Electromagnetic Theory

Vector Analysis: Gradient, Divergence and Curl

Cartesian Coordinates

Gradient of (x, y, z) : $\varphi(x, y, z)$ can be any scalar function like $\varphi(x, y, z) = x^2 - xy + z$

$$\boldsymbol{\nabla}\varphi(x, y, z) = \frac{\partial \varphi}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial \varphi}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial \varphi}{\partial z}\widehat{\boldsymbol{k}}$$

Divergence of $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\widehat{\boldsymbol{k}}$: $\boldsymbol{A}$ can be any vector with coefficients A_x, A_y, A_z as functions of x , y and z. $\boldsymbol{A} = (x^2 - xy + z)\hat{\boldsymbol{\imath}} + (x^3 - xz + x)\hat{\boldsymbol{\jmath}} + (y^2 - y + z)\widehat{\boldsymbol{k}}$

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial A_y}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial A_z}{\partial z}\widehat{\boldsymbol{k}}$$

Curl of $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\widehat{\boldsymbol{k}}$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical coordinates (ρ, ϕ, z)

$x = \rho\cos\phi \; y = \rho\sin\phi \; z = z$

$$\boldsymbol{\nabla}\varphi(\rho, \phi, z) = \frac{\partial \varphi}{\partial \rho}\widehat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial \varphi}{\partial \phi}\widehat{\boldsymbol{\phi}} + \frac{\partial \varphi}{\partial z}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{\rho}\left[\frac{\partial\left(\rho A_\rho\right)}{\partial \rho} + \frac{\partial A_\phi}{\partial \phi} + \frac{\partial(\rho A_z)}{\partial z}\right]$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{vmatrix}$$

Spherical coordinates (r, θ, ϕ)

$x = r\sin\theta\cos\phi \; y = r\sin\theta\sin\phi \; z = r\cos\theta$

$$\boldsymbol{\nabla}\varphi(\rho, \theta, \phi) = \frac{\partial \varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi}\hat{\phi}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial(r^2 \sin\theta\, A_r)}{\partial r} + \frac{\partial(r \sin\theta\, A_\theta)}{\partial\theta} + \frac{\partial\left(rA_\phi\right)}{\partial\phi}\right]$$

$$\nabla \times \boldsymbol{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

Electric field E is a gradient of electrostatic potential V

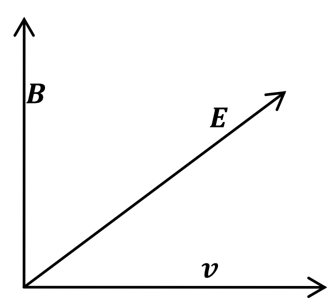
$$\boldsymbol{E} = -\nabla V$$

It is only for symmetrical cases of V like for a point charge we can write $\boldsymbol{E} = -\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}$ or for a capacitor we can write $\boldsymbol{E} = \frac{V}{d}$

Lorentz force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Motion of charged particle in crossed-electric and magnetic field



We consider a situation where uniform constant fields act in perpendicular directions

$$\boldsymbol{E} = E_x\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 0\widehat{\boldsymbol{k}}$$

$$\boldsymbol{B} = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + B_z\widehat{\boldsymbol{k}}$$

$$\boldsymbol{E} \perp \boldsymbol{B} \Rightarrow \boldsymbol{E} \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

$$m\frac{d\boldsymbol{v}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

$$\boldsymbol{v} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ v_x & v_y & v_z \\ 0 & 0 & B_z \end{vmatrix} = v_y B_z \hat{\boldsymbol{\imath}} - v_x B_z \hat{\boldsymbol{\jmath}}$$

$$m\frac{dv_x}{dt} = qE_x + qv_y B_z \quad m\frac{dv_y}{dt} = -qv_x B_z \qquad m\frac{dv_z}{dt} = 0$$

Substitute $\frac{qB_z}{m} = \omega$ (called the gyro frequency).

$$\frac{dv_x}{dt} = \frac{qE_x}{m} + \omega v_y \quad \frac{dv_y}{dt} = -\omega v_x \qquad \frac{dv_z}{dt} = 0$$

Differentiate the first equation and substitute the second in the first

$$\frac{d^2 v_x}{dt^2} = -\omega^2 v_x$$

$$v_x = Asin(\omega t + \phi)$$

Substituting the above solution in the equation $\frac{dv_x}{dt} = \frac{qE_x}{m} + \omega v_y$

$$\omega Acos(\omega t + \phi) = \frac{qE_x}{m} + \omega v_y$$

$$v_y = Acos(\omega t + \phi) - \frac{E_x}{B_z}$$

The constants $A \; and \; \phi$ can be determined from the initial conditions:

Let the particle be at rest at origin initially i.e., at t = 0

$$x = 0; y = 0; z = 0; \; and \; v_x = v_y = v_z$$

$$0 = Asin(\omega 0 + \phi)$$

$$\Longrightarrow \phi = 0$$

$$0 = Acos(\omega 0 + 0) - \frac{E_x}{B_z}$$

$$A = \frac{E_x}{B_z}$$

$$v_x = \frac{E_x}{B_z} sin(\omega t)$$

$$v_y = \frac{E_x}{B_z}(cos(\omega t) - 1)$$

$$v_z = 0$$

Integrating

$$x = \frac{-E_x}{B_z \omega} cos(\omega t) + \ C_1$$

$$y = \frac{E_x}{B_z \omega}(sin(\omega t) - \omega t) + C_2$$

$$z = 0$$

Using the initial conditions we get

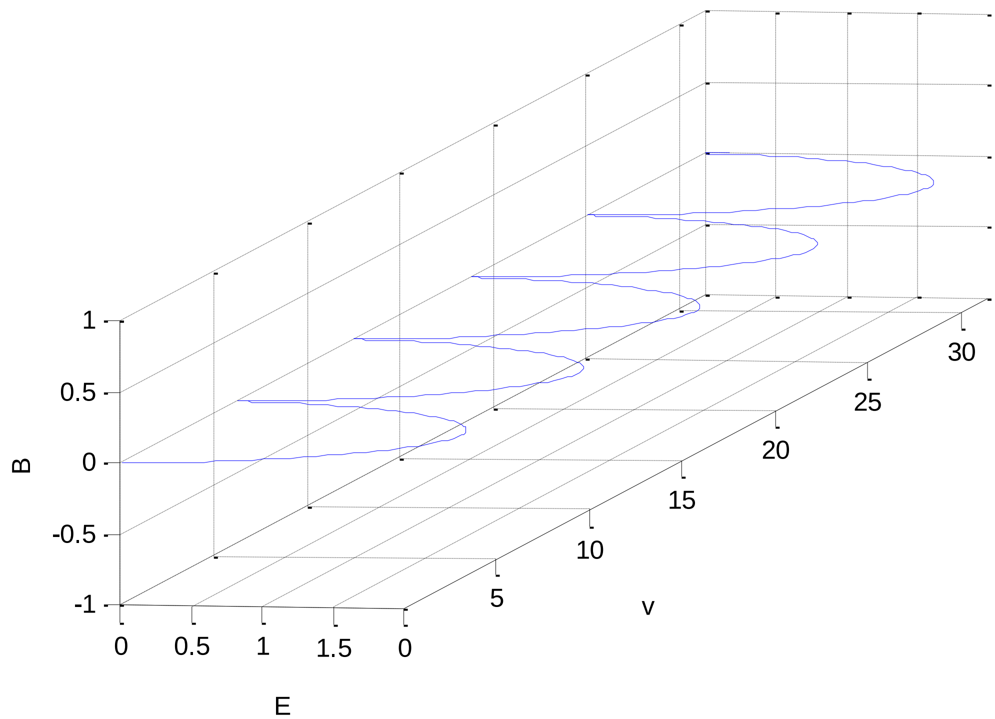
$$C_1 = \frac{E_x}{B_z \omega} \qquad C_2 = 0$$

$$x = \frac{E_x}{B_z \omega}\big(1 - cos(\omega t)\big)$$

$$y = \frac{E_x}{B_z \omega}(sin(\omega t) - \omega t)$$

$$z = 0$$

These equations represent a cycloid. The particle moves in x-y plane.



If $v_y(0) = v_0$ then

$$v_0 = A cos(\omega 0 + 0) - \frac{E_x}{B_z}$$

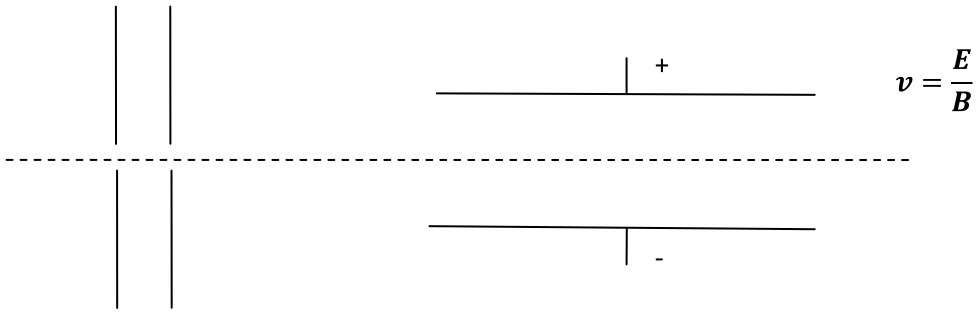
$$A = v_0 - \frac{E_x}{B_z}$$

$$v_x = \left(v_0 - \frac{E_x}{B_z}\right) sin(\omega t + \phi)$$

Thus , if the initial velocity $v_0$is equal to $\frac{E_x}{B_z}$ then the particle will go undeflected or in other words all those particles having velocity $\frac{E_x}{B_z}$ will pass undeflected. This can be useful method to select particles with particular velocity (*velocity selector*).

Velocity selector

A device used to filter out a beam of charged particles of a particular velocity *v*.



Suppose a narrow beam of charged particles is collimated by a slit along z-axis. After passing through the slit the particles enter the region where uniform constant electric and magnetic fields exist along x and y axes respectively.

$$\boldsymbol{E} = E\hat{\boldsymbol{\imath}}$$

$$\boldsymbol{B} = B\hat{\boldsymbol{\jmath}}$$

The electric field produces a force $F_E = qE\hat{\boldsymbol{\imath}}$ along +x-axis whereas the magnetic field produces a force along $F_B = -qvB\hat{\boldsymbol{\imath}}$ - x-axis. There is a unique velocity

$$v_0 = \frac{\boldsymbol{E}}{\boldsymbol{B}}$$

for which the two forces cancel each other. Hence we get a velocity selector by adjusting the values of electric and magnetic fields.

Bainbrige mass spectrograph

A beam of electrons is collimated through slits and passed through a velocity selector, which consists of a transverse electric field and a magnetic field perpendicular to the plane of the paper. The former is produced by maintaining the plate at a suitable potential difference, while the latter by an electromagnet represented by the dotted circle. The two fields are so arranged that the deflections they produce are in opposite axes.