Tutorial Sheet:Multivariable Calculus

1. Examine if the limits as $(x,y)\to(0,0)$ exist?

   $(a)\ \begin{cases}\frac{x^3+y^3}{x^2-y^2} & x\neq y\\ 0 & x=y\end{cases}\qquad (b)\ xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right)\qquad (c)\ \begin{cases}x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x} & xy\neq 0\\ 0 & xy=0\end{cases}$

   $(d)\ \dfrac{\sin(xy)}{x^2+y^2}.$

2. Examine the continuity of the following functions.

   $(a)\ \begin{cases}\dfrac{xy^3}{x^2+y^6}\,, & (x,y)\neq(0,0)\\ 0\,, & \text{otherwise.}\end{cases}\qquad (b)\ \begin{cases}x^2+y^2\,, & x^2+y^2\leq 1\\ 0\,, & \text{otherwise.}\end{cases}$

   $(c)\ \begin{cases}\dfrac{\sin^2(x-y)}{|x|+|y|}\,, & (x,y)\neq(0,0)\\ 0\,, & \text{otherwise.}\end{cases}\qquad (d)\ \begin{cases}\dfrac{x^2y^2}{x^2y^2+(x-y)^2}\,, & (x,y)\neq(0,0)\\ 0\,, & \text{otherwise.}\end{cases}$

3. Discuss the differentiability of the following functions at $(0,0)$.

   $(a) f(x,y)=\begin{cases}x\sin\dfrac{1}{x}+y\sin\dfrac{1}{y} & xy\neq 0\\ 0 xy=0\end{cases}\qquad (b)\ \begin{cases}\frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2\neq 0\\ 0 & x=y=0\end{cases}$

   $(c)\ \begin{cases}\frac{x^6-2y^4}{x^2+y^2} & x^2+y^2\neq 0\\ 0 & x=0, y=0\end{cases}$

4. Let $f(x,y)=\dfrac{y}{|y|}\sqrt{x^2+y^2}$, $y\neq 0$ and $f(x,0)=0$. Show that f has all directional derivatives at $(0,0)$ but it is not differentiable at $(0,0)$.

5. Let $f(x,y)=\Big|\,|x|-|y|\,\Big|-|x|-|y|$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$? Give reasons.

6. Let $f(x,y)=\dfrac{1}{2}\ln(x^2+y^2)+tan^{-1}(\dfrac{y}{x})$, $P=(1,3)$. Find the direction in which $f(x,y)$ is increasing the fastest at P. Find the derivative of $f(x,y)$ in this direction.

7. A heat-seeking bug is a bug that always moves in the direction of the greatest increase in heat. Discuss the behavior of a heat seeking bug placed at a point $(2,1)$ on a metal plate heated so that the temperature at (x,y) is given by $T(x,y)=50y^2e^{\frac{-1}{5}(x^2+y^2)}$.

8. Suppose the gradient vector of the linear function $z=f(x,y)$ is $\nabla z=(5,-12)$. If $f(9,15)=17$, what is the value of $f(11,11)$?

9. Suppose $f(8,3)=24$ and $f_x(8,3)=-3.4$, $f_y(8,3)=4.2$. Estimate the values $f(9,3)$, $f(8,5)$, and $f(9,5)$. Explain how you got your estimates.

10. Find the quadratic Taylor's polynomial approximation of $e^{-x^2-2y^2}$ near $(0,0)$.