Algorithmen in der Direkten Anwendung

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Bewegunsgleichungen

$$\rho \frac{D\mathbf{u}}{Dt} = - 2\boldsymbol{\Omega} \times \mathbf{u} - \nabla p + \nabla \cdot \mathbf{T} + \rho \mathbf{g}$$
$$- \mathit{Coriolis} - \mathit{Druckgrad.} + \mathit{Reibung} + \mathit{Gravi.} \quad (1)$$

$$\frac{Dm}{Dt} = 0 \quad (2)$$

Approximationen/Manipulationen

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  $w = 0$
- vertikal integrieren
  $\int_H \mathbf{u} \, \mathrm{d}z = \mathbf{U}$

Viereckiges flaches Honigbecken

brig bleibt...

$$\rho \frac{D\mathbf{u}}{Dt} = - 2\mathbf{\Omega} \times \mathbf{u} - \quad \nabla p \qquad + \nabla \cdot \mathbf{T} \quad + \quad \rho \mathbf{g}$$

$$\Downarrow$$

$$0 = -(f_0 + \beta y)\,\hat{\mathbf{z}} \times \mathbf{U} - gH\boldsymbol{\nabla} h + \tau_s - \tau_b + A\boldsymbol{\nabla}^2\mathbf{U} \tag{3}$$

$$\frac{D\mathbf{U}}{Dt} = 0 \tag{4}$$

Curl Operation

$$0 = \nabla \times \left[\quad -\mathbf{f} \times \mathbf{U} - gH\nabla h + \tau_s - \tau_b + A\nabla^2\mathbf{U} \quad \right]$$

Curl Operation

$$
\begin{aligned}
0 = & \boldsymbol{\nabla} \times \left[\quad -\mathbf{f}(y) \times \mathbf{U} - gH\boldsymbol{\nabla} h + \tau_s - \tau_b + A\boldsymbol{\nabla}^2\mathbf{U} \quad \right] \\
= & \quad - \beta V + \boldsymbol{\nabla} \times (\boldsymbol{\tau}_{\mathbf{s}} - \boldsymbol{\tau}_{\mathbf{b}}) + A\boldsymbol{\nabla} \times \boldsymbol{\nabla}^2\mathbf{U}
\end{aligned}
$$

Stromfunktion

$$\mathbf{U} = \underset{\neg}{\boldsymbol{\nabla}} \psi(x, y) \tag{5}$$

soll heien...

$$\begin{aligned} U &= -\frac{\partial \psi}{\partial y} \\ V &= \quad \frac{\partial \psi}{\partial x} \end{aligned}$$

Stromfunktion

$$0 = -\beta V + \boldsymbol{\nabla} \times (\boldsymbol{\tau}_{\mathbf{s}} - \boldsymbol{\tau}_{\mathbf{b}}) + A\boldsymbol{\nabla} \times \boldsymbol{\nabla}^2 \mathbf{U}$$

$$\Downarrow$$

$$0 = -\beta \frac{\partial \psi}{\partial x} + \boldsymbol{\nabla} \times (\boldsymbol{\tau}_{\mathbf{s}} - \boldsymbol{\tau}_{\mathbf{b}}) + A\boldsymbol{\nabla}^4 \psi \tag{6}$$

weitere Approximationen

- Bodenreibung proportional zu $\mathbf{U}$
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- Bodenreibung proportional zu $\mathbf{U}$
  $\boldsymbol{\tau}_\mathbf{b} = R\mathbf{U}$
- keine laterale Reibung!
  $A\boldsymbol{\nabla}^4\psi = 0$
- Windstress sinusoidale Funktion von y
  $\boldsymbol{\tau}_\mathbf{s} = F\cos\left(\pi y/B\right)$

weitere Approximationen

$$0 = -\beta\frac{\partial\psi}{\partial x} + \boldsymbol{\nabla}\times(\boldsymbol{\tau_s} - \boldsymbol{\tau_b}) + A\boldsymbol{\nabla}^4\psi$$

$$\Downarrow$$

$$\begin{aligned} 0 =& -\beta\frac{\partial\psi}{\partial x} - R\boldsymbol{\nabla}\times\mathbf{U} - F\boldsymbol{\nabla}\times\cos(\pi y/B) \\ =& -\beta\frac{\partial\psi}{\partial x} - R\boldsymbol{\nabla}^2\psi + \frac{F\pi}{B}\sin(\pi y/B) \\ R\boldsymbol{\nabla}^2\psi + \beta\frac{\partial\psi}{\partial x} =& F\alpha\sin(\alpha y) \end{aligned} \tag{7}$$

...mit $\alpha = \frac{\pi}{B}$

Analytische Lsung mglich

Inhomogene partielle Differentialgleichung 2er Ordnung.

$$R\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = F\alpha\sin(\alpha y)$$

Numerische Lsung

- variable Beckenform
- variabler Windstress

Finite Differenzen

TODO bild aus BSC gitter!

Finite Differenzen

Beispiel $\frac{\partial\psi(x)}{\partial x}$ durch Taylorreihe diskretisiert:

$$\psi(x+\delta x) = \psi(x) + \frac{\delta x}{1!}\frac{\partial\psi}{\partial x} + \frac{\delta x^2}{2!}\frac{\partial^2\psi}{\partial x^2} + \dots \tag{8}$$

$$\psi(x-\delta x) = \psi(x) - \frac{\delta x}{1!}\frac{\partial\psi}{\partial x} + \frac{\delta x^2}{2!}\frac{\partial^2\psi}{\partial x^2} + \dots \tag{9}$$

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(8) - (9):

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi(x+\delta x) - \psi(x-\delta x)}{2\delta x}$$

Hhere Ordnungen

Beispiel: $\boldsymbol{\nabla}^2$ an der Stelle $(0, 0)$

$$\boldsymbol{\nabla}^2\psi(0,0) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$
$$\approx \frac{\psi(1,0) - 2\psi(0,0) + \psi(-1,0)}{\delta x^2} + \frac{\psi(0,1) - 2\psi(0,0) + \psi(0,-1)}{\delta y^2}$$

Numerische Lsung

$$R\boldsymbol{\nabla}^2\psi + \beta\frac{\partial\psi}{\partial x} = W(x,y)$$

...erst mal ohne $\beta\frac{\partial\psi}{\partial x}$

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mit Randbedingung (willkrlich)

$$\psi_{bndry} = 0$$

Jacobi-Methode

first guess: $\psi^{k=0} = random$

$$\nabla^2 \psi^k_{i,j} = W_{i,j}$$

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$$\nabla^2 \psi^k_{i,j} - W_{i,j} = \Phi_{i,j} \tag{10}$$

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$k+1$:

$$\nabla^2 \psi^{k+1}_{i,j} - W_{i,j} = 0 \tag{11}$$

Jacobi-Methode

(11) - (10):

$$\nabla^2\psi_{i,j}^{k+1} - \nabla^2\psi_{i,j}^{k} = -\Phi_{i,j} \tag{12}$$

Jacobi-Methode

einsetzen :

$$
\begin{aligned}
\frac{\psi^k_{i+1,j} - 2\psi^{k+1}_{i,j} + \psi^k_{i-1,j}}{\delta x^2} - \frac{\psi^k_{i+1,j} - 2\psi^k_{i,j} + \psi^k_{i-1,j}}{\delta x^2} + \ldots &= -\Phi_{i,j} \\
\frac{-2\psi^{k+1}_{i,j} + 2\psi^k_{i,j}}{\delta x^2} + \frac{-2\psi^{k+1}_{i,j} + 2\psi^k_{i,j}}{\delta y^2} &= -\Phi_{i,j} \\
2\delta y^2 \left(-\psi^{k+1}_{i,j} + \psi^k_{i,j}\right) + 2\delta x^2 \left(-\psi^{k+1}_{i,j} + \psi^k_{i,j}\right) &= -\Phi_{i,j}\delta y^2 \delta x^2 \\
2\left(\delta y^2 + \delta x^2\right)\left(-\psi^{k+1}_{i,j} + \psi^k_{i,j}\right) &= -\Phi_{i,j}\delta y^2 \delta x^2 \\
\psi^{k+1}_{i,j} &= \psi^k_{i,j} + \Phi_{i,j}\frac{\delta y^2 \delta x^2}{2\left(\delta y^2 + \delta x^2\right)}
\end{aligned}
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