1) Problem 1 (outer product of 2 vectors)

a) Let $x_1 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$. Calculate matrix $A_1 = x_1 . x_1^T$. Note that A_1 is called the outer product of x_1.

→ Here, Given, $x_1 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}_{3\times1}$

Now, $x_1^T = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}_{1\times3}$

Then, $A_1 = x_1 . x_1^T$

$$= \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}_{3\times1} \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}_{1\times3}$$

[outcome is 3x3 matrix]

$$= \begin{bmatrix} -1.(-1) & -1.1 & -1.-2 \\ 1.-1 & 1.1 & 1.(-2) \\ -2.(-1) & -2.1 & -2.-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

b) In order to identify whether a matrix is PD or PSD, we can perform a determinant test.

\# If all the upper left determinants ≥ 0, it is PSD

\# If all the upper left determinants > 0, it is PD.

we calculate the determinant as:

$[1]$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$

↓ $|1| > 0$

↓ $1\times1 - \{(-1).(-1)\}$

$1-1$

$0 \geq 0$

↓ $1\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} + 1\begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} +$

$2\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix}$

$$1\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} + 1\begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix}$$

$$= 1(4-4) + 1(-4+4) + 2(2-2)$$

$$= 0+0+0$$

$$= 0$$

Since, Det is strictly ≥ 0, A_1 is Positive semi definite.

Test 2

Pivot Test

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

$R_2 \Rightarrow R_1 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$$

$R_3 \Rightarrow 2R_1 - R_3$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$\left.\begin{matrix} a=1 \\ b=0 \\ c=0 \end{matrix}\right\}$ Diagonal elements

Since, $c=0$, $\neq 0$, The matrix is strictly PSD, not PD because all pivots are ≥ 0.

c) Let $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. calculate $A_2 = x_2 \cdot x_2^T$

→ Here,

$$A_2 = x_2 \cdot x_2^T$$

$$= \underset{3\times1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}$$

[∵ Result matrix is 3×3]

$$= \begin{bmatrix} 1.1 & 1.1 & 1.0 \\ 1.1 & 1.1 & 1.0 \\ 0.1 & 0.1 & 0.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d) Define $A = A_1 + A_2$. Is matrix A PSD ? Is matrix A PSD ?

→ Here, $A_1 = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $A = A_1 + A_2 = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

(I) Determinant test

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

$[1]$ ↓ $= 1 > 0$

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ↓ $4 - 0$ $= 4 > 0$

Determinant of : $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix}$

$$\Rightarrow 2\begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & 2 \\ 2 & -2 \end{vmatrix}$$

$$= 2(8-4) + 2(0-4)$$
$$= 2\times 4 + 2(-4)$$
$$= 8\cancel{4} - 8$$
$$= \cancel{4}\ 0 = 0$$

Also, A is PSD if all eigenvalues of A are non-negative i.e $A \geq 0$ & A is PD if $A > 0$, if all eigenvalues of A are +ve.

we have, $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix}$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\text{Now, } A - \lambda I = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & -2 \\ 2 & -2 & 4-\lambda \end{bmatrix}$$

Now, find Determinant of above,

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & 2-\lambda \\ 2 & -2 \end{vmatrix}$$

$$= (2-\lambda)\{(2-\lambda)(4-\lambda) - 4\} + 2(0 - 2(2-\lambda))$$
$$= (2-\lambda)(8 - 2\lambda - 4\lambda + \lambda^2 - 4) + 2(0 - 4 + 2\lambda)$$
$$= (2-\lambda)(\lambda^2 - 6\lambda + 4) + 2(2\lambda - 4)$$
$$= (2-\lambda)(\lambda^2 - 6\lambda + 4) + 4\lambda - 8$$
$$= 2\lambda^2 - 12\lambda + 8 - \lambda^3 + 6\lambda^2 - \cancel{4\lambda} \cancel{+4\lambda} - 8$$

$\Rightarrow = 2\lambda^2 - 12\lambda \cancel{+8} - \lambda^3 + 6\lambda^2 \cancel{-8}$

$= 2\lambda^2 - 12\lambda - \lambda^3 + 6\lambda^2$

Now, To find Eigen values,

$\det(A - \lambda I) = 0$

or, $2\lambda^2 - 12\lambda - \lambda^3 + 6\lambda^2 = 0$

$\Rightarrow 8\lambda^2 - 12\lambda - \lambda^3 = 0$

$\Rightarrow \lambda^3 - 8\lambda^2 + 12\lambda = 0$

So, $P(\lambda) = \lambda^3 - 8\lambda^2 + 12\lambda = 0$

$\lambda(\lambda^2 - 8\lambda + 12) = 0 \longrightarrow \lambda = 0$

$\Rightarrow \lambda^2 - 8\lambda + 12 = 0$

$\Rightarrow \lambda^2 - (6+2)\lambda + 12 = 0$

$\Rightarrow \lambda^2 - 6\lambda - 2\lambda + 12 = 0$

$\Rightarrow \lambda(\lambda - 6) - 2(\lambda - 6) = 0$

$(\lambda - 2)(\lambda - 6) = 0$

$\left.\begin{array}{l}\lambda = 2 \\ \lambda = 6\end{array}\right\}$ Eigen values

$\lambda = 0$

As, Eigenvalues of A are non negative i.e $A \geq 0$, It is PSD not PD //

e) Define $B = A + 0.2I$, where matrix I is a 3×3 Identity matrix. Is B positive definite? why?

→ Here,

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$$0.2I = 0.2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2\times1 & 0.2\times0 & 0.2\times0 \\ 0.2\times0 & 0.2\times1 & 0.2\times0 \\ 0.2\times0 & 0.2\times0 & 0.2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Then,

$$B = A + 0.2I$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2 & 0 & 2 \\ 0 & 2.2 & -2 \\ 2 & -2 & 4.2 \end{bmatrix}$$

To check whether B is PSD or not, we perform determinant Test first.

$$[2.2] \quad \begin{bmatrix} 2.2 & 0 \\ 0 & 2.2 \end{bmatrix} \quad \begin{bmatrix} 2.2 & 0 & 2 \\ 0 & 2.2 & -2 \\ 2 & -2 & 4.2 \end{bmatrix}$$

Det of $[2.2] = |2.2| = 2.2 > 0$

Determinant of $\begin{bmatrix} 2.2 & 0 \\ 0 & 2.2 \end{bmatrix} = \begin{vmatrix} 2.2 & 0 \\ 0 & 2.2 \end{vmatrix}$

$= 4.84 - 0 = 4.84 > 0$

Determinant of : $\begin{bmatrix} 2.2 & 0 & 2 \\ 0 & 2.2 & -2 \\ 2 & -2 & 4.2 \end{bmatrix}$

$$= \begin{vmatrix} 2.2 & 0 & 2 \\ 0 & 2.2 & -2 \\ 2 & -2 & 4.2 \end{vmatrix}$$

$$= 2.2 \begin{vmatrix} 2.2 & -2 \\ -2 & 4.2 \end{vmatrix} + 2 \begin{vmatrix} 0 & 2.2 \\ 2 & -2 \end{vmatrix}$$

$$= 2.2\{9.24 + 4\} + 2\{0 - 4.4\}$$

$$= 2.2 \times 13.24 + 2(-4.4)$$

$$= 29.128 - 8.8$$

$$= 20.328 > 0$$

Since, all the values of Determinants are strictly greater than zero, matrix B is positive Definite.