

## Module 4.7: Backpropagation: Computing Gradients w.r.t. Parameters

## Quantities of interest (roadmap for the remaining part):

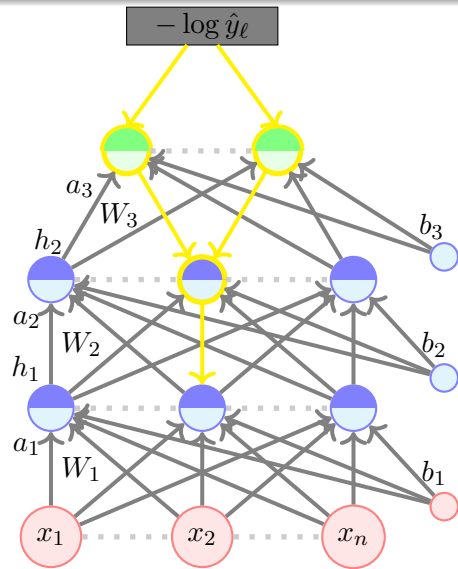
- Gradient w.r.t. output units
- Gradient w.r.t. hidden units
- Gradient w.r.t. weights and biases

$$\underbrace{\frac{\partial \mathcal{L}(\theta)}{\partial W_{111}}}_{\text{Talk to the weight directly}} = \underbrace{\frac{\partial \mathcal{L}(\theta)}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial a_3}}_{\text{Talk to the output layer}} \underbrace{\frac{\partial a_3}{\partial h_2} \frac{\partial h_2}{\partial a_2}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_2}{\partial h_1} \frac{\partial h_1}{\partial a_1}}_{\text{Talk to the previous hidden layer}} \underbrace{\frac{\partial a_1}{\partial W_{111}}}_{\text{and now talk to the weights}}$$

- Our focus is on *Cross entropy loss* and *Softmax* output.

Recall that,

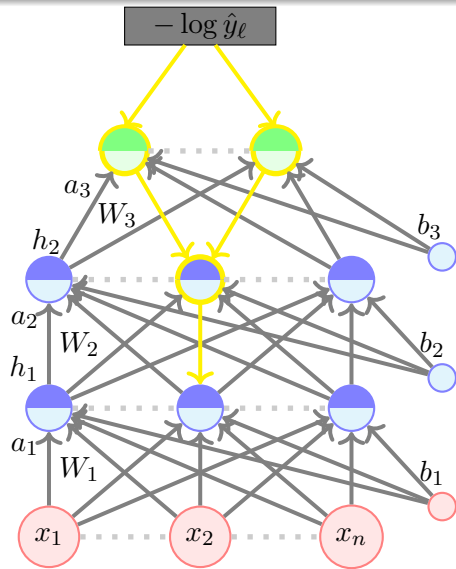
$$\mathbf{a}_k = \mathbf{b}_k + W_k \mathbf{h}_{k-1}$$



Recall that,

$$\mathbf{a}_k = \mathbf{b}_k + W_k \mathbf{h}_{k-1}$$

$$\frac{\partial a_{ki}}{\partial W_{kij}} = h_{k-1,j}$$

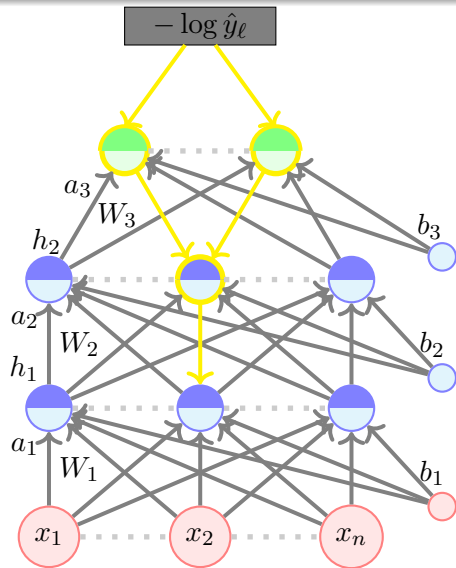


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$$\frac{\partial \mathcal{L}(\theta)}{\partial W_{kij}}$$

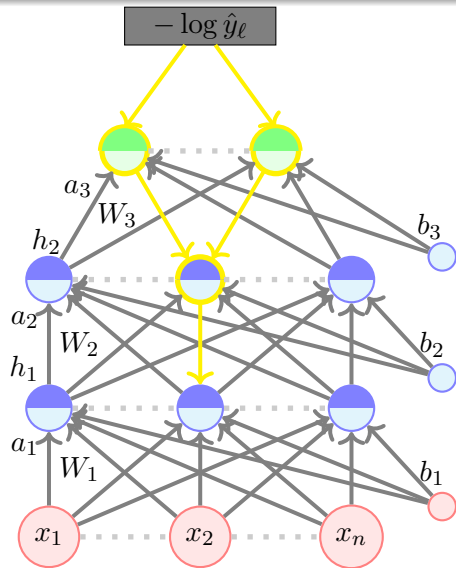


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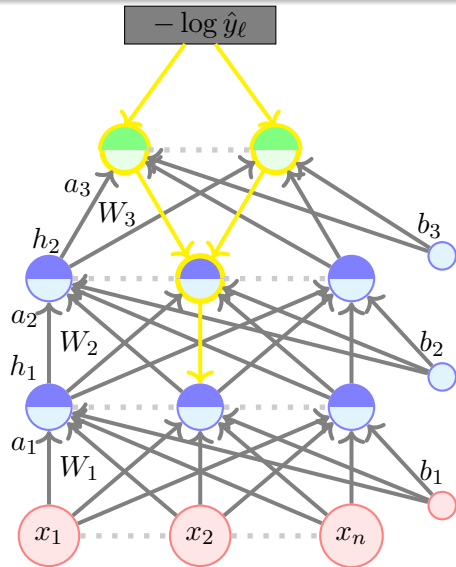


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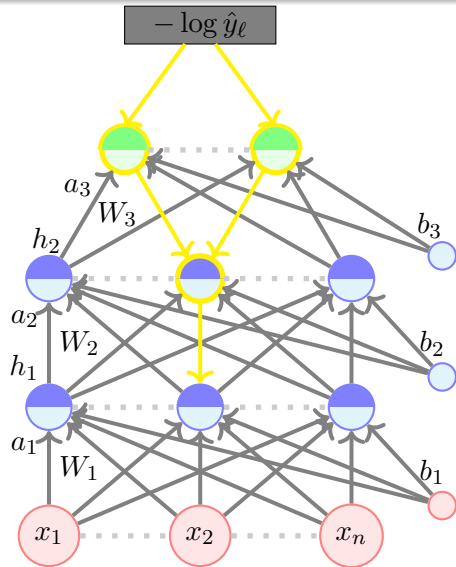
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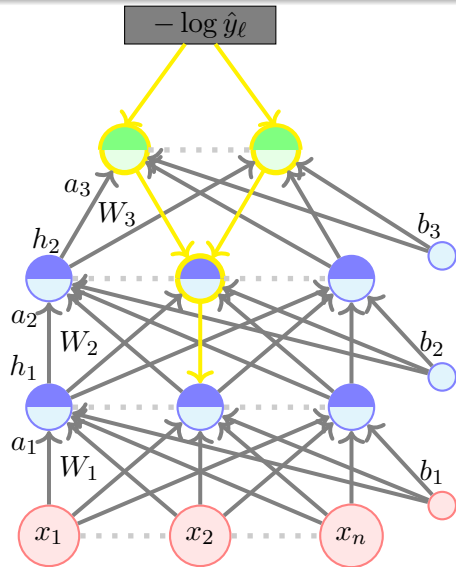
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Lets take a simple example of a  $W_k \in \mathbb{R}^{3 \times 3}$  and see what each entry looks like

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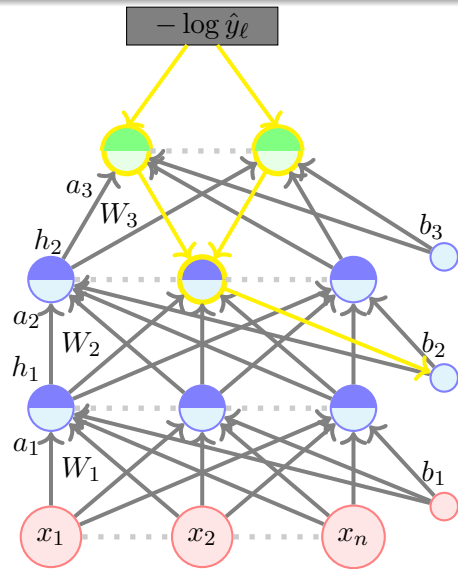
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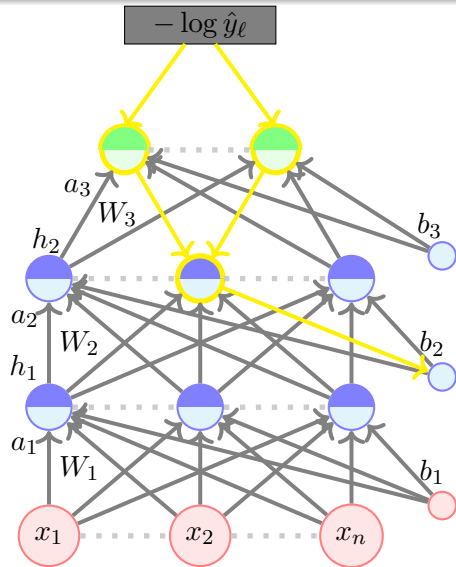
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Finally, coming to the biases



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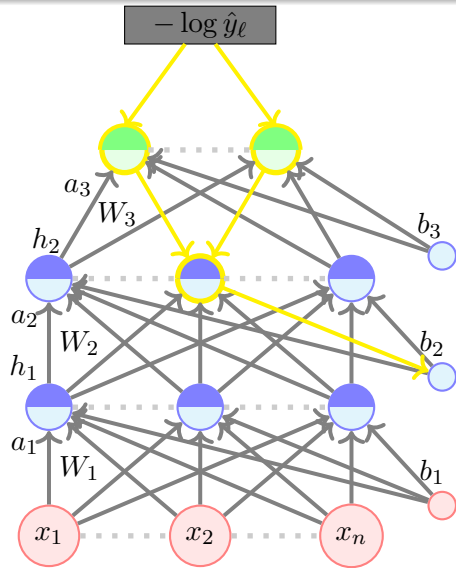
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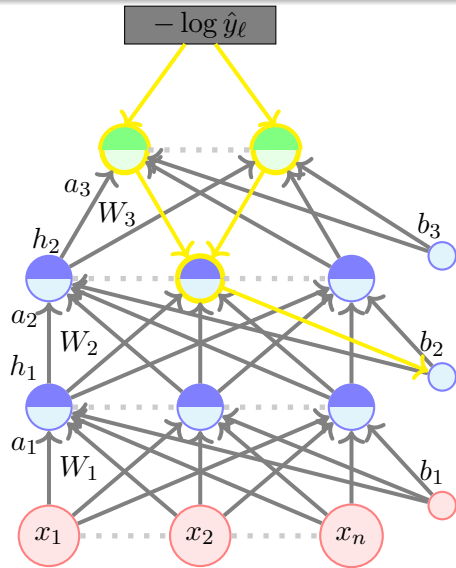
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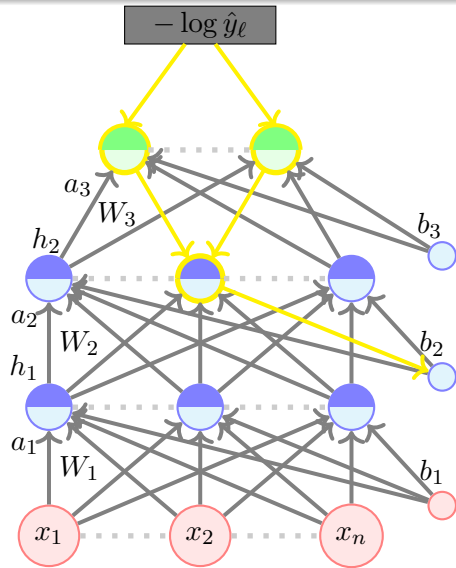


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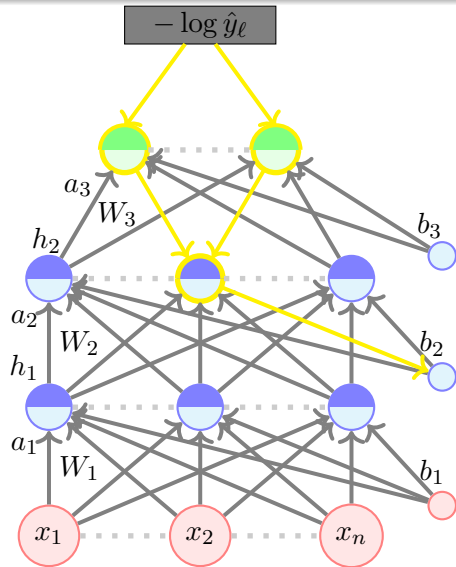
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$$\nabla_{\mathbf{b}_k} \mathcal{L}(\theta) = \begin{bmatrix} \frac{\partial \mathcal{L}(\theta)}{\partial a_{k0}} \\ \frac{\partial \mathcal{L}(\theta)}{\partial a_{k1}} \\ \vdots \\ \frac{\partial \mathcal{L}(\theta)}{\partial a_{kn}} \end{bmatrix}$$





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