Module 8.4 : L_2 regularization

Different forms of regularization

- L_2 regularization
- Dataset augmentation
- Parameter Sharing and tying
- Adding Noise to the inputs
- Adding Noise to the outputs
- Early stopping
- Ensemble methods
- Dropout

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- Let us see the geometric interpretation of this

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- Notice that if $\alpha \to 0$ then $\widetilde{w} \to w^*$ [no regularization]

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- Let us analyse the case when $\alpha \neq 0$

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where $D = (\Lambda + \alpha\mathbb{I})^{-1}\Lambda$, is a diagonal matrix which we will see in more detail soon

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- Each element i of $Q^T w^*$ gets scaled by $\frac{\lambda_i}{\lambda_i+\alpha}$ before it is rotated back by Q

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$$(\Lambda + \alpha\mathbb{I})^{-1} = \begin{bmatrix} \frac{1}{\lambda_1+\alpha} & & & \\ & \frac{1}{\lambda_2+\alpha} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n+\alpha} \end{bmatrix}$$

$$D = (\Lambda + \alpha\mathbb{I})^{-1}\Lambda$$

$$(\Lambda + \alpha\mathbb{I})^{-1}\Lambda = \begin{bmatrix} \frac{\lambda_1}{\lambda_1+\alpha} & & & \\ & \frac{\lambda_2}{\lambda_2+\alpha} & & \\ & & \ddots & \\ & & & \frac{\lambda_n}{\lambda_n+\alpha} \end{bmatrix}$$

- Each element i of $Q^T w^*$ gets scaled by $\frac{\lambda_i}{\lambda_i+\alpha}$ before it is rotated back by Q
- if $\lambda_i >> \alpha$ then $\frac{\lambda_i}{\lambda_i+\alpha} = 1$

$$\widetilde{w} = Q(\Lambda + \alpha\mathbb{I})^{-1}\Lambda Q^T w^*$$
$$= QDQ^T w^*$$

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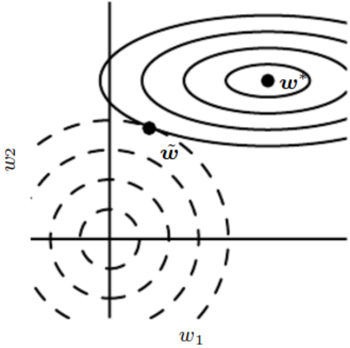
$$(\Lambda + \alpha \mathbb{I})^{-1} = \begin{bmatrix} \frac{1}{\lambda_1+\alpha} & & & \\ & \frac{1}{\lambda_2+\alpha} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n+\alpha} \end{bmatrix}$$

$$D = (\Lambda + \alpha \mathbb{I})^{-1} \Lambda$$

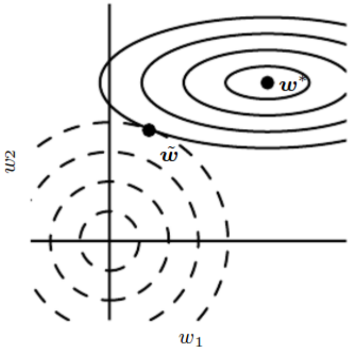
$$(\Lambda + \alpha \mathbb{I})^{-1} \Lambda = \begin{bmatrix} \frac{\lambda_1}{\lambda_1+\alpha} & & & \\ & \frac{\lambda_2}{\lambda_2+\alpha} & & \\ & & \ddots & \\ & & & \frac{\lambda_n}{\lambda_n+\alpha} \end{bmatrix}$$

- Each element i of $Q^T w^*$ gets scaled by $\frac{\lambda_i}{\lambda_i+\alpha}$ before it is rotated back by Q
- if $\lambda_i >> \alpha$ then $\frac{\lambda_i}{\lambda_i+\alpha} = 1$
- if $\lambda_i << \alpha$ then $\frac{\lambda_i}{\lambda_i+\alpha} = 0$
- Thus only significant directions (larger eigen values) will be retained.

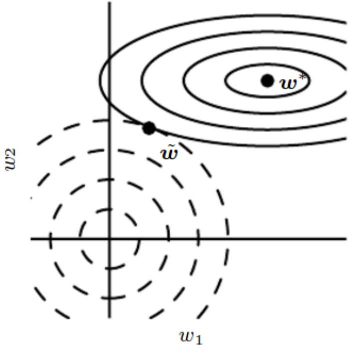
$$\text{Effective parameters} = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \alpha} < n$$



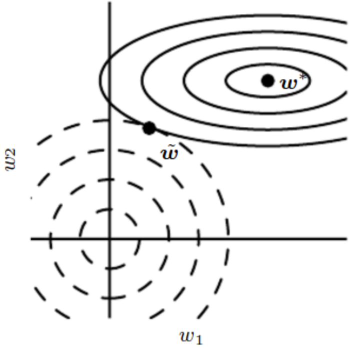
w^*
$\tilde{w}$
w_2
w_1



- The weight vector(w^*) is getting rotated to ($\tilde{w}$)



- The weight vector(w^*) is getting rotated to ($\tilde{w}$)
- All of its elements are shrinking but some are shrinking more than the others



- The weight vector(w^*) is getting rotated to ($\tilde{w}$)
- All of its elements are shrinking but some are shrinking more than the others
- This ensures that only important features are given high weights