Shri Vile Parle Kelavani Mandal's

DWARKADAS J. SANGHVI COLLEGE OF ENGINEERING

(Autonomous College Affiliated to the University of Mumbai)

NAAC Accredited with "A" Grade (CGPA : 3.18)

RUBRICS INDEX

Academic Year 2022-23

Name: Prerna Sunil Jadhav
SAP ID: 60004220127

Department /Branch: Computer Engineering
Division:

Course: Engineering Mathematics-III Tutorial
Course Code: DJ19CET301

At the end of the course, the student will be able to...		PO'S	Blooms Level
DJ19CET301.1 CO1	Use Laplace and inverse Laplace Transform to the Ordinary Differential Equations.	PO1 PO2	Understand Apply
DJ19CET301.2 CO2	Expand the periodic function by using Fourier series and complex form of Fourier series.	PO1	Understand
DJ19CET301.3 CO3	Apply Fourier Transform in the future subjects like signal processing.	PO1 PO2	Understand Apply
DJ19CET301.4 CO4	Apply the concept of Z- transformation and its inverse of the given sequence.	PO1	Understand

Performance Indicators (Maximum 5 marks per indicator)	Tut 1	Tut 2	Tut 3	Tut 4	Tut 5	Tut 6	Tut 7	Tut 8	Tut 9	Tut 10	Total
Course Outcome	CO1	CO1	CO1	CO1	CO2	CO2	CO3	CO4			
1. Knowledge											
2. Describe											
3. Demonstration											
4. Interpret / Develop											
5. Attitude towards learning											
Total (out of 25 marks)	21	21	21	21	21	21	21	21			21

Outstanding (5), Excellent (4), Good (3), Fair (2), Needs improvement (1)

8

Signature of the Teacher   Head of the Department   Principal

Name of the Teacher: Dr. (Mrs) Gayatri Pandey

Date: 12/12/22

TUTORIAL 1 : LAPLACE TRANSFORM

Q1 Find the laplace transform of

$$f(t) = \begin{cases} \cos t & , 0 < t < \pi \\ \sin t & , t > \pi \end{cases}$$

Soln:

By definition,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cos t \, dt + \int_\pi^\infty e^{-st} \sin t \, dt \quad \ldots \text{(I)}$$

Now,

$$\int e^{at} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$\int e^{at} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\therefore \int_0^\pi e^{-st} \cos t \, dt = \left[\frac{e^{-st}}{s^2+1}\left[-s\cos t + \sin t\right]\right]_0^\pi$$

$$= \left[\frac{e^{-\pi s}}{s^2+1}(s) - \frac{e^0}{s^2+1}(-s)\right]$$

$$= \frac{s}{s^2+1}\left(e^{-\pi s} + 1\right)$$

$$\int_{\pi}^{\infty} e^{-st} \sin t \, dt = \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_{\pi}^{\infty}$$

$$= \left[0 - \frac{e^{-\pi s}}{s^2+1}\left(-s(0) - (-1)\right)\right]$$

$$= -\frac{e^{-\pi s}}{s^2+1}(1)$$

$$= \frac{-e^{-\pi s}}{s^2+1}$$

From I,

$$L\{f(t)\} = \frac{s}{(s^2+1)}(e^{-\pi s}+1) + \left(-\frac{e^{-\pi s}}{s^2+1}\right)$$

$$= \frac{se^{-\pi/s} + s - e^{-\pi/s}}{s^2+1}$$

$$= \frac{1}{s^2+1}\left[s^2 + (s-1)e^{-\pi/s}\right]$$

Q2 $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

Soln: $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{t^{1/2}} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\left\{\frac{1}{t^{1/2}} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots\right\}$$

Now $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$$= \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \cdots$$

$$= \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1/2\,\Gamma(1/2)}{2!\,s^{3/2}} + \frac{1}{4!}\frac{3/2\cdot 1/2\,\Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2\cdot 3/2\cdot 1/2\,\Gamma(1/2)}{s^{7/2}}$$

$$= \frac{\Gamma(1/2)}{s^{1/2}}\left[1 - \frac{1/2}{2!\,s} + \frac{3/2\times 1/2}{4!\cdot s^2} - \frac{1}{6!}\left(\frac{5/2\times 3/2\times 1/2}{s^3}\right)\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1}{1!(4s)} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + \cdots\right]$$

$$= \sqrt{\frac{\pi}{s}} \left(e^{-1/4s} \right) \left[\because e^{-x} = 1 - \frac{1}{x} + \frac{1}{2!x^2} - \frac{1}{3!x^3} + \cdots \right]$$

Q3 Evaluate by Laplace Transform $\int_0^\infty e^{-3t} \sin^3 t \, dt$

Soln: Now,

$$\int_0^\infty e^{-3t} \sin^3 t \, dt = L\{\sin^3 t\} \quad \ldots s = 3$$

$\because$ By definition

We know,

$$\sin^3 t = \frac{3 \sin t - \sin 3t}{4}$$

So,

$$L\{\sin^3 t\}_{s=3} = L\left\{\frac{3 \sin t - \sin 3t}{4}\right\}$$

$$= \frac{1}{4}\left(L\{3 \sin t\}_{s=3} - L\{\sin 3t\}_{s=3} \right)$$

$$= \frac{1}{4}\left[\left(\frac{3}{s^2+1}\right)_{s=3} - \left(\frac{3}{s^2+9}\right)_{s=3} \right]$$

$$= \frac{3}{4}\left(\frac{18-10}{180}\right)$$

$$= \frac{\cancel{3} \times \cancel{8}^{2}}{\cancel{4} \times \cancel{180}_{30}} = \frac{1}{30}$$

Q4 $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-1/4s}$. Find $L\{\sin(2\sqrt{t})\}$

Soln: $L\{\sin(2\sqrt{t})\} = L\{\sin(\sqrt{4t})\}$

By change of order by scalar

$$= \left(\frac{1}{2 s/4}\sqrt{\frac{\pi}{s/4}}\, e^{-1/4 \times s/4}\right) \times \frac{1}{4}$$

$$= \left(\frac{2}{s} \times 2\sqrt{\frac{\pi}{s}} \times e^{-1/s}\right) \times \frac{1}{4}$$

$$= \frac{1}{s}\sqrt{\frac{\pi}{s}} \times e^{-1s}$$

Here, $f(s) = \frac{1}{2s}\sqrt{\frac{\pi}{s}}$

So, By change of order by scalar

$$L\{\sin(2\sqrt{t})\} = \frac{1}{4} f\left(\frac{s}{4}\right)$$

Q5 $L\{e^{-3t}\cosh 4t \sin 3t\}$

Soln: $L\{\sin 3t\} = \dfrac{3}{s^2+9}$ ....by formula

$L\{e^{-3t}\cosh 4t \sin 3t\}$

$= L\left\{e^{-3t}\left(\dfrac{e^{4t}+e^{-4t}}{2}\right)\sin 3t\right\}$

$= \dfrac{1}{2} L\left\{(e^{t}+e^{-7t})\sin 3t\right\}$

$= \dfrac{1}{2}\left[L(e^{t}\sin 3t) + L(e^{-7t}\cdot\sin 3t)\right]$

By first shifting property

$= \dfrac{1}{2}\left(\dfrac{3}{(s-1)^2+9} + \dfrac{3}{(s+7)^2+9}\right)$

$= \dfrac{1}{2}\left(\dfrac{3}{s^2-2s+10} + \dfrac{3}{s^2+14s+58}\right)$

$= \dfrac{3}{2}\left(\dfrac{s^2+14s+58+s^2-2s+10}{(s^2-2s+10)(s^2+14s+58)}\right)$

$= \dfrac{3}{2}\left(\dfrac{2s^2+12s+68}{(s^2-2s+10)(s^2+14s+58)}\right)$

$$= \frac{3\ (s^2 + 6s + 34)}{(s^2 - 2s + 10)(s^2 + 14s + 58)}$$

TUTORIAL 2 : LAPLACE TRANSFORM

Q1 Evaluate $\int_0^\infty \frac{t^2 \sin t}{e^{2t}}\, dt$

Soln:

$$= \int_0^\infty e^{-2t}\, t^2 . \sin t \, dt$$

$$= L\{t^2 \sin t\}$$

We know,

$$L\{\sin t\} = \frac{1}{s^2+1} \quad \text{.... By definition}$$

By Multiplication of order of power of t

$$L\{t^2 \sin t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{1+s^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s}{(1+s^2)^2}\right)$$

21 d

$$= -1 \left(\frac{(1+s^2)^2(2) - 2(1+s^2)(2s)(2s)}{(1+s^2)^4}\right)$$

$$= -\left(\frac{2 + 2s^2 - 8s^2}{(1+s^2)^3}\right)$$

$$= -\left(\frac{2 + 2s^2 - 8s^2}{(1+s^2)^3}\right)$$

$$= -\left(\frac{2 - 6s^2}{(1+s^2)^3}\right)$$

$$= \frac{6s^2 - 2}{(1+s^2)^3} = f(s)$$

Substituting $s = 2$ in $f(s)$

$$= \frac{6(2)^2 - 2}{(1+2^2)^3} = \frac{24-2}{125} = \frac{22}{125}$$

Q2 Find $L\{e^{-2t} t^4 \sinh(4t)\}$

Soln: we know,

$$\sinh(4t) = \frac{e^{4t} - e^{-4t}}{2}$$

$$L\{e^{-2t} t^4 \sinh(4t)\} = L\left[t^4 e^{-2t}\left(\frac{e^{4t} - e^{-4t}}{2}\right)\right]$$

$$= \frac{1}{2} L\{t^4(e^{2t} - e^{-6t})\}$$

$L\{t^4\} = \frac{4!}{s^5}$ ... By definition

By first shifting property.

$$= \frac{1}{2}\left[\frac{4!}{(s-2)^5} - \frac{4!}{(s+6)^5}\right]$$

$$= \frac{4!}{2}\left[\frac{1}{(s-2)^5} - \frac{1}{(s+6)^5}\right]$$

$$= 12\left[\frac{1}{(s-2)^5} - \frac{1}{(s+6)^5}\right]$$

Q3

Soln:

Now, $L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$

By scalar property,

$$L\{J_0(at)\} = \frac{1}{\sqrt{1+\left(\frac{s}{a}\right)^2}} \times \frac{1}{a}$$

$$= \frac{a}{\sqrt{a^2+s^2}} \times \frac{1}{a} = \frac{1}{\sqrt{s^2+a^2}}$$

By Multiplication by t

$$L\{t\, J_0(at)\} = (-1)\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right)$$

$$= (-1)\left(\frac{-1}{2} \times \frac{2s}{(s^2+a^2)^{3/2}}\right)$$

$$= \frac{s}{(s^2+a^2)^{3/2}}$$

Q4 Prove: $\int_0^\infty e^{-\sqrt{2}t}\, \frac{\sin t \cdot \sinh t}{t}\, dt = \frac{\pi}{8}$

Soln: $\int_0^\infty e^{-\sqrt{2}t}\, \frac{\sin t \cdot \sinh t}{t}\, dt$

$$= \int_0^\infty \frac{e^{-\sqrt{2}t}}{t}\left(\frac{e^{t}-e^{-t}}{2}\right)\sin t\, dt$$

$$= \int_0^\infty \frac{e^{(-\sqrt{2}+1)t}}{2t}\sin t - \frac{e^{(-\sqrt{2}-1)t}}{2t}\sin t\, dt$$

$$= L\left\{\frac{\sin t}{2t}\right\}_{s=\sqrt{2}-1} - L\left\{\frac{\sin t}{2t}\right\}_{s=\sqrt{2}+1} \quad \ldots \text{I}$$

for $L\left\{\frac{\sin t}{2t}\right\}$,

$$L\{\sin t\} = \frac{1}{1+s^2}$$

$$\therefore L\left\{\frac{\sin t}{2t}\right\} = \frac{1}{2}\int_s^\infty \frac{1}{1+s^2}\,ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s)\right]$$

Sustituting $s=\sqrt{2}-1$ & $s=\sqrt{2}+1$ in Eq. I

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(\sqrt{2}-1) - \frac{\pi}{2} + \tan^{-1}(\sqrt{2}+1)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

Q5 $L\left\{\int_0^t u e^{-3u} \sin 4u \, du\right\}$

Soln: for $L\{u e^{-3u} \sin 4u\}$

Now,

$$L\{\sin 4u\} = \frac{4}{s^2+16} \quad \text{.... By definition}$$

So,

$$L\{e^{-3u}\sin 4u\} = \frac{4}{(s+3)^2+16}$$ .... By first shifting property

$$L\{ue^{-3u}\sin 4u\} = (-1)\frac{d}{ds}\left(\frac{4}{s^2+6s+25}\right)$$

... By Multiplication by power of t

$$= -1 \times \frac{(-4(2s+6))}{(s^2+6s+25)^2}$$

$$= \left[\frac{8s+24}{(s^2+6s+25)^2}\right]$$

By Laplace of Integration

$$L\left\{\int_0^t ue^{-3u}\sin 4u\,du\right\} = \frac{1}{s}\left[\frac{8s+24}{(s^2+6s+25)^2}\right]$$

TUTORIAL 3 : INVERSE LAPLASE TRANSFORM

Q1 $L^{-1}\left\{\dfrac{3s+5}{9s^2-25}\right\}$, find the Inverse laplace.

Soln:

$$L^{-1}\left\{\frac{3s+5}{9s^2-25}\right\} = L^{-1}\left\{\frac{3s}{9s^2-25}\right\} + L^{-1}\left\{\frac{5}{9s^2-25}\right\}$$

$$= 3\,L^{-1}\left\{\frac{s}{9(s^2-25/9)}\right\} + 5\,L^{-1}\left\{\frac{1}{9(s^2-25/9)}\right\}$$

$$= \frac{1}{3}\,L^{-1}\left\{\frac{s}{s^2-25/9}\right\} + \frac{5}{9}\,L^{-1}\left\{\frac{1}{s^2-25/9}\right\}$$

$$= \frac{1}{3}\,L^{-1}\left\{\frac{s}{s^2-(5/3)^2}\right\} + \frac{5}{9}\,L^{-1}\left\{\frac{1}{s^2-(5/3)^2}\right\}$$

We know,

$$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh(at)}{a}, \quad L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$$

2

$$L^{-1}\left\{\frac{3s+5}{9s^2-25}\right\} = \frac{1}{3}\cosh\left(\frac{5}{3}t\right) + \frac{5}{9}\sinh\left(\frac{5}{3}t\right) \times \frac{1}{5/3}$$

$$= \frac{1}{3}\cosh\left(\frac{5t}{3}\right) + \frac{1}{3}\sinh\left(\frac{5t}{3}\right)$$

$$L^{-1}\left\{\frac{3s+5}{9s^2-25}\right\} = \frac{1}{3}\left[\cosh\left(\frac{5t}{3}\right) + \sinh\left(\frac{5t}{3}\right)\right]$$

$$= \frac{1}{3} e^{5/3 t}$$

Q2 find $L^{-1}\left\{\frac{s+2}{s^2+4s+7}\right\}$

Soln: $$L^{-1}\left\{\frac{s+2}{s^2+4s+7}\right\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+3}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s+2)^2+(\sqrt{3})^2}\right\}$$

By first shifting property

$$L^{-1}\left\{\frac{s+2}{s^2+4s+7}\right\} = e^{-2t} \, L^{-1}\left\{\frac{s}{s^2+(\sqrt{3})^2}\right\}$$

Now,

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

$$L^{-1}\left\{\frac{s+2}{s^2+4s+7}\right\} = e^{-2t} \cos(\sqrt{3}\,t) \quad (\because a = \sqrt{3})$$

Q3 find $L^{-1}\left\{\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$

Soln: SQ, using partial ~~fractions~~ functions.

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1}$$

$$\therefore 5s^2+8s-1 = A(s^2+1)+(Bs+C)(s+3)$$

Put $s=(-3)$

$$20 = 10(A) \Rightarrow \boxed{A=2}$$

Put $s=0$

$$-1 = A + C(3)$$

$$3C = -3 \Rightarrow \boxed{C=-1}$$

Put $s=1$

$$12 = 2(A) + (B+C)(4)$$

$$12 = 4+(B-1)4$$

$$4(B-1) = 8 \Rightarrow B-1 = 2 \Rightarrow \boxed{B=3}$$

$$\therefore \frac{5s^2+8s-1}{(s+3)(s^2+1)} = \frac{2}{s+3} + \frac{3s-1}{s^2+1}$$

So,

$$L^{-1}\left\{\frac{2}{s+3} + \frac{3s-1}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2}{s+3}\right\} + 3L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

... By Linearity Property

Now,

$$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}, \quad L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at,$$

$$L^{-1}\left\{\frac{a1}{s^2+a^2}\right\} = \frac{\sin(at)}{a}$$

$$\therefore L^{-1}\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\} = 2e^{-3t} + 3\cos t - \sin t$$

$$\therefore L^{-1}\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\} = 2e^{-3t} + 3\cos t - \sin t$$

Q4 Find $L^{-1}\{\cot^{-1}(s+1)\}$.

Soln: We know,

$$L^{-1}\{f(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds} f(s)\right\}$$

$$\therefore L^{-1}\{\cot^{-1}(s+1)\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\cot^{-1}(s+1)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left(\frac{\pi}{2} - \tan^{-1}(s+1)\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{-1}{1+(s+1)^2}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

By first shifting property

$$= \frac{1}{t} e^{-t} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

We know, $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

$$= \frac{1}{t} e^{-t} \sin t$$

$$\therefore L^{-1}\left\{\cot^{-1}(s+1)\right\} = \frac{e^{-t}\sin t}{t}$$

Q5 Find using Convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\}$$

Soln:

$$L^{-1}\left\{\frac{s}{(s^2+4)} \times \frac{1}{(s^2+1)}\right\}$$

Here,

$$F(s) = \frac{s}{s^2+4} \quad \& \quad G(s) = \frac{1}{s^2+1}$$

so, $f(t) = \cos(2t)$ & $g(t) = \sin t$

$\therefore$ By convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} = f(t) * g(t) \quad \ldots \text{(I)}$$

where

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$\therefore \cos(2t) * \sin t = \int_0^t \cos(2u) \sin(t-u)\, du$$

$$= \int_0^t \cos(2u)\left[\sin t \cos u - \cos t \sin u\right] du$$

$$= \int_0^t \cos 2u \cos u \sin t\, du - \int_0^t \cos 2u \sin u \cos t\, du$$

$$= \frac{\sin t}{2}\left[\frac{\sin 3x}{3} + \frac{\sin x}{1}\right]_0^t$$

$$- \frac{\cos t}{2}\left[-\frac{\cos 3u}{3} + \frac{\cos x}{1}\right]_0^t$$

$$= \frac{\sin t}{2}\left[\frac{\sin 3t}{3} + \frac{\sin t}{1}\right] - \frac{\cos t}{2}\left[\left(\cos t - \frac{\cos 3t}{3}\right) \frac{-2}{3}\right]$$

$$= \frac{\sin t \sin 3t}{2 \times 3} + \frac{\sin^2 t}{2} - \frac{(\cos t)^2}{2} + \frac{\cos t \cos 3t}{6} + \frac{\cos t}{3}$$

$$= \frac{\cos 2t}{6} + \frac{\cos t}{3} - \frac{\cos 2t}{2}$$

$$= \frac{\cos t}{3} - \frac{\cos 2t}{3}$$

$$= \frac{1}{3}(\cos t - \cos 2t)$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} = \frac{1}{3}(\cos t - \cos 2t)$$

Q6 Find using convolution theorem

$$L^{-1}\left\{\frac{1}{(s-2)^4(s+3)}\right\}$$

Soln:

$$L^{-1}\left\{\frac{1}{(s-2)^4(s+3)}\right\}$$

Here,

$$f(s) = \frac{1}{(s-2)^4}, \quad G(s) = \frac{1}{(s+3)}$$

So, $f(t) = e^{2t} L^{-1}\left\{\frac{1}{s^4}\right\}$, $G(t) = e^{-3t}$

..... By first shifting property

i.e. $f(t) = e^{2t}\dfrac{t^3}{6}$, $g(t) = e^{-3t}$

So, By convolution theorem

$$L^{-1}\left\{\frac{1}{(s-2)^4(s+3)}\right\} = f(t) * g(t)$$

where,

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$\therefore \left(e^{2t}\frac{t^3}{6}\right) * \left(e^{-3t}\right) = \int_0^t e^{2u}\frac{u^3}{6} \times e^{-3(t-u)}\, du$$

$$= \int_0^t e^{2u}\frac{u^3}{6}\left(e^{-3t}e^{3u}\right) du$$

$$= e^{-3t}\int_0^t \frac{u^3}{6} e^{5u}\, du$$

$$= e^{-3t}\left\{\left[\frac{u^3}{6}\left(\frac{e^{5u}}{5}\right)\right]_0^t - \int_0^t \frac{3u^2}{6}\left(\frac{e^{5u}}{5}\right) du\right\}$$

$$= e^{-3t}\left\{\left[\frac{u^3}{6}\left(\frac{e^{5u}}{5}\right)\right]_0^t - \left[\frac{1}{2}u^2\frac{e^{5u}}{25}\right]_0^t + \int_0^t u\frac{e^{5u}}{25}\, du\right\}$$

$$= e^{-3t}\left\{\left[\frac{u^3}{6}\times\frac{e^{5u}}{5}\right]_0^t - \left[\frac{u^2}{2}\,\frac{e^{5u}}{25}\right]_0^t + \left[u\frac{e^{5u}}{125}\right]_0^t - \left[\frac{e^{5u}}{625}\right]_0^t\right\}$$

$$= e^{-3t}\left\{\frac{t^3e^{5t}}{30} - \frac{t^2e^{5t}}{50} + \frac{te^{5t}}{125} - \frac{e^{5t}}{625} + \frac{1}{625}\right]$$

$$= \frac{e^{-3t}}{625} + e^{2t}\left[\frac{t^3}{30} - \frac{t^2}{50} + \frac{t}{125} - \frac{1}{625}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$$

$$= \frac{e^{-3t}}{625} + e^{2t}\left[\frac{t^3}{30} - \frac{t^2}{50} + \frac{t}{125} - \frac{1}{625}\right]$$

TUTORIAL 4: APPLICATION, HEAVISIDE & DIRECT DELTA

Q1

Soln: $(D^2 - D - 2)y = 20\sin 2t$, $y(0) = 1$, $y'(0) = 2$

Taking Laplace Transform on both side

$$L\{(D^2 - D - 2)y\} = L\{20 \sin 2t\}$$

$$L\{D^2y - Dy - 2y\} = L\{20 \sin 2t\}$$

$$L\{D^2y\} - L\{Dy\} - 2L\{y(t)\} = \frac{20 \times 2}{s^2+4}$$

$$s^2 L\{y(t)\} - sy(0) - y'(0) - sL[y(t)] + y(0) - 2L\{y(t)\} = \frac{40}{s^2+4}$$

$$L\{y(t)\}[s^2 - s - 2] - s - 2 + 1 = \frac{40}{s^2+4}$$

$$L\{y(t)\}(s^2 - s - 2) = \frac{40}{s^2+4} + (s+1)$$

$$L\{y(t)\} = \frac{40}{(s^2+4)(s^2-s-2)} + \frac{(s+1)}{(s^2-s-2)}$$

$$= \frac{40 + (s+1)(s^2+4)}{(s^2+4)(s^2-s-2)}$$

$$= \frac{40 + s^3 + 4s + s^2 + 4}{(s^2+4)(s^2-s-2)}$$

$$= \frac{s^3 + s^2 + 4s + 44}{(s^2+4)(s-2)(s+1)}$$

Applying partial function

$$\frac{s^3+s^2+4s+44}{(s+1)(s-2)(s^2+4)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4}$$

$$s^3+s^2+4s+44 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (s+D)(s+1)(s-2)$$

Put $s=2$,

$$8+4+8+44 = B(3)(8) \Rightarrow B = \frac{64}{24} = \frac{8}{3} \Rightarrow \boxed{B = \frac{8}{3}}$$

Put $s=-1$,

$$-1+1-4+44 = A(-3)(5) \Rightarrow A = \frac{40}{5\times(-3)} \Rightarrow \boxed{A = -\frac{8}{3}}$$

Put $s=0$

$$44 = A(-2)(4) + B(4) + D(-2)$$

$$44 = -\frac{8}{3}\times -8 + \frac{8}{3}\times 4 - 2D$$

$$12 = -20 \Rightarrow \boxed{D = -6}$$

Put $s=1$

$$50 = A(-1)(5) + B(2)(5) + (C+D)(2)(-1)$$

$$50 = -5A + 10B + (C-6)(-2)$$

$$50 = \frac{-5\times -8}{3} + \frac{10\times 8}{3} + (-2C) + 12$$

$$-2 = -2C \Rightarrow \therefore \boxed{C=1}$$

$$\therefore L\{y(t)\} = \frac{-8}{3(s+1)} + \frac{8}{3(s-2)} + \frac{s-6}{s^2+4}$$

Taking inverse Laplace Transform

$$y(t) = \frac{-8}{3} L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{8}{3} L^{-1}\left\{\frac{1}{s-2}\right\} + L^{-1}\left\{\frac{s-6}{s^2+4}\right\}$$

$$y(t) = \frac{-8}{3} e^{-t} + \frac{8}{3} e^{2t} + \cos 2t - 3\sin 2t$$

$$\left[\because L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right.$$

$$\left. L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at,\ L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}\right]$$

Q2

Soln: $2x^2 + y' = 5e^t$, $x(0) = 0$, $y(0) = 0$
$y' - 3x' = 5$
Solving both equations simultaneously
Taking Laplace transform on both side of the equation

$$L\{2x' + y'\} = L\{5e^t\}, \quad L\{3x' + y'\} = L\{5\} \quad \ldots (1)$$

$$2L\{x'\} + L\{y\} = 5L\{e^t\}$$

$$2\{sL[x(t)] - x(0)] + sL[y(t) - y(0)] = \frac{5}{s-1}$$

$$\because L[e^{at}] = \frac{1}{s-a}$$

$$2sL\{x(t)\} + sL\{y(t)\} = \frac{5}{s-1}$$

Let $L\{x(t)\}$ be X & $L\{y(t)\}$ be Y

$$\therefore 2sX + sY = \frac{5}{s-1} \quad \ldots (2)$$

from 1,

$$L\{-3x' + y'\} = L\{5\}$$

$$-3L\{x'\} + L\{y'\} = 5L\{1\}$$

$$-3\{sL[x(t)] - x(0)] + [sL[y(t)] - y(0)] = \frac{5}{s}$$

$$-3sL[x(t)] - sL[y(t)] = \frac{5}{s}$$

Let $L[x(t)]$, $L[y(t)]$ be X, Y respectively

$$-3s(X) + s(Y) = \frac{5}{s} \quad \ldots (3)$$

Subtracting (2) & (3)

$$\begin{array}{rcl} X(2s) + Y(s) & = & 5/s - 1 \\ X(-3s) + Y(s) & = & 5/s \\ - \quad\quad - & & - \\ \hline X(5s) & = & \frac{5}{s-1} - \frac{5}{s} \end{array}$$

Now, $X = L\{x(t)\}$

$$L\{x(t)\}(5s) = \frac{5}{s-1} - \frac{5}{s}$$

$$L\{x(t)\} = \frac{1}{s(s-1)} - \frac{1}{s^2}$$

$$= \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \quad \ldots (4)$$

Taking Inverse Laplace Transform.

$$x(t) = L^{-1}\left[\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right]$$

$$x(t) = e^{-t} - 1 - t$$

i.e. $x(t) = e^{-t} - t - 1$

from 2,

$$Y(s) = \frac{5}{s-1} - X(2s)$$

Putting Y, X as $L\{y(t)\}$ & $L\{x(t)\}$ resp.

$$(s) L\{y(t)\} = \frac{5}{s-1} - L\{x(t)\}(2s)$$

As $L\{x(t)\} = \dfrac{1}{s^2(s-1)}$

$$(s) L\{y(t)\} = \frac{5}{s-1} - \frac{1}{s^2(s-1)}(2s)$$

$$(s) L\{y(t)\} = \frac{5}{s-1} - \frac{2}{s(s-1)}$$

$$(s) L\{y(t)\} = \frac{5}{s-1} - 2\left[\frac{1}{s-1} - \frac{1}{s}\right]$$

$$L\{y(t)\} = \frac{3}{s(s-1)} + \frac{2}{s^2}$$

$$L\{y(t)\} = 3\left[\frac{1}{s-1} - \frac{1}{s}\right] + \frac{2}{s^2}$$

Taking Inverse Laplace Transform

$$y(t) = 3\,L^{-1}\left[\frac{1}{s-1} - \frac{1}{s}\right] + 2\,L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = 3e^{t} - 3 + 2t$$

$$y(t) = 3e^{t} + 2t - 3$$

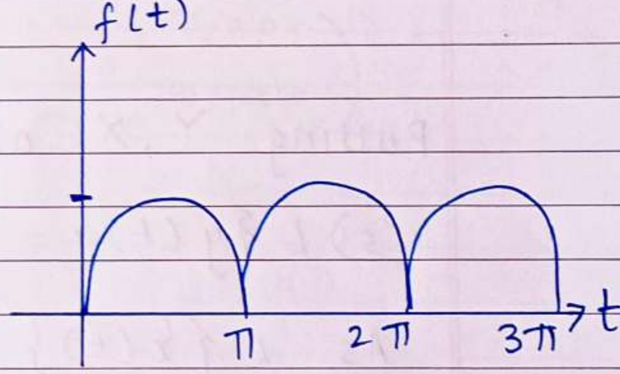
$$\therefore\ x(t) = e^{t} - t - 1$$

$$\&\ y(t) = 3e^{t} + 2t - 3$$

Q3

Soln: $f(t) = |\sin pt|,\ t \geq 0$

here $|\sin pt|$ is a periodic function



$$f(t + \pi/p) = f(t),\ \forall t$$

$$\therefore\ \text{period } (T) = \pi/p$$

For periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-s\pi/p}}\int_0^{\pi/p} e^{-st}\sin pt\,dt$$

$$= \frac{1}{1-e^{-s\pi/p}}\left[\frac{e^{-st}}{s^2+p^2}(-s\sin pt - p\cos pt)\right]_0^{\pi/p}$$

$$= \frac{1}{1-e^{-s\pi/p}}\left[\frac{e^{-\pi s/p}}{s^2+p^2}[p] - \frac{(-p)}{s^2+p^2}\right]$$

$$= \frac{1}{1-e^{-s\pi/p}} \times \frac{p}{s^2+p^2}\left[e^{-\pi s/p}+1\right]$$

$$= \frac{p}{s^2+p^2} \frac{(1+e^{-\pi s/p})}{(1-e^{-\pi s/p})}$$

$$= \frac{p}{s^2+p^2}\left(\frac{e^{\pi s/2p}+e^{-\pi s/2p}}{e^{\pi s/2p}-e^{-\pi s/2p}}\right)$$

$$= \frac{p}{s^2+p^2} \cosh\left(\frac{\pi s}{2p}\right)$$

Q4

Soln: $\int_0^\infty e^{-t}(1+2t-t^2+t^3)\cdot H(t-1)\,dt$

By definition,

$L\{(1+2t-t^2+t^3)\cdot H(t-1)\}_{s=1}$

Now, $L\{f(t)\cdot H(t-a)\} = e^{-as}[f(t-a)]$

$\therefore L\{(1-2t-t^2+t^3)\cdot H(t-1)\}$

$= e^{-s}\, L\{1+2(t+1)-(t+1)^2+(t+1)^3\}$

$= e^{-s} \mathcal{L}\{1 + 2t + 2 - t^2 - 2t - 1 + t^3 + 1 + 3t^2 + 3t\}$

$= e^{-s} \mathcal{L}\{t^3 + 2t^2 + 3t + 3\}$

$= e^{-s}\left[\frac{3!}{s^4} + 2\cdot\frac{2!}{s^3} + 3\cdot\frac{1!}{s^2} + \frac{3}{s}\right]$

Now putting $s = 1$

$= e^{-1}[3! + 4 + 3 + 3] = e^{-1}(16) = 16/e$

$\therefore \int_0^\infty e^{-t}(t^3 - t^2 + 2t + 1)\cdot H(t-1)\,dt = \frac{16}{e}$

Q5

Soln: $\mathcal{L}\{\cos t\,[H(t-\pi/2) - H(t - 3\pi/2)]\}$

$= \mathcal{L}\{\cos t\,[H(t-\pi/2)]\} - \mathcal{L}\{\cos t\,[H(t-3\pi/2)]\}$

Now $\mathcal{L}\{f(t)\,H(t-a)\} = e^{-as} f(s+a)$

$= e^{-s\pi/2} \mathcal{L}\{\cos(t + \pi/2)\} - e^{-3\pi s/2} \mathcal{L}\{\cos(t - 3\pi/2)\}$

$= e^{-s\pi/2} \mathcal{L}\{-\sin t\} - e^{-3\pi s/2} \mathcal{L}\{\sin t\}$

$= -e^{-s\pi/2}\left(\frac{1}{s^2+1}\right) - e^{-3\pi s/2}\left(\frac{1}{s^2+1}\right)$

$$= \frac{-1}{s^2+1}\left[e^{\pi s/2} + e^{3\pi s/2} \right]$$

Q6

Soln: $L\{t^2 \cdot H(t-2) - \cosh t\ \delta(t-4)\}$

Now,

$L\{f(t)\ H(t-a)\} = e^{-as} L\{f(t-a)\}$

$L\{f(t)\ \delta(t-a)\} = e^{-as} f(a)$

$\therefore L\{t^2\ H(t-2)\} - L\{\cosh t\ \delta(t-4)\}$

$= e^{-2s} L\{(t+2)^2\} - e^{-4s}\cosh 4$

$= e^{-2s} L\{t^2+2t+4\} - e^{-4s}\cosh 4$

$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cosh 4$

$\therefore L\{t^2\ H(t-2) - \cosh t\ \delta(t-4)]$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s}\cosh 4$$

TUTORIAL 5: FOURIER SERIES

Q1 Find fourier series for $f(x)$ in $(0, 2\pi)$

$$f(x) = \begin{cases} x & 0 < x \leq \pi \\ 2\pi - x & \pi \leq x \leq 2\pi \end{cases}$$

Soln:

$$f(x) = \begin{cases} x & 0 < x \leq \pi \\ 2\pi - x & \pi \leq x \leq 2\pi \end{cases}$$

Here fourier series is $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{x^2}{2} \right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2} \right]_{\pi}^{2\pi} \right]$$

21

$$= \frac{1}{\pi} \left\{ \left[\frac{\pi^2}{2} \right] + \left[4\pi^2 - \frac{4\pi^2}{2} - \left(2\pi^2 - \frac{\pi^2}{2} \right) \right] \right\}$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right] = \frac{1}{\pi}\times\pi^2 = \boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\, dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\, dx + \left[\frac{(2\pi-x)\sin nx}{n}\right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}\frac{\sin nx}{n}\, dx\right\}$$

$$= \frac{1}{\pi}\left\{0 + \left[\frac{\cos nx}{n^2}\right]_0^{\pi} + 0 - \left[\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \left[\frac{\cos 2n\pi}{n^2} - \frac{\cos n\pi}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{2\cos(n\pi)}{n^2} - \frac{\cos(2n\pi)}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n - 2}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1.\right]$$

$$= \begin{cases} \dfrac{-4}{\pi n^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right.$$

$$\left. - \left[\frac{(2\pi - x)\cos nx}{n}\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n\pi)}{\pi} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi} + \frac{\pi\cos n\pi}{\pi}\right.$$

$$\left. - \left[\frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[0\right] = \boxed{0 = b_n}$$

$\therefore$ fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$f(x) = \frac{\pi}{2} + \left[\frac{-4}{\pi}\cos x - \frac{4}{\pi 9}\cos 3x - \frac{4}{\pi 25}\cos 5x \ldots\right]$$

(If n is odd

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

As for n = even

$$\sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx = 0$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Q2 Obtain fourier series

$$f(x) = \begin{cases} \pi/2 + x & -\pi < x < 0 \\ \pi/2 - x & 0 < x < \pi \end{cases}$$

Deduce in

(1) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

(2) $\frac{\pi^2}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

Solu:

$$f(x) = \begin{cases} \pi/2 + x, & -\pi < x < 0 \\ \pi/2 - x, & 0 < x < \pi \end{cases} \quad \cdots (1)$$

$$f(-x) = \begin{cases} \pi/2 - x, & \pi > x > 0 \\ \pi/2 + x, & 0 > x > -\pi \end{cases}$$

~~ff~~ Now Here

$f(x) = f(-x)$

$\therefore f(x)$ is even in $(-\pi, \pi)$ $\boxed{\therefore b_n = 0}$

So, fourier series for $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cancel{+} \cancel{b_n \sin nx}$$

where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{\pi}{2} - x \, dx$$

$$= \frac{2}{\pi} \left[\frac{\pi x}{2} - \frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{2} - \frac{\pi^2}{2} \right] = 0$$

$\boxed{\therefore a_0 = 0}$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x \right) \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}-x\right)\frac{\sin nx}{n} - \left(\cancel{\frac{x}{2}} - 1\right)\frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\cos n\pi}{n^2} + \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-(-1)^n + 1}{n^2}\right]$$

$$= \frac{-2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 4/\pi n^2 & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases}$$

$\therefore$ fourier series for $f(x)$ in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= 0 + \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \cos nx$$

$$\left[\text{as for } n = \text{even} \quad \sum_{n=1}^{\infty} a_n \cos nx = 0\right]$$

$$f(x) = \frac{4}{\pi}\cos x + \frac{4}{\pi 3^2}\cos 3x + \frac{4}{\pi 5^2}\cos 5x + \dots \quad -(2)$$

Put $x = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x \to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{\pi}{2}$$

$$\therefore \frac{\pi}{2} = \frac{4}{\pi} + \frac{4}{\pi 3^2} + \frac{4}{\pi 5^2} + \dots \quad \text{from } 2$$

(1) $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Hence proved

(2) By Parseval's identity for even function

$$\frac{2}{\pi}\int_0^{\pi} (f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)^2 dx = \frac{16}{\pi^2} + \frac{16}{\pi^2 3^4} + \frac{16}{\pi^2 5^4} + \dots$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{4}+x^2-\pi x\right)dx = \frac{16}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$$

$$\therefore \frac{\pi}{8}\left[\frac{\pi^3}{4}+\frac{\pi^3}{3}-\frac{\pi^3}{2}\right] = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

(2) $$\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

Hence deduced.

Q3 Find fourier ~~ser~~ expansion of $f(x) = 4 - x^2$ in $(0,2)$. Also state the values for $x = 0, 1, 2, 10, 11$. Hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

Solu:

Here $2l = 2$, $\boxed{l=1}$

fourier series in $(0,2)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right)+b_n\sin\left(\frac{n\pi x}{l}\right)\right] \quad \cdots (1)$$

where,

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\int_0^2 4-x^2\,dx$$

$$= \frac{1}{2}\left[4x-\frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[8-\frac{8}{3}\right] = \boxed{\frac{8}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 1\int_0^2 (4-x^2)\cos(n\pi x)\,dx$$

$$= \left[(4-x^2)\frac{\sin n\pi x}{n\pi} - (-2x)\frac{-\cos n\pi x}{n^2\pi^2} + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[-2x\frac{\cos n\pi x}{n^2\pi^2}\right]_0^2$$

$$= \frac{-2}{n^2\pi^2}\left[x\cos n\pi x\right]_0^2$$

$$= \frac{-2}{n^2\pi^2}\left[2\cos 2n\pi - 0\right]$$

$$= \frac{-2}{n^2\pi^2}\left[2(1)\right] = \frac{-4}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^2 (4-x^2)\sin n\pi x\,dx$$

$$= \left[(4-x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{+\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left\{ \left[(4-(2)^2)\left(-\frac{\cos 2n\pi}{n\pi}\right) - 2\frac{\cos 2n\pi}{n^3\pi^3} \right] - \right.$$

$$\left. \left[4 - 0^2 \left(\frac{-\cos 0}{n\pi}\right) - 2\frac{\cos 0}{n^3\pi^3} \right] \right\}$$

$$= \left\{ \left[0 - \frac{2(1)}{n^3\pi^3} \right] - \left[\frac{-4}{n\pi} - \frac{2}{n^3\pi^3} \right] \right\}$$

$$= \left[\cancel{\frac{-2}{n^3\pi^3}} + \frac{4}{n\pi} + \cancel{\frac{2}{n^3\pi^3}} \right]$$

$$\boxed{b_n = \frac{4}{n\pi}}$$

from 1,

$$f(x) = \frac{8}{3} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2\pi^2} \cos(n\pi x) + \frac{4}{n\pi} \sin(n\pi x) \right)$$

$$= \frac{8}{3} + \left(\frac{-4}{\pi^2}\right)\left[\frac{1}{1^2}\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \dots \right]$$

$$+ \frac{4}{\pi}\left[1(\sin \pi x) + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \dots \right]$$

Now $f(x) = 4 - x^2$

$$4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\left[1\cos\pi x + \frac{1}{4}\cos 2\pi x + \cdots\right]$$
$$+ \frac{4}{\pi}\left[1\sin\pi x + \frac{1}{2}\sin 2\pi x + \cdots\right]$$

Put $x = 0$

$$4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$
$$\frac{-1}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \cdots (2)$$

Put $x = 1$

$$\frac{1}{3} = \frac{-4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$
$$\frac{-\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Put $x = 10$

$$4 - 100 - 8/3 = -4/\pi^2\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$
$$\frac{-246}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

Put $x = 11$

$$4 - 121 - \frac{8}{3} = -\frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$
$$\frac{359}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

Add eq ② & eq ③

$$\frac{-1}{3}+\frac{2}{3}=\frac{2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{1}{3}=\frac{2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

Hence deduced.

Q4 Obtain half range sine series to represent

$$f(x)=\begin{cases} 2x/3 & , \quad 0\leq x\leq \pi/3 \\ \pi-x/3 & , \quad \pi/3\leq x\leq \pi \end{cases}$$

Soln:

$$f(x)=\begin{cases} 2x/3 & 0\leq x\leq \pi/3 \\ \pi-x/3 & \pi/3\leq x\leq \pi \end{cases}$$

Half Range sine series in $(0,\pi)$ is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin(nx) \quad \cdots ①$$

where,

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n=\frac{2}{\pi}\left[\int_0^{\pi/3}\frac{2x}{3}\sin(nx)\,dx+\int_{\pi/3}^{\pi}\left(\frac{\pi-x}{3}\right)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{3}\left[x\frac{\cos nx}{n}\right]_0^{\pi/3} + \int_0^{\pi/3}\frac{\cos nx}{n}\right] +$$

$$\frac{1}{3}\left[\frac{2\pi}{3n}\cos\frac{n\pi}{3} - \left[\frac{\sin(nx)}{n^2}\right]_{\pi/3}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{3}\left[\frac{-\pi}{3n}\cos\left(\frac{n\pi}{3}\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi/3}\right] +\right.$$

$$\left.\frac{1}{3}\left[\frac{2\pi}{3n}\cos\left(\frac{n\pi}{3}\right) - \left[\frac{\sin nx}{n^2}\right]_{\pi/3}^{\pi}\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{3}\left[\frac{-\pi}{3n}\cos\left(\frac{n\pi}{3}\right) + \frac{\sin n\pi/3}{n^2}\right] + \frac{1}{3}\left[\frac{2\pi}{3n}\cos\frac{n\pi}{3}\right.\right.$$

$$\left.\left. + \frac{\sin n\pi/3}{n^2}\right]\right]$$

$$b_n = \frac{2}{\pi}\left[\left[\frac{-2\pi}{9n}\cos\left(\frac{n\pi}{3}\right) + \frac{2}{3n^2}\sin n\pi/3\right] +\right.$$

$$\left.\frac{2\pi}{9n}\cos\left(\frac{n\pi}{3}\right) + \frac{\sin n\pi/3}{3n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{\pi n}{3}\right)\right]$$

$\therefore$ Half range sine series is

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\sin\left(\frac{n\pi}{3}\right)\sin(nx) \quad (\text{from } 2)$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{3}\right)\sin(nx)$$

Q5 Find half range cosine series for $f(x) = e^x$, $0 < x < 1$

Soln:

$f(x) = e^x$

Half Range cosine series in $(0,1)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \text{ Here } l = 1$$

where,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{1}\int_0^1 e^x\,dx = \left[2e^x\right]_0^1$$

$$\boxed{\therefore a_0 = 2e - 2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 e^x \cos n\pi x\,dx$$

$$= 2\left[\frac{e^x}{1+(n\pi)^2}\left[\cos(n\pi x) + n\pi \sin(n\pi x)\right]\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}(-1)^n - \frac{1}{1+n^2\pi^2}\right]$$

$$= \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]$$

$$\therefore f(x) = (e-1) + \sum_{n=1}^{\infty} \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]\cos(n\pi x)$$

$$= (e-1) + 2\sum_{n=1}^{\infty} \frac{\left[(-1)^n e - 1\right]}{1+n^2\pi^2}\cos(n\pi x)$$

TUTORIAL 6:

Q1 Find the complex form of fourier series of

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 1, & 1 < x < 2 \end{cases}$$

Soln: ~~f(x)~~ In the range $x \in (c, c+2l)$, complex form of fourier integral is given as

$$f(x) = c_0 + \sum_{n=-\infty}^{n=\infty} c_n \, e^{in\pi x/l}, \quad \dots (1)$$

where

$$c_n = \frac{1}{2l} \int_c^{c+2l} f(x) \, e^{-in\pi x/l} \, dx$$

$$c_n = \frac{1}{2l} \int_c^{c+2l} f(x) \, e^{-in\pi x/l} \, dx, \text{ here } l = 1$$

$$c_n = \frac{1}{2} \int_0^2 f(x) \, e^{-in\pi x} \, dx$$

$$c_n = \frac{1}{2} \left[\int_0^1 x^2 e^{-in\pi x} \, dx + \int_1^2 e^{-in\pi x} \, dx \right]$$

$$c_n = \frac{1}{2} \left[\left[\frac{x^2 e^{-in\pi x}}{-in\pi} - \frac{2x e^{-in\pi x}}{(-in\pi)^2} + \frac{2 e^{-in\pi x}}{(-in\pi)^3} \right]_0^1 + \left[\frac{e^{-in\pi x}}{-in\pi} \right]_1^2 \right]$$

$$C_n = \frac{1}{2}\left[-\frac{e^{-in\pi x}}{in\pi} + \frac{2e^{-in\pi}}{n^2\pi^2} - \frac{2e^{-in\pi}}{in^3\pi^3} + \frac{2}{in^3\pi^3} - \frac{e^{-i2n\pi}}{in\pi} + \frac{e^{-in\pi}}{in\pi}\right]$$

$$C_n = \frac{1}{2}\left[\frac{2e^{-in\pi}}{n^2\pi^2} + \frac{2ie^{-in\pi} - 2i}{n^3\pi^3} + \frac{ie^{-i2n\pi}}{n\pi}\right]$$

Now, $e^{-in\pi} = (-1)^n$ & $e^{-2in\pi} = +1$

$$C_n = \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i((-1)^n - 1)}{n^3\pi^3} + \frac{i}{n\pi}\right]$$

and

$$C_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 x^2\,dx + \int_1^2 dx\right]$$

$$= \frac{1}{2}\left[\left[\frac{x^3}{3}\right]_0^1 + [x]_1^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} + 1\right] = \frac{2}{3}$$

Substituting C_0 & C_n in eq ①

$$f(x) = \frac{2}{3} + \sum_{n=-\infty}^{n=\infty} \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i((-1)^n-1)}{n^3\pi^3} + \frac{i}{n\pi}\right] e^{in\pi x/l}$$

Here $n \neq 0$ as numerator $\to \infty$ at $n=0$

$$\therefore f(x) = \frac{2}{3} + \sum_{n=-\infty}^{n=\infty} \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} + \frac{2i((-1)^n-1)}{n^3\pi^3} + \frac{i}{n\pi}\right] e^{in\pi x/l}$$

Q2 Show that set of functions $\phi_n(x) = \sin \frac{\pi n x}{l}$, $n = 1, 2, 3 \ldots$ is orthogonal set on interval $0 \leq x \leq l$ and find corresponding orthonormal set

Sol:

Let $f_1(x) = \sin(n\pi x/l)$, $n = 1, 2, 3, \ldots$

& $f_2(x) = \sin(m\pi x/l)$, $m = 1, 2, 3, \ldots$

for the functions $\phi_n(x)$ to be orthogonal to each other in interval $0 \leq x \leq l$ the inner product i.e. $(f_1(x), f_2(x))$ should be zero

i.e. $$(f_1(x), f_2(x)) = \int_0^l f_1(x) f_2(x)\, dx = 0$$

So,

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right) dx \quad , \quad \text{Here } m \neq n$$

$$= \int_0^l \frac{1}{2} \times \left[\cos\left(\frac{(m-n)\pi x}{l}\right) - \cos\left(\frac{(m-n)\pi x}{l}\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin((m-n)\pi x/l)}{\pi(m-n)/l} - \frac{\sin((m+n)\pi x/l)}{\pi(m+n)/l}\right]_0^l$$

$$= \frac{1}{2}(0-0) = 0$$

Hence, the given set of functions is orthogonal in $0 \leq x \leq l$ as inner product is zero.

Now, their corresponding orthonormal sets,

$$\left\|\sin\left(\frac{n\pi x}{l}\right)\right\|^2 = \int_0^l \left[\sin\left(\frac{n\pi x}{l}\right)\right]^2 dx$$

$$= \int_0^l \left[\frac{1-\cos\left(\frac{2n\pi x}{l}\right)}{2}\right] dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2\pi n x/l)}{2\,n\pi/l}\right]_0^l$$

$$= \frac{1}{2}\left[l - 0\right] = \frac{l}{2}$$

Corresponding orthonormal set is

$$\left\{ \sqrt{\frac{2}{l}}\, \phi_n x \right\}$$

Q3 Find fourier integral representation of function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

Soln: fourier integral representation of $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(u) \cos \lambda (u - x)\, du\, d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left\{ \int_{-\infty}^{0} 0\, du + \int_0^{\infty} e^{-u} \cos \lambda (u - x)\, du + 0 \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left\{ 0 + \int_0^{\infty} e^{-u} (\cos \lambda u \cos \lambda x + \sin \lambda u \sin \lambda x)\, du + 0 \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left\{ \int_0^{\infty} e^{-u} \cos \lambda u \cos \lambda x\, du + \int_0^{\infty} e^{-u} \sin \lambda u \sin \lambda u\, du \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left\{ \left[\frac{\cos \lambda x\, e^{-u}}{1 - \lambda^2} [-\cos \lambda u + \lambda \sin \lambda u] \right]_0^{\infty} + \left[\frac{\sin \lambda x}{1 + \lambda^2} e^{-u} [-\sin \lambda u - \lambda \cos \lambda u] \right]_0^{\infty} \right\} d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\cos \lambda x}{1 + \lambda^2} + \frac{\sin \lambda x\, (\lambda)}{1 + \lambda^2}\, d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\cos \lambda x + \lambda \sin \lambda x}{1 + \lambda^2}\, d\lambda$$

TUTORIAL 7: Fourier Transform, Fourier sine Transform, Fourier cosine transform.

Q1 Find fourier transform of

$$f(x) = \begin{cases} 1+\frac{x}{a}, & -a<x<0 \\ 1-\frac{x}{a}, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$$

Soln:

Put $x = -x$ in definition of $f(x)$

$$f(-x) = \begin{cases} 1+\frac{x}{a}, & -a<x<0 \\ 1-\frac{x}{a}, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$$

$\because f(x) = f(-x) \Rightarrow$ The function is even.

Function is even, the fourier transform of $f(x)$ = fourier cosine transform of $f(x)$ denoted by $F_c(\alpha)$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty f(t)\cos\alpha t\, dt\right]$$

21 10 12/11/21

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^a \left(1-\frac{t}{a}\right)\cos\alpha t + \int_0^\infty \cancel{0}\, dt\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \cos \alpha t \, dt - \frac{1}{a} \int_0^a t \cos \alpha t \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ \frac{\sin \alpha t}{\alpha} \right\}_0^a - \frac{1}{a} \left\{ \frac{t \sin \alpha t}{\alpha} - \frac{1}{\alpha} \int_0^a \sin \alpha t \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha a}{\alpha} - \frac{1}{a} \left[\frac{t \sin \alpha t}{\alpha} + \frac{\cos \alpha t}{\alpha^2} \right]_0^a \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin a\alpha}{\alpha} - \frac{\sin a\alpha}{\alpha} - \frac{\cos a\alpha}{a\alpha^2} + \frac{1}{a\alpha^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{a\alpha^2} \right] (1 - \cos a\alpha)$$

But,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, dx$$

Taking $\sqrt{\frac{2}{\pi}}$ with $F_c(\alpha)$

$$F_c(\alpha) = \frac{2}{\pi} \times \frac{1}{a\alpha^2} \times 2 \sin^2\left(\frac{\alpha}{2}\right)$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \left(\frac{4}{a\alpha^2} \sin^2\left(\frac{\alpha}{2}\right) \right) \cos \alpha x \, dx$$

$$\boxed{\therefore F_c(\alpha) = \frac{4}{a\alpha^2} \sin^2\left(\frac{\alpha}{2}\right)}$$

Q2 Find the fourier sine and cosine transform of

$$f(x) = \begin{cases} x &, 0<x<1 \\ 2-x &, 1<x<2 \\ 0 &, x>2 \end{cases}$$

Soln:

Let the fourier cosine and sine transforms be denoted by $F_c(\alpha)$ and $F_s(\alpha)$ respectively.
Now, using formula for $F_c(\alpha)$ we get

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos\alpha t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 t\cos\alpha t \, dt + \int_1^2 (2-t)\cos\alpha t \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{t\sin\alpha t}{\alpha} - \int \frac{\sin\alpha t}{\alpha} \right]_0^1 + \left[\frac{2\sin\alpha t}{\alpha} - \left\{ \frac{t\sin\alpha t}{\alpha} - \int \frac{\sin\alpha t}{\alpha} \right\} \right]_1^2 \right.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{t\sin\alpha t}{\alpha} + \frac{\cos\alpha t}{\alpha^2} \right]_0^1 + \left[\frac{2\sin\alpha t}{\alpha} - \frac{t\sin\alpha t}{\alpha} - \frac{\cos\alpha t}{\alpha^2} \right]_1^2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha} - \frac{1}{\alpha^2} + \left[\frac{2\sin 2\alpha}{\alpha} - \frac{2\sin 2\alpha}{\alpha} - \frac{\cos 2\alpha}{\alpha^2} - \left(\frac{2\sin\alpha}{\alpha} - \frac{\sin\alpha}{\alpha} - \frac{\cos\alpha}{\alpha^2} \right) \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2} - \frac{\cos 2\alpha}{\alpha^2} - \frac{2\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} \right\}$$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \left\{ \frac{2\cos\alpha - 1 - \cos 2\alpha}{\alpha^2} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{2\cos\alpha - 2\cos 2\alpha}{\alpha^2} \right\}$$

$$= 2\sqrt{\frac{2}{\pi}} \frac{\cos\alpha(1-\cos\alpha)}{\alpha^2}$$

However, $f(x) = \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos\alpha x \, d\alpha$ .... $\frac{2}{\pi}$ was distributed below

$F_c(\alpha) = \frac{2}{\alpha^2} \cos\alpha(1-\cos\alpha)$ inner & outer integral combining inner $\sqrt{\frac{2}{\pi}}$ with outer, we get

Also $F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\alpha t \, dt$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin\alpha t \, dt + \int_1^2 (2-x) \sin\alpha t \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[-\frac{x\cos\alpha t}{\alpha} - \int -\frac{\cos\alpha t}{\alpha} \right]_0^1 + \left[-\frac{(2-x)\cos\alpha t}{\alpha} - \int \frac{(-1)(-\cos\alpha t)}{\alpha} \right]_1^2 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left(-\frac{x\cos\alpha t}{\alpha} + \frac{\sin\alpha t}{\alpha^2} \right)_0^1 + \left\{ \frac{(x-2)\cos\alpha t}{\alpha} - \frac{\sin\alpha t}{\alpha^2} \right\}_1^2 \right\}$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\left\{\frac{-\cos\alpha}{\alpha} + \frac{\sin\alpha}{\alpha^2} - 0 + \left\{0 - \frac{\sin 2\alpha}{\alpha^2} - \left[\frac{-\cos\alpha}{\alpha} - \frac{\sin\alpha}{\alpha^2}\right]\right\}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{2\sin\alpha}{\alpha^2} + \frac{\sin 2\alpha}{\alpha^2}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{2\sin\alpha}{\alpha^2}(1+\cos\alpha)$$

But $\sqrt{\frac{2}{\pi}}$ is part of outer integral

$$\boxed{F_s(\alpha) = \frac{2}{\alpha^2}\sin\alpha(1+\cos\alpha)}$$

Q3 Find the fourier sine and cosine transform of (i) x^{m-1} (ii) $\frac{1}{\sqrt{x}}$

Soln: Let $F_s(\alpha)$ be the fourier transform

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty x^{m-1}\sin\alpha x\,dx$$

Now, $\sqrt{\frac{2}{\pi}}\int_0^\infty x^{m-1}(\cos\alpha t - i\sin\alpha t)\,dt$

using $e^{-i\alpha t} = \cos\alpha t - i\sin\alpha t$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\alpha) - F_s(\alpha)\,dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\,[\cos\alpha t - i\sin\alpha t]\,dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty t^{m-1}\, e^{-i\alpha t}$$ Let $i\alpha t = s$

$dt = \frac{ds}{i\alpha}$

t	0	∞
s	0	∞

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{s}{i\alpha}\right)^{m-1} e^{-s}\, \frac{ds}{i\alpha} = \sqrt{\frac{2}{\pi}}\, \frac{1}{(i\alpha)^m} \int_0^\infty e^{-s}\, s^{m-1}\, ds$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{1}{(i\alpha)^m}\, \Gamma m = \sqrt{\frac{2}{m}}\, \frac{\Gamma m}{\alpha^m} \left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right)$$

Real part $= F_c(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma m}{\alpha^m} \cos\frac{m\pi}{2}$

Imaginary part $= F_s(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma m}{2^m} \sin\frac{m\pi}{2}$

put $m = \frac{1}{2}$ for $f(x) = \frac{1}{\sqrt{x}}$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma 1/2}{\alpha^{1/2}} \cos\frac{\pi}{9} = \frac{1}{\sqrt{\alpha}}$$

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{\Gamma 1/2}{\alpha^{1/2}} \sin\frac{\pi}{9} = \frac{1}{\sqrt{\alpha}}$$

TUTORIAL 8: Z Transform

Q1 Find $Z\{C^k \sin\alpha k\}$ from $Z\{\sin\alpha k\}$

Soln:

$$Z\{\sin\alpha k\} = Z\{f(k)\}$$

$$= \sum_{k=-\infty}^{\infty} f(k) Z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} \sin\alpha t \; z^{-k}$$

$$= \sum_{k=0}^{\infty} \sin\alpha k \; z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha t} - e^{-i\alpha t}}{2i} \right) z^{-k}$$

$$= \frac{1}{2i} \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha}}{z} \right)^k - \frac{1}{2i} \sum_{k=0}^{\infty} \left(\frac{1}{e^{i\alpha} z} \right)^k$$

$$= \frac{1}{2i} \cdot \frac{1}{1 - \frac{e^{i\alpha}}{z}} - \frac{1}{2i} \cdot \frac{1}{1 - \frac{1}{e^{i\alpha} z}}$$

$$= \frac{1}{2i} \frac{z}{z - e^{i\alpha}} - \frac{1}{2} \frac{e^{i\alpha} \cdot z}{e^{i\alpha} \cdot z - 1}$$

Q1

ok 12/12/22

$$= \frac{z}{2i} \left(\frac{1}{z - e^{i\alpha}} - \frac{1}{z \cdot e^{i\alpha} - 1} \right)$$

$$= \frac{z}{2i} \left(\frac{z \cdot e^{i\alpha} - 1 - z + e^{i\alpha}}{(z - e^{i\alpha})(z e^{i\alpha} - 1)} \right)$$

$$= \frac{z \cdot \sin\alpha}{z^2 - 2z\cos\alpha + 1} \qquad |z| > 1$$

By change of scalar property

$$Z\{c^k \sin\alpha k\} = \frac{\frac{z}{c}\sin\alpha}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1}$$

$$= \frac{cz\sin\alpha}{z^2 - 2zc\cos\alpha + c^2}$$

Q2 Find z transform of $\{ke^{-ak}\}$ $k \geq 0$

Soln:

$$Z\{e^{-ak}\} = \sum_{k=0}^{\infty} e^{-ak} \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{e^a \cdot z}\right)^k$$

$$= 1 + \frac{1}{e^a \cdot z} + \left(\frac{1}{e^a \cdot z}\right)^2 + \cdots$$

$$= \frac{1}{1 - \frac{1}{z \cdot e^a}} = \frac{z \cdot e^a}{z \cdot e^a - 1} \qquad |z \cdot e^a| > 1$$

$$|z| > \frac{1}{|e^a|}$$

$$Z\{k \cdot e^{-ak}\} = -z\frac{d}{dz}\left(\frac{e^a \cdot z}{e^a \cdot z - 1}\right)$$

$$z\{ke^{-ak}\} = -ze^{a}\left[\frac{(e^{a}.z-1) - z.e^{a}}{(e^{a}z-1)^2}\right]$$

$$\boxed{z\{k\cdot e^{-ak}\} = \frac{z\cdot e^{a}}{(z\cdot e^{a}-1)^2}}$$

Q3 Find $Z\{{}^nC_k\}$ $\quad 0 \le k \le n$

Soln: $$Z\{f(k)\} = \sum_{k=0}^{n} {}^nC_k Z^{-k}$$

$$= {}^nC_0 + \frac{{}^nC_1}{z} + \frac{{}^nC_2}{z^2} + \dots + \frac{{}^nC_n}{z^n}$$

$$= \left(1+\frac{1}{z}\right)^n \dots\dots z \ne 0$$

Q4 Find inverse z-transform of

$$F(z) = \frac{2z^2-10z+13}{(z-3)^2(z-2)} \qquad 2 < |z| < 3$$

Soln: Let $$\frac{2z^2-10z+13}{(z-3)^2(z-2)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2}$$

$$2z^2-10z+13 = A(z-3)^2 + B(z-2)(z-3) + C(z-2)$$

Put,

$z = 2$

$1 = A(1)^2 \qquad \boxed{\therefore A = 1}$

Put, $z = 3$

$\boxed{\therefore C = 1}$

$\boxed{\therefore B = 1}$

$$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$= \frac{1}{2}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\dots\right] - \frac{1}{3}\left[1+\frac{z}{3}+\frac{z^2}{3^2}+\dots\right]$$

$$+\frac{1}{9}\left[1+\frac{2z}{3}+\frac{3z^2}{3^2}+\dots\right]$$

from first series coefficient of z^{-k} is 2^{k-1} from second is $\frac{-1}{3^{k+1}}$ and from third is $\frac{k+1}{3^{k+2}}$

$\therefore$ from first series, coefficient of $z^{-k} = 2^{k-1}$ $k \geq 9$. from 2nd & 3rd series

coefficient of $z^{k} = \frac{k+1}{3^{k+2}} - \frac{1}{3^{k+1}}$

$$= \frac{k+1}{3^{k+2}} - \frac{3}{3^{k+2}}$$

$$= \frac{k-2}{3^{k+2}} \qquad k \geq 0$$

Coeff. of $z^{-k} = \frac{-k-2}{3^{-k+2}} \qquad k \leq 0$

$$z^{-1}[F(z)] = \{2^{k-1}\} \qquad k \geq 1$$

$$= \left\{\frac{-k-2}{3^{-k+2}}\right\} \qquad k \leq 0$$