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CARTESIAN PRODUCTS OF SETS

The ***ordered n-tuple*** is the ordered collection $(a_1, a_2, \dots, a_n)$ that has a_1 as its first element, a_2 as its second element,…, and a_n as its *n*th element.
Two ordered *n*-tuples are equal if and only if each corresponding pair of their elements is equal.
In other words, $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \dots, n$.

Definition

Let ***P*** and ***Q*** be two sets. The Cartesian product ***P*** × ***Q*** is the set of all ordered pairs $(\boldsymbol{p}, \boldsymbol{q})$**, where** $\boldsymbol{p} \in \boldsymbol{P}$ **and** $\boldsymbol{q} \in \boldsymbol{Q}$

i.e. $\boldsymbol{P} \times \boldsymbol{Q} = \{(\boldsymbol{p}, \boldsymbol{q}) : \boldsymbol{p} \in \boldsymbol{P}, \boldsymbol{q} \in \boldsymbol{Q}\}$

Note the following:

- The *Cartesian product* of the sets $A_1, A_2, \dots, A_n$ denoted by $A_1 \times A_2 \times \dots \times A_n$ is the set of ordered *n*-tuples $(a_1, a_2, \dots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \dots, n$.
  In other words, $A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) | a_i \in A_i, for\ i = 1,2, \dots, n\}$The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.
- If either ***P*** or ***Q*** is the null set, then ***P*** × ***Q*** will also be empty set.
- If ***A*** and ***B*** are non-empty sets and either ***A*** or ***B*** is an infinite set, then so is ***A*** × ***B***.
- If there are **p** elements in ***A*** and **q** elements in ***B*,** then there will be **pq** elements in ***A*** × ***B***, i.e., if ***n(A) = p*** and ***n(B) = q*,** then ***n (A × B) = pq.***
- $A^n = A \times A \times \dots \times A = \{(a_1, a_2, \dots, a_n) | a_i \in A, for\ i = 1,2, \dots, n\}$

Illustration

Consider the two sets: $\boldsymbol{A} = \{DL, MP, KA\}$, where DL, MP, KA represent Delhi, Madhya Pradesh and Karnataka, respectively and $\boldsymbol{B} = \{01, 02, 03\}$ representing codes for the licence plates of vehicles issued by DL, MP and KA.

If the three states, Delhi, Madhya Pradesh and Karnataka were making codes for the licence plates of vehicles, with the restriction that the code begins with an element from set $\boldsymbol{A}$, which are the pairs available from these sets and how many such pairs will there be ?

The available pairs are:

(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03) and the product of set ***A*** and set ***B*** is given by

A × ***B*** = {(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03)}.

It can easily be seen that there will be 9 such pairs in the Cartesian product, since there are 3 elements in each of the sets A and B. This gives us 9 possible codes.

Note: The order in which these elements are paired is crucial.

For example, the code (DL, 01) will not be the same as the code (01, DL).

RELATIONS AND THEIR PROPERTIES

Definition

A relation *R* from a non-empty set *A* to a non-empty set *B* is a subset of the Cartesian product A × *B*. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in *A* × *B*.

The second element is called the image of the first element.

For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$.

A relation from a set A to itself is called a relation on A.

Domain-Definition

The set of all first elements of the ordered pairs in a relation ***R*** from a set ***A*** to a set ***B*** is called the **domain** of the relation ***R***.

Co-domain Definition

The set of all second elements in a relation ***R*** from a set ***A*** to a set ***B*** is called the ***range*** of the relation ***R***. The whole set ***B*** is called the **codomain** of the relation R.

Note the following:

- Range ⊆ Codomain.
- If $(a, b) \in R$, then we say that $a\ is\ related\ to\ b$, which can also be written as aRb.
- The total number of relations that can be defined from a set ***A*** to a set ***B*** is the number of possible subsets of ***A* × *B***.
- If **n(*A*) = p** and **n(*B*) = *q***, then **n (*A* × *B*) = pq** and **the total number of relations =** $\mathbf{2^{pq}}$
- A relation ***R*** in a set ***A*** is called **empty relation (void relation)**, if no element of A is related to any element of A, i.e., ***R* = φ⊂*A* × *A***.
- A relation ***R*** in a set ***A*** is called **universal relation**, if each element of ***A*** is related to every element of ***A***, i.e., ***R* = *A* × *A***.
- Both the empty relation and the universal relation are sometimes called **trivial relations**.
- A relation ***R*** in a set ***A*** is called **identity relation**, if each element of ***A*** is related to itself only. i.e., $\boldsymbol{R = \{(a, a) | a \in A\}}$
- In case of relations from a set A to a set B, $A \times B$ is considered as the universal relation. The complement relation of a relation R is denoted and given as $R' = (A \times B) - R$.
- If $R = \{(a, b) | a \in A, b \in B\}$ then the inverse relation is denoted and given as $R^{-1} = \{(b, a) | a \in A, b \in B\}$.
- Union, intersection, difference and other operations of sets are all applicable to relations as they are for sets.

Example-1:

What is the largest possible relation from the set $A = \{1,2,3,4,5\}$ to the set $B = \{1,2,3\}$?

Write the relations from A to B in each of the following cases when

1) a is related to b if and only if $a \geq b$
2) a is related to b if and only if $a > b$
3) a is related to b if and only if $a < b$
4) aRb if and only if $a + b > 4$
5) $(a, b) \in R$ if and only if $a\ is\ a\ divisor\ of\ b$

Solution:

The largest possible relation from A to B is

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3),$$
$$(4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$$

1) $R = \{(a, b) | a \geq b, a \in A, b \in B\}$

$$= \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$$

2) $R = \{(a, b) | a > b, a \in A, b \in B\} = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$

3) $R = \{(a, b) | a < b, a \in A, b \in B\} = \{(1,2), (1,3), (2,3)\}$

4) $R = \{(a, b) | a + b > 4, a \in A, b \in B\} =$

$\{(2,3), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$

5) $R = \{(a, b) | a\ is\ a\ devisor\ of\ b, a \in A, b \in B\} = \{(1,1), (1,2), (1,3), (2,2), (3,3)\}$

Example-2: [Summer 2023 – 24] [Winter 2022 - 23]

Let $A = \{1, 2, 5\}$ and $B = \{3, 5,7\}$ and let $R = \{(a, b) | 7 \leq a + b < 10, a \in A, b \in B\}$.

1) Write all the elements of R and R'

2) Write the inverse relation of R.

3) Find the Domain and Range of R and R^{-1} .

Solution:

1) $R = \{(1,7), (2,5), (2,7), (5,3)\}$

$R' = \{(1,3), (1,5), (2,3), (5,5), (5,7)\}$

2) $R^{-1} = \{(7,1), (5,2), (7,2), (3,5)\}$

3) $Dom(R) = \{1,2,5\}, Range(R) = \{3,7,5\}$

$Dom(R^{-1}) = \{3,7,5\}, Range(R) = \{1,2,5\}$

Example-3:

Consider these relations on the set of integers:

$R1 = \{(a, b) \mid a \leq b\}$,

$R2 = \{(a, b) \mid a > b\}$,

$R3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R4 = \{(a, b) \mid a = b\}$,

$R5 = \{(a, b) \mid a = b + 1\}$,

$R6 = \{(a, b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,−1), and (2,2)?

Solution:

The pair (1,1) is in R_1, R_3, R_4, and R_6; (1,2) is in R_1 and R_6; (2,1) is in R_2, R_5, and R_6; (1,−1) is in R_2, R_3, and R_6; and finally, (2,2) is in R_1, R_3, and R_4.

Exercise-1:

1. How many relations are possible from A to B if A = {1, 2, 3} and B = {0, 2}. **[Winter 2017 – 18]**

OR

{Number of possible relations from A = {1, 2, 3} to B = {0, 2} is________ **[Winter 2018 – 19]}**

2. How many relations are possible from A to B if A = {1, 2} and B = {a, b, c}?
   [Summer 2018 – 19]
3. If A = {1,2,3,4,5,6} and R = {(x, y)/ |x − y| = 2, x ∈ A, y ∈ A}. Then, the relation set R = _____________. **[Summer/Winter 2023 – 24]**
   A) {(3,1), (4,2), (5,3), (6,4)}
   B) {(1,3), (2,4), (5,3), (4,6)}
   C) {(1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4)}
   D) {(1,2), (2,1), (2,4), (4,2), (3,6), (6,3)}
4. Let $A = \{1, 2, 3, \ldots, 14\}$. Let a relation R on A be defined as $R = \{(x, y) : 3x - y = 0, where\ x, y \in A\}$. Write down its domain, co-domain and range.
5. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A to B. Which of the following is not a relation from A to B? Justify your answer.
   i) $\{(x, 1), (y, 2), (z, 3)\}$
   ii) $\{(x, 1), (x, 2)\}$
   iii) $\{(x, 2), (y, 2), (z, 2)\}$
   iv) $\{(1, x), (2, x)\}$

COMPOSITE OF RELATIONS

Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C.

The composite of R and Sgiven by $S \circ R$ is the relation from A to C consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Thus, $S \circ R = \{(a, c) | (a, b) \in R, (b, c) \in S, for\ some\ b \in B\}$.

In other words, $a(S \circ R)c$ if and only if aRb and bSc for some $b \in B$.

Remark:

1. The powers of a relation R can be recursively defined from the definition of a composite of two relations.

   Let $\boldsymbol{R}$ be a relation on the set $\boldsymbol{A}$. The powers $\boldsymbol{R^n}, n = 1,2,3, \ldots$are defined recursively by $\boldsymbol{R^1} = \boldsymbol{R}$ and $\boldsymbol{R^{n+1}} = \boldsymbol{R^n} \circ \boldsymbol{R}$.

   Thus, $\boldsymbol{R^2} = \boldsymbol{R} \circ \boldsymbol{R}$, $\boldsymbol{R^3} = \boldsymbol{R^2} \circ \boldsymbol{R} = (\boldsymbol{R} \circ \boldsymbol{R}) \circ \boldsymbol{R}$, and so on.

Example-4:

Let $R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$ and $S = \{ (1,0), (2,0), (3,1), (3,2), (4,1)\}$ be two relations on some sets. Check if $S \circ R$ is possible or not. If it is possible then write the elements of the relation $S \circ R$.

Solution

Here, $codom(R) = \{1,3,4\}$ is a subset of $dom(S) = \{1,2,3,4\}$

Hence, $S \circ R$ is Possible.

Further, $S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

Example-5:

Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find the powers R^n, n= 2, 3, 4,,.. **[Winter 2022-23]**

Solution

$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$

Further, $R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$.

Similarly, $R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$.

It follows that $R^n = R^3$ for $n = 5,6,7, \ldots$

Exercise-2:

1. Let R_1 = {(1, 2) (1, 6) (2, 4) (3, 4) (3, 6) (3, 8)} and R_2 = {(2, x) (4, y) (4, z) (6, z) (8, x)}. Then $R_1\ o\ R_2$ = ________. **[Winter 2021 – 22]**
2. Let $R = \{(0,1), (1,2), (1,4), (2,3), (3,1), (4,3)\}$ and $S = \{ (1,0), (2,1), (3,1), (3,2)\}$ be two relations on some sets. Check if $R \circ S$ is possible or not. If it is possible then write the elements of the relation $R \circ S$.
3. let R = {(1, b), (2, a), (2, c)} be a relation from A = {1, 2, 3} to B = {a, b, c} and let S = {(a, y), (b, x), (c, y), (c, z)} be a relation from B to C = {x, y, z}. Check if the composition relation $S \circ R$ and $R \circ S$. can exist. Write the relation as set if they exist. If not, give the reason.
   [Winter 2021 – 22]

PROPERTIES OF RELATIONS

Let A be a set. Let R be a relation on it.

- The relation $\boldsymbol{R}$ is said to be **reflexive** if $(a, a) \in R$, for every $a \in A$
  In other words, a relation on A is reflexive if every element of A is related to itself.
- The relation $\boldsymbol{R}$ is said to be **transitive,** if $(a, b) \in R\ and\ (b, c) \in R \Rightarrow (a, c) \in R$, for all $a, b, c \in A$.
- The relation $\boldsymbol{R}$ is said to be **symmetric** $if\ (a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$
- The relation $\boldsymbol{R}$ is said to be **anti-symmetric** if $(a, b) \in R, (b, a) \in R \Rightarrow a = b$,for all $a, b \in A$
  In other words, a relation R on a set A is anti-symmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a.
  i.e. the only way to have $\boldsymbol{a}$ related to $\boldsymbol{b}$ and $\boldsymbol{b}$ related to $\boldsymbol{a}$ is for $\boldsymbol{a}$ and $\boldsymbol{b}$ to be the same element.

Equivalence relation-Definition:

A relation R on a set A is said to be ***equivalence relation***, if R is reflexive, transitive and symmetric.

Partially ordered relation-Definition:

A relation R on a set A is said to be ***partially ordered relation***, if R is reflexive, transitive and anti-symmetric.

A set A with a partially ordered operation R,(i.e. (A, R)) is said to be ***Partially Ordered Set (POSet).***

Note:

(i) The terms symmetric and antisymmetric are not opposites.

(ii) If R is an equivalence relation, and $(a, b) \in R$, then a and b are called equivalent. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example-6: Consider $A = \{1,2,3\}$ and a relation R on A in each of the following cases.

Check whether they are reflexive, symmetric, anti-symmetric or transitive.

Also check which of them is equivalence relation or partially ordered relation.

(1) $aRb, if\ a = b$ **[Reflexive, Transitive, Symmetric, Anti-Symmetric]**

(2) $aRb\ if\ a \leq b$ **[Reflexive, Transitive, Anti-Symmetric]**

(3) $aRb\ if\ a \neq b$ **[Symmetric]**

(4) $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ **[Reflexive, Transitive, Symmetric]**

(5) $R = \{(1,1), (2,2), (1,2), (2,1)\}$ **[Transitive, Symmetric]**

(6) $R = \{(1,1), (2,2)\}$ **[Transitive, Symmetric, Anti-Symmetric]**

(7) $R = \{(1,1), (2,2), (3,3), (2,1), (1,3)\}$ **[Reflexive]**

Example-7:

Show that the “greater than or equal” relation (≥) is a partial ordering on the set of integers.

[Summer 2017 – 18] [Winter 2021 – 22]

Solution:

Because a ≥ a for every integer a, ≥ is reflexive.

If a ≥ b and b ≥ a, then a = b.

Hence, ≥is antisymmetric.

Finally, ≥is transitive because a ≥ b and b ≥ c implies that a ≥ c.

It follows that ≥is a partial ordering on the set of integers and **(Z, ≥) is a poset**.

Example-8:

Show that the inclusion relation ⊆ is a partial ordering on the power set of a set *S*.

Solution:

Because A ⊆A whenever A is a subset of ***S***, ⊆ is reflexive.

It is antisymmetric because A ⊆ B and B ⊆ A imply that A = B.

Finally, ⊆is transitive, because A ⊆ B and B ⊆ C imply that A ⊆ C.

Hence, ⊆is a partial ordering on **P(S)**, and **(P(S), ⊆) is a poset**.

Exercise-3

1. Check whether the following relations are equivalence relation or not on the set of all integers where aRb if and only if 1) $a \neq b$ 2) $ab \geq 0$
2. Check whether from the following relation sets, which are satisfying the transitive, reflexive or symmetric property which relation is an Equivalence relation and partially ordered relation.
   1) R1 = {(1, 1), (2, 2), (3, 3)}
   2) R2 = {(1, 1), (2, 2), (1, 2), (2, 1), (3, 1)}
   3) R3= {(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)}
   4) R4= {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)}
3. Prove that $(\mathbb{Z}, \leq)$ is a partially ordered set where Z is the set of integers.
4. Check if ℕ with the '$divides$' relation is a Poset.
5. If a relation R satisfies reflexive, anti- symmetric and Transitive property then the relation is called ______________ relation.
   [Winter 2017 – 18] [Winter 2018 – 19] [Summer 2018 – 19] [Summer 2023 – 24]
6. If a relation satisfies reflexive, symmetric and Transitive property then the relation is called __________ relation. **[Winter 2021 – 22]**
7. Determine whether the relation R on the set of natural numbers is reflexive, symmetric, antisymmetric, and/or transitive, where (a, b) ∈ R if and only if ab ≥1. And hence check it is an equivalence relation or a partial ordering relation. **[Summer 2018 – 19]**
8. A relation R on a set A is said to be equivalence relation, if R is _______. **[Winter 2019 – 20]**
   (a) Reflexive, transitive and symmetric (b) anti-symmetric, transitive and symmetric
   (c) Anti-symmetric, reflexive, symmetric (d) reflexive, transitive and anti-symmetric
9. If $(a, b) \in R$, $(b, a) \Rightarrow a = b$, for all $a, b \in A$, then relation R is said to be ___________.
   [Summer 2023 – 24]
10. If $(a, a) \in R$ for every $a \in R$, then relation R is said to be ___________. **[Winter 2022 - 23]**

11. For the relation {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)} on the set {1,2,3,4}, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive. (Justify your answer if the property is not satisfied**) [Winter 2023 – 24]**

DIFFERENCE BETWEEN RELATION AND FUNCTION

Function

We can, visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as 'map' or 'mapping' used to denote a function.

Function-Definition

A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.

In other words,

A relation, f is a function from a non-empty set A to a non-empty set B if

(i) the domain of f is A

(ii) no two distinct ordered pairs in f have the same first elements.

Remark:

If f is a function from A to B and $(a, b) \in f$, then we write $f\ (a) = b$, where b is called the image of a under f and a is called the preimage of b under f.

Example-9:

Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not on the given domain?

(i)	R = {(2,1),(3,1), (4,2)}, Domain= {1,2,3,4}	**Not a Function**
(ii)	R = {(2,2),(2,4),(3,3), (4,4)} , Domain= {2,3,4}	**Not a Function**
(iii)	R = {(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)}, Domain= {1,2,3,4,5,7}	**Not a Function**

REPRESENTION OF RELATIONS

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix.

Suppose that $\boldsymbol{R}$ is a relation from $\boldsymbol{A} = \{ a_1, a_2, \ldots, a_m\}$ to $\boldsymbol{B} = \{ b_1, b_2, \ldots, b_n\}$.

(Here the elements of the sets $\boldsymbol{A}$ and $\boldsymbol{B}$ have been listed in a particular, but arbitrary, order. Furthermore, when $\boldsymbol{A} = \boldsymbol{B}$ we use the same ordering for $\boldsymbol{A}$ and $\boldsymbol{B}$.)

The relation $\boldsymbol{R}$ can be represented by the matrix $\boldsymbol{M_R} = [\boldsymbol{m_{ij}}]$, where $\boldsymbol{m_{ij}} = \begin{cases} 1 \text{ } if \ \ (a_i, b_j) \in \boldsymbol{R} \\ 0 \text{ } if \ \ (a_i, b_j) \notin \boldsymbol{R} \end{cases}$

In other words, the zero–one matrix representing $\boldsymbol{R}$ has a **1** as its(i, j)th entry when $\boldsymbol{a_i}$**is related to** $\boldsymbol{b_j}$, and a **0** in this position if $\boldsymbol{a_i}$**is not related to** $\boldsymbol{b_j}$.

Remark: Such a representation depends on the orderings used for $\boldsymbol{A}$ and $\boldsymbol{B}$.

Example-10:

Suppose that $A = \{1,2,3\}$and $B = \{ 1,2\}$.

Let R be the relation from A to B containing (a, b) if $a \in A, b \in B, and\ a > b$.

What is the matrix representing R?

Solution:

Because $R = \{(2,1), (3,1), (3,2)\}$, the matrix for $\boldsymbol{R}$ is $\mathbf{M_R} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

Example-11:

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1 , b_2, b_3, b_4, b_5\}$.

Which ordered pairs are in the relation R represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$?

Solution:

Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Remarks:

- R is reflexive if and only if $m_{ii} = 1$ for $i = 1,2, \ldots, n$.

  In other words, R is reflexive if all the elements on the main diagonal of M_R are equal to 1.

  Note that the elements off the main diagonal can be either 0 or 1.

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

(The matrix for a reflexive relation)

- The relation $\boldsymbol{R}$ is **symmetric** if and only if $\boldsymbol{m_{ji} = 1}$ whenever $\boldsymbol{m_{ij} = 1}$**.**

  This also means $\boldsymbol{m_{ji} = 0}$ whenever $\boldsymbol{m_{ij} = 0}$.

  Consequently, $\boldsymbol{R}$ is symmetric if and only if $\boldsymbol{m_{ij} = m_{ji}}$, for all i and j.

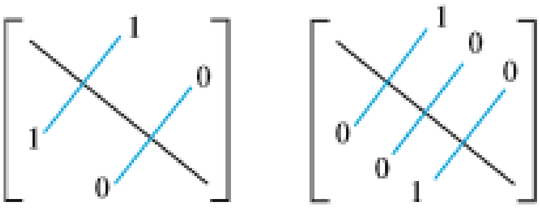
  i.e. $\boldsymbol{R}$ is symmetric if and only if $M_R = M_R^T$,

  i.e, $\boldsymbol{R}$ is symmetric if M_R is a symmetric matrix.

- The relation R is anti-symmetric if and only if $m_{ij} = 1$ with $i \neq j$,then $m_{ji} = 0$.

  In other words, if $i \neq j$ then either $m_{ij} = 0 \ or \ m_{ji} = 0$.

  The form of the matrix for an antisymmetric relation is illustrated in Figure.



(a) Symmetric (b) Antisymmetric

Matrix of Union and Intersection of two relations

Suppose that $\boldsymbol{R}$ and $\boldsymbol{S}$ are relations on a set **A** represented by the matrices $\boldsymbol{M_R}$ and $\boldsymbol{M_S}$, respectively.

The matrix representing the **union** of these relations has a 1 in the positions where **either** $\boldsymbol{M_R}$ **or** $\boldsymbol{M_S}$ **has a 1.**

The matrix representing the **intersection** of these relations has a 1 in the positions **where both** $\boldsymbol{M_R}$ **and** $\boldsymbol{M_S}$ **have a 1.**

Thus, the matrices representing the union and intersection of these relations are

$\boldsymbol{M_{R\cup S} = M_R \vee M_S}$ and $\boldsymbol{M_{R\cap S} = M_R \wedge M_S}$**.**

Example-12:

Suppose that the relations R and S on a set A are represented by the matrices

$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. What are the matrices representing $R \cup S$ and $R \cap S$?

Solution:

The matrices of these relations are

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } M_{\boldsymbol{R\cap S}} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix of Composite of two relations:

Matrix of Composite of two relations can be found using the Boolean product of the matrices for these relations.

In particular, suppose that ***R*** is a relation from ***A to B*** and ***S*** is a relation from ***B to C.***

Suppose that ***A***, ***B***, and ***C*** have $\boldsymbol{m}, \boldsymbol{n}$, and $\boldsymbol{p}$ elements, respectively.

Let the zero– one matrix for $S \circ R, R$, and S be $M_{S\circ R} = [t_{ij}], M_R = [r_{ij}], and\ M_S = [\ s_{ij}]$, respectively.

(Note that these matrices have sizes $m \times p, m \times n$, and $n \times p$, respectively).

The ordered pair (a_i, c_j) belongs to $\boldsymbol{S} \circ \boldsymbol{R}$ if and only if there is an element b_k such that $(a_i, b_k) \in \boldsymbol{R}$ and $(b_k, c_j) \in \boldsymbol{S}$.

It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k.

In other words, $t_{ij} = 1$ if and only if i^{th} row of M_R and j^{th} column of M_S has 1 at a same position.

From the definition of the Boolean product, this means that $\boldsymbol{M_{S\circ R}} = \boldsymbol{M_R} \odot \boldsymbol{M_S}$.

Example-13:

Find the matrix representing the relation $S \circ R$, where the matrices representing R and S are

$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Solution:

The matrix for S∘R is $M_{\boldsymbol{S\circ R}} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Exercise-4:

1. Let R be the relation represented by the matrix $M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find the matrix representing

   (a) R^{-1} (**b**) R' ($\boldsymbol{c}$) R^2

2. Let R and S be relations on a set A represented by the matrices $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and

   $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Find the matrices representing the following relations. **[Winter 2019 – 20]**

   (a) $R \cup S$ (b) $R \cap S$ (c) $S \circ R$ (d) $R \circ \mathrm{S}$ (e) $R \oplus S$

Representing Relations Using Digraphs

There is another important way of representing a relation using a pictorial representation.

Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow.

We use such pictorial representations when we think of relations on a finite set as **directed graphs**, or **digraphs**.

Directed Graph/ Diagraph-Definition:

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).

Initial and Terminal Edge-Definition:

The vertex 'a' is called the initial vertex of the edge (a,b), and the vertex ' b' is called the terminal vertex of this edge.

Loop-Definition:

An edge of the form (a,a) is represented using an arc from the vertex 'a' back to itself. Such an edge is called a loop.

Remarks:

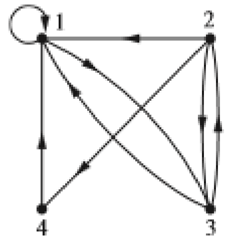
- A relation R is **reflexive** if and only if **there is a loop at every vertex** of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation.

- A relation is **transitive** if and only if **whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z** (completing a triangle where each side is a directed edge with the correct direction).
- A relation is **symmetric** if and only if **for every edge between distinct vertices in its digraph there is an edge in the opposite direction**, so that (y, x) is in the relation whenever (x, y) is in the relation.
- A relation is **antisymmetric** if and only if **there are never two edges in opposite directions between distinct vertices**.

Example-14: Draw the directed graph of the relation **[Winter 2022-23]**

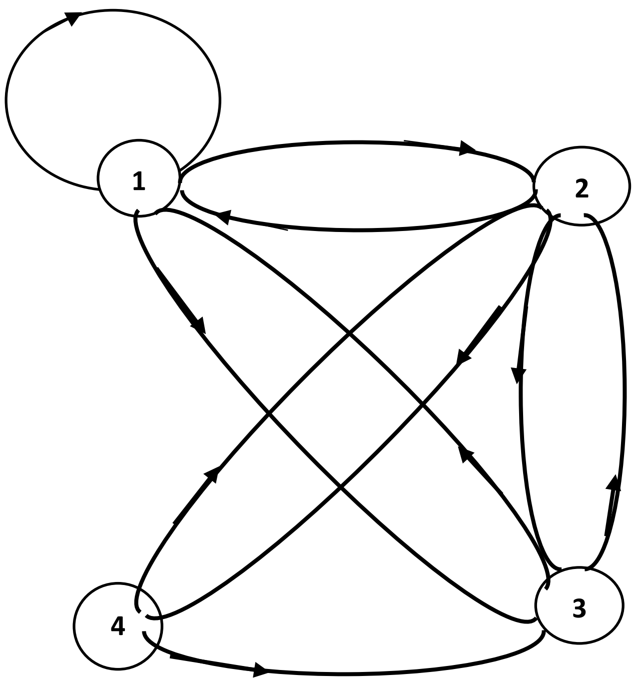
$$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$$

Solution:

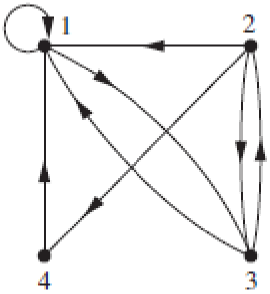


Example-15: Draw the directed graph of the relation $R =$
$\{(1, 1), (1, 3), (1, 2), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3)\}$

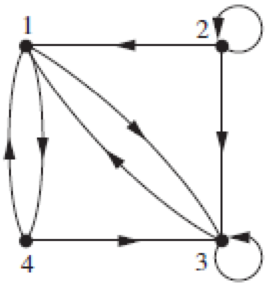
Solution:



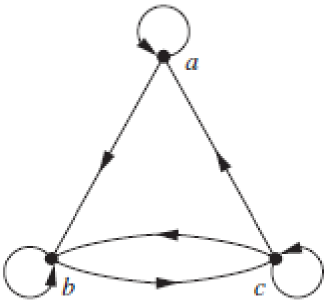
Example-16: Determine whether the relations for the directed graphs shown in the following figure are reflexive, symmetric, antisymmetric, and/or transitive.



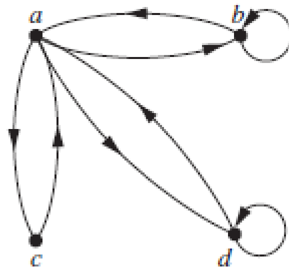
(a)



(b)



(c)



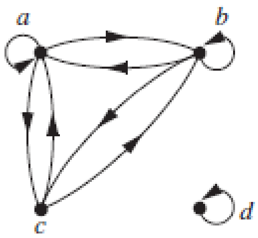
(d)

Solution:

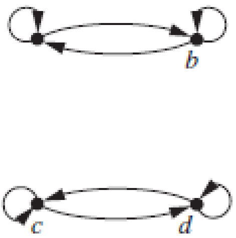
no	Reflexive	Symmetric	Antisymmetric	Transitive
(a)	NO	NO	NO	NO
(b)	NO	NO	NO	NO
(c)	YES	NO	NO	YES
(d)	NO	YES	NO	NO

Example-17: Write the relation represented by the following digraph and also write the matrix representing this relation.

(a)



(b)



Solution:

(a) $R = \{(a,a),(a,b),(b,a),(b,b),(b,c),(c,b),(c,a),(a,c).(d,d)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $R = \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Exercise-5:

1. Find the matrix representing the relations $S°R$, hence draw the digraph of where the matrices representing R and S are $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

   [Winter 2018 – 19] [Winter 2017 – 18]

   Check whether the relation SoR is reflexive, symmetric. **[Winter 2023 – 24]**

   Check whether the relation R∘ S is reflexive, symmetric or anti-symmetric.

   [Winter 2023 – 24]

2. Draw the directed graph and write a matrix representation of the relation R on $A = \{1,2,3,4\}$ where $R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (2, 2), (3,1), (3,2), (3,3), (4,1)\}$ Also, check that the relation R is reflexive, symmetric, transitive, and anti-symmetric? **[Winter 2022 - 23]**
3. Draw the directed graph for the relation and check whether it satisfies Reflexive, Anti-symmetric Relation. $R = \{(a, b)/a \neq b; a, b \in A, A = \{1,2,3,4\}$ **[Winter 2019 – 20]**
4. In directed graph if there are never two edges in opposite directions between distinct vertices then relation is

   A) Symmetric B) Anti-Symmetric C) Reflexive D) Transitive

   [Summer 2021-2022]
5. Write the relation represented by the following matrices also draw the corresponding digraph.

(a) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

CLOSURES OF RELATIONS

Introduction

Let R be a relation on a set A.

R may or may not have some property P, such as reflexivity, symmetry, or transitivity. If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R.

In other words, S is the smallest superset of R with the property P.

Reflexive closure of R

For given a relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$, the **reflexive closure of R** can be formed by adding to $\boldsymbol{R}$ all pairs of the form (a, a) with $a \in \boldsymbol{A}$, not already in $\boldsymbol{R}$.

The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R. Consequently, it is the reflexive closure of R.

Thus, the reflexive closure of R can be given by R∪Δ, where $\Delta = \{(a, a) \mid a \in A\}$is the diagonal relation on A.

Example-18:

The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the set $A = \{1,2,3\}$ is not reflexive. Obtain the reflexive closure of R.

Solution:

Here, diagonal relation on A is $\Delta = \{(1,1), (2,2), (3,3)\}$

Therefore, the reflexive closure of R is $S = R \cup \Delta = \{(1,1), (1,2), (2,1), (2,2), (3,2), (3,3)\}$

Example-19:

What is the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$on the set of integers?

Solution:

The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in Z\} = \{(a, b) \mid a \leq b\}.$$

Symmetric Closure of R

The symmetric closure of a relation $\boldsymbol{R}$ can be constructed by adding all ordered pairs of the form (b, a), for all (a, b) that are not already present in $\boldsymbol{R}$.

Adding these pairs produces a relation that is symmetric, that contains $\boldsymbol{R}$, and that is contained in any symmetric relation that contains $\boldsymbol{R}$. Consequently, it is the symmetric closure of R.

The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse i.e., $R \cup R^{-1}$is the symmetric closure of R, where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

Example-20:

Find the symmetric closure of the relation$\{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)\}$ on $\{1,2,3\}$.

Solution:

$$R^{-1} = \{(1,1),(2,1),(2,2),(3,2),(1,3),(2,3)\}$$

Therefore, symmetric closure of R is

$$S = R \cup R^{-1} = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,1),(1,3)\}$$

Example-21:

What is the symmetric closure of the relation $R = \{(a,b) \mid a > b\}$ on the set of positive integers?

Solution:

$$R^{-1} = \{(b,a) \mid a > b\} = \{(a,b)|b > a\} = \{(a,b)|a < b\}$$

The symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a,b)|a > b\} \cup \{(a,b)|a < b\} = \{(a,b)|a \neq b\}$$

<u>**Transitive closure of R:**</u>

Suppose that a relation R is not transitive.

Let M_R be the zero–one matrix of the relation R on a set with n elements.

Let R^* be the transitive closure of R.

Then the zero–one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}} \vee \cdots \vee M_{R^{[n]}} .$$

Here R^* is known as connectivity relation.

Example-22:

Find the zero–one matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

The zero–one matrix of $M_R$is $M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}}$.

Now, $M_{R^{[2]}}$ is the matrix of the composite relation $R \circ R$.

$$\Rightarrow \boldsymbol{M_{R^{[2]}}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and $M_{R^{[3]}}$ is the matrix of the composite relation $R \circ R^2$

$$\Rightarrow \boldsymbol{M_{R^{[3]}}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence,

$$M_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise-6:

1. Find the zero–one matrix of the transitive closure of the relation R where,

   $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ **[Summer 2021-2022]**

2. Let R be the relation on the set {0,1,2,3} containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0). Find a) reflexive closure of R. b) symmetric closure of R.

3. Find the reflexive and symmetric closure of the relation R = {(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)} on {1,2,3}. **[Winter 2022-23]**

DATABASE & RELATIONS

Concepts of relations have a strong application in the theory of relational databases.

Definition:

Let $A1, A2, \ldots, An$ be sets. An n-ary relation on these sets is a subset of $A1 \times A2 \times \ldots \times An$.

The sets $A1, A2, \ldots, An$ are called the *domains* of the relation, and *n* is called its degree.

Example-23:

Let R be the relation on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ consisting of triples (a, b, c), where $a, b, and\ c$ are integers with $a < b < c$. Then find its domain and degree of this relation.

Solution:

Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

Example-24:

Let R be the relation on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ consisting of all triples of integers (a, b, c) in which $a, b, and\ c$ form an arithmetic progression. That is, $(a, b, c) \in R$ if and only if there is an integer k such that $b = a + k$ and $c = a + 2k$, or equivalently, such that $b - a = k$ and $c - b = k$. Find the domain and degree of this relation.

Solution:

Note that $(1, 3, 5) \in R$ because $3 = 1 + 2$ and $5 = 1 + 2 \cdot 2$, but $(2, 5, 9) \notin R$ because $5 - 2 = 3$ while $9 - 5 = 4$. This relation has degree 3 and its domains are all equal to the set of integers.

Example-25:

Let R be the relation consisting of $5 - tuples\ (A, N, S, D, T\ )$ representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time.

For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) belongs to R. Find the domain and degree of the relation.

Solution:

The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

Database & relations:

The time required to manipulate information in a database depends on how this information is stored. The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database.

Because of the importance of these operations, various methods for representing databases have been developed. We will discuss one of these methods, called the relational data model, based on the concept of a relation.

A database consists of records, which are n-tuples, made up of fields. The fields are the entries of the n-tuples. For instance, a database of student records may be made up of fields containing the name, student number, major, and grade point average of the student. The relational data model represents a database of records as an n-ary relation.

Selection operator-Definition:

Let R be an $n-ary$ relation and C be a condition that elements in R may satisfy. Then the selection operator S_c maps the $n-ary$ relation R to the $n-ary$ relations of all n-tuples from R that satisfy the condition C.

Projection operator-Definition:

The *projection* $P_{i_1 i_2,\ldots,i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

In other words, the projection $P_{i_1 i_2,\ldots,i_m}$ deletes $n - m$ of the components of an n-tuple, leaving the $i_1^{th}, i_2^{th}, \ldots$, and i_m^{th} components.

Example-26:

Consider the student records given by the following table.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

These student records can be given using $4-tuple$ of the form $(Studen\ name, ID\ number, Major, GPA)$. Explain the representation of various records

Solution:

A sample database of six such records is

$(Ackermann, 231455, Computer\ Science, 3.88)$

$(Adams, 888323, Physics, 3.45)$

$(Chou, 102147, Computer\ Science, 3.49)$

$(Goodfriend, 453876, Mathematics, 3.45)$

$(Rao, 678543, Mathematics, 3.90)$

$(Stevens, 786576, Psychology, 2.99)$.

To find the records of computer science majors in the $n-ary$ relation R shown in the above table, we use the operator S_{c_1}, where c_1 is the condition $Major = “Computer\ Science”$ The result is the two 4-tuples $(Ackermann, 231455, Computer\ Science, 3.88)$ and $(Chou, 102147, Computer\ Science, 3.49)$.

Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator S_{c_2}, where c_2 is the condition $GPA > 3.5$. The result is the two 4-tuples $(Ackermann, 231455, Computer\ Science, 3.88)$ and $(Rao, 678543, Mathematics, 3.90)$.

Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator S_{c_3}, where C_3 is the condition $(Major = “Computer\ Science” \wedge GPA > 3.5)$. The result consists of the single 4-tuple $(Ackermann, 231455, Computer\ Science, 3.88)$.

When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and grade point averages are obtained. The following table displays the results of this projection.

Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

Join operator- Definition:

Let R be a relation of degree m and S be a relation of degree n. The join $J_P(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p) - tuples$ $(a_1, a_2, \ldots a_{m-p}, c_1, c_2, \ldots c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots a_{m-p}, c_1, c_2, \ldots c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S. In other words, the join operator J_P produces a new relation from two relations by combining all m-tuples of the first relation with all n-tuples of the second relation, where the last p components of the m-tuples agree with the first p components of the n-tuples

Example-27: What relation results when the join operator $J2$ is used to combine the relation displayed in the following tables?

Teaching_assignments.		
Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

Class_schedule.			
Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

The join J_2 produces a member of relation as $(Cruz, Zoology, 335, A100, 9{:}00 A.M.)$ by joining the members $(Cruz, Zoology, 335)$ and $(Zoology, 335, A100, 9{:}00 A.M.)$

The relation thus produced is shown in the following table.

Teaching_schedule.				
Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

Exercise-7:

1. Consider the following Tables.

   a. What do you obtain when you apply the selection operator S_C, where C is the condition $(Project = 2) \wedge (Quantity \geq\ 50)$, to the database in the table of Parts_inventory.?

   b. Construct the table obtained by applying the join operator *J*2 to the relations in the following tables

Part_needs.		
Supplier	*Part_number*	*Project*
23	1092	1
23	1101	3
23	9048	4
31	4975	3
31	3477	2
32	6984	4
32	9191	2
33	1001	1

Parts_inventory.			
Part_number	*Project*	*Quantity*	*Color_code*
1001	1	14	8
1092	1	2	2
1101	3	1	1
3477	2	25	2
4975	3	6	2
6984	4	10	1
9048	4	12	2
9191	2	80	4

CANTOR'S DIAGONAL ARGUMENT

Finite and infinite sets:

Let *S* be a set. If there are exactly *n* distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is a *finite set* and that *n* is the *cardinality* of *S*. The cardinality of *S* is denoted by $|S|$.

A set is said to be *infinite* if it is not finite.

Countable and uncountable sets:

A set that is either finite or has the same cardinality as the set of positive integers is called *countable.*

A set that is not countable is called *uncountable.*

Cantor's diagonal argument

A set S is finite iff there is a bijection between S and $\{1, 2, \ldots, n\}$ for some positive integer n, and infinite otherwise. (i.e., if it makes sense to count its elements.)

Two sets have the same cardinality iff there is a bijection between them.

A set S is called countably infinite if there is a bijection between S and $\mathbb{N}$. Such a set is countable because elements can be counted, but unlike a finite set, counting never ends.

On the other hand, not all infinite sets are countably infinite. In fact, there are infinitely many sizes of infinite sets.

Georg Cantor proved this astonishing fact in 1895 by showing that the set of real numbers is not countable. That is, it is impossible to construct a bijection between ℕ and ℝ. In fact, it's impossible to construct a bijection between ℕ and the interval $[0, 1]$.

Theorem:

The set of real numbers is not countable

Proof:

Suppose that $f : \mathbb{N} \to [0,1]$ is any function.

Make a table of values of f, where the 1st row contains the decimal expansion of $f(1)$, the 2nd row contains the decimal expansion of $f(2)$, . . . the nth row contains the decimal expansion of $f(n)$, …

Perhaps, $f(1) = \pi/10$, $f(2) = 37/99, f(3) = 1/7, f(4) = \sqrt{2}/2, f(5) = 3/8$, and so on, so that the table starts out like this.

n	$f(n)$												
1	0	.	3	1	4	1	5	9	2	6	5	3	...
2	0	.	3	7	3	7	3	7	3	7	3	7	...
3	0	.	1	4	2	8	5	7	1	4	2	8	...
4	0	.	7	0	7	1	0	6	7	8	1	1	...
5	0	.	3	7	5	0	0	0	0	0	0	0	...
⋮	⋮												

Highlighting the digits in the main diagonal of the table.

n	$f(n)$												
1	0	.	**3**	1	4	1	5	9	2	6	5	3	...
2	0	.	3	**7**	3	7	3	7	3	7	3	7	...
3	0	.	1	4	**2**	8	5	7	1	4	2	8	...
4	0	.	7	0	7	**1**	0	6	7	8	1	1	...
5	0	.	3	7	5	0	**0**	0	0	0	0	0	...
⋮	⋮												

The highlighted digits are 0.37210 . . . . Suppose that we add 1 to each of these digits, to get the number

0.48321 . . . . then this number can't be in the table. Because

- it differs from f(1) in its first digit;
- it differs from f(2) in its second digit;
- . . .
- it differs from f(n) in its nth digit;
- . . .

So it can't equal f(n) for any n — that is, it can't appear in the table

This looks like a trick, but in fact there are lots of numbers that are not in the table.

As long as we highlight at least one digit in each row and at most one digit in each column, we can change each the digits to get another number not in the table.

Therefore, there does not exist a bijection between $\mathbb{N}$ and $[0, 1]$.

Hence, [0,1] is not a countable set.

Since, cardinality of ℝ and [0,1] is same, ℝ is also uncountable.

The Power set theorem

Statement: For every set S, |S| < |P(S)|

Proof:

Let $f: S \to P(S)$ be any function and define $X = \{\, s \in S \mid s \notin f(s)\}$.

Suppose that $X = f(s)$ for some $s \in S$

If so, then either s belongs to X or it doesn't.

But by the very definition of X, if s belongs to X then it doesn't belong to $X = f(s)$.

And if it doesn't belong to X then it belong to $X = f(s)$.

This situation is impossible.

Hence,X cannot equal $f(s)$ for any s.

Using the Cantor's diagonal argument, this proves that f cannot be onto.|

Hence, $|S| \neq |P(S)|$. Which gives |S| < |P(S)|.

Schroder-Bernstein Theorem

If *A* and *B* are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions *f* from *A* to *B* and *g* from *B* to *A*, then there is a one-to-one correspondence between *A* and *B*.

Hasse Diagrams

- Many edges in the directed graph for a finite poset do not have to be shown because they must be present.
- For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set {1, 2, 3, 4}, shown in Figure-1(a).
- Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices.

- Consequently, we do not have to show these loops because they must be present; in Figure-1(b) loops are not shown.
- Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity.

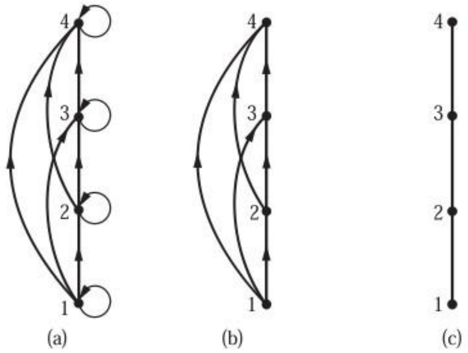


Figure -1

- For example, in Figure 2(c) the edges *(*1*,* 3*)*, *(*1*,* 4*)*, and *(*2*,* 4*)* are not shown because they must be present. If we assume that all edges are pointed "upward" (as they are drawn in the figure), we do not have to show the directions of the edges;
- Figure 2(c) does not show directions.
- In general, we can represent a finite poset $(S, \leq)$ as a Hasse Diagram, using this procedure:

Step-1: Start with the directed graph for this relation.

Step-2: Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops.

Step-3: Remove all edges that must be in the partial ordering because of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec x$.

Step-4: Finally, arrange each edge so that its initial vertex is below its terminal vertex.

Step-5: Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

These steps are well defined, and only a finite number of steps need to be carried out for a finite poset. When all the steps have been taken, the resulting diagram contains sufficient information to find the partial ordering, as we will explain later. The resulting diagram is called the **Hasse diagram** of $(S, \leq)$.

Example-28: Draw the Hasse diagram representing the partial ordering {*(a, b)* |*a* divides *b*} on {1, 2, 3, 4, 6, 8, 12}.

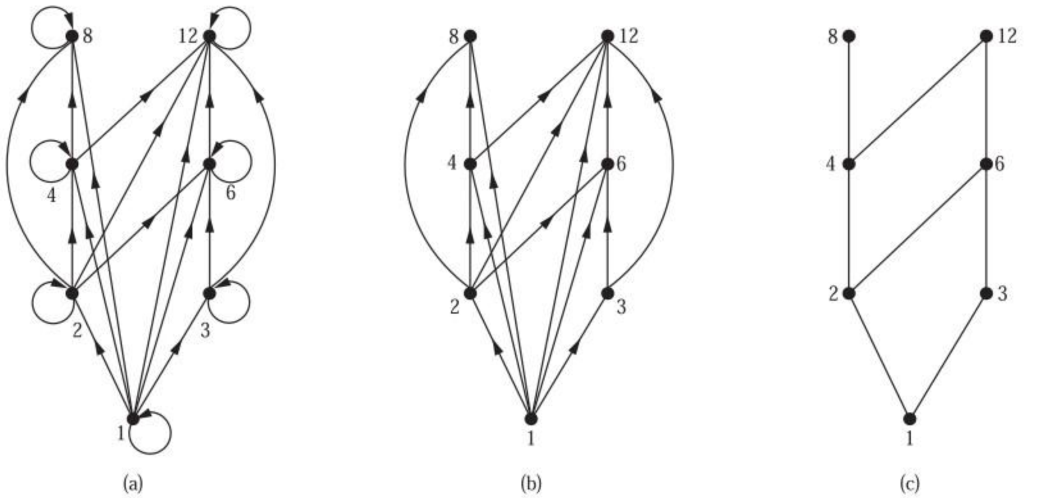


Figure- 2

Solution:

Step-1: Begin with the digraph for this partial order, as shown in Figure 2(a).

Step-2: Remove all loops, as shown in Figure 2(b). Then delete all the edges implied by the transitive property. These are *(*1, 4*)*, *(*1, 6*)*, *(*1, 8*)*, *(*1, 12*)*, *(*2, 8*)*, *(*2, 12*)*, and *(*3, 12*)*.

Step-3: Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 2(c).

Example-29: Draw the Hasse diagram for the partial ordering {*(A, B)* | *A* ⊆ *B*} on the power set *P(S)* where**,** *S* = {*a, b, c*}.

Solution:

The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, *(*∅, {*a, b*}*)*, *(*∅, {*a, c*}*)*, *(*∅, {*b, c*}*)*, *(*∅, {*a, b, c*}*)*, *(*{*a*}, {*a, b, c*}*)*, *(*{*b*}, {*a, b, c*}*)*, and *(*{*c*}, {*a, b, c*}*)*. Finally, all edges point upward, and arrows are deleted.

The resulting Hasse diagram is illustrated in Figure 3.

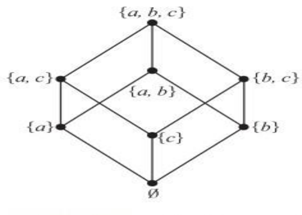


Figure-3

Maximal and Minimal Elements:

Maximal element:

- An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset $(S, \preceq)$ if there is no $b \in S$ such that $a \prec b$.

Minimal element:

An element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$.

Remark: Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

Example-30: Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal?

Solution:

The Hasse diagram in Figure-4 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element.

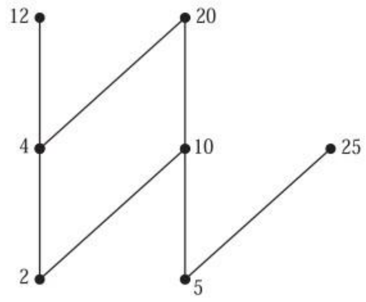


Figure-4

Greatest Element:

- Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, b is the **greatest element** of the poset $(S, \preceq)$ if $a < b$ for all $b \in S$.

Least Element:

- An element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of $(S, \preceq)$ if $a < b$ for all $b \in S$.

Remark: The greatest and least elements are unique when it exists.

Example-31: Determine whether the posets represented by each of the Hasse diagrams in Figure-5 have a greatest element and a least element.

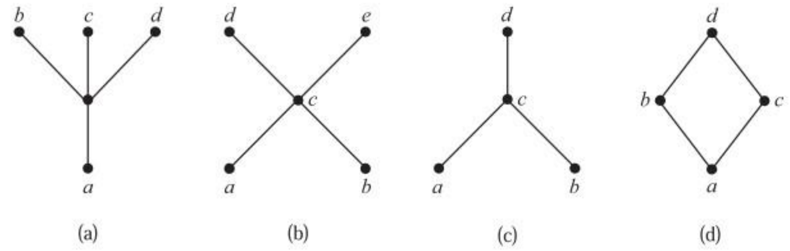


Figure-5

Solution:

- The least element of the poset with Hasse diagram (a) is *a*. This poset has no greatest element.
- The poset with Hasse diagram (b) has neither a least nor a greatest element.
- The poset with Hasse diagram (c) has no least element. Its greatest element is *d*.
- The poset with Hasse diagram (d) has least element *a* and greatest element *d*.

Example-32: Let *S* be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

Solution:

The least element is the empty set, because $\emptyset \subseteq T$ for any subset *T* of *S*. The set *S* is the greatest element in this poset, because $T \subseteq S$ whenever *T* is a subset of *S*.

Example-33: Is there a greatest element and a least element in the poset $(\mathbf{Z}^+, |)$?

Solution:

The integer 1 is the least element because $1|n$ whenever *n* is a positive integer. because there is no integer that is divisible by all positive integers, there is no greatest element.

Upper Bound:

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset *A* of a poset $(S, \leq)$. If *u* is an element of *S* such that $a \leq u$ for all elements $a \in A$, then *u* is called an **upper bound** of *A*.

Lower Bound:

- There may be an element less than or equal to all the elements in *A*. If *l* is an element of *S* such that $l \leq a$ for all elements $a \in A$, then *l* is called a **lower bound** of *A*.

Example-34: Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

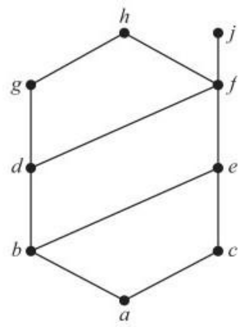


Figure-7

Solution:

- The upper bounds of $\{a, b, c\}$ are *e, f, j,* and *h*, and its only lower bound is *a*.
- There are no upper bounds of $\{j, h\}$, and its lower bounds are *a, b, c, d, e*, and *f*.
- The upper bounds of $\{a, c, d, f\}$ are *f, h,* and *j*, and its lower bound is *a*.

Least Upper Bound:

The element *x* is called the **least upper bound** of the subset *A* if *x* is an upper bound that is less than every other upper bound of *A*. Because there is only one such element, if it exists, it makes the element the least upper bound.

- That is, *x* is the least upper bound of *A* if $a \leq x$ whenever $a \in A$, and $x \leq z$ whenever *z* is an upper bound of *A*.
- The least upper bound of a subset *A* are denoted by *lub(A).*

Greatest Lower Bound:

- The element *y* is called the **greatest lower bound** of *A* if *y* is a lower bound of *A* and $z \leq y$ whenever *z* is a lower bound of *A*.
- The greatest lower bound of a subset *A* are denoted by *glb(A).*

Remarks:

- The greatest lower bound and least upper bound of *A* is unique if it exists.
- If *A* has an upper bound, *A* is bounded above, and if *A* has a lower bound, *A* is bounded below and *A* is bounded if *A* has an upper and lower bound.

Example-35: Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 7.

Solution:

The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound.

Example-9: Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Solution:

An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to | is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$.

Lattices:

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

Example-36: Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

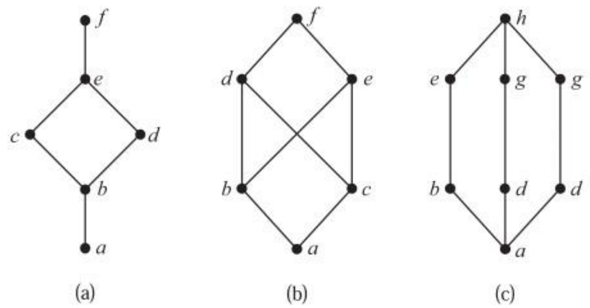


Figure-8

Solution:

The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements *b* and *c* have no least upper bound. To see this, note that each of the elements *d*, *e*, and *f* is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset.

Example-37: Is the poset *(*$\mathbf{Z}^+$*, |)* a lattice?

Solution:

Let *a* and *b* be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice.

Example-38: Determine whether the posets *(*{1, 2, 3, 4, 5}*, |)* and *(*{1, 2, 4, 8, 16}*, |)* are lattices.

Solution:

Because 2 and 3 have no upper bounds in *(*{1, 2, 3, 4, 5}*, |)*, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second poset is a lattice.

Example-39: Determine whether *(P (S), ⊆)* is a lattice where *S* is a set.

Solution:

Let *A* and *B* be two subsets of *S*. The least upper bound and the greatest lower bound of *A* and *B* are $A \cup B$ and $A \cap B$, respectively. Hence, *(P (S), ⊆)* is a lattice.

Exercise-8:

1. Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by the relation "x divides y." Construct Hasse diagram for the relation have a greatest element and a least element.
2. Draw the Hasse diagram for the "greater than or equal to" relation on set $S = \{0, 1, 2, 3, 4, 5\}$.
3. Draw the Hasse diagram for the divisibility on the sets

a. $S=\{2,3,5,10,11,15,25\}$.

b. $S=\{1,2,3,5,7,11,13\}$.

c. $S=\{1,2,3,6,12,24,36,48\}$

4. Answer the following questions for the poset

$(\{\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\},\subseteq)$

a. Find the minimal elements.

b. Find the maximal elements.

c. Is there a greatest element?

d. Is there a least element?

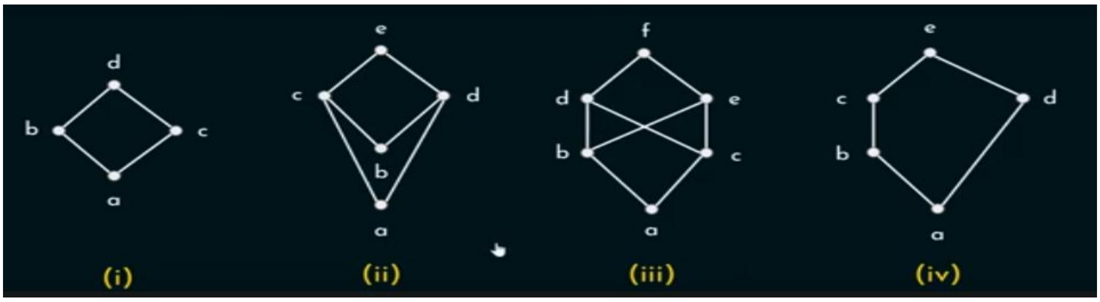
e. Find all upper bounds of $\{\{2\},\{4\}\}$.

f. Find the least upper bound of $\{\{2\},\{4\}\}$, if it exists.

g. Find all lower bounds of $\{\{1,3,4\},\{2,3,4\}\}$.

h. Find the greatest lower bound of $\{\{1,3,4\},\{2,3,4\}\}$, if it exists.

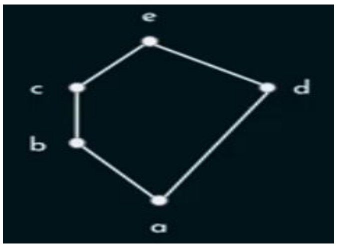
5. Consider the following Hasse diagrams. Which of the following represents a Lattice?



6. Consider the set $X=\{a,b,c,d,e\}$ under partial ordering

$$R=\left\{\begin{matrix}(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,e),(c,c),\\(c,e),(d,d),(d,e),(e,e)\end{matrix}\right\}$$

The Hasse diagram of the partial order (X,R) is shown below:



The minimum number of ordered pairs that need to be added to R to make (X,R) a lattice is?