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A new mathematical model and a Lagrangean decomposition for the point-feature cartographic label placement problem

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ABSTRACT

This paper proposes a 0-1 integer linear programming model for the point-feature cartographic label placement problem based on labeling of the largest number of free labels. In addition, one non-trivial valid inequality is presented to strengthen this proposed model. Even with the strengthened model, a commercial solver was not able to solve a representative sample of known instances presented in the literature. Thus, we also present a Lagrangean decomposition technique based on graph partitioning. Our added approaches established optimal solutions for practically all the used instances and the results significantly improved the ones presented in recent studies concerning the problem.

1. Introduction

Label placement problems appear in several situations such as cartographic maps or in medical image analysis [1]. They can be defined by: given a set of graphical features, we must define positions for their labels such that each feature is uniquely identified. However, to label each feature, we can have an explicit enumerated list of candidate positions where one is selected (discrete approach) or we can slide the label around the feature until finding the best position possible (slider approach). In both cases, the label must be placed adjacent to the feature avoiding overlaps.

Graphical features can be points, lines or polygons. In a map, cities can be represented by points, roadways by lines and states by polygons (areas). There are different approaches to delineate each component's individual trait or quality.

A major problem in map labeling is point-feature cartographic label placement (PFCLP) [2]. A good review about approaches for problems with lines or areas can be found at the "map-labeling bibliography web site" [3], where we can see an illustrative chart of map labeling publications over the last 50 years.

In the PFCLP the overlapping labels may be accepted or not. When overlaps are not accepted, we may attempt to either label a maximum number of points or determine the largest possible font size such that all points can be labeled. These problems are known as label number maximization problem (LNMP) and label size maximization problem (LSMP), respectively [1,2].

Certain limitations that need to be addressed relate to the LNMP's inability to apply labels completely when presented with a conflict. This problem is generally represented in a conflict graph where each node represents a candidate position for a label and each edge a potential conflict (overlap) between two candidate positions. Now, considering its objective (label number maximization), this problem can be seen as the traditional maximum independent set problem (MISP) [4,5].

When overlaps are acceptable, all points must be labeled and scaling is not allowed. Two problems are identified: the maximum number of conflict free labels problem (MNCFLP) and the minimum number of conflicts problem (MNCP). The MNCFLP [6] is also known as the label overlap minimization problem [7] and the number of labels obstructed by at least one other label [8]. The MNCP was recently presented by Ribeiro and Lorena [6,9] and this approach "spread" the overlaps to minimize conflicts (edges) between candidate positions.

In this paper, we concentrate on the MNCFLP with discrete positions, presenting a new 0-1 optimization model for the MNCFLP. To the best of our knowledge, this is the first model using this approach. In addition, one non-trivial valid inequality is presented to strengthen this proposed model. Even with this strengthened model, commercial solvers have difficulties in solving the MNCFLP large-scale instances available in the literature. We also present a Lagrangean decomposition that has generated feasible solutions, and has outperformed recent results reported in the literature.

The remainder of the paper is organized as follows; Section 2 presents a brief review of PFCLP approaches, Sections 3 and 4 describe the proposed model and the Lagrangean decomposition are described in, computational results are reported in Section 5 and the conclusions are summarized in Section 6.

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2. Literature review of the PFCLP with discrete candidate positions

The PFCLP is an optimization problem shown to be NP-hard [10,11]. Exact approaches are limited to solve only small instances [4,5], therefore heuristics and metaheuristics have been proposed.

Many approaches, exact or not, have their strategies of solution based on conflict graphs. Let N be the number of points to be labeled and P_i a set of discrete positions for the label of point i (candidate positions). A conflict graph for the PFCLP can be defined by $G=(V,E)$, where $V=\{v_{1,p_1^1}, v_{1,p_2^1}, \ldots, v_{1,p_{|P_1|}^1}, \ldots, v_{N,p_1^N}, v_{N,p_2^N}, \ldots, v_{N,p_{|P_N|}^N}\}$ is a set of nodes (all candidate positions) and $E=\{(v_{i,j}, v_{t,u}): v_{i,j} \text{ and } v_{t,u} \in V,\ i \neq t\}$ a set of potential conflicts (overlaps) between candidate positions. Atamtürk et al. [12] presented a good review about conflict graphs.

For the PFCLP with discrete candidate positions, Christensen et al. [8] have proposed a cartographic pattern (see Fig. 1) with eight available candidate positions, where each one has a number to indicate a cartographic preference. In Fig. 1, the position 1 is the most suitable, i.e., the lower number indicates the best position. Starting from this pattern, the PFCLP can be defined as the problem of assigning the labels to one of its available candidate positions subject to conflict constraints and minimizing or maximizing an objective function.

Considering a problem with two points and four candidate positions for each one, we can get a conflict graph as presented in Fig. 2 where dashed edges indicate conflicts from vertex $v_{1,3}$ with $v_{2,2}$ and $v_{1,4}$ with $v_{2,1}$ and $v_{2,2}$. The proportion of conflict free labels indicates the quality of the labeling [4,8,13]. So, if the labels are placed in positions 1,1 and 2,1, labeling is improved and free labels are available.

Considering that the conflict graph can be large and become hard to deal with, Wagner et al. [21] presented an approach to reduce the conflict graph. They proposed three rules to reduce the graph size without altering the set of optimal solutions. For the MNCFLP, the following rules are applicable:

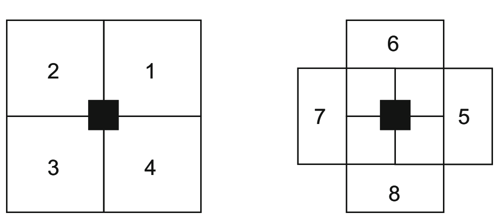

Fig. 1. Cartographic pattern proposed by Christensen et al. [8].

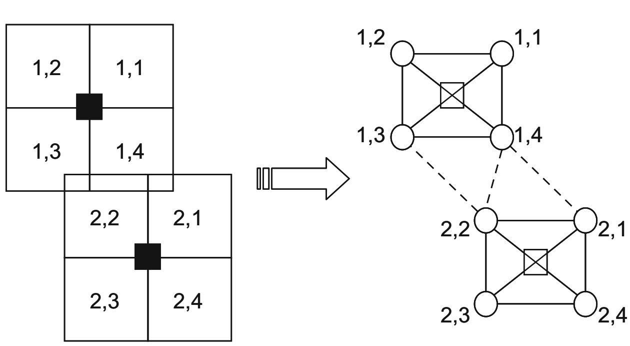

Fig. 2. Example of a conflict graph [6].

- Rule 1: If a point p has only one candidate position p_i without any conflicts, declare p_i to be part of the solution, and eliminate all other candidate positions of p
- Rule 2: If a point p has a candidate position p_i that is only in conflict with a candidate position q_k, and q has a candidate position q_j ($j \neq k$) that is only overlapped by candidate position p_l ($l \neq i$), then add p_i and q_j to the solution and eliminate all other candidate positions of p and q
- Rule 3: If p has only one candidate position p_i left, and the candidate positions overlapping p_i form a clique, then declare p_i to be part of the solution and eliminate all candidate positions that overlap p_i.

These rules are applied exhaustively. After eliminating a candidate p_i, we must check recursively whether the rules can be applied in the neighborhood of p_i.

Considering the PFCLP as a MISP, many studies are reported in the literature. But in the mathematical models field, Zoraster [5] and Strijk et al. [4] have presented interesting contributions. Zoraster [5] formulated mathematically the PFCLP working with conflict constraints and dummy candidate positions of high cost if the points could not be labeled. He also proposed a Lagrangean relaxation for the problem and obtained some computational results on small-scale instances. Strijk et al. [4] proposed new mathematical formulations based on the so-called clique inequalities [14], implemented a branch-and-cut algorithm and tested several heuristics such as Tabu search and simulated annealing. The authors used instances up to 950 points with four candidate positions.

Now if we look at the PFCLP as a MNCFLP, many researchers have proposed heuristics and metaheuristics. Christensen et al. [8,15] presented a good review about the PFCLP and proposed a local search technique based on a discrete form of the gradient descent and a simulated annealing algorithm. Verner et al. [16] applied a genetic algorithm with mask such that if a label has a conflict, the changing of positions is allowed by crossover operators. Yamamoto et al. [13] proposed a Tabu search algorithm. Yamamoto and Lorena [17] developed a constructive genetic algorithm and applied it to a set of large-scale instances.

Recently, Alvim and Taillard [18] presented a partial optimization metaheuristic under special intensification conditions (POPMUSIC) frame for the MNCFLP. POPMUSIC was proposed by Taillard and Voss [19] and its basic idea consists of locally optimizing sub-parts of a solution once a solution of the problem is available. The local optimizations are repeated until no further improvements are found. For the local optimizations, the authors implemented a new version of the Tabu search proposed by Yamamoto et al. [13].

Alvim and Taillard [18] have applied POPMUSIC to instances proposed in the literature by Yamamoto et al. [13] and to real instances with 13 206 points obtained from the Switzerland road network. POPMUSIC has presented good solutions, improved approaches, and diminished computational times.

Finally, looking at PFCLP from a MNCP point of view, Ribeiro and Lorena [6,20] introduced this approach to minimize the number of conflicts (edges in the remaining conflict graph). The authors have proposed two 0-1 optimization models and a Lagrangean heuristic. Considering the optimization models, the last one proposed in 2008 [20] has a compact number of constraints.

To visualize the difference between the approaches presented in this section for the PFCLP, see Fig. 3. In this figure, there are three hypothetical solutions for a problem with four points. Note that in the context of the MNCP approach, both solutions (a) and (b) presented two conflicts (edges), i.e., both could be optimal. However,

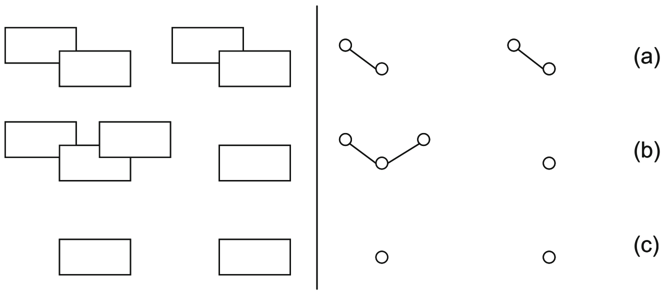

Fig. 3. Solutions for the many approaches to the PFCLP.

note that in the context of the MNCFLP approach, solution (b) is better than solution (a) because it provided one free label. Finally, in the context of the MISP, solution (c) provides two free labels, but it did not label all four points.

For more details and algorithms, see "map-labeling bibliography web site" at http://i11www.iti.uni-karlsruhe.de/~awolff/map-labeling/bibliography/.

3. The proposed model

In this section we propose a 0-1 integer linear programming model for the PFCLP approached as a MNCFLP. Remember that we are considering label sizes fixed, discrete potential positions of the labels, all points must be labeled and we are looking for the maximum number of conflict free labels.

Let $x_{i,j}$ be a binary variable to represent the candidate position j of point i for all $i \in \{1,\ldots,N\}$ and $j \in P_i$. If $x_{i,j}=1$ the label of point i must be placed at candidate position j, and $x_{i,j}=0$ otherwise. For each candidate position of point i is associated a profit represented by $w_{i,j}$. Now let $S_{i,j}$ be a set of pairs $(t,u): t \neq i$ composed by candidate positions $x_{t,u}$ that present potential conflicts with $x_{i,j}$.

The MNCFLP 0-1 optimization model is

MNCFLP: $v(MNCFLP)$=Maximize:

$$\sum_{i=1}^{N}\sum_{j \in P_i} w_{i,j}x_{i,j} - \sum_{i=1}^{N} z_i \tag{1}$$

Subject to

$$\sum_{j \in P_i} x_{i,j} = 1 \quad \forall i = 1,\ldots,N \tag{2}$$

$$x_{i,j}+x_{t,u}-z_i \leq 1 \quad \forall i=1,\ldots,N; \quad \forall j \in P_i; i \neq t; (t,u) \in S_{i,j} \tag{3}$$

$$x_{i,j}, x_{t,u}, z_i \in \{0,1\} \quad \forall i=1,\ldots,N; \quad \forall j \in P_i; (t,u) \in S_{i,j} \tag{4}$$

where $z_i=1$ means that some candidate position is overlapping point i, $z_i=0$ otherwise.

Constraints (2) ensure that each point i must be labeled, i.e., some candidate position $x_{i,j}$ must be equal to 1. Constraints (3) ensure the correct assignment to variables z when overlaps (conflicts) are inevitable and constraints (4) ensure that all variables are binary. Constraints (2) also can be seen as "conflicts" but between candidate positions of the same point.

The variables z_i have negative signs in the objective function to be maximized, and so, these variables should take value zero. Remember that profits $w_{i,j}$ indicate cartographic preferences. For the objective function (1) it is important to place labels on the best positions avoiding overlapped points.

The model (1)–(4) is similar to the one proposed by Zoraster [5] and Ribeiro and Lorena [9] but it allows allocating all labels maximizing the number of conflict free labels. Fig. 4a presents a conflict graph with respective constraints (3) in Fig. 4b.

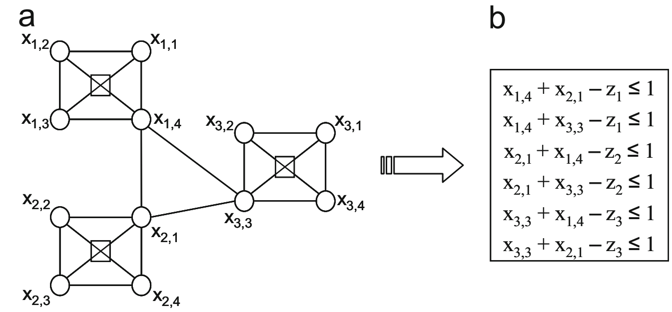

Fig. 4. Conflict graph and conflict constraints for the *MNCFLP* model.

Thus, if $x_{1,4}=x_{2,1}=1$, the label of point 1 is overlapped by the label of point 2, consequently point 1, represented by z_1 is overlapped by point 2, and point 2 represented by z_2 is overlapped by point 1. Finally, $z_1=z_2=1$ (ensured by constraint 3 presented in Fig. 4b) indicating two overlapped labels or overlapped points.

The objective function maximizes the number of conflict free labels. The profited sum of the variables $x_{i,j}$ (Eq. (1)) ignoring the cartographic preferences $w_{i,j}$ will be always equal to the total number of points (ensured by constraints (2)). The number of overlapped labels will be given by the sum of variables z_i. Therefore, setting all w_{ij} to 1, the value for the objective function will be exactly the number of conflict free labels.

In order to strengthen formulation (1)–(4) we may add non-trivial valid inequalities that are not satisfied by the linear relaxation solutions. Let $C_{i,j}$ be a set with all points that contain at least one candidate position in conflict with $x_{i,j}$. The valid inequalities presented below replace constraints (3):

$$x_{i,j}+\sum_{(c,u) \in S_{ij}} x_{c,u}-z_i \leq 1 \quad \forall i=1,\ldots,N; \quad \forall j \in P_i \forall c \in C_{ij} : i \neq c$$

Thus, a strengthened formulation can be presented:

$MNCFLP^{STR}$: $v(MNCFLP^{STR})$=Maximize:

$$\sum_{i=1}^{N}\sum_{j \in P_i} w_{i,j}x_{i,j} - \sum_{i=1}^{N} z_i \tag{1*}$$

Subject to (2), (4)

$$x_{i,j}+\sum_{(c,u) \in S_{ij}} x_{c,u}-z_i \leq 1 \quad \forall i=1,\ldots,N; \quad \forall j \in P_i \forall c \in C_{ij} : i \neq c \tag{5}$$

4. Lagrangean decomposition

The Lagrangean decomposition is a special case of Lagrangean relaxation that consists of partitioning the original problem into several sub-problems creating a copy of the decision variables in each one of the generated sub-problems. These "clones" are used in the sub-problems' constraints and new constraints ensure the equality between them and the original variables. Thus, the Lagrangean decomposition appears when we relax in a Lagrangean way these new constraints [22,23]. Therefore, it is important to copy variables as little as possible to reduce the number of new constraints.

It is important to define which strategy must be used to partition our problem. Ribeiro and Lorena [6] proposed a Lagrangean relaxation with clusters for the PFCLP as a minimum number of conflicts problem. The idea presented by the authors consists in: represent the PFCLP by a conflict graph; partition this graph into clusters; and relax, in a Lagrangean way, all constraints with vertices (decision variables) in different clusters.

Our decomposition is based on the same idea presented by Ribeiro and Lorena [6], however, instead of relaxing all constraints with vertices in different clusters, we make copies of the decision variables to include them into the sub-problems and relax those necessary additional constraints to make sure that the original variables and the copies are equal. This approach reduces the number of constraints to be relaxed, providing a stronger relaxation than the one proposed by Ribeiro and Lorena [6].

For the *MNCFLP* model (1)–(4), the conflict graph G will be partitioned into m $(m \le N)$ clusters of vertices with $V = V_1 \cup V_2 \cup \ldots \cup V_m$, and $V_i \cap V_j = \varnothing$, $\forall i,j \in \{1,\ldots,m\}$ forming sub-graphs $G_k = (V_k, E_k) \forall k = 1,\ldots,m$. Now, let $X_k = V - V_k$ be a set of the vertices not included in cluster k and C_k be the set of copied variables in cluster k.

To copy variables as little as possible and to adequately partition the conflict graph is challenging. But for the conflict graph provided by *MNCFLP*, we can use the technique of vertices contraction which consists on grouping all candidate positions of the same point i to form a single vertex (see Fig. 5b). In Fig. 5 the squares and circles indicate the points to be labeled and its candidate positions, respectively.

The vertices contraction generates a conflicts graph G between points and not between candidate positions. Therefore, the conflict graph partitioning between points will generate sub-problems (Fig. 5c) where conflicts between candidate positions of the same point are preserved (constraint (2)) resulting in a stronger relaxation for the PFCLP. In this step it is important to cut edges as little as possible. It also helps to strengthen the relaxation.

After the partitioning of G, the contractions are expanded (Fig. 5d) resulting in the original conflict graph with the inter-cluster edges (dashed edges on Fig. 5d) and the sub-problems (clusters 1 and 2).

Now, we must determine what binary variables (vertices) must be copied. A good strategy was proposed by Sachdeva [24] which copies vertices with the greatest number of inter-clusters edges. First, the approach proposed by Sachdeva [24] selects that vertex $x_{i,j}$ with the largest number of inter-clusters edges, next this vertex is copied to that cluster with highest number of inter-cluster edges connected to it. After this, the approach redefines the number of inter-clusters edges and a new vertex is selected to be copied. This process is repeated until all the necessary copies are carried out, i.e., all the inter-clusters edges are eliminated.

As shown in Fig. 5e, the first vertex copied presents three inter-clusters edges (black vertex is a copy of the gray one). The simple copy of the gray vertex to cluster 1 removes three inter-cluster edges. The process is repeated and finished in Fig. 5f. Now we have two independent clusters. If we ensure that all copies and original variables (black and gray vertices, respectively,) are equal, the problem can be decomposed into two clusters. To proceed, we present the *MNCFLP* decomposition into $m(m \le N)$ clusters:

$MNCFLP^m$: $v(MNCFLP^m)$ = Maximize:

$$\sum_{k=1}^{m} \Big(\sum_{(i,j) \in V_k} w_{i,j} x_{i,j} - \sum_{(i,j) \in V_k} z_i \Big) \tag{6}$$

Subject to

$$\sum_{j \in P_i} x_{i,j} = 1 \quad \forall i \in V_k; k = 1,\ldots,m \tag{7}$$

$$x_{i,j} + x_{t,u} - z_i \le 1 \quad \forall (i,j) \in V_k; i \ne t; (t,u) \in S_{i,j}; k = 1,\ldots,m \tag{8}$$

$$x_{i,j} + x^k_{t,u} - z_i \le 1 \quad \forall (i,j) \in V_k; (t,u) \in C_k \cap S_{i,j}; k = 1,\ldots,m \tag{9}$$

$$x_{i,j} + x^k_{t,u} - z^k_t \le 1 \quad \forall (i,j) \in V_k; (t,u) \in C_k \cap S_{i,j}; k = 1,\ldots,m \tag{10}$$

$$x_{t,u} = x^k_{t,u} \quad \forall (t,u) \in C_k \cap X_k; k = 1,\ldots,m \tag{11}$$

$$z_t = z^k_t \quad \forall (t,u) \in C_k \cap X_k; k = 1,\ldots,m \tag{12}$$

$$x_{i,j}, x_{t,u}, z_i, z_t, x^k_{t,u}, z^k_t \in \{0,1\} \quad \forall (i,j) \in V_k; (t,u) \in V_k \cup C_k; k = 1,\ldots,m \tag{13}$$

The variables $x^k_{t,u}$ and z^k_t represent, respectively, the copies of $x_{t,u}$ and z_t into cluster k, respectively. The copies are not considered in the objective function, i.e., its coefficients are zero. Constraints (7) and (8) deal only with the edges whose vertices are internal to each cluster (sub-problem) k. Constraints (9) and (10) indicate the edges whose vertices are in different clusters (inter-clusters edges) and constraints (11) and (12) ensure the equality between original and copy variables.

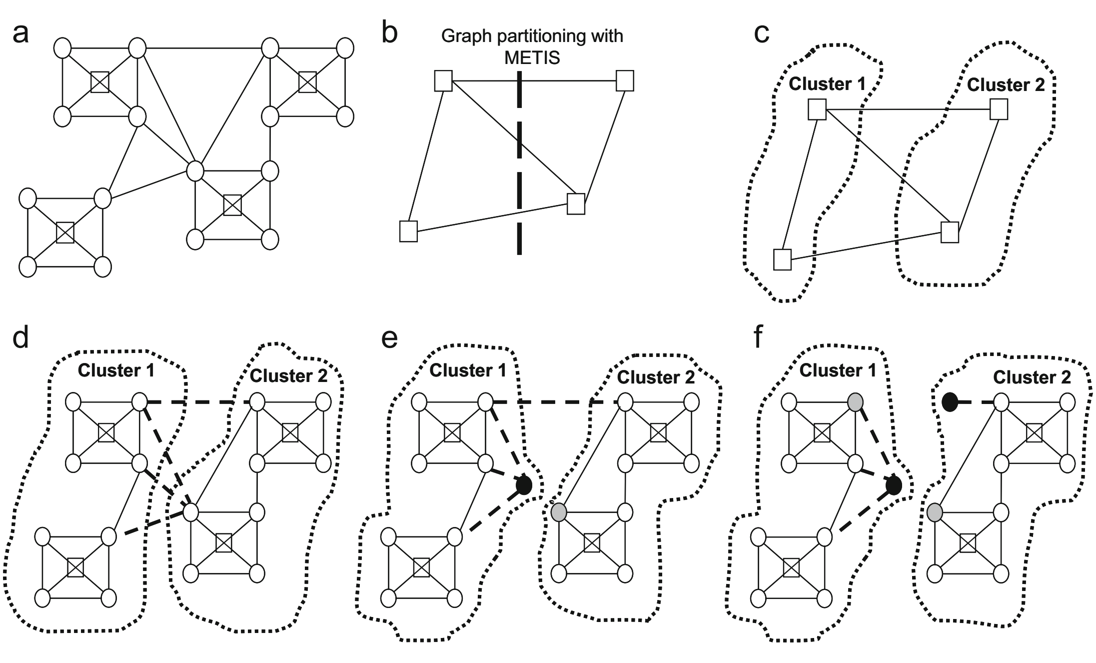

Fig. 5. Graph decomposition for the *PFCLP*.

Relaxing constraints (11) and (12) in a Lagrangean way with vectors of unrestricted multipliers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the problem $MNCFLP^m$ (indirectly the $MNCFLP$) can be divided into m independent sub-problems. Each sub-problem k can be defined as follows:

$LD_{\alpha\beta}MNCFLP_k$: $v(LD_{\alpha\beta}MNCFLP_k)$=Maximize:

$$\sum_{(i,j)\in V_k}(w_{i,j}-\sum_{d\neq k}\alpha_{i,j}^d)x_{i,j}+\sum_{(t,u)\in C_k}(\alpha_{t,u}^k)x_{t,u}^k-\sum_{(i,j)\in V_k}(z_i-\sum_{d\neq k}\beta_i^d)z_i+\sum_{(t,u)\in C_k}(\beta_t^k)z_t^k \tag{14}$$

Subject to

$$\sum_{j\in P_i}x_{i,j}=1 \quad \forall i\in V_k \tag{15}$$

$$x_{i,j}+x_{t,u}-z_i\leq 1 \quad \forall (i,j)\in V_k;\, i\neq t;\,(t,u)\in S_{i,j} \tag{16}$$

$$x_{i,j}+x_{t,u}^k-z_i\leq 1 \quad \forall (i,j)\in V_k;\,(t,u)\in C_k\cap S_{i,j} \tag{17}$$

$$x_{i,j}+x_{t,u}^k-z_t^k\leq 1 \quad \forall (i,j)\in V_k;\,(t,u)\in C_k\cap S_{i,j} \tag{18}$$

$$x_{i,j},x_{t,u},z_i,x_{t,u}^k,z_t^k\in\{0,1\} \quad \forall (i,j)\in V_k;\,(t,u)\in V_k\cup C_k \tag{19}$$

Finally, the $MNCFLP$ relaxation in m sub-problems is given by the expression (20) and the corresponding Lagrangean dual is presented in expression (21).

```
LAGRANGEAN HEURISTIC
x_{i,j}*: value for x_{i,j} on integer solution (viable);
x_{i,j}: value for x_{i,j} on sub-problem for vertex (i,j);
p: selected candidate position;
r: candidate position selected before the best change;
q: candidate position selected after the best change;
FOR i ← 1 TO N DO
   FOR all j ∈ P_i DO
      x_{i,j}* ← x_{i,j};
   END-FOR;
END-FOR;
fo* ← COMPUTE (the objective function value);
DO
    improve ← false;
    FOR i ← 1 TO N DO
       FOR all j ∈ P_i DO
          IF (x_{i,j}* = 1) THEN
             p ← j;
          END-IF;
       END-FOR;
       FOR all j ∈ P_i DO
          IF (j ≠ p) THEN
             x_{i,p} ← 0;
             x_{i,j} ← 1;
             fo = COMPUTE (the objective function value);
             IF (fo > fo*) THEN
                r ← p;
                q ← j;
                fo* ← fo;
                improve ← true;
             END-IF;
             x_{i,p} ← 1;
             x_{i,j} ← 0;
          END-IF;
       END-FOR;
    END-FOR;
    IF (improve) THEN
       x_{i,r}* ← 0;
       x_{i,q}* ← 1;
    END-IF;
WHILE (improve);
```

Fig. 6. Lagrangean heuristic for the *MNCFLP*.

$$LD_{\alpha\beta}MNCFLP^m: \quad v(LD_{\alpha\beta}MNCFLP^m) = \sum_{k=1}^{m} v(LD_{\alpha\beta}MNCFLP_k) \tag{20}$$

$$DLD_{\alpha\beta}MNCFLP^m: \quad v(DLD_{\alpha\beta}MNCFLP^m) = \underset{\alpha,\beta\ unrestricted}{\textbf{Min}} \{v(LD_{\alpha\beta}MNCFLP^m)\} \tag{21}$$

Now, we have m small and independent sub-problems that can be solved by a commercial solver. Consequently, a subgradient algorithm must be implemented to manage these sub-problems and update the subgradient steps.

CPLEX 10 [25] was used for solving the sub-problems and the heuristic METIS [26] was used for the graph partitioning task. According to Warrier et al. [27] this heuristic presented good results at minimizing the number of edges with endings in different clusters.

We have implemented a subgradient algorithm to solve the Lagrangean dual (21) based on the one proposed by Narciso and Lorena [28]. Our algorithm is similar to the one proposed by Held and Karp [29] and it updates the Lagrangean multipliers considering step sizes based on the relaxed solutions and the feasible solutions obtained with a Lagrangean heuristic (Fig. 6).

The heuristic shown in Fig. 6 is a greedy heuristic that verifies the better changing of candidate positions to generate an improved feasible solution. Initially (lines 6–10) the solution obtained by relaxation in m sub-problems is mounted. The candidate position selected for each point i is verified on lines 15–19 and changed for all the other ones (lines 20–34). A new candidate position is stored if a better solution is obtained, otherwise the change is discarded (lines 31 and 32). This procedure is repeated for all the points (lines 14–35). The whole process is repeated when a better solution is found (lines 12–40).

5. Computational results

Several computational experiments were performed to evaluate our model and our decomposition, considering a set of instances with 100, 250, 500, 750 and 1000 points as proposed by Yamamoto et al. [13]. These instances have been used in the several works pertaining to PFCLP (see Section 2) and are available at http://www.lac.inpe.br/~lorena/instancias.html. The Lagrangean decomposition was implemented in C++ and the experiments performed on a PC with a Pentium Dual Core processor of 1.73 GHz with 1 GB of RAM memory.

In accord with [5,6,8,9,16–18] and others, the cartographic preferences were not considered ($w_{i,j}=1$) for all candidate positions and the total number of candidate positions was equal to 4 ($P_i=\{1,2,3,4\} \forall i=1,\ldots,N$). The reduction heuristic proposed by Wagner et al. [21] was used to reduce the initial conflict graph, i.e., the conflict graph (problem) was reduced before we apply the METIS heuristic. However let us mention that Rule 3 is not applied to our computational experiments because all points have four candidate positions and Rule 3 is applied only for points with one candidate position. In addition, we have implemented the heuristic proposed by Sachdeva [24] to determine the vertices to be copied.

Tables 1–3 present the results for the instances with 500, 750 and 1000 points, respectively. CPLEX with *MNCFLP*, CPLEX with $MNCFLP^{STR}$ and our Lagrangean decomposition presented optimal solutions (100% of conflict free labels) in a computational time less than 1 s for the instances with 100 and 250 points. CPLEX and Lagrangean decomposition are not able to solve the instances with 13 206 points, even CPLEX with model $MNCFLP^{STR}$. The CPLEX fails due to an out of memory state and Lagrangean decomposition due to heuristic of Sachdeva [24] that was not able to determine the copies in a reasonable computational time.

Table 1
Results for instances with 500 points.

Inst.	$DLD_{\alpha\beta}MNCFLP^m$								CPLEX+*MNCFLP*				CPLEX+$MNCFLP^{STR}$			
	m	LB	UB	Gap (%)	Time (s)				LB	UB	Gap (%)	Time (s)	LB	UB	Gap(%)	Time (s)
					G&P	SP	SG	Total								
1	2	500*	500	0	0.03	0.01	0.32	0.36	500*	500	0	0.83	500*	500	0	0.08
2	2	500*	500	0	0.03	0.02	0.27	0.32	500*	500	0	0.45	500*	500	0	0.13
3	2	498*	498	0	0.00	0.00	0.22	0.22	498*	498	0	0.94	498*	498	0	0.14
4	2	500*	500	0	0.05	0.02	0.23	0.30	500*	500	0	0.83	500*	500	0	0.06
5	2	496*	496	0	0.02	0.00	0.56	0.58	496*	496	0	2.98	496*	496	0	0.25
6	2	500*	500	0	0.02	0.01	0.14	0.17	500*	500	0	0.55	500*	500	0	0.06
7	2	498*	498	0	0.03	0.01	0.66	0.70	498*	498	0	1.56	498*	498	0	0.09
8	2	498*	498	0	0.00	0.02	0.42	0.44	498*	498	0	1.69	498*	498	0	0.11
9	2	496*	496	0	0.03	0.00	0.95	0.98	496*	496	0	6.75	496*	496	0	0.20
10	2	500*	500	0	0.02	0.00	0.06	0.08	500*	500	0	0.37	500*	500	0	0.05
11	2	500*	500	0	0.01	0.02	0.14	0.17	500*	500	0	0.89	500*	500	0	0.06
12	2	500*	500	0	0.05	0.00	0.44	0.49	500*	500	0	1.42	500*	500	0	0.06
13	2	498*	498	0	0.01	0.00	0.39	0.40	498*	498	0	0.98	498*	498	0	0.11
14	2	498*	498	0	0.00	0.00	0.34	0.34	498*	498	0	1.16	498*	498	0	0.09
15	2	498*	498	0	0.05	0.00	0.78	0.83	498*	498	0	0.97	498*	498	0	0.09
16	2	496*	496	0	0.02	0.00	0.47	0.49	496*	496	0	4.74	496*	496	0	0.42
17	2	493*	493	0	0.05	0.00	0.94	0.99	493*	493	0	25.12	493*	493	0	0.41
18	2	500*	500	0	0.00	0.00	0.16	0.16	500*	500	0	0.53	500*	500	0	0.05
19	2	500*	500	0	0.00	0.01	0.16	0.17	500*	500	0	0.45	500*	500	0	0.06
20	2	500*	500	0	0.01	0.01	0.17	0.19	500*	500	0	0.53	500*	500	0	0.08
21	2	498*	498	0	0.06	0.00	0.77	0.83	498*	498	0	2.29	498*	498	0	0.14
22	2	500*	500	0	0.00	0.01	0.30	0.31	500*	500	0	1.25	500*	500	0	0.05
23	2	495*	495	0	0.00	0.01	0.39	0.40	495*	495	0	7.43	495*	495	0	0.36
24	2	498*	498	0	0.00	0.02	0.23	0.25	498*	498	0	1.00	498*	498	0	0.08
25	2	500*	500	0	0.05	0.00	0.09	0.14	500*	500	0	0.47	500*	500	0	0.05
Average	**2**	**498.40**	**498.40**	**0**	**0.02**	**0.01**	**0.38**	**0.41**	**498.40**	**498.40**	**0**	**2.65**	**498.40**	**498.40**	**0**	**0.13**

*The optimal solution was proven.

Table 2
Results for instances with 750 points.

Inst.	$DLD_{\alpha\beta}MNCFLP^m$								CPLEX+*MNCFLP*				CPLEX+$MNCFLP^{STR}$			
	m	LB	UB	Gap (%)	Time (s)				LB	UB	Gap (%)	Time (s)	LB	UB	Gap (%)	Time (s)
					G&P	SP	SG	Total								
1	10	739*	739	0	0.08	0.02	1.00	1.10	739	741.83	0.38	10 800.06	739*	739.00	0	44.67
2	10	736*	736	0	0.03	0.02	1.66	1.71	735	744.46	1.29	10 800.86	736*	736.00	0	2033.56
3	10	731*	731	0	0.16	0.02	32.00	32.18	730	743.55	1.86	6532.46	731	733.78	0.38	8957.94
4	10	741*	741	0	0.06	0.02	1.58	1.66	741	742.00	0.13	10 800.34	741*	741.00	0	10.69
5	10	739*	739	0	0.06	0.08	1.24	1.38	739	743.78	0.65	5392.55	739*	739.00	0	143.73
6	10	730*	730	0	0.06	0.01	22.75	22.82	730	743.50	1.85	7550.23	730	731.09	0.15	15210.70
7	10	737*	737	0	0.06	0.02	2.36	2.44	737	742.00	0.68	7887.57	737*	737.00	0	70.05
8	10	736*	736	0	0.06	0.02	1.41	1.49	736	743.02	0.95	7657.6	736*	736.00	0	1913.95
9	10	726*	726	0	0.06	0.01	8.55	8.62	724	744.67	2.85	6264.66	726	731.21	0.72	5183.67
10	10	743*	743	0	0.03	0.19	40.01	40.23	743*	743.00	0	635.98	743*	743.00	0	2.13
11	10	733*	733	0	0.16	0.02	2.47	2.65	732	744.00	1.64	7665.78	733*	733.00	0	2992.88
12	10	734*	734	0	0.06	0.01	1.80	1.87	733	742.50	1.30	8357.19	734*	734.00	0	20.08
13	10	743*	743	0	0.05	0.01	2.02	2.08	743*	743.00	0	1142.99	743*	743.00	0	1.94
14	10	728*	728	0	0.08	0.02	5.05	5.15	728	743.58	2.14	7367.05	728	731.09	0.42	8911.52
15	10	730*	730	0	0.06	0.02	32.48	32.56	728	744.25	2.23	7651.08	730*	730.00	0	15 095.80
16	10	729*	729	0	0.05	0.03	396.02	396.10	729	744.00	2.06	7355.9	729*	729.00	0	11 111.20
17	10	729*	729	0	0.05	0.02	6.70	6.77	728	744.62	2.28	7388.97	729	731.74	0.38	10 142.30
18	10	737*	737	0	0.08	0.02	1.59	1.69	737	742.67	0.77	10 800.51	737*	737.00	0	154.44
19	10	740*	740	0	0.12	0.06	3.41	3.59	740	742.50	0.34	10 800.1	740*	740.00	0	9.97
20	10	737*	737	0	0.06	0.02	2.83	2.91	737	742.67	0.77	7327.87	737*	737.00	0	81.36
21	10	731*	731	0	0.08	0.02	19.81	19.91	731	744.67	1.87	7070.45	731	733.48	0.34	7324.94
22	10	744*	744	0	0.05	0.02	0.84	0.91	744*	744.00	0	470.09	744*	744.00	0	3.00
23	10	731*	731	0	0.06	0.03	3.81	3.90	731	742.00	1.50	7614.33	731*	731.00	0	36.77
24	10	732*	732	0	0.06	0.02	8.25	8.33	732	744.40	1.69	5800.79	732	734.95	0.40	5569.06
25	10	732*	732	0	0.06	0.01	5.66	5.73	732	743.33	1.55	7355.57	732*	732.00	0	13 494.50
Average	**10**	**734.72**	**734.72**	**0**	**0.07**	**0.03**	**24.21**	**24.31**	**734.36**	**743.36**	**1.23**	**7139.64**	**734.72**	**735.53**	**0.11**	**4340.83**

*The optimal solution was proven.

Table 3
Results for instances with 1000 points.

Inst.	$DLD_{\alpha\beta}MNCFLP^m$								CPLEX+*MNCFLP*				CPLEX+$MNCFLP^{STR}$			
	m	LB	UB	Gap (%)	Time (s)				LB	UB	Gap (%)	Time (s)	LB	UB	Gap (%)	Time (s)
					G&P	SP	SG	Total								
1	25	939*	939.98	0.10	0.34	0.95	1347.61	1348.90	918	995.27	8.42	8234.54	939	961.03	2.35	8217.50
2	25	933	939.62	0.71	0.14	1.77	14 405.16	14 407.07	892	996.56	11.72	7944.66	934	962.59	3.06	9372.64
3	20	934*	934.00	0	0.14	0.50	2751.06	2751.70	911	994.81	9.20	8010.25	933	961.14	3.02	7993.61
4	40	929	939.90	1.17	0.16	9.45	14 446.88	14 456.49	882	996.94	13.03	7675.42	933	962.43	3.15	9234.03
5	15	960	962.33	0.24	0.14	0.56	14 413.56	14 414.26	945	997.00	5.50	7585.55	961	976.20	1.58	8834.52
6	20	932*	932.92	0.10	0.16	0.16	1400.39	1400.71	920	995.42	8.20	8165.56	932	960.52	3.06	8020.03
7	25	928	931.99	0.43	0.14	1.02	14 418.14	14 419.30	897	996.92	11.14	8102.22	928	960.06	3.45	8819.64
8	20	940	942.21	0.24	0.14	0.77	14 418.98	14 419.89	908	998.33	9.95	8037.76	939	968.44	3.14	8312.06
9	20	923	930.32	0.79	0.14	0.77	15 143.19	15 144.10	893	996.08	11.54	8074.16	925	959.52	3.73	8894.97
10	20	943	946.48	0.37	0.14	0.70	14 703.78	14 704.62	901	997.14	10.67	8220.80	944	971.54	2.92	8861.16
11	25	947*	947.90	0.10	0.14	1.11	3332.97	3334.22	938	995.41	6.12	7889.20	947	967.78	2.19	8521.02
12	25	934	936.44	0.26	0.17	2.11	14 562.00	14 564.28	903	996.22	10.32	7858.28	934	956.23	2.38	7860.77
13	25	954	956.03	0.21	0.12	1.39	14 528.36	14 529.87	928	997.00	7.44	7549.22	955	972.99	1.88	8964.73
14	25	930	941.15	1.20	0.17	6.42	14 523.11	14 529.70	897	996.15	11.05	8154.78	933	960.52	2.95	9293.41
15	25	932	937.00	0.54	0.19	2.98	16 450.00	16 453.17	904	994.85	10.05	8148.46	934	963.99	3.21	9248.70
16	25	928	936.01	0.86	0.16	2.27	14 450.20	14 452.63	876	997.17	13.83	8223.61	931	959.58	3.07	9647.72
17	25	937	939.63	0.28	0.14	0.52	14 467.31	14 467.97	929	996.75	7.29	8105.20	937	964.61	2.95	11 243.55
18	25	946	947.27	0.13	0.16	1.30	14 507.63	14 509.09	923	995.75	7.88	7927.24	946	968.91	2.42	8934.28
19	25	950	953.25	0.34	0.16	1.58	15 262.58	15 264.32	842	996.83	18.39	8724.84	949	970.46	2.26	9214.44
20	25	929	945.84	1.81	0.19	2.56	14 416.59	14 419.34	909	997.00	9.68	7910.86	934	966.22	3.45	9253.99
21	25	928	932.22	0.45	0.17	1.09	14 435.02	14 436.28	842	997.75	18.50	8219.96	930	955.38	2.73	8707.73
22	25	952	955.41	0.36	0.14	2.33	14 403.81	14 406.28	928	997.25	7.46	8141.83	952	974.27	2.34	8638.53
23	25	933	940.23	0.77	0.14	2.36	14 608.95	14 611.45	853	995.39	16.69	8516.06	934	960.13	2.80	9103.09
24	25	929	934.39	0.58	0.14	2.77	14 461.00	14 463.91	877	995.25	13.48	8664.67	932	964.74	3.51	9178.91
25	25	945*	945.70	0.07	0.12	0.62	3512.50	3513.24	917	995.50	8.56	8181.32	944	964.70	2.19	8601.40
Average	**24.20**	**937.40**	**941.93**	**0.48**	**0.16**	**1.92**	**12 214.83**	**12 216.91**	**901.32**	**996.35**	**10.64**	**8090.66**	**938.40**	**964.56**	**2.79**	**8918.90**

*The optimal solution was proven.

In Tables 1–3, m indicates the number of clusters used; LB and UB indicate the upper and lower bounds, respectively; Gap indicates the difference between LB and UB ($Gap=((UB-LB)/LB)\times 100$); $G\&P$ indicates the time required to generate and partitioning the graph; SP presents the time to prepare the sub-problems and the column SG indicates the processing time for the subgradient algorithm; the Total column shows the total processing time and the last line presents the arithmetic averages to each column.

From Table 1 we can observe that both CPLEX (with *MNCFLP* and $MNCFLP^{STR}$) and our Lagrangean decomposition found and proved the optimal solutions for all instances within less that an average of 3 s. The proportion of conflict free labels (Eq. (22)) in both cases was 99.68%.

$$\text{Conflicts free labels}(\%)=\frac{LB}{N}\times 100 \tag{22}$$

Table 2 presents the results obtained for instances with 750 points. Our Lagrangean decomposition found the optimal solutions for all the instances with an average computational time of 24.31 s. CPLEX with *MNCFLP* presented an average residual gap of 1.23% proving the optimal solution only for 3 of 25 instances (see instances with asterisk) with an average time of 7139.64 s. CPLEX with $MNCFLP^{STR}$ presented an average residual gap of 0.11% proving the optimal for 18 of 25 instances with an average time of 4340.83 s. Our Lagrangean decomposition found 97.96% of conflict free labels while CPLEX found 97.91% and 97.96% with *MNCFLP* and $MNCFLP^{STR}$, respectively.

The results obtained for the instances with 1000 points are reported in the Table 3. Our Lagrangean decomposition has proved the optimality for 5 of 25 instances with an average residual gap of 0.48%. CPLEX with *MNCFLP* and $MNCFLP^{STR}$ cannot show the optimality for any instance and has presented a gap of 10.64% and 2.79%, respectively. Our Lagrangean decomposition times are in magnitude similar to the ones provided by CPLEX and it has provided 93.74% of conflict free labels while CPLEX 90.13% and 93.84% with *MNCFLP* and $MNCFLP^{STR}$, respectively.

Table 4
Comparison with the literature: proportion (%) of conflict free labels.

Methods	Points-N			
	250	500	750	1000
$DLD_{\alpha\beta}MNCFLP^{m}$	**100.00***	**99.68***	**97.96***	93.74
CPLEX+$MNCFLP^{STR}$	**100.00***	**99.68***	**97.96**	**93.84**
CPLEX+*MNCFLP*	**100.00***	**99.68***	97.91	90.13
Pop(asc) [18]	**100.00**	99.67	97.72	92.68
Pop(10) [18]	**100.00**	99.67	97.46	91.94
Pop(30) [18]	**100.00**	99.67	97.72	92.54
Pop(70) [18]	**100.00**	99.67	97.73	92.58
Tabu(50n) [18]	**100.00**	99.57	97.53	91.54
Tabu(100n) [18]	**100.00**	99.57	97.54	91.54
Tabu(500n) [18]	**100.00**	99.57	97.55	91.59
Column Generationgeneration [6]	**100.00**	99.67	97.67	92.40
LagClus [9]	**100.00**	99.67	97.65	91.42
GRASP(6) [30]	**100.00**	99.67	97.72	92.20
GRASP(5) [30]	**100.00**	99.67	97.70	92.02
CGA [17]	**100.00**	99.60	97.10	90.70
Tabu SeachSearch [13]	**100.00**	99.26	96.76	90.00
GA with masking [16]	99.98	98.79	95.99	88.96
GA [16]	98.40	92.59	82.38	65.70
Simulated Annealingannealing [8]	99.90	98.30	92.30	82.09
3-opt Gradient Descentgradient descent [8]	99.76	97.34	89.44	77.83
2-opt Gradient Descentgradient descent [8]	99.36	95.62	85.60	73.37
Gradient Descentdescent [8]	95.47	86.46	72.40	58.29
Greedy Algorithmalgorithm [8]	88.82	75.15	58.57	43.41

*The optimal solution was proven.

Considering Tables 1–3, we can see that CPLEX with $MNCFLP^{STR}$ provided better solutions and gaps than it with *MNCFLP*. It shows that the proposed valid inequality is efficient.

Table 4 shows a comparison among the conflict free labels proportion obtained by a set of the main works about the *MNCFLP*. We can observe that our Lagrangean decomposition present better results proving the optimality for several instances. We also can note a slight improvement among the methods found in the literature detaching the improvements presented by our Lagrangean decomposition.

The average computational time was not placed on Table 4 due to the diversity of the computers used. However, we note that our computational times (see Tables 1–3) are higher. This fact is acceptable because we consider exact methods to solve the *MNCFLP*.

As mentioned before, our Lagrangean decomposition was not able to deal with instances of 13 206 points presented by Alvim and Taillard [18], but efforts are in course to reduce the time used to copy the binary variables.

6. Conclusions

In the literature, we can note a tough dispute for the best solutions to the point-feature cartographic label placement problem. However, most of the methods are based on heuristics and metaheuristics that do not allow us to verify how close the solution is from an optimal one, particularly when the PFCLP is modeled according to the maximum number of conflict free labels approach.

Thus, trying to reduce this problem, this paper has presented a new 0-1 integer linear programming model to the *MNCFLP*. In addition, one non-trivial valid inequality is presented to strengthen this proposed model. A method based on Lagrangean decomposition presented tight gaps for a set of instances up to 1000 points. The decomposition method has proved the optimality for several instances of the literature for the first time, with gaps smaller than the ones proposed by CPLEX even with the strengthened model. We believe that our Lagrangean decomposition is an interesting tool for solving problems represented by conflict graphs.

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