Applied Linear Algebra in Data Analysis: Course Notes

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LIST OF DEFINITIONS

Preface

0.1 Features of this template

TeX, stylized within the system as LaTeX, is a typesetting system which was designed and written by Donald Knuth and first released in 1978. TeX is a popular means of typesetting complex mathematical formulae; it has been noted as one of the most sophisticated digital typographical systems.

- Wikipedia

0.1.1 crossref

different styles of clickable definitions and theorems

- nameref: **??**
- autoref: **??**, algorithm 1.9.1
- cref: **??**,
- hyperref: Gaussian,

0.1.2 ToC (Table of Content)

- mini toc of sections at the beginning of each chapter
- list of theorems, definitions, figures
- the chapter titles are bi-directional linked

0.1.3 header and footer

fancyhdr

- right header: section name and link to the beginning of the section
- left header: chapter title and link to the beginning of the chapter
- footer: page number linked to ToC of the whole document

0.1.4 bib

- titles of reference is linked to the publisher webpage e.g., [Kit+02]
- backref (go to the page where the reference is cited) e.g., [Chi09]
- customized video entry in reference like in [Bab16]

0.1.5 preface, index, quote (epigraph) and appendix

index page at the end of this document...

0.1.6 symbol and glossary (abbreviation)

examples: $\mathbb{R}$, SVM, $\vec{v}$

usage

- glossary package

```
pdflatex notes_template.tex
makeglossaries notes_template
pdflatex notes_template.tex
```

- glossary-extra package and bib2gls

```
pdflatex notes_template.tex
bib2gls notes_template
pdflatex notes_template.tex
```

0.2 Related Tools

0.2.1 VSCode

Extension: Latex Workshop by James Yu

settings

0.2.2 lualatex and latexmk

.latexmkrc configuration file

```
$pdflatex␣=␣'lualatex␣-synctex=1␣-interaction=nonstopmode␣--shell-escape␣%O␣%S';
@generated_exts␣=␣(@generated_exts,␣'synctex.gz');
$pdf_mode␣=␣1;

add_cus_dep('glo',␣'gls',␣0,␣'makeglo2gls');
sub␣makeglo2gls␣{
system("makeindex␣-s␣'$_[0]'.ist␣-t␣'$_[0]'.glg␣-o␣'$_[0]'.gls␣'$_[0]'.glo");
}
```

To explain ....

```
# Also delete the *.glstex files from package glossaries-extra. Problem is,
# that that package generates files of the form "basename-digit.glstex" if
# multiple glossaries are present. Latexmk looks for "basename.glstex" and so
# does not find those. For that purpose, use wildcard.
$clean_ext = "%R-*.glstex";

push @generated_exts, 'glstex', 'glg';

add_cus_dep('aux', 'glstex', 0, 'run_bib2gls');

# PERL subroutine. $_[0] is the argument (filename in this case).
# File from author from here: https://tex.stackexchange.com/a/401979/120853
sub run_bib2gls {
    if ( $silent ) {
    #    my $ret = system "bib2gls --silent --group '$_[0]'"; # Original version
        my $ret = system "bib2gls --silent --group $_[0]"; # Runs in PowerShell
    } else {
    #    my $ret = system "bib2gls --group '$_[0]'"; # Original version
        my $ret = system "bib2gls --group $_[0]"; # Runs in PowerShell
    };

    my ($base, $path) = fileparse( $_[0] );
    if ($path && -e "$base.glstex") {
        rename "$base.glstex", "$path$base.glstex";
    }

    # Analyze log file.
    local *LOG;
    $LOG = "$_[0].glg";
    if (!$ret && -e $LOG) {
        open LOG, "<$LOG";
    while (<LOG>) {
            if (/^Reading (.*\.bib)\s$/) {
        rdb_ensure_file( $rule, $1 );
        }
    }
    close LOG;
    }
    return $ret;
}
```

0.3 Copyright and License

- GitHub Repo: https://github.com/Jue-Xu/Latex-Template-for-Scientific-Style-Book

- Overleaf template: https://www.overleaf.com/latex/templates/latex-template-for-scientific-styl
ntprxjksmqxx

Part I

Linear Algebra

Chapter 1

Vectors

1.1 n-Vectors

A collection of an ordered list of n numbers is called an n-vector. We will use bold lower case alphabets to represent such vectors, and we will represent these as a column of numbers, which is referred to as a *column vector*. We will look at *row vectors* at a later stage. Consider the following example:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The elements of the n-vector $x_1, x_2, \ldots, x_n$ are called the *components* of the vector $\mathbf{x}$; x_i is the i^{th} component of the vector $\mathbf{x}$. If these components are all real numbers, the set of all such n-vectors is the set $\mathbb{R}^n$.

Where do we come across such n-vectors? In many places, such as in physics, engineering, economics, medicine, etc. Any application where we deal with multiple pieces of information that can be represented as a list of numbers can be represented as an n-vector. When we deal with systems with multiple inputs, multiple output, or multiple states, we can represent these as n-vectors. We talk about the state of a system in a later chapter.

1.2 Visualizing n-vectors

The n-vectors can be visualized as points in n-dimensional space. For example, A 1-vector or just single real number or a *scalar* can be thought of as a point on the real line. The 1-vector $x = 2.45$ is shown in Figure 1.1 is the red point. But we will find it useful to visualize a 1-vector as an arrow starting at the origin and ending at the point on the real line. The arrow is shown in blue in Figure 1.1.

The elements of $\mathbb{R}^2$ are points on the plane, and we can visualize them as points in the plane. The 2-vectors $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ are shown in Figure 1.2a. A similar visualization is shown for $\mathbb{R}^3$

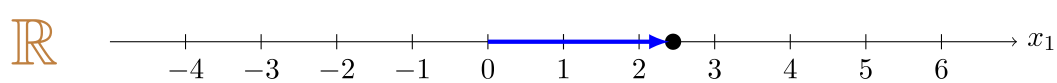
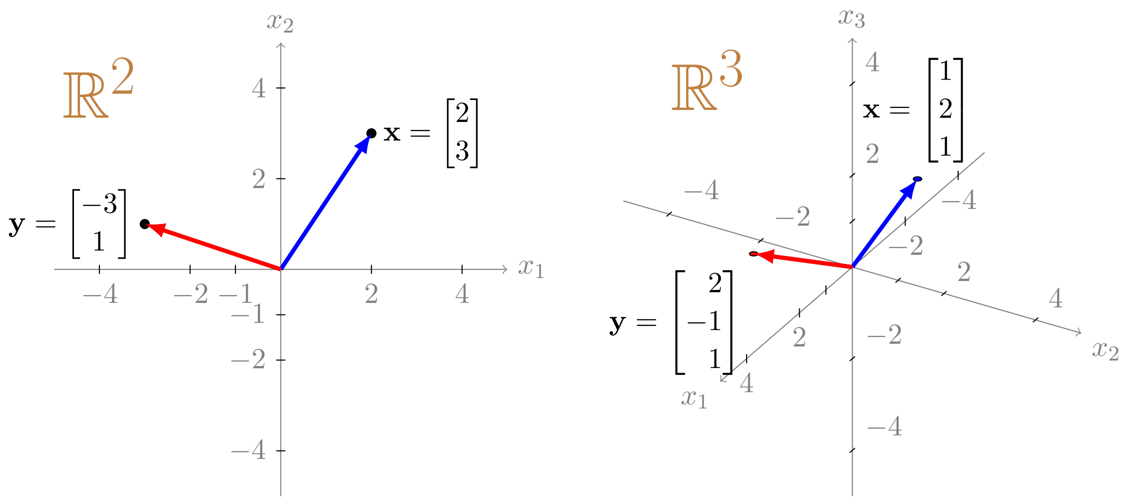


Figure 1.1: The real line $\mathbb{R}$ contains the 1-vectors.



(a) The $\mathbb{R}^2$ plane contains the 2-vectors.

(b) $\mathbb{R}^3$ contains the 3-vectors.

Figure 1.2: The $\mathbb{R}^2$ and $\mathbb{R}^3$ sets.

(Figure 1.2b), and for $\mathbb{R}^4$ and beyond you simply pretend that you can visualize things in your head like your instructor does.

1.3 Some Commonly Used n-vectors

We will now define a some commonly used n-vectors that we will use in the course.

- **Zero vector:** The n-vector whose components are all zeros is called the *zero vector*. $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

- **One vector:** The n-vector whose components are all ones is called the *one vector*. $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

- **Unit vectors:** The n-vectors whose components are all zeros except for one component which is 1. These are called the *standard basis vectors* and are denoted by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The n-vector $\mathbf{e}_i$ has all components as zeros except for the i^{th} component which is 1. For example, the unit vectors in $\mathbb{R}^2$ are:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1.4 Operations on n-vectors

There are many operations we can perform on n-vectors, but we will only focus on two operations for this:

- **Scalar multiplication:** Given a scalar $c \in \mathbb{R}$ and an n-vector $\mathbf{x}$. The scalar multiplication operation produces another n-vector $c\mathbf{x}$ whose components are $cx_1, cx_2, \ldots, cx_n$.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow 2\mathbf{x} = \begin{bmatrix} 2\,(1) \\ 2\,(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 8.2 \end{bmatrix}$$

- **Vector Addition:** Given two n-vectors $\mathbf{x}$ and $\mathbf{y}$, the vector addition operation, represented by $\mathbf{x} + \mathbf{y}$, producs another n-vector whose components are $x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n$.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow \mathbf{x} + \mathbf{y} = \begin{bmatrix} 1+2 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

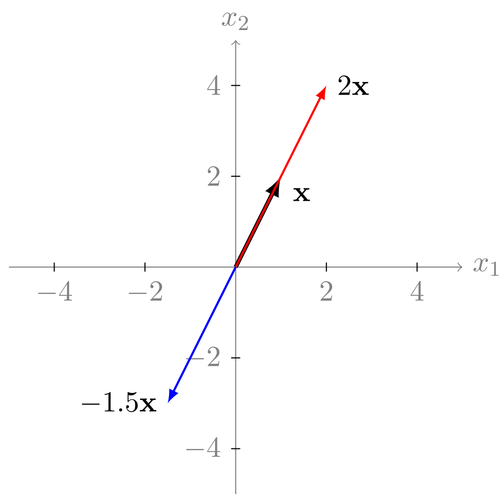


Figure 1.3: Scalar multiplication of a vector.

The geometric interpretation of these operations is shown in Figure 1.4 and Figure **??**. Scalar multiplication stretches or shrinks the vector without rotating the vector. When the scalar is positive the direction of the scaled vector is the same as the original vector, and when the scalar is negative the direction is opposite. When the scalar is zero, the scaled vector is the zero vector $\mathbf{0}$.

Vector addition moves the vector to a new location without changing its direction. The vector $\mathbf{x} + \mathbf{y}$ is the vector that starts at the origin and ends at the point where the vector $\mathbf{x}$ ends and the vector $\mathbf{y}$ ends.

Vector addition moves the vector to a new location without changing its direction. The vector $\mathbf{x} + \mathbf{y}$ is the vector that starts at the origin and ends at the point where the vector $\mathbf{x}$ ends and the vector $\mathbf{y}$ ends.

gls example

- Greatest Common Divisor (GCD); Greatest Common Divisor; GCD; Greatest Common Divisor (GCD)

1.5 Proof

Lemma 1.1.

Claim 1.1.

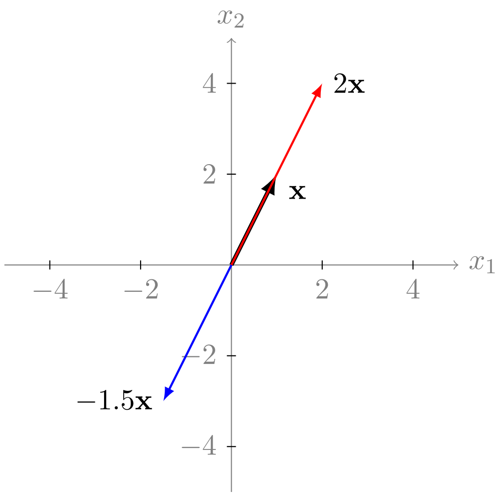


Figure 1.4: Scalar multiplication of a vector.

Theorem 1.1.

Example 1.1.

Fact 1.1.

Remark 1.1.

Exercise 1.1. Prove A iff B

Solution. By induction: □

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1.6 Quantifier

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1.7 Graph

“Graph Isomorphism in Quasipolynomial Time” [Bab16]

1.8 Number theory

Figure example

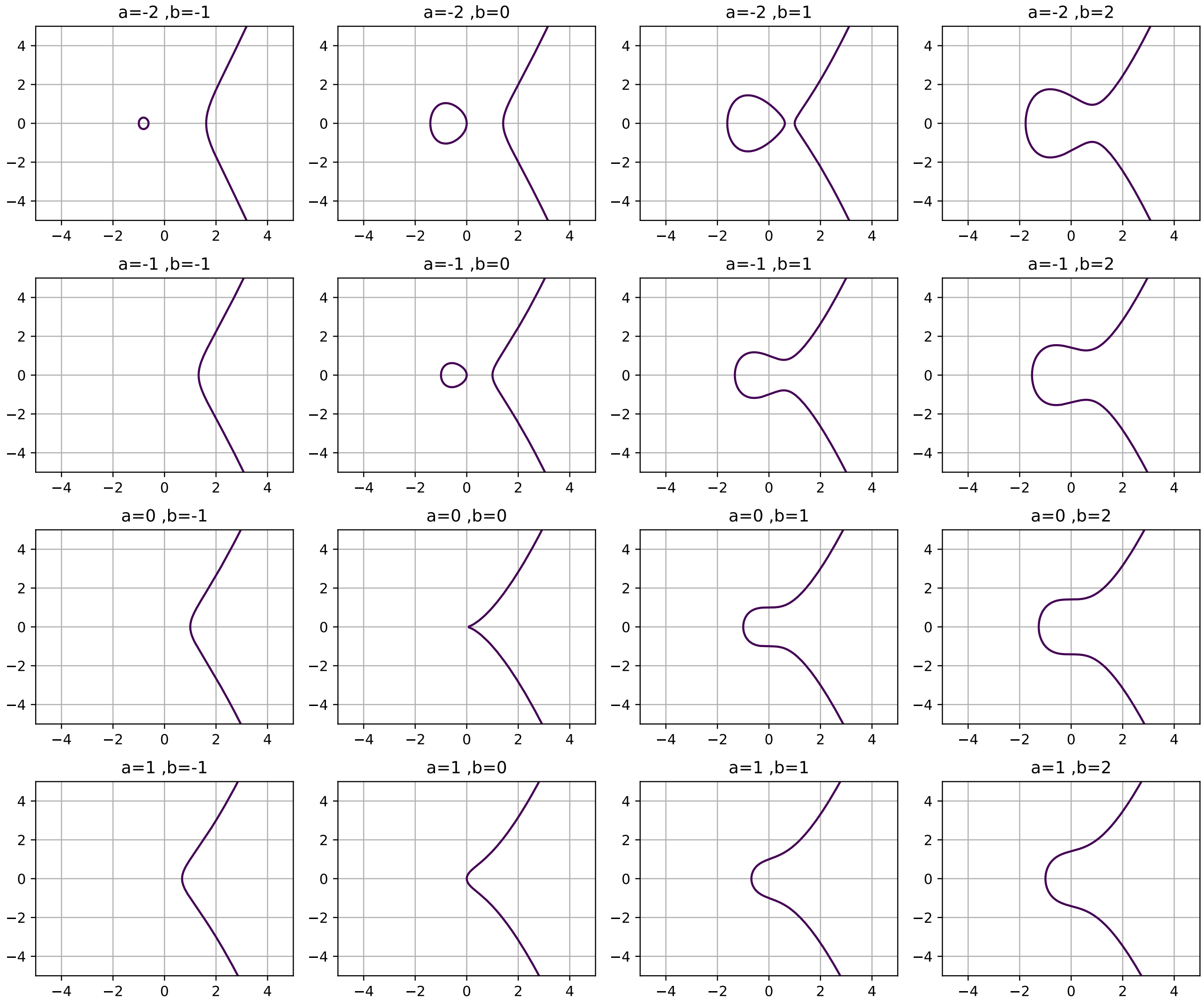

Figure 1.5: Elliptic curves [Chi09]

1.9 Algorithm

Algorithm 1.9.1: Primality testing - first attempt

input : Integer N and parameter 1^t

output: A decision as to whether N is prime or composite

```
for i = 1, 2, ..., t do
    a ← {1, ..., N_1};
    if a^(N−1) ≠ 1 mod N then
        return "composite"
return "prime"
```

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BIBLIOGRAPHY

Alphabetical Index