Shooting Method for Boundary Value Problem

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Introduction

The shooting method is a powerful numerical technique employed to solve boundary value problems (BVPs) for ordinary differential equations (ODEs) by reducing it to a sequence of initial value problems. It involves finding solutions to the initial value problem for different initial conditions until one finds the solution that also satisfies the boundary conditions of the boundary value problem.

Motivation

- **Nonlinear 2nd order ODE (BVP):** Using normal discretization and replacing y'' and y' using terms of y, may give quite complex equation in y to be solved using conventional method of solving n unknowns using matrix.
- **Handling Complex Boundary Conditions:** The shooting method is particularly useful when dealing with boundary value problems (BVPs) that involve complex or non-standard/ nonlinear boundary conditions.
- **Versatility with Nonlinear Problems:** Effectively handles both linear and nonlinear boundary value problems, making it a versatile tool.
- **Efficiency with Stiff Problems:** More effective for stiff differential equations, as after changing problem to IVPs, various methods having high order of convergence rate can be applied.

Requirements

- **Continuity of Coefficients:** The coefficients of the ODE (e.g., $p(x)$, $q(x)$, $r(x)$) must be continuous functions over the interval of interest.
- **Linear Independence of Boundary Conditions:** The boundary conditions should be linearly independent. Linear dependence can lead to an ill-posed problem with multiple solutions or no solution at all.
- **Smoothness of the Solution:** The solution of the problem *y(x)* should at least be triple continuously differentiable. I.e. $y(x) \in C^3$
- **Lipschitz Condition:** For a second-order ODE of the form: $y''(x) = f(x, y(x), y'(x))$, it is required that the function f satisfies a Lipschitz condition in y and y′ over a specific domain.
  This condition can be expressed as: $|f(x,y1,v1)-f(x,y2,v2)| \leq L(|y1-y2|+|v1-v2|)$

Shooting Method Steps

- **Step1: Formulating the BVP**

  Define the second-order ODE along with the specified boundary conditions at both ends of the interval. For eg: O.D.E. :- $y''(x)=f(x,y(x),y'(x))$ &
  boundary conditions:- $a_1y(a)+a_2y'(a)=A$, $b_1y(b)+b_2y'(b)=B$

- **Step2: Converting BVP to sequence of IVP**

  We need to get formulate a sequence of IVPs using the given BVP, which can be mostly done using replacing y'(x) by some another variable and defining IVP on that variable too.

Shooting Method Steps

- **Step3: Getting the required initial conditions**

  For the found IVPs, we need to get required initial conditions at initial point (on one boundary point) to solve those, which are mostly $y(x_0)$ and $y'(x_0)$. This can be achieved through multiple guesses for initial conditions and checking if solution obtained through that satisfies the given boundary conditions within certain tolerance limits.

- **Step4: Approximating the solution**

  After we have found the initial conditions for IVPs, then we can solve those IVPs using any finite difference method with right step size.

  In above two steps, we can parallelize the computation during

  1. Checking of multiple initial guesses
  2. Computation of next step for different IVPs can be done in parallel (if methods are explicit)

Example Problem

The boundary value problem is linear if *f* has the form

$$f(t, y(t), y'(t)) = p(t)y'(t) + q(t)y(t) + r(t).$$

In this case, the solution to the boundary value problem is usually given by:

$$y(t) = y_{(1)}(t) + \frac{y_1 - y_{(1)}(t_1)}{y_{(2)}(t_1)} y_{(2)}(t)$$

where $y_{(1)}(t)$ is the solution to the initial value problem:

$$y''_{(1)}(t) = p(t)y'_{(1)}(t) + q(t)y_{(1)}(t) + r(t), \quad y_{(1)}(t_0) = y_0, \quad y'_{(1)}(t_0) = 0,$$

and $y_{(2)}(t)$ is the solution to the initial value problem:

$$y''_{(2)}(t) = p(t)y'_{(2)}(t) + q(t)y_{(2)}(t), \quad y_{(2)}(t_0) = 0, \quad y'_{(2)}(t_0) = 1.$$

Consistency, Convergence & Stability

Consistency, convergence & stability depend upon the sub methods used for solving BVP, which are:

- to guess the required initial value
- to solve the sequence of IVPs.

So the error and instability is lower bounded by the lesser efficient method of all methods we're using.

Benefits

- **Utilization of Established Initial Value Problem Solvers:** Leverages powerful algorithms designed for solving initial value problems.
- **Handling of Complex Boundary Value Problems:** Can be used to solve efficiently nonlinear ODEs with nonlinear boundary conditions.
- **Natural Convergence Diagnostics:** Provides built-in convergence assessment during the iterative process, ensuring solution accuracy.
- **Parallelization:** Can be implemented in parallel to utilize modern computing resources, potentially leading to faster solutions.

Limitations

- Multiple Initial Guesses: The shooting method's convergence heavily relies on selecting an appropriate initial guess for the unknown derivative. Finding this initial value can be challenging and may require many trials.
- **Limited to One-Dimensional Problems:** The shooting method is primarily designed for one-dimensional problems. Adapting it to multi-dimensional problems can be complex and computationally expensive.
- **Not Suitable for Singular Points:** The shooting method may face difficulties when dealing with problems that involve singular points, as these can lead to numerical instability.
- **Lack of General Convergence Theory:** Unlike some other numerical methods, the shooting method does not have a general convergence theory, which means its convergence behavior may not be guaranteed for all types of problems.

References and Further Examples

- https://en.wikipedia.org/wiki/Shooting_method'
- **An Introduction to Numerical Analysis** -by Endre Süli and David F. Mayers
- M. Kiehl, Parallel multiple shooting for the solution of initial value problems, Parallel Computing, Volume 20, Issue 3, 1994, https://www.sciencedirect.com/science/article/pii/S016781910680013X
- Examples problems that uses shooting method:
    - Stieltjes integral boundary value problems - https://boundaryvalueproblems.springeropen.com/articles/10.1186/s13661-015-0359-8
    - Heat conduction in rod problem - https://www.youtube.com/watch?v=ObWbtg30Vo8
- Example of problem where it fails:
    - plug flow reactor problem - https://www.youtube.com/watch?v=iX3sCtzfaOQ