Applied Linear Algebra in Data Analysis: Course Notes

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Update on June 12, 2024

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LIST OF DEFINITIONS

Preface

0.1 Features of this template

> *TeX, stylized within the system as LaTeX, is a typesetting system which was designed and written by Donald Knuth and first released in 1978. TeX is a popular means of typesetting complex mathematical formulae; it has been noted as one of the most sophisticated digital typographical systems.*
>
> \- Wikipedia

0.1.1 crossref

different styles of clickable definitions and theorems

- nameref: **??**
- autoref: **??**, **??**
- cref: **??**,
- hyperref: Gaussian,

0.1.2 ToC (Table of Content)

- mini toc of sections at the beginning of each chapter
- list of theorems, definitions, figures
- the chapter titles are bi-directional linked

0.1.3 header and footer

fancyhdr

- right header: section name and link to the beginning of the section
- left header: chapter title and link to the beginning of the chapter
- footer: page number linked to ToC of the whole document

0.1.4 bib

- titles of reference is linked to the publisher webpage e.g., [Kit+02]
- backref (go to the page where the reference is cited) e.g., [Chi09]
- customized video entry in reference like in [Bab16]

0.1.5 preface, index, quote (epigraph) and appendix

index page at the end of this document...

0.1.6 symbol and glossary (abbreviation)

examples: $\mathbb{R}$, SVM, $\vec{v}$

usage

- glossary package

  ```
  pdflatex notes_template.tex
  makeglossaries notes_template
  pdflatex notes_template.tex
  ```

- glossary-extra package and bib2gls

  ```
  pdflatex notes_template.tex
  bib2gls notes_template
  pdflatex notes_template.tex
  ```

0.2 Related Tools

0.2.1 VSCode

Extension: Latex Workshop by James Yu

settings

0.2.2 lualatex and latexmk

.latexmkrc configuration file

```
$pdflatex␣=␣'lualatex␣-synctex=1␣-interaction=nonstopmode␣--shell-escape␣%O␣%S';
@generated_exts␣=␣(@generated_exts,␣'synctex.gz');
$pdf_mode␣=␣1;

add_cus_dep('glo',␣'gls',␣0,␣'makeglo2gls');
sub␣makeglo2gls␣{
system("makeindex␣-s␣'$_[0]'.ist␣-t␣'$_[0]'.glg␣-o␣'$_[0]'.gls␣'$_[0]'.glo");
}
```

To explain ....

```
# Also delete the *.glstex files from package glossaries-extra. Problem is,
# that that package generates files of the form "basename-digit.glstex" if
# multiple glossaries are present. Latexmk looks for "basename.glstex" and so
# does not find those. For that purpose, use wildcard.
$clean_ext = "%R-*.glstex";

push @generated_exts, 'glstex', 'glg';

add_cus_dep('aux', 'glstex', 0, 'run_bib2gls');

# PERL subroutine. $_[0] is the argument (filename in this case).
# File from author from here: https://tex.stackexchange.com/a/401979/120853
sub run_bib2gls {
    if ( $silent ) {
    #    my $ret = system "bib2gls --silent --group '$_[0]'"; # Original version
        my $ret = system "bib2gls --silent --group $_[0]"; # Runs in PowerShell
    } else {
    #    my $ret = system "bib2gls --group '$_[0]'"; # Original version
        my $ret = system "bib2gls --group $_[0]"; # Runs in PowerShell
    };

    my ($base, $path) = fileparse( $_[0] );
    if ($path && -e "$base.glstex") {
        rename "$base.glstex", "$path$base.glstex";
    }

    # Analyze log file.
    local *LOG;
    $LOG = "$_[0].glg";
    if (!$ret && -e $LOG) {
        open LOG, "<$LOG";
    while (<LOG>) {
            if (/^Reading (.*\.bib)\s$/) {
        rdb_ensure_file( $rule, $1 );
        }
    }
    close LOG;
    }
    return $ret;
}
```

0.3 Copyright and License

- GitHub Repo: https://github.com/Jue-Xu/Latex-Template-for-Scientific-Style-Book

- Overleaf template: https://www.overleaf.com/latex/templates/latex-template-for-scientific-styl
ntprxjksmqxx

Part I

Linear Algebra

Chapter 1

Vectors

1.1 n-Vectors

A collection of an ordered list of n numbers is called an n-vector. We will use bold lower case alphabets to represent such vectors, and we will represent these as a column of numbers, which is referred to as a *column vector*. We will look at *row vectors* at a later stage. Consider the following example:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The elements of the n-vector $x_1, x_2, \ldots, x_n$ are called the *components* of the vector $\mathbf{x}$; x_i is the i^{th} component of the vector $\mathbf{x}$. If these components are all real numbers, the set of all such n-vectors is the set $\mathbb{R}^n$.

Where do we come across such n-vectors? In many places, such as in physics, engineering, economics, medicine, etc. Any application where we deal with multiple pieces of information that can be represented as a list of numbers can be represented as an n-vector. When we deal with systems with multiple inputs, multiple output, or multiple states, we can represent these as n-vectors. We talk about the state of a system in a later chapter.

1.2 Visualizing n-vectors

The n-vectors can be visualized as points in n-dimensional space. For example, A 1-vector or just single real number or a *scalar* can be thought of as a point on the real line. The 1-vector $x = 2.45$ is shown in Figure 1.1 is the red point. But we will find it useful to visualize a 1-vector as an arrow starting at the origin and ending at the point on the real line. The arrow is shown in blue in Figure 1.1.

The elements of $\mathbb{R}^2$ are points on the plane, and we can visualize them as points in the plane. The 2-vectors $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ are shown in Figure 1.2(a). A similar visualization is shown for $\mathbb{R}^3$

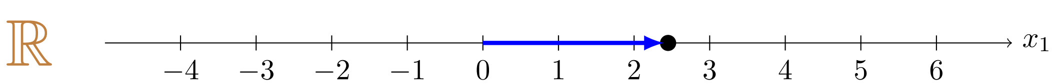
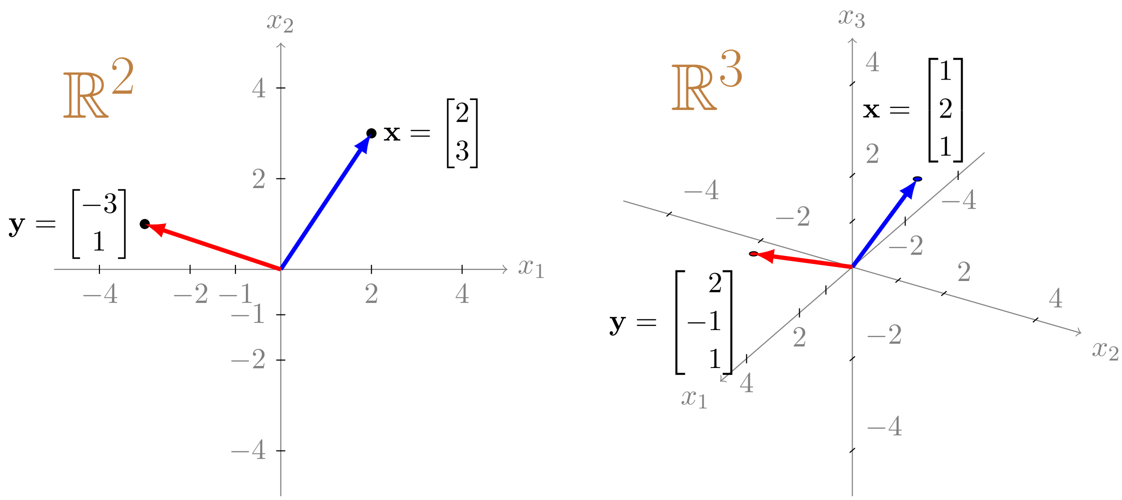


Figure 1.1: The real line $\mathbb{R}$ contains the 1-vectors.



((a)) The $\mathbb{R}^2$ plane contains the 2-vectors.

((b)) $\mathbb{R}^3$ contains the 3-vectors.

Figure 1.2: The $\mathbb{R}^2$ and $\mathbb{R}^3$ sets.

(Figure 1.2(b)), and for $\mathbb{R}^4$ and beyond you simply pretend that you can visualize things in your head like your instructor does.

1.3 Some Commonly Used n-vectors

We will now define a some commonly used n-vectors that we will use in the course.

- **Zero vector:** The n-vector whose components are all zeros is called the *zero vector*. $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

- **One vector:** The n-vector whose components are all ones is called the *one vector*. $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

- **Unit vectors:** The n-vectors whose components are all zeros except for one component which is 1. These are called the *standard basis vectors* and are denoted by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The n-vector $\mathbf{e}_i$ has all components as zeros except for the i^{th} component which is 1. For example, the unit vectors in $\mathbb{R}^2$ are:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1.4 Operations on n-vectors

There are many operations we can perform on n-vectors, but we will only focus on two operations for this:

- **Scalar multiplication:** Given a scalar $c \in \mathbb{R}$ and an n-vector $\mathbf{x}$. The scalar multiplication operation produces another n-vector $c\mathbf{x}$ whose components are $cx_1, cx_2, \ldots, cx_n$.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow 2\mathbf{x} = \begin{bmatrix} 2\,(1) \\ 2\,(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 8.2 \end{bmatrix}$$

The geometric interpretation scalar multiplication is shown in Figure 1.3. Scalar multiplication stretches or shrinks the vector without rotating it. When the scalar is positive the direction of the scaled vector is the same as the original vector, and when the scalar is negative the direction is opposite. When the scalar is zero, the scaled vector is the zero vector $\mathbf{0}$.

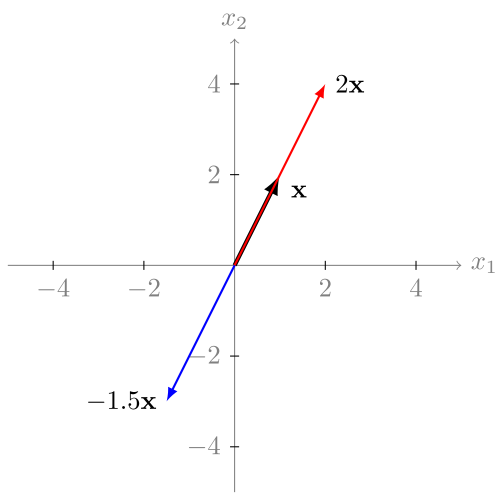


Figure 1.3: Scalar multiplication of a vector.

- **Vector Addition:** Given two n-vectors $\mathbf{x}$ and $\mathbf{y}$, the vector addition operation, represented by $\mathbf{x}+\mathbf{y}$, producs another n-vector whose components are $x_1+y_1, x_2+y_2, \ldots, x_n+y_n$.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow \mathbf{x}+\mathbf{y} = \begin{bmatrix} 1+2 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

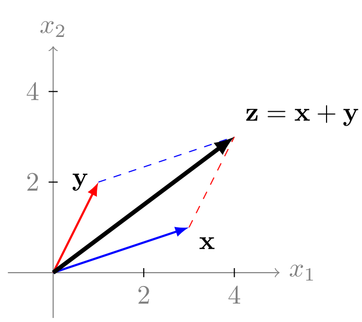


Figure 1.4: Vector addition.

The geometric interpretation the vector addition operation is shown in Figure 1.4. Geometrically, the vector addition operation follows the parallelogram law of addition, where the resulting vector $\mathbf{x}+\mathbf{y}$ is a diagonal of the parallelogram formed by the two vectors $\mathbf{x}$ and $\mathbf{y}$. Another way to think about this, is that you first move along $\mathbf{x}$ to its end point, and starting from there then move along $\mathbf{y}$ to its end point or vice versa.

You can add more than two vectors to obtain a new vector, like below:

$$\mathbf{w} = \mathbf{x} + \mathbf{y} + \mathbf{z}$$

Geometrically, we can first apply the parallelogram law to $\mathbf{x}$ and $\mathbf{y}$, and then apply the parallelogram law to $\mathbf{x} + \mathbf{y}$ and $\mathbf{z}$ to get $\mathbf{w}$.

1.5 Vector spaces

Vector spaces are *sets* with some special properites. More specifically, a vector space is a set V of elements called *vectors* that are closed under two operations called *addition* and *scalar multiplication*. This simply means that if you perfom these operations using elements from the set V, the result is also an element of the set V. A vector space must satisfy the following properties:

- **Closure under addition:** For any two vectors $\mathbf{x}, \mathbf{y} \in V$, the sum $\mathbf{x} + \mathbf{y} \in V$.
- **Closure under scalar multiplication:** For any scalar $c \in \mathbb{R}$ and any vector $\mathbf{x} \in V$, the product $c\mathbf{x} \in V$.
- **Additive identity:** There exists a vector $\mathbf{0} \in V$ such that for any vector $\mathbf{x} \in V$, $\mathbf{x} + \mathbf{0} = \mathbf{x}$.
- **Additive inverse:** For any vector $\mathbf{x} \in V$, there exists a vector $-\mathbf{x} \in V$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
- **Commutativity of addition:** For any two vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
- **Associativity of addition:** For any three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
- **Distributive property:** For any scalar $c \in \mathbb{R}$ and any two vectors $\mathbf{x}, \mathbf{y} \in V$, $c(\mathbf{x}+\mathbf{y}) = c\mathbf{x}+c\mathbf{y}$.
- **Distributive property:** For any two scalars $c, d \in \mathbb{R}$ and any vector $\mathbf{x} \in V$, $(c+d)\mathbf{x} = c\mathbf{x}+d\mathbf{x}$.
- **Associativity of scalar multiplication:** For any two scalars $c, d \in \mathbb{R}$ and any vector $\mathbf{x} \in V$, $(cd)\mathbf{x} = c(d\mathbf{x})$.
- **Multiplicative identity:** For any vector $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$.

These properties are satisfied by the set $\mathbb{R}^n$ of n-vectors, and hence $\mathbb{R}^n$ is a vector space. Geometrically, the concept of a vector space corresponds to flat spaces that contain the origin. This will become more clear when we talk about subspaces. Notice that definition of the vector space given above does not make any specific mention of n-vectors. The definition is general and can be applied to any set of elements that satisfy the properties listed above. The following are some examples of vector spaces with the addition and scalar multiplication operations defined on them.

Example 1.1. Set of $m \times n$ matrices. The set M of all $m \times n$ matrices of real numbers is a vector space.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}, \ a_{ij} \in \mathbb{R}$$

We define scalar multiplication and addition of matrices as follows:

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{11} & ca_{12} & \cdots & ca_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{11} & ca_{12} & \cdots & ca_{1n} \end{bmatrix}, \; c \in \mathbb{R}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \end{bmatrix}, \; \mathbf{A}, \mathbf{B} \in M$$

Since each element of $c\mathbf{A}$ and $\mathbf{A} + \mathbf{B}$ is a real number, M is a vector space.

Example 1.2. Set of polynomials of order $\leq n$. Now we look at strange example of a vector space. The set P_n of all polynomials of degree at most n with real coefficients, defined over an interval $[a, b]$.

$$p(x) = \sum_{k=0}^{n-1} a_k x^k, \; x \in [a, b], \, a_k \in \mathbb{R}$$

The set P_n contains all polynomials of the form shown above. We define scalar multiplication and addition of polynomials as follows:

$$cp(x) = c \sum_{k=0}^{n-1} a_k x^k = \sum_{k=0}^{n-1} c a_k x^k, \; p(x) \in P$$

$$p(x) + q(x) = \sum_{k=0}^{n-1} a_k x^k + \sum_{k=0}^{n-1} b_k x^k = \sum_{k=0}^{n-1} (a_k + b_k) x^k, \; p(x), q(x) \in P_n$$

The set P_n is a vector space because the sum and product of any two polynomials from P_n is also a polynomial of degree at most n with real coefficients.

Example 1.3. Set of continuous functions. The set $C[0, 1]$ of all continuous functions $f(x)$ over the time interval $x \in [0, 1]$ is a vector space. We define scalar multiplication and addition of functions as follows:

$$cf(x) = cf(x), \; f(x) \in C(0, 1)$$

$$f(x) + g(x) = f(x) + g(x), \; f(x), g(x) \in C(0, 1)$$

The set $C(0, 1)$ is a vector space because the sum and product of any two continuous functions from $C(0, 1)$ is also a continuous function.

1.6 Subspaces – "Little" Vector Spaces

These are little subspaces in the sense that they are subsets of a larger vector space that are themselves vector spaces. More formally, a subspace U of a vector space V is a subset of V that is itself a vector space. The subspace U of the vector space V must satisfy the following properties:

- **Closure under addition:** For any two vectors $\mathbf{x}, \mathbf{y} \in U$, the sum $\mathbf{x} + \mathbf{y} \in U$.

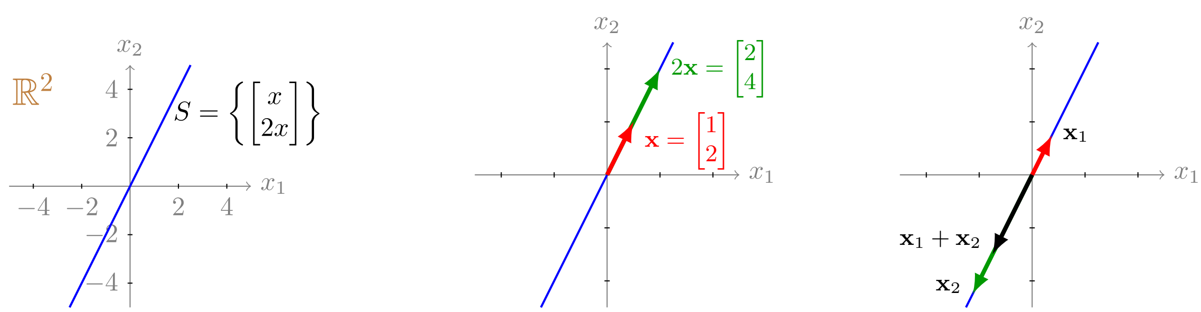
- **Closure under scalar multiplication:** For any scalar $c \in \mathbb{R}$ and any vector $\mathbf{x} \in U$, the product $c\mathbf{x} \in U$.

One immediate consequence of the above definition is that hte zero element of the vector space V must be present in every subspace of V. If the zero element is not in a subset, then it cannot be a subspace. Geometrically subspace capture the idea of flat spaces that contain the origin, and extend infinitely. Let's look at some examples of subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$, which are easier to visualize.

Example 1.4. A straight line through the origin. We know that $\mathbb{R}^2$ is a vector space. Now consider the set of all points in $\mathbb{R}^2$ that lie on a straight line passing through the origin, defined as follows:

$$S = \left\{ \mathbf{x} \,:\, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2,\, x_1 = m \cdot x_2,\, m \in \mathbb{R} \right\}$$

How do we verfiy this is a subspace of $\mathbb{R}^2$? The definition above shows that any x in S comes from $\mathbb{R}^2$, which means its a subset of $\mathbb{R}^2$. Figure 1.5(a) shows the set S for $m = 2$.



((a)) A subspace of $\mathbb{R}^2$. **((b))** Vector scaling. **((c))** Vector addition.

Figure 1.5: Example of a subspace of $\mathbb{R}$. (a) Shows the set of all points in $\mathbb{R}^2$ corresponding to the subset $S = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \right\} \subset \mathbb{R}^2$. (b) Shows that the set S is closed under scalar multiplication. Take any vecotr from the line, and scale it and it remains on that blue line. (c) Shows that S is closed under vector addition. If we take any two vectors from the blue line and add them, the resulting vector remains in the blue line.

How do we verify if S is a subspace of $\mathbb{R}^2$? We need to now verify that S satisfies the properties of a vector space.

1. First, let's check if S contains the zero vector. If it does not contain the zero vector, then it cannot be a subspace. The elements from S are of the form $\begin{bmatrix} x \\ mx \end{bmatrix}$, thus if we choose $x = 0$, then we get $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$. So, S contains the zero vector. This means that S can be a subspace space of $\mathbb{R}^2$.

2. Let's verify vector scaling. Scaling the element $\begin{bmatrix} x \\ mx \end{bmatrix} \in S$ by a scalar c we get,

$$c \begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} cx \\ cmx \end{bmatrix} = \begin{bmatrix} cx \\ m\,(cx) \end{bmatrix} = \begin{bmatrix} y \\ my \end{bmatrix}, \quad \text{where } y = cx \in \mathbb{R}$$

This means that $c \begin{bmatrix} x \\ mx \end{bmatrix}$ belongs to S, this the set S is closed under scalar multiplication. This still means that S can be a subspace of $\mathbb{R}^2$.

3. Let's verify vector addition. Adding two elements $\begin{bmatrix} x_1 \\ mx_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ mx_2 \end{bmatrix} \in S$ we get,

$$\begin{bmatrix} x_1 \\ mx_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ mx_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ mx_1 + mx_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ my_1 \end{bmatrix}, \quad \text{where } y_1 = x_1 + x_2 \in \mathbb{R}$$

This means that $\begin{bmatrix} y_1 \\ my_1 \end{bmatrix}$ belongs to S, this the set S is closed under vector addition. This means that S is a subspace of $\mathbb{R}^2$.

Since, the subset S is closed under both vector addition and scalar multiplication, it is a subspace of $\mathbb{R}^2$.

Example 1.5. A straight line not through the origin. Consider the set of all points in $\mathbb{R}^2$ of the following form:

$$S = \left\{ \mathbf{x} \,:\, x = \begin{bmatrix} x \\ mx + c \end{bmatrix} \in \mathbb{R}^2,\, m, c \in \mathbb{R} \right\}$$

This is shown in the Figure 1.6(a).

How do we verfiy this is a subspace of $\mathbb{R}^2$? The definition above shows that any x in S comes from $\mathbb{R}^2$, which means its a subset of $\mathbb{R}^2$. Figure 1.6(a) shows the set S for $m = -\frac{1}{2}$ and $c = 1$. How do we verify if S is a subspace of $\mathbb{R}^2$? We need to now verify that S satisfies the properties of a vector space.

1. First, let's check if S contains the zero vector. If it does not contain the zero vector, then it cannot be a subspace. The elements from S are of the form $\begin{bmatrix} x \\ mx + c \end{bmatrix}$, thus if we choose $x = 0$, then we get $\begin{bmatrix} 0 \\ c \end{bmatrix} \in S$. So, S does not contain the zero vector, which implies that S is not a subspace space of $\mathbb{R}^2$. We need not check the other two conditions; but we will test them just to see which of these two fails.

2. Scaling the element $\begin{bmatrix} x \\ mx + c \end{bmatrix} \in S$ by a scalar d we get,

$$d \begin{bmatrix} x \\ mx + c \end{bmatrix} = \begin{bmatrix} dx \\ dmx + dc \end{bmatrix} = \begin{bmatrix} dx \\ m\left(dx\right) + dc \end{bmatrix} \neq \begin{bmatrix} y \\ my + c \end{bmatrix}, \quad \text{where } y = dx \in \mathbb{R}$$

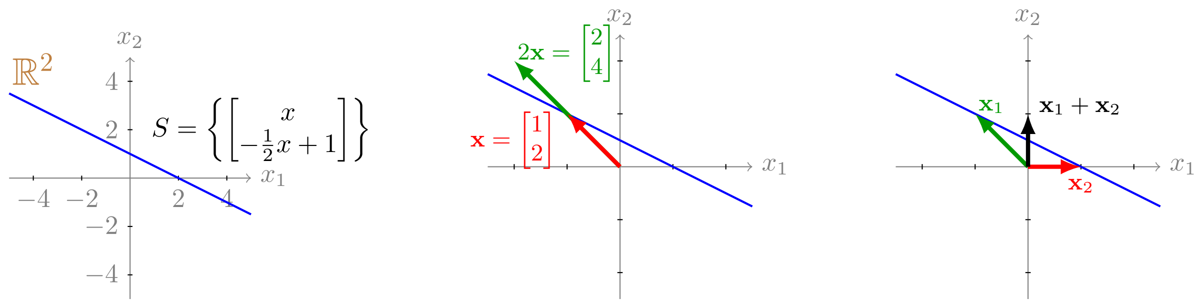
This means that $d \begin{bmatrix} x \\ mx + c \end{bmatrix} \notin S$. Thus, the set S is closed under scalar multiplication. Another confirmation that it is not a subspace. This can be seen in Figure 1.6(b), which shows that when we choose an elment from $\mathbf{x}$ (red arrow) from S (blue line), the scaled version of this vector leaves the set S, i.e., the tip of the green arrow does not stay on the blue line.

3. Let's verify vector addition. Adding two elements $\begin{bmatrix} x_1 \\ mx_1 + c \end{bmatrix}, \begin{bmatrix} x_2 \\ mx_2 + c \end{bmatrix} \in S$ we get,

$$\begin{bmatrix} x_1 \\ mx_1 + c \end{bmatrix} + \begin{bmatrix} x_2 \\ mx_2 + c \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ mx_1 + mx_2 + 2c \end{bmatrix} \neq \begin{bmatrix} y_1 \\ my_1 + c \end{bmatrix}, \quad \text{where } y_1 = x_1 + x_2 \in \mathbb{R}$$

This means that $\begin{bmatrix} x_1 \\ mx_1 + c \end{bmatrix} + \begin{bmatrix} x_2 \\ mx_2 + c \end{bmatrix} \notin S$. Thus the set S is closed under vector addition. We see this geometrically in Figure (c), where the sum of two vectors in S does not stay in the set S. Even though the tips of the green and red arrow are on the blue line, the tip of the black arrow is not on the blue line.

Since, the subset S is closed under both vector addition and scalar multiplication, it is a subspace of $\mathbb{R}^2$.



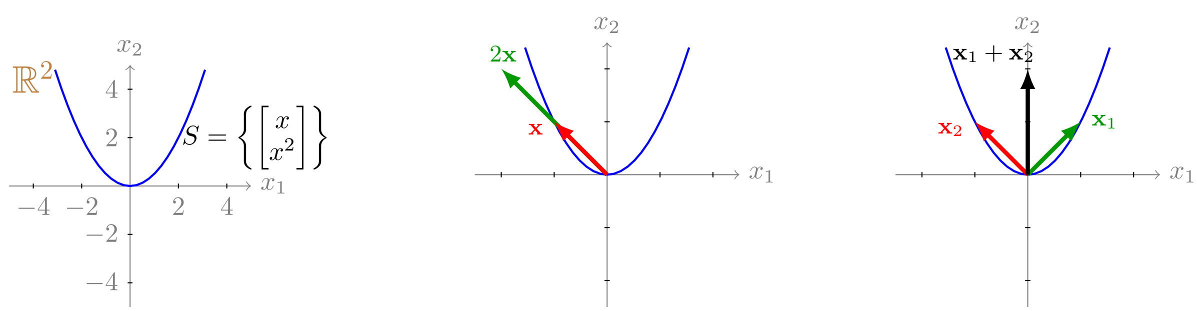
((a)) Not a subspace of $\mathbb{R}^2$. **((b))** Vector scaling. **((c))** Vector addition.

Figure 1.6: Example of a subspace of $\mathbb{R}$. (a) Shows the set of all points in $\mathbb{R}^2$ corresponding to the subset $S = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \right\} \subset \mathbb{R}^2$. (b) Shows that the set S is closed under scalar multiplication. Take any vecotr from the line, and scale it and it remains on that blue line. (c) Shows that S is closed under vector addition. If we take any two vectors from the blue line and add them, the resulting vector remains in the blue line.

Example 1.6. A parabola through the origin. Consider the set of all points in $\mathbb{R}^2$ of the following form:

$$S = \left\{ \mathbf{x} \,:\, x = \begin{bmatrix} x \\ \frac{1}{2}x^2 \end{bmatrix} \in \mathbb{R}^2,\, m, c \in \mathbb{R} \right\}$$

This is not a subspace of $\mathbb{R}^2$. This is geometrically depicted in Figure 1.7(a), Figure 1.7(b) and Figure 1.7(c). You are encouraged to verify this algebraically by checking the properties of a vector space.



((a)) Not a subspace of $\mathbb{R}^2$. **((b))** Vector scaling. **((c))** Vector addition.

Figure 1.7: Example of a subspace of $\mathbb{R}$. (a) Shows the set of all points in $\mathbb{R}^2$ corresponding to the subset $S = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \right\} \subset \mathbb{R}^2$. (b) Shows that the set S is closed under scalar multiplication. Take any vecotr from the line, and scale it and it remains on that blue line. (c) Shows that S is closed under vector addition. If we take any two vectors from the blue line and add them, the resulting vector remains in the blue line.

1.7 Linear combinations and others

Linear combination is an *algebraic operation* performed on a set of vectors. We can combine the two fundamental operations on vectors into a single operation called the *linear combination* of a set of vectors. Given a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$, the linear combination of the vectors is given by:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m \in \mathbb{R}^n \tag{1.1}$$

Notice that the linear cominations of single vector $\mathbf{v}_1$ are simply different scaled versions of the vector $c_1\mathbf{v}_1$. Linear cominations are the bread-and-butter of linear algebra and we will encounter them again and again. An informal way to think of a linear combination of a set of vectors as process of mixing the set of vectors together with the corresponding scalar c_i determining the amount of a vector in the mixture. There are other types of combinations of vectors, which we will not discuss further in this book.

- **Affine combination:** $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m, \quad \sum_{i=1}^m c_i = 1$
- **Convex combinations:** $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m, \quad c_i \geq 0,\ \sum_{i=1}^m c_i = 1$
- **Conic combinations:** $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m, \quad c_i \geq 0$

1.8 Linear independece of a set of vectors

Linear independece is a *property* of a set of vectors; a set of vector is either linearly independent or linearly dependent. The concept of linear independence is easy to understand the but the algebraic condition for independence can seem a bit unintuitive. A set of vectors is said to be linearly independent if no vector in the set can be expressed as a linear combination of the other vectors in the set.

More formally, a set of vectors $V = \{\mathbf{v}_i\}_{i=1}^m$ is said to be linearly independent if and only if the only way to produce the zero vector $\mathbf{0}$ through the linear combination of the set V is by setting all the scalars to zero, i.e.,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m = \mathbf{0} \quad \text{if and only if} \quad c_1 = c_2 = \cdots = c_m = 0 \tag{1.2}$$

To understand this better, let's assume that the set V is linear dependent and let's assume that the vector $\mathbf{v}_m$ can be represented as the linear combination of the vectors $\mathbf{v}_1, \mathbf{v} - 2, \cdots \mathbf{v}_{m-1}$. This means that there exist a set of scalar $\alpha_i,\ 1 \leq i \leq m-1$, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{m-1}\mathbf{v}_{m-1} = \mathbf{v}_m$$

Multiplying both sides by a scalar $c_m \neq 0$ we get,

$$\begin{aligned} c_m\alpha_1\mathbf{v}_1 + c_m\alpha_2\mathbf{v}_2 + \cdots + c_m\alpha_{m-1}\mathbf{v}_{m-1} &= c_m\mathbf{v}_m \\ \implies c_m\alpha_1\mathbf{v}_1 + c_m\alpha_2\mathbf{v}_2 + \cdots + c_m\alpha_{m-1}\mathbf{v}_{m-1} - c_m\mathbf{v}_m &= \mathbf{0} \end{aligned}$$

This implies that there exist a set of scalar $c_i = c_m\alpha_i,\ 1 \leq i \leq m-1$, and c_m such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$, where not all c_i are zero. So when a set is linearly dependent, then there are scalars c_i, not all zero, such that the linear combination of the vectors from V with these scalars produces the zero vector.

Now, let's assume that the set V is linearly independent, that is no vector in the set V can be expressed as a linear combination of other vectors in that set. And let's assume that there are scalars c_i, not all zero, such that the linear combination of the vectors from V with these scalars produces the zero vector, i.e.,

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m &= \mathbf{0} \\ \implies \frac{c_1}{c_m}\mathbf{v}_1 + \frac{c_2}{c_m}\mathbf{v}_2 + \cdots + \frac{c_{m-1}}{c_m}\mathbf{v}_{m-1} &= \mathbf{v}_m,\ c_m \neq 0 \end{aligned}$$

But this a contradiction because we have just expressed $\mathbf{v}_m$ is expressed as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_{m-1}$.

1.9 Span of a set of vectors

So, linear combinations of a set of vectors $V = \{\mathbf{v}_i\}_{i=1}^m$ is a way of generating new vectors not in that set. All we need to do is choose a random set of real numbers $\{c_i\}_{i=1}^m$, and "mix" these vectors from the set using these as weights. What all can we generate by taking all possible linear combinations of the set V? Clearly there are infinite number of vectors we could generate through this process, and we can put them all together in a set. And this set has a name – the *span* of the set V. The span of a set of vectors $V = \{\mathbf{v}_i\}_{i=1}^m$ is denoted by $\text{span}(V)$ and is defined as:

$$\text{span}(V) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m \,:\, c_i \in \mathbb{R}\} \tag{1.3}$$

It turns out that the set of all possible linear combinations of the set V is a subspace of $\mathbb{R}^n$ called the *span* of the set V. The span of a set of vectors $V = \{\mathbf{v}_i\}_{i=1}^m$ is denoted by $\text{span}(V)$ and is defined as:

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