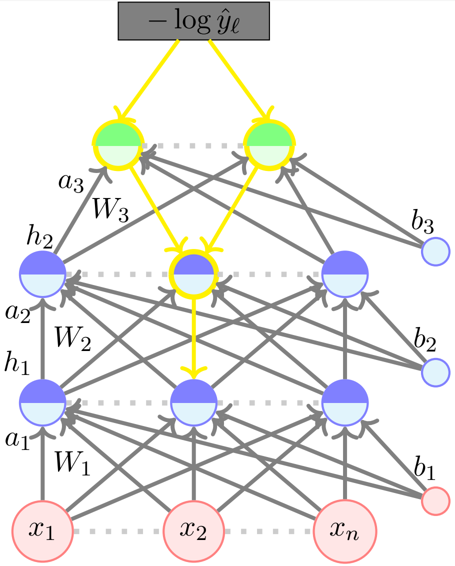

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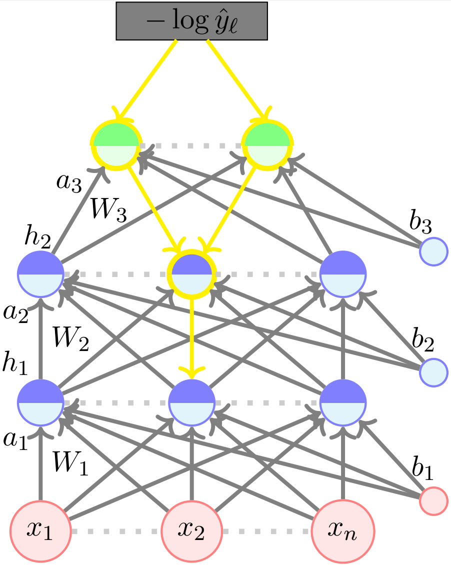

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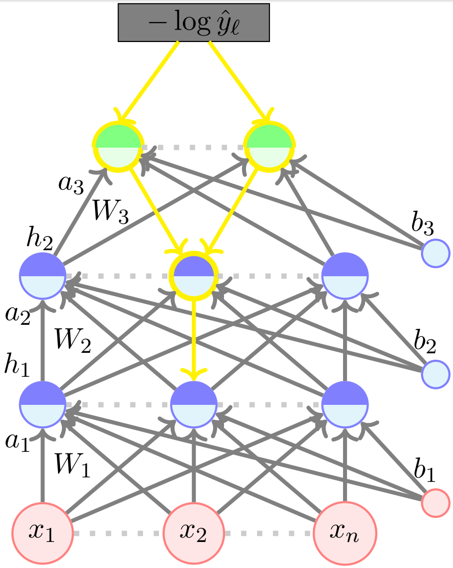
$$\frac{\partial \mathscr{L}(\theta)}{\partial W_{kij}}$$



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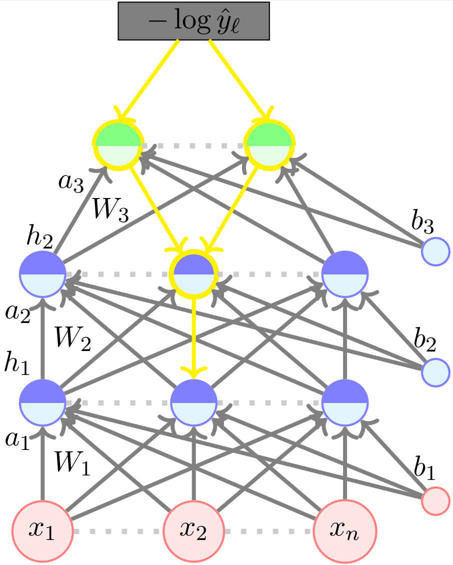

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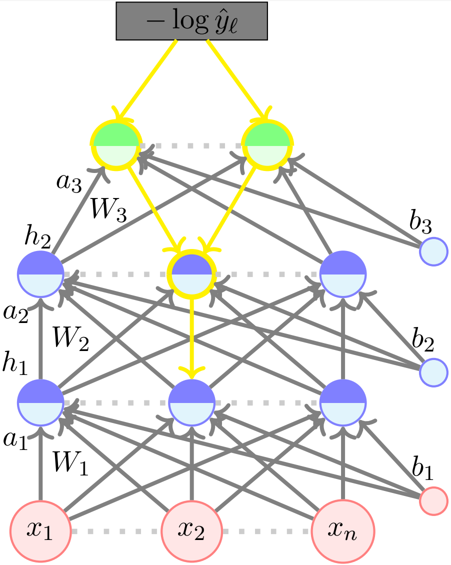

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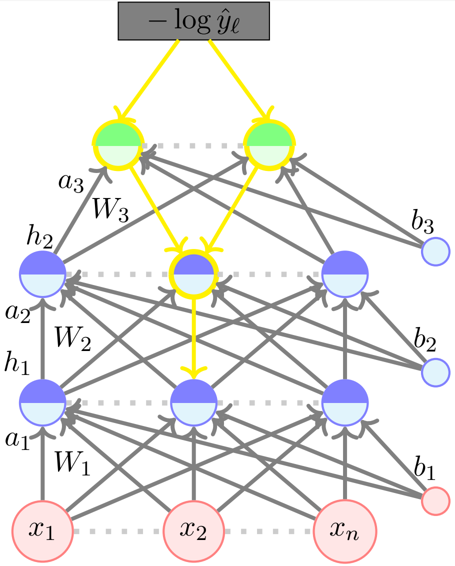
$$\nabla_{W_K} \mathscr{L}(\theta) =$$



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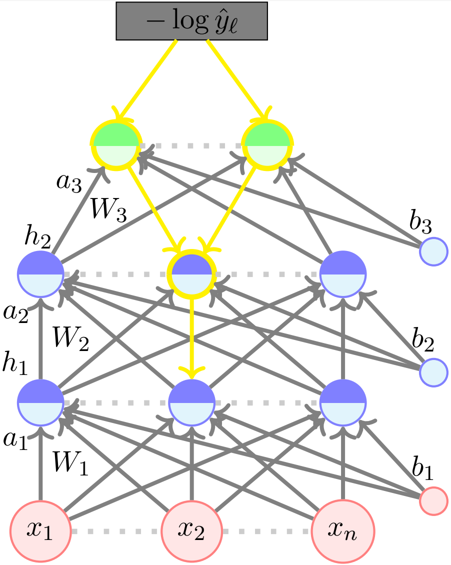

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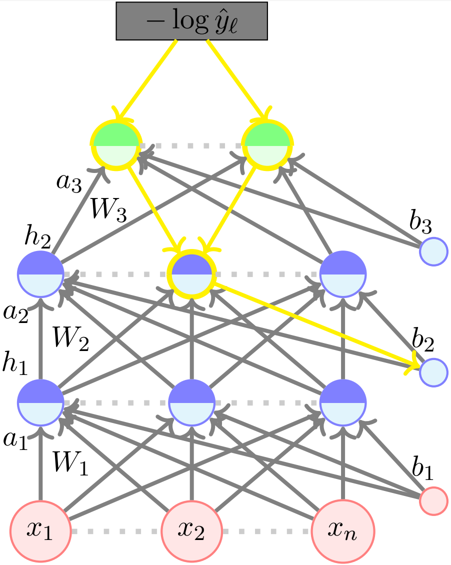
Finally, coming to the biases



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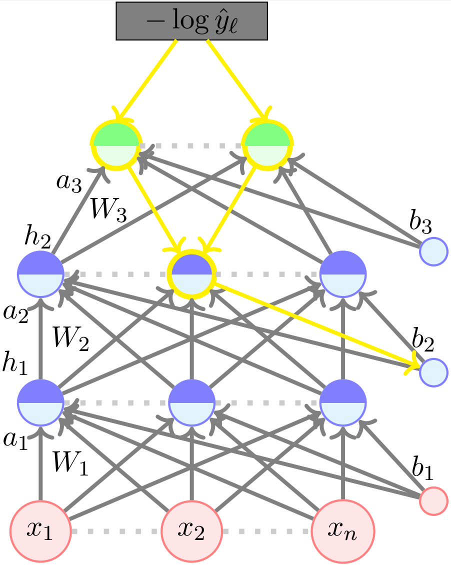

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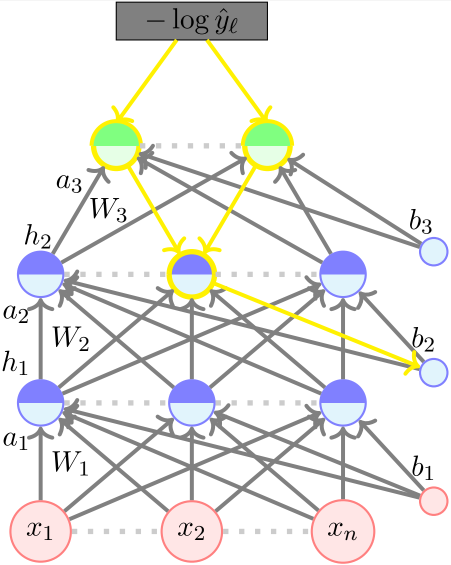

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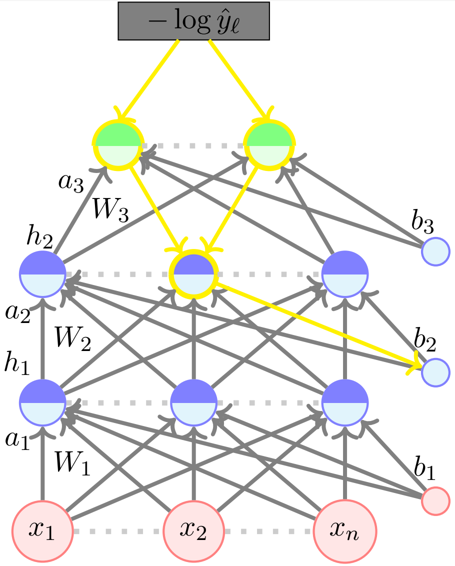

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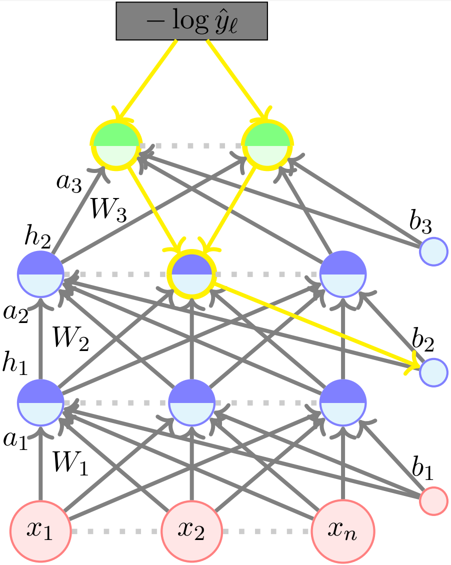
We can now write the gradient w.r.t. the vector b_k



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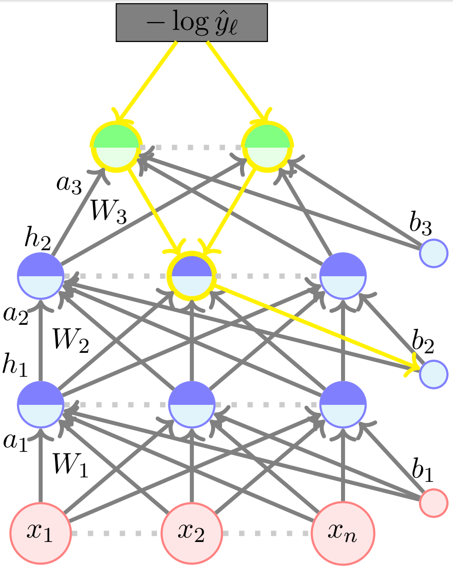

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