MTL100 - Calculus

Tutorial 4

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Question 1. Examine if the limits as $(x, y) \to (0, 0)$ exist?

(a) $\begin{cases} \dfrac{x^3 + y^3}{x^2 - y^2} & x \neq y \\ 0 & x = y. \end{cases}$

(b) $xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right), \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$

(c) $\begin{cases} x \sin \dfrac{1}{x} + y \sin \dfrac{1}{y} & xy \neq 0 \\ 0 & xy = 0. \end{cases}$

(d) $\dfrac{\sin(xy)}{x^2 + y^2}, \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$

Let us recall the following definition:

Definition

Let Ω be a open set in $\mathbb{R}^2$, $(a, b) \in \Omega$ and let f be a real valued function defined on Ω except possibly at (a, b). Then we say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x, y) - L| < \epsilon.$$

If $\lim_{(x,y)\to(a,b)} f(x, y)$ has different values along two different paths then the limit does not exist. (Why?)

Question 1 (a). Examine if the limit as $(x, y) \to (0, 0)$ exist?

$$\begin{cases} \dfrac{x^3 + y^3}{x^2 - y^2} & x \neq y \\ 0 & x = y. \end{cases}$$

Solution:

- Let $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 - y^2} & x \neq y \\ 0 & x = y. \end{cases}$
- Then along the curves $y = mx$; $m \neq 1$, compute

$$f(x, mx) = \frac{x^3 + m^3x^3}{x^2 - m^2x^2} = \frac{x^3(1 + m^3)}{x^2(1 - m^2)} = \frac{x(1 + m^3)}{(1 - m^2)}.$$

- Therefore, the limit along this path is

$$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x(1 + m^3)}{(1 - m^2)} = 0.$$

1 (a) Contd..

- Now, take another path $y = x(x+1)$ and compute

$$f(x, x(x+1)) = \frac{x^3 + x^3(1+x)^3}{x^2 - x^2(1+x)^2} = \frac{x^3(1+(1+x)^3)}{x^2(1-(1+x)^2)}$$
$$= \frac{(1+(1+x)^3)}{-x-2}.$$

- Then

$$\lim_{x\to 0} f(x, x(1+x)) = \lim_{x\to 0} \frac{(1+(1+x)^3)}{-x-2} = -1.$$

- We get two different paths $y = mx$ and $y = x(1+x)$ such that limits are different. Hence at $(0,0)$ limit of the function does not exist.

Question 1 (b). Examine if the limit as $(x, y) \to (0, 0)$ exist?

$$xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right).$$

Solution:

- Let $f(x, y) = xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right)$.
- Take the path $y = x$. Clearly, $\lim\limits_{x \to 0} f(x, x) = \lim\limits_{x \to 0} x^2\left(\frac{x^2 - x^2}{x^2 + x^2}\right) = 0$.
- Hence if the limit of this function at $(0, 0)$ exists then it should be 0.
- Now we will prove the limit to be 0 with the help of definition.

1 (b) Contd..

- We have

$$\left| xy\left(\frac{x^2-y^2}{x^2+y^2}\right) - 0 \right| = |x||y| \left|\frac{x^2-y^2}{x^2+y^2}\right| \leq |x||y|,$$

as $\left|\frac{x^2-y^2}{x^2+y^2}\right| \leq 1.$

- Now $|x|, |y| \leq \sqrt{x^2+y^2}$ implies

$$\left| xy\left(\frac{x^2-y^2}{x^2+y^2}\right) - 0 \right| \leq \sqrt{x^2+y^2} \cdot \sqrt{x^2+y^2}.$$

- Let $\epsilon > 0$. Choosing $\delta = \sqrt{\epsilon}$,

$$\sqrt{x^2+y^2} < \delta \implies \left| xy\left(\frac{x^2-y^2}{x^2+y^2}\right) - 0 \right| < \epsilon.$$

- Hence, $\displaystyle\lim_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) = 0.$

Question 1 (c). Examine if the limit as $(x, y) \to (0, 0)$ exist?

$$\begin{cases} x \sin \dfrac{1}{x} + y \sin \dfrac{1}{y} & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

Solution:

- We have

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} - 0 \right| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2}.$$

- Let $\epsilon > 0$ be arbitrary and $\delta = \dfrac{\epsilon}{2}$.
- Clearly,

$$\sqrt{x^2 + y^2} < \delta \implies \left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

- Hence, the limit of given function at $(0, 0)$ is 0.

Question 1 (d). Examine if the limit as $(x, y) \to (0, 0)$ exist?

$\dfrac{\sin(xy)}{x^2 + y^2}$.

Solution:

- Let $f(x, y) = \dfrac{\sin(xy)}{x^2 + y^2}$ and take the path $y = mx$; $m \in \mathbb{R}$.
- Along this path we have,

$$f(x, mx) = \frac{\sin mx^2}{x^2 + mx^2} = \frac{\sin mx^2}{x^2(1 + m^2)}.$$

- Therefore, the limit along this path is

$$\lim_{x \to 0} \frac{\sin mx^2}{x^2(1 + m^2)} = \lim_{x \to 0} \frac{\sin mx^2}{mx^2} . \frac{m}{1 + m^2} = \frac{m}{1 + m^2}.$$

- Note that the limit depends on the parameter m.
- So, the limit of the given function at $(0, 0)$ does not exist.

Question 2

Examine the continuity of the following functions.

(a) $\begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$

(b) $\begin{cases} x^2+y^2 & x^2+y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

(c) $\begin{cases} \dfrac{\sin^2(x-y)}{|x|+|y|} & (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$

(d) $\begin{cases} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} & (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$

First recall the following definition:

Continuity of a function of two variables

Let f be a real valued function defined in a ball around (a, b). Then f is said to be continuous at a point (a, b) if
1) $f(a, b)$ exists.
2) $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and
3) $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Question 2 (a). Examine the continuity of the following function

$$\begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

- First let us examine the continuity of the given function for all points $(x,y) \neq (0,0)$.
- It is known that the multi-variable polynomials are continuous at every point. Further, As $x^2+y^6 \neq 0$ for every $(x,y) \neq (0,0)$, we have that $\dfrac{xy^3}{x^2+y^6}$ is continuous at all $(x,y) \neq (0,0)$.

Remark: Note that the existence of $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ implies that along all possible paths approaching (a,b), the limit exists and its value is same along all paths and equal to its functional value.

2 (a) Contd.

- Now let us examine the continuity of the given function at $(x, y) = (0, 0)$.
- Consider approaching the point $(0, 0)$ through the path $x = my^3$. Then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{y\to 0} f(my^3, y)$$
$$= \lim_{y\to 0} \frac{my^6}{m^2y^6 + y^6} = \frac{m}{m^2 + 1}.$$

- For different values of m, we get different limits.
- Therefore, limit of f at $(0, 0)$ does not exist and hence f is not continuous at $(0, 0)$.

Question 2 (b). Examine the continuity of the following function

$$\begin{cases} x^2 + y^2 & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

- Note that for all the points (a, b) satisfying $a^2 + b^2 < 1$ (i.e. inside the unit disc), $f(x, y) = x^2 + y^2$ is a polynomial in two variables and so is continuous.
- Note that for all the points (a, b) satisfying $a^2 + b^2 > 1$ (i.e. outside the unit disc), $f(x, y) = x^2 + y^2$ is a constant function and hence continuous.
- Now let us examine the continuity of the function at the points (a, b) with $a^2 + b^2 = 1$.

2 (b) Contd.

- There exists some $\phi \in [0, 2\pi)$ such that $a = \cos\phi$ and $b = \sin\phi$.
- Take the path $x = r\cos\theta$ and $y = r\sin\theta$, where $\theta \in [0, 2\pi)$ and $r < 1$.

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(r,\theta)\to(1^-,\phi)} f(r\cos\theta, r\sin\theta)$$
$$= \lim_{r\to 1^-} r^2 = 1.$$

- Now take the path $x = r\cos\theta$ and $y = r\sin\theta$, where $\theta \in [0, 2\pi)$ and $r > 1$.

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(r,\theta)\to(1^+,\phi)} f(r\cos\theta, r\sin\theta) = 0.$$

- Therefore, the limit doesn't exist. Hence f is not continuous at points (a, b) when $a^2 + b^2 = 1$.

Question 2 (c). Examine the continuity of the following function

$$\begin{cases} \dfrac{\sin^2(x-y)}{|x|+|y|} & (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

- Clearly, the function is continuous at all $(x,y) \neq (0,0) \in \mathbb{R}^2$.
- Now to check its continuity at $(0,0)$, fix any real number $\epsilon > 0$. Then

$$\begin{aligned} |f(x,y) - f(0,0)| &= \left| \frac{\sin^2(x-y)}{|x|+|y|} - 0 \right| \\ &\leq \frac{|x-y|^2}{|x|+|y|} \quad (\because |\sin z| \leq |z|) \\ &\leq |x|+|y| \quad (\because |x-y| \leq |x|+|y|) \\ &\leq 2\sqrt{x^2+y^2} < \epsilon. \end{aligned}$$

2 (c) Contd.

- Hence $|f(x,y) - f(0,0)| < \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$, where $\delta = \dfrac{\epsilon}{2}$.
- So by the definition of limit,

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

- Hence the function is continuous at every $(x,y) \in \mathbb{R}^2$.

Question 2 (d). Examine the continuity of the following function.

$$\begin{cases} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} & (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

- As multi-variable polynomial functions are continuous, f is clearly continuous at every $(x,y) \in \mathbb{R}^2$ except $(0,0)$.
- Consider the limit approaching to $(0,0)$ along two different paths.
- First along the straight line $y = x$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2} = \lim_{x\to 0} \frac{x^4}{x^4} = 1.$$

2 (d) Contd.

- Now along the straight line $y = 2x$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2 + (x-y)^2} = \lim_{x\to 0} \frac{4x^4}{4x^4 + x^2} = 0.$$

- Since we get different limits along different paths, the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist at $(0, 0)$.
- Hence the function is continuous at all $(x, y) \in \mathbb{R}^2$ except $(0, 0)$.

Question 3

Discuss the differentiability of the following functions at $(0,0)$.

(a) $$f(x,y) = \begin{cases} x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0. \end{cases}$$

(b) $$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x = y = 0. \end{cases}$$

(c) $$f(x,y) = \begin{cases} \dfrac{x^6-2y^4}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x = y = 0. \end{cases}$$

Theorem

- Let Ω be an open subset of $\mathbb{R}^2$.
- Let $f(x,y) : \Omega \to \mathbb{R}$ be a function and $(a,b) \in \Omega$.
- The partial derivative of f with respect to x is defined as

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

- The partial derivative of f with respect to y is defined as

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}.$$

- Set $\Delta f = f(a+h,b+k) - f(a,b)$ and $df = hf_x(a,b) + kf_y(a,b)$.

Then f is differentiable at (a,b) if and only if

$$\lim_{\rho\to 0} \frac{\Delta f - df}{\rho} = 0, \text{ where } \rho = \sqrt{h^2+k^2}.$$

Question 3 (a)

Discuss the differentiability of the following function at $(0,0)$:

$$f(x,y) = \begin{cases} x\sin\dfrac{1}{x} + y\sin\dfrac{1}{y} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0. \end{cases}$$

Solution:

- The partial derivative of f with respect to x is

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

- The partial derivative of f with respect to y is

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

- Note that $\Delta f = f(0+h, 0+k) - f(0,0) = f(h,k) = h\sin\dfrac{1}{h} + k\sin\dfrac{1}{k}$ and $df = hf_x(0,0) + kf_y(0,0) = 0$.

- Therefore,

$$\lim_{\rho \to 0} \frac{\Delta f - df}{\rho} = \lim_{(h,k)\to(0,0)} \frac{h \sin \frac{1}{h} + k \sin \frac{1}{k}}{\sqrt{h^2 + k^2}}.$$

- Along the path $k = h$, the above limit reduces to $\lim_{h \to 0} \sqrt{2} \sin \dfrac{1}{h}$.
- Since $\lim_{h \to 0} \sqrt{2} \sin \dfrac{1}{h}$ does not exist (see Tutorial 3), $\lim_{\rho \to 0} \dfrac{\Delta f - df}{\rho}$ does not exist.
- Hence, f is not differentiable at $(0, 0)$.

Question 3 (b)

Discuss the differentiability of the following functions at $(0,0)$:

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } x^2+y^2\neq 0\\ 0 & \text{if } x=y=0.\end{cases}$$

Solution:

- The partial derivative of f with respect to x is

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0.$$

- The partial derivative of f with respect to y is

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0.$$

- Here, $\Delta f=f(0+h,0+k)-f(0,0)=\dfrac{hk}{\sqrt{h^2+k^2}}-0=\dfrac{hk}{\sqrt{h^2+k^2}}$ and $df=hf_x(0,0)+kf_y(0,0)=0.$

3 (b) Contd..

- Therefore,

$$\lim_{\rho \to 0} \frac{\Delta f - df}{\rho} = \lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}\sqrt{h^2+k^2}}.$$

- Along the path $k = h$, the above limit reduces to $\lim\limits_{h\to 0} \dfrac{h^2}{2h^2} = \dfrac{1}{2}$.
- Along the path $k = 2h$, the above limit reduces to $\lim\limits_{h\to 0} \dfrac{2h^2}{5h^2} = \dfrac{2}{5}$.
- Thus,

$$\lim_{\rho \to 0} \frac{\Delta f - df}{\rho} \text{ does not exist.}$$

- Hence, f is not differentiable at $(0, 0)$.

Question 3 (c)

Discuss the differentiability of the following functions at $(0, 0)$:

$$f(x,y) = \begin{cases} \dfrac{x^6 - 2y^4}{x^2 + y^2} & \text{if } x^2 + y^2 \neq 0 \\ 0 & \text{if } x = y = 0. \end{cases}$$

Solution:

- The partial derivative of f with respect to x is

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^6}{h^3} = 0.$$

- The partial derivative of f with respect to y is

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{-2k^4}{k^3} = 0.$$

- Here, $\Delta f = f(0+h, 0+k) - f(0,0) = \dfrac{h^6 - 2k^4}{h^2 + k^2} - 0 = \dfrac{h^6 - 2k^4}{h^2 + k^2}$ and $df = hf_x(0,0) + kf_y(0,0) = 0.$

3 (c) Contd..

- Therefore,

$$\frac{\Delta f - df}{\rho} = \frac{h^6 - 2k^4}{(h^2+k^2)\sqrt{h^2+k^2}} = \frac{h^6 - 2k^4}{(h^2+k^2)^{\frac{3}{2}}}.$$

- Put $h = r\cos\theta, k = r\sin\theta$ such that $\cos\theta \neq 0, \sin\theta \neq 0$ we get

$$\lim_{\rho\to 0} \frac{\Delta f - df}{\rho} = \lim_{r\to 0} \frac{r^6\cos^6\theta - 2r^4\sin^4\theta}{r^3} = 0.$$

- Hence, f is differentiable at $(0,0)$.

Question 4

Let $f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}, y \neq 0$ and $f(x,0) = 0$. Show that f has all directional derivatives at (0,0) but it is not differentiable at (0,0).

Solution:

- Let $\hat{p} = (p_1, p_2)$ be any unit vector.
- Then, by the definition, the directional derivative in the direction $\hat{p}$ at $(0,0)$ is

$$D_{\hat{p}} f(0,0) = \lim_{s \to 0} \frac{f(0+sp_1, 0+sp_2) - f(0,0)}{s}$$
$$= \lim_{s \to 0} \frac{f(sp_1, sp_2) - f(0,0)}{s}.$$

- For $p_2 = 0$, we have $f(sp_1, sp_2) = f(0,0) = 0$ and consequently $D_{\hat{p}} f(0,0) = 0$.

- Assume $p_2 \neq 0$.
- Then we get

$$D_{\hat{p}} f(0,0) = \lim_{s \to 0} \frac{sp_2\sqrt{s^2{p_1}^2 + s^2{p_2}^2} - 0}{|sp_2|s} = \frac{p_2\sqrt{{p_1}^2 + {p_2}^2}}{|p_2|} = \frac{p_2}{|p_2|}.$$

- Therefore, f has all directional derivatives at (0,0).

- Note that,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{1-0}{k} = 1.$$

- Now

$$\lim_{\rho\to 0}\frac{\Delta f - df}{\rho} = \lim_{\rho\to 0}\frac{\dfrac{k}{|k|}\sqrt{h^2+k^2} - 0 - 0h - 1k}{\rho}$$

$$= \lim_{(h,k)\to(0,0)}\frac{\dfrac{k}{|k|}\sqrt{h^2+k^2} - k}{\sqrt{h^2+k^2}}.$$

- Along $h = k$ the above limit is $\lim_{k\to 0}\dfrac{(\sqrt{2}-1)k}{|k|}$, which does not exist.
- Therefore, f is not differentiable at $(0,0)$.

Question 5

Let $f(x, y) = ||x| - |y|| - |x| - |y|$. Is f continuous at $(0, 0)$? Which directional derivatives of f exist at $(0, 0)$? Is f differentiable at $(0, 0)$? Give reasons.

- We have

$$
\begin{aligned}
|f(x,y) - 0| &= |||x| - |y|| - |x| - |y|| \\
&\leq ||x| - |y|| + |x| + |y| \\
&\leq 2(|x| + |y|) \\
&\leq 4\sqrt{x^2 + y^2}.
\end{aligned}
$$

- Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{4}$. Then

$$\sqrt{x^2 + y^2} < \delta \implies |f(x, y) - 0| < \epsilon.$$

- Therefore,

$$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0.$$

- Hence f is continuous at $(0, 0)$.

- Let $\hat{p} = (p_1, p_2)$ be a unit vector.
- We have

$$\begin{aligned}\frac{f(hp_1, hp_2) - f(0,0)}{h} &= \frac{||hp_1| - |hp_2|| - |hp_1| - |hp_2| - 0}{h}\\ &= \frac{|h|}{h}(||p_1| - |p_2|| - |p_1| - |p_2|).\end{aligned}$$

- Note that for any non-zero real number a,

$$\lim_{h\to 0^+} \frac{a|h|}{h} = a \neq -a = \lim_{h\to 0^-} \frac{a|h|}{h}.$$

- Therefore,

$$
\begin{aligned}
&\lim_{h\to 0}\frac{f(hp_1,hp_2)-f(0,0)}{h} \text{ exists}\\
&\Leftrightarrow \lim_{h\to 0}\frac{|h|}{h}(||p_1|-|p_2||-|p_1|-|p_2|) \text{ exists}\\
&\Leftrightarrow ||p_1|-|p_2||-|p_1|-|p_2| = 0\\
&\Leftrightarrow |p_1|+|p_2| = ||p_1|-|p_2||\\
&\Leftrightarrow \text{either } |p_1|+|p_2| = |p_1|-|p_2| \text{ or } |p_1|+|p_2| = -|p_1|+|p_2|\\
&\Leftrightarrow \text{either } |p_2| = 0 \text{ or } |p_1| = 0.
\end{aligned}
$$

- So, the directional derivatives exist only in the direction of $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$.

- The partial derivative of f with respect to x is

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

- The partial derivative of f with respect to y is

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

- Here, $\Delta f = f(0+h, 0+k) - f(0,0) = ||h| - |k|| - |h| - |k|$ and $df = hf_x(0,0) + kf_y(0,0) = 0.$

5 Contd..

- Therefore,
$$\frac{\Delta f - df}{\rho} = \frac{||h| - |k|| - |h| - |k|}{\sqrt{h^2 + k^2}}.$$
- Along $h = k$, $\dfrac{\Delta f - df}{\rho} \to -\sqrt{2}$ as $h \to 0$.
- Hence, f is not differentiable at $(0, 0)$.

Question 6

Let $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, $P = (1,3)$. Find the direction in which $f(x,y)$ is increasing the fastest at P. Find the derivative of $f(x,y)$ in this direction.

Solution:

- We have, $f_x(x,y) = \dfrac{x-y}{x^2+y^2}$ and $f_y(x,y) = \dfrac{x+y}{x^2+y^2}$.
- $\nabla f(x,y)\big|_{(1,3)} = (f_x(1,3), f_y(1,3)) = \left(-\frac{1}{5}, \frac{2}{5}\right)$.
- $f(x,y)$ is increasing the fastest at $P = (1,3)$ in the direction of the gradient, $\dfrac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.
- The derivative at $P = (1,3)$ in the direction $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ is

$$\frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} \cdot \nabla f(x,y)\big|_{(1,3)} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \cdot \left(-\frac{1}{5}, \frac{2}{5}\right) = \frac{1}{\sqrt{5}}.$$

Question 7

A heat-seeking bug is a bug that always moves in the direction of the greatest increase in heat. Discuss the behavior of a heat seeking bug placed at a point $(2, 1)$ on a metal plate heated so that the temperature at (x, y) is given by $T(x, y) = 50y^2e^{-\frac{1}{5}(x^2+y^2)}$.

Solution:

- Since the direction of greatest increase of temperature is in the direction of gradient. We first compute the gradient of temperature function T.
- We have

$$
\begin{aligned}
T_x(x, y) &= 50y^2 e^{\frac{-(x^2+y^2)}{5}} \left(\frac{-2x}{5}\right) \\
\implies T_x(x, y) &= -20xy^2 e^{\frac{-(x^2+y^2)}{5}} \\
\implies T_x(2, 1) &= -\frac{40}{e}.
\end{aligned}
$$

- Now

$$T_y(x,y) = 50\left(2ye^{\frac{-(x^2+y^2)}{5}} - \frac{2y^3}{5}e^{\frac{-(x^2+y^2)}{5}}\right)$$
$$\implies T_y(2,1) = \frac{80}{e}.$$

- $\nabla T(x,y)\big|_{(2,1)} = (T_x(2,1), T_y(2,1)) = \left(-\frac{40}{e}, \frac{80}{e}\right).$
- So the direction of the heat seeking bug is

$$\frac{\nabla T(2,1)}{\|\nabla T(2,1)\|} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$

Question 8

Suppose the gradient vector of the linear function $z = f(x, y)$ is $\nabla z = (5, -12)$. If $f(9, 15) = 17$, what is the value of $f(11, 11)$?

Solution:

- Let the linear function be $z = f(x, y) = ax + by + c$.
- We are given that $\nabla z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = (5, -12)$.
- Hence, $a = \frac{\partial z}{\partial x} = 5$ and $b = \frac{\partial z}{\partial y} = -12$.
- Since $f(9, 15) = 17$, we have that

  $$17 = f(9, 15) = 5(9) - 12(15) + c$$

  and so $c = 152$.
- Thus $f(11, 11) = 5 \times 11 - 12 \times 11 + 152 = 75$.

Question 9

Suppose $f(8,3) = 24$ and $f_x(8,3) = -3.4$, $f_y(8,3) = 4.2$. Estimate the values $f(9,3)$, $f(8,5)$ and $f(9,5)$. Explain how you got your estimates.

Solution: To estimate $f(x,y)$, we use linear approximation. So let us first recall:

Linear approximation

Let Ω be an open subset of $\mathbb{R}^2$ and $(a,b) \in \Omega$. Let $f : \Omega \to \mathbb{R}$ be a function which is differentiable at (a,b). Then the standard linear approximation of $f(x,y)$ at (a,b) is the function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

9 Contd..

- Here $(a, b) = (8, 3)$. Therefore,

$$\begin{aligned} f(9,3) &\approx f(8,3) + f_x(8,3)(9-8) + f_y(8,3)(3-3) \\ &= 24 - 3.4 + 0 \\ &= 20.6 \end{aligned}$$

$$\begin{aligned} f(8,5) &\approx f(8,3) + f_x(8,3)(8-8) + f_y(8,3)(5-3) \\ &= 24 + 0 + 8.4 \\ &= 32.4 \end{aligned}$$

and

$$\begin{aligned} f(9,5) &\approx f(8,3) + f_x(8,3)(9-8) + f_y(8,3)(5-3) \\ &= 24 - 3.4 + 8.4 \\ &= 29. \end{aligned}$$

Question 10

Find the quadratic Taylor's polynomial approximation of $e^{-x^2-2y^2}$ near $(0,0)$.

Taylor's polynomial approximation

Theorem

Let R be an open rectangular region centered at a point (a, b). Suppose $f(x, y)$ and its partial derivatives upto order $n+1$ are continuous throughout R. Then, for all $(c, d) \in R$,

$$f(c,d) = f(a,b) + \frac{1}{1!}\left((c-a)\frac{\partial}{\partial x} + (d-b)\frac{\partial}{\partial y}\right)^1 f\Big|_{(a,b)} + \cdots$$
$$+ \frac{1}{n!}\left((c-a)\frac{\partial}{\partial x} + (d-b)\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} +$$
$$\frac{1}{(n+1)!}\left((c-a)\frac{\partial}{\partial x} + (d-b)\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(u,v)},$$

where (u, v) is a point on the line segment joining (a, b) and (c, d).

10 Contd..

Solution:

- The quadratic Taylor's polynomial approximation of f near $(0,0)$ is

$$\begin{aligned} f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) \\ + \frac{x^2}{2}f_{xx}(0,0) + xyf_{xy}(0,0) + \frac{y^2}{2}f_{yy}(0,0). \end{aligned}$$

- Note that $f_x(x,y) = e^{-x^2-2y^2}(-2x)$, $f_y(x,y) = e^{-x^2-2y^2}(-4y)$, $f_{xx}(x,y) = e^{-x^2-2y^2}(-2+4x^2)$, $f_{xy}(x,y) = e^{-x^2-2y^2}(8xy)$ and $f_{yy}(x,y) = e^{-x^2-2y^2}(-4+16y^2)$.
- Therefore, we have $f_x(0,0) = 0$, $f_y(0,0) = 0$, $f_{xx}(0,0) = -2$, $f_{xy}(0,0) = 0$ and $f_{yy}(0,0) = -4$.
- Hence, the quadratic Taylor's polynomial approximation of f near $(0,0)$ is

$$f(x,y) = 1 - x^2 - 2y^2.$$