Algorithmen in der Direkten Anwendung

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Bewegunsgleichungen

$$\rho \frac{D\mathbf{u}}{Dt} = -\, 2\boldsymbol{\Omega} \times \mathbf{u} - \quad \nabla p \qquad + \nabla \cdot \mathbf{T} \quad + \quad \rho \mathbf{g}$$
$$-\, Coriolis - Druckgrad. + Reibung + Gravi. \tag{1}$$

$$\frac{Dm}{Dt} = 0 \tag{2}$$

Approximationen/Manipulationen

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- Horizontale Geschwindigkeiten viel grer als vertikale
  $w = 0$
- vertikal integrieren
  $\int_H \mathbf{u} \, \mathrm{d}z = \mathbf{U}$

Viereckiges flaches Honigbecken

brig bleibt...

$$\rho \frac{D\mathbf{u}}{Dt} = -\, 2\boldsymbol{\Omega} \times \mathbf{u} - \quad \nabla p \qquad + \nabla \cdot \mathbf{T} \quad + \quad \rho \mathbf{g}$$

$$\Downarrow$$

$$0 = -\,(f_0 + \beta y)\,\hat{\mathbf{z}} \times \mathbf{U} - gH\boldsymbol{\nabla} h + \tau_s - \tau_b + A\boldsymbol{\nabla}^2\mathbf{U} \tag{3}$$

$$\frac{D\mathbf{U}}{Dt} = 0 \tag{4}$$

Curl Operation

$$0 = \boldsymbol{\nabla} \times \left[\quad -\mathbf{f} \times \mathbf{U} - gH\boldsymbol{\nabla} h + \tau_s - \tau_b + A\boldsymbol{\nabla}^2\mathbf{U} \quad \right]$$

Curl Operation

$$
\begin{aligned}
0 =& \boldsymbol{\nabla} \times \left[\quad -\mathbf{f}(y) \times \mathbf{U} - gH\boldsymbol{\nabla} h + \tau_s - \tau_b + A\boldsymbol{\nabla}^2\mathbf{U} \quad \right] \\
=& \quad -\beta V + \boldsymbol{\nabla} \times (\boldsymbol{\tau_s} - \boldsymbol{\tau_b}) + A\boldsymbol{\nabla} \times \boldsymbol{\nabla}^2\mathbf{U}
\end{aligned}
$$

Stromfunktion

$$\mathbf{U} = \underset{\neg}{\boldsymbol{\nabla}} \psi(x, y) \tag{5}$$

soll heien...

$$U = -\frac{\partial \psi}{\partial y}$$
$$V = \frac{\partial \psi}{\partial x}$$

Stromfunktion

$$0 = -\beta V + \boldsymbol{\nabla} \times (\boldsymbol{\tau}_\mathbf{s} - \boldsymbol{\tau}_\mathbf{b}) + A\boldsymbol{\nabla} \times \boldsymbol{\nabla}^2\mathbf{U}$$

$$\Downarrow$$

$$0 = -\beta\frac{\partial\psi}{\partial x} + \boldsymbol{\nabla} \times (\boldsymbol{\tau}_\mathbf{s} - \boldsymbol{\tau}_\mathbf{b}) + A\boldsymbol{\nabla}^4\psi \tag{6}$$

weitere Approximationen

- Bodenreibung proportional zu $\mathbf{U}$
  $\boldsymbol{\tau}_{\mathbf{b}} = R\mathbf{U}$

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weitere Approximationen

- Bodenreibung proportional zu **U**
  $\boldsymbol{\tau_b} = R\mathbf{U}$
- keine laterale Reibung!
  $A\boldsymbol{\nabla}^4\psi = 0$
- Windstress sinusoidale Funktion von y
  $\boldsymbol{\tau_s} = F\cos(\pi y/B)$

weitere Approximationen

$$0 = -\beta\frac{\partial\psi}{\partial x} + \boldsymbol{\nabla}\times(\boldsymbol{\tau}_{\mathbf{s}} - \boldsymbol{\tau}_{\mathbf{b}}) + A\boldsymbol{\nabla}^4\psi$$

$$\Downarrow$$

$$\begin{aligned} 0 &= -\beta\frac{\partial\psi}{\partial x} - R\boldsymbol{\nabla}\times\mathbf{U} - F\boldsymbol{\nabla}\times\cos(\pi y/B) \\ &= -\beta\frac{\partial\psi}{\partial x} - R\boldsymbol{\nabla}^2\psi + \frac{F\pi}{B}\sin(\pi y/B) \\ R\boldsymbol{\nabla}^2\psi + \beta\frac{\partial\psi}{\partial x} &= F\alpha\sin(\alpha y) \end{aligned} \tag{7}$$

...mit $\alpha = \frac{\pi}{B}$

Analytische Lsung mglich

Inhomogene partielle Differentialgleichung 2er Ordnung.

$$R\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = F\alpha\sin(\alpha y)$$

Numerische Lsung

- variable Beckenform
- variabler Windstress

Finite Differenzen

TODO bild aus BSC gitter!

Finite Differenzen

Beispiel $\frac{\partial\psi(x)}{\partial x}$ durch Taylorreihe diskretisiert:

$$\psi(x+\delta x) = \psi(x) + \frac{\delta x}{1!}\frac{\partial\psi}{\partial x} + \frac{\delta x^2}{2!}\frac{\partial^2\psi}{\partial x^2} + \dots \tag{8}$$

$$\psi(x-\delta x) = \psi(x) - \frac{\delta x}{1!}\frac{\partial\psi}{\partial x} + \frac{\delta x^2}{2!}\frac{\partial^2\psi}{\partial x^2} + \dots \tag{9}$$

Finite Differenzen

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(8) - (9):

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi(x + \delta x) - \psi(x - \delta x)}{2\delta x}$$

Hhere Ordnungen

Beispiel: $\boldsymbol{\nabla}^2$ an der Stelle $(0,0)$

$$\boldsymbol{\nabla}^2\psi(0,0) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$
$$\approx \frac{\psi(1,0) - 2\psi(0,0) + \psi(-1,0)}{\delta x^2} + \frac{\psi(0,1) - 2\psi(0,0) + \psi(0,-1)}{\delta y^2}$$

Numerische Lsung

$$R\mathbf{\nabla}^2\psi + \beta\frac{\partial\psi}{\partial x} = W(x,y)$$

...erst mal ohne $\beta\frac{\partial\psi}{\partial x}$

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mit Randbedingung (willkrlich)

$$\psi_{bndry} = 0$$

Jacobi-Methode

first guess: $\psi^{k=0} = random$

$$\boldsymbol{\nabla}^2 \psi^k_{i,j} = W_{i,j}$$

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k:

$$\nabla^2 \psi^k_{i,j} - W_{i,j} = \Phi_{i,j} \tag{10}$$

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$k+1$:

$$\nabla^2 \psi^{k+1}_{i,j} - W_{i,j} = 0 \tag{11}$$

Jacobi-Methode

(11) - (10):

$$\nabla^2\psi_{i,j}^{k+1} - \nabla^2\psi_{i,j}^{k} = -\Phi_{i,j} \tag{12}$$

Jacobi-Methode

einsetzen :

$$\frac{\psi^{k}_{i+1,j} - 2\psi^{k+1}_{i,j} + \psi^{k}_{i-1,j}}{\delta x^2} - \frac{\psi^{k}_{i+1,j} - 2\psi^{k}_{i,j} + \psi^{k}_{i-1,j}}{\delta x^2} + term_y = -\Phi_{i,j}$$

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einsetzen :

$$\frac{\psi^{k}_{i+1,j} - 2\psi^{k+1}_{i,j} + \psi^{k}_{i-1,j}}{\delta x^2} - \frac{\psi^{k}_{i+1,j} - 2\psi^{k}_{i,j} + \psi^{k}_{i-1,j}}{\delta x^2} + term_y = -\Phi_{i,j}$$

$$\psi^{k+1}_{i,j} = \frac{\delta x}{2}\Phi_{i,j} + \psi^{k}_{i,j}$$

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