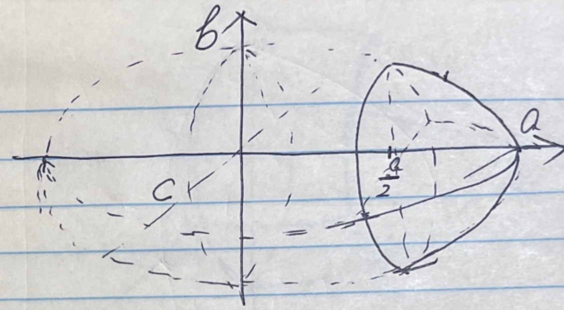
General formula for the volume of sectional ellipsoid from ka to a. The ellipsoid is expressed as (x/a)^2 + (y/b)^2 + (z/c)^2 = 1

Truncated Ellipsoid



If $z=0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$y=b\sqrt{1-\frac{x^2}{a^2}}$$

If $y=0$ $\quad \frac{x^2}{a^2}+\frac{z^2}{c^2}=1 \Rightarrow z=c\sqrt{1-\frac{x^2}{a^2}}$

$$S_s=\pi\cdot y\cdot z=\pi\cdot b\sqrt{1-\frac{x^2}{a^2}}\cdot c\sqrt{1-\frac{x^2}{a^2}}$$

$$V=\int_{\frac{a}{2}}^{a} S_s\,dx=\int_{\frac{a}{2}}^{a}\pi\cdot b\cdot c\left(1-\frac{x^2}{a^2}\right)dx$$

$$=\int_{\frac{a}{2}}^{a}\pi bc\,dx-\int_{\frac{a}{2}}^{a}\frac{\pi\cdot b\cdot c}{a^2}x^2\,dx=\pi bcx\Big|_{\frac{a}{2}}^{a}-\frac{\pi bc}{a^2}\frac{x^3}{3}\Big|_{\frac{a}{2}}^{a}$$

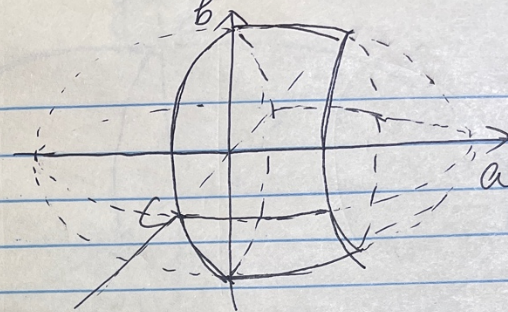
$$=\pi\cdot b\cdot c\,\frac{a}{2}-\frac{\pi bc}{a^2}\cdot\frac{a^3-\left(\frac{a}{2}\right)^3}{3}=\frac{5\pi a\cdot b\cdot c}{24}$$

$$=\frac{\pi\cdot a\cdot b\cdot c}{3}\cdot\frac{5}{8}$$

If $l=k\cdot a$, $0<k<1$ $\quad \int_{ka}^{a} \Rightarrow V=\frac{\pi\cdot a\cdot b\cdot c}{3}\cdot(2-3k+k^3)$

General formula for the volume of sectional ellipsoid from 0 to ka. The ellipsoid is expressed as (x/a)^2 + (y/b)^2 + (z/c)^2 = 1

Truncated Ellipsoid



If $z=0$ $\quad y = b\sqrt{1-\frac{x^2}{a^2}}$

If $y=0$ $\quad z = c\sqrt{1-\frac{x^2}{a^2}}$

$$S_s = \pi \cdot y' \cdot z' = \pi \cdot b\sqrt{1-\frac{x^2}{a^2}} \cdot c\sqrt{1-\frac{x^2}{a^2}} = \pi bc\left(1-\frac{x^2}{a^2}\right)$$

$$V = \int_0^{\frac{a}{2}} S_s \cdot dx = \int_0^{\frac{a}{2}} \pi bc\left(1-\frac{x^2}{a^2}\right)\cdot dx$$

$$= \int_0^{\frac{a}{2}} \pi bc \cdot dx - \int_0^{\frac{a}{2}} \pi \frac{bc}{a^2} \cdot x^2 dx = \pi bcx\Big|_0^{\frac{a}{2}} - \frac{\pi bc}{a^2} \cdot \frac{x^3}{3}\Big|_0^{\frac{a}{2}}$$

$$= \frac{11}{24}\pi \cdot a \cdot b \cdot c = \frac{\pi \cdot a \cdot b \cdot c}{3} \cdot \frac{11}{8}$$

If $L = ka$ $\quad V = \int_0^{ka} S_s \cdot dx$

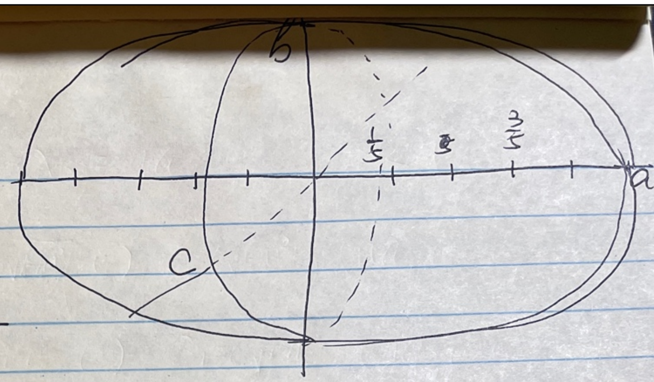
$0 \le k \le 1$

$$= \int_0^{ka} \pi bc dx - \int_0^{ka} \frac{\pi b \cdot c}{a^2} x^2 dx$$

$$= \frac{\pi a \cdot b \cdot c}{3} \cdot (3k - k^3)$$

☆ $V = \frac{\pi \cdot a \cdot b \cdot c}{3}\left[3(n-m) - (n^3 - m^3)\right]$ $\qquad \int_{ma}^{na}$

General formula for the volume of sectional ellipsoid from ma to na. The ellipsoid is expressed as (x/a)^2 + (y/b)^2 + (z/c)^2 = 1



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

If $z=0$ $\quad y=b\sqrt{1-\frac{x^2}{a^2}}$

If $y=0$ $\quad z=c\sqrt{1-\frac{x^2}{a^2}}$

$$V=\int_{ma}^{na} S_s\,dx=\int_{ma}^{na}\pi\cdot b\sqrt{1-\frac{x^2}{a^2}}\cdot c\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$=\int_{ma}^{na}\pi bc\left(1-\frac{x^2}{a^2}\right)dx=\pi bcx\Big|_{ma}^{na}-\pi bc\frac{x^3}{3a}\Big|_{ma}^{na}$$

$$=\frac{\pi abc}{3}\left[3(n-m)-(n^3-m^3)\right] \qquad (n>m)$$

I $\quad m=0 \quad n=\frac{1}{5} \quad k=3\cdot\frac{1}{5}-\left(\frac{1}{5}\right)^3=\frac{74}{125}$

$$V=\frac{\pi\cdot a\cdot b\cdot c}{3}\cdot\frac{74}{125}$$

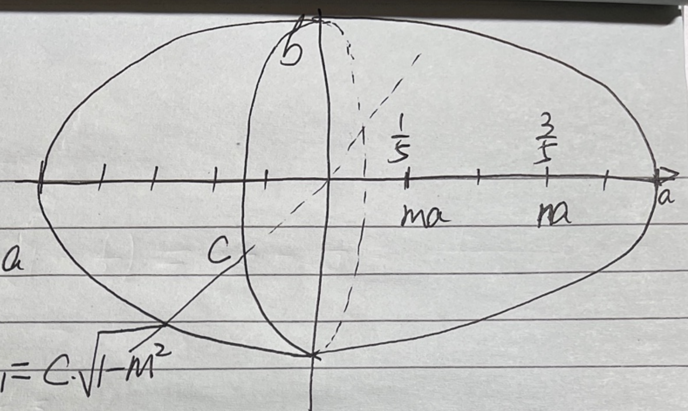
II $\quad m=\frac{1}{5} \quad n=\frac{3}{5} \quad k=3\left(\frac{3}{5}-\frac{1}{5}\right)-\left[\left(\frac{3}{5}\right)^3-\left(\frac{1}{5}\right)^3\right]=124/125$

$$V=\frac{\pi\cdot a\cdot b\cdot c}{3}\quad\frac{124}{125}$$

III $\quad m=\frac{3}{5} \quad n=1 \quad k=3\left(1-\frac{3}{5}\right)-\left(1^3-\left(\frac{3}{5}\right)^3\right)=52/125$

$$V=\frac{\pi\cdot a\cdot b\cdot c}{3}\quad\frac{52}{125}$$

The Volume when ellipsoid is sectioned into several truncated cones. The ellipsoid is expressed as (x/a)^2 + (y/b)^2 + (z/c)^2 = 1



$m < n$

$x_1 = ma \qquad x_2 = na$

$x_1 = ma \qquad y_1 = b\sqrt{1-m^2} \qquad z_1 = c\cdot\sqrt{1-m^2}$

$x_2 = ma \qquad y_2 = b\sqrt{1-n^2} \qquad z_2 = c\sqrt{1-n^2}$

$$V_{TCM} = 2\left(V_1' + V_2' + V_3'\right) \qquad \begin{cases} V = \frac{1}{3}\cdot h \cdot A \\ V' = \frac{1}{3}\cdot h\cdot\left(A_1 + A_2 + \sqrt{A_1 A_2}\right) \end{cases}$$

$$V_1' = \frac{1}{3}\cdot ma\left(\pi\cdot b\cdot c + \pi\cdot b\sqrt{1-m^2}\cdot c\sqrt{1-m^2} + \sqrt{\pi\cdot b\cdot c\cdot \pi\cdot b\sqrt{1-m^2}\, c\sqrt{1-m^2}}\right)$$

$$= \frac{\pi\cdot b\cdot c\cdot a}{3} m\left(2 - m^2 + \sqrt{1-m^2}\right) = \frac{\pi\cdot b\cdot c\, a}{3}\left(2m - m^3 + m\sqrt{1-m^2}\right)$$

$$V_2' = \frac{1}{3}(n-m)a\left[\pi b\sqrt{1-n^2}\, c\sqrt{1-n^2} + \pi b\sqrt{1-m^2}\, c\sqrt{1-m^2} + \sqrt{\pi b\sqrt{1-m^2}\, c\sqrt{1-m^2}\cdot \pi b\sqrt{1-n^2}\, c\sqrt{1-n^2}}\right]$$

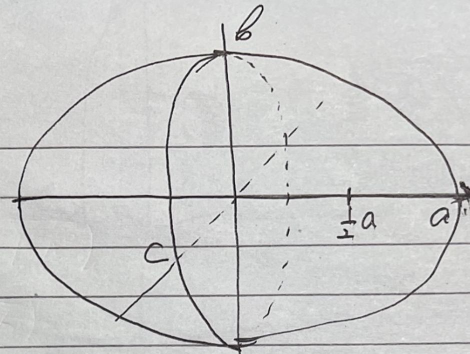
$$= \frac{\pi\cdot a\cdot b\, c}{3}(n-m)\left(2 - n^2 - m^2 + \sqrt{(1-n^2)(1-m^2)}\right)$$

$$V_3' = \frac{1}{3}(a - na)\,\pi\, b\sqrt{1-n^2}\cdot c\sqrt{1-n^2}$$

$$= \frac{\pi\cdot abc}{3}(1-n)(1-n^2)$$

The Volume when ellipsoid is sectioned into two truncated cones at a/2. The ellipsoid is expressed as (x/a)^2 + (y/b)^2 + (z/c)^2 = 1

$m = 0 \quad < \quad n = \frac{1}{2}$



$V_1 = 0$

$$V'_{JCM} = 2(V'_2 + V'_3)$$

$$V'_2 = \frac{\pi abc}{3} \cdot n\left(2 - n^2 + \sqrt{1-n^2}\right)$$

$$= \frac{\pi abc}{3} \cdot \frac{(7+2\sqrt{3})}{8}$$

$$V'_3 = \frac{\pi abc}{3} (1-n)(1-n^2)$$

$$= \frac{\pi \cdot abc}{3} \cdot \frac{3}{8}$$

$$V_{JCM} / V_{real} = (5+\sqrt{3})/8 = 0.84$$

The Volume when ellipsoid is sectioned into two truncated cones at a/5 and 3a/5. The ellipsoid is expressed as (x/a)^2 + (y/b)^2 + (z/c)^2 = 1

$$m = \frac{1}{5} < n = \frac{3}{5}$$

$$V_1' = \frac{\pi \cdot abc}{3} \cdot m\left(2 - m^2 + \sqrt{1-m^2}\right)$$

$$= \frac{\pi \cdot abc}{3} \cdot \frac{(49 + 10\sqrt{6})}{125}$$

$$V_2' = \frac{\pi \cdot abc}{3}(n-m)\left(2 - n^2 - m^2 + \sqrt{(1-n^2)(1-m^2)}\right)$$

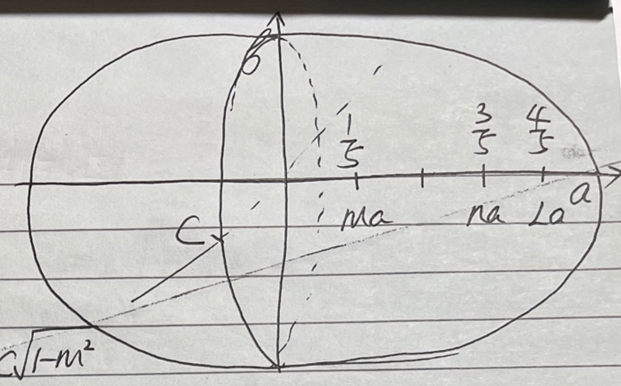
$$= \frac{\pi \cdot abc}{3} \cdot \frac{(80 + 16\sqrt{6})}{125}$$

$$V_3 = \frac{\pi \cdot abc}{3}(1-n)(1-n^2)$$

$$= \frac{\pi \cdot abc}{3} \cdot \frac{32}{125}$$

$$V_{FEM} \Big/ V_{real} = (161 + 26\sqrt{6}) \big/ 250 = 0.90$$

The Volume when ellipsoid is sectioned into two truncated cones at a/5, 3a/5, and 4a/5. The ellipsoid is expressed as (x/a)^2 + (y/b)^2 + (z/c)^2 = 1



8 Sections

$ma < na < La$

$x_1 = ma \quad y_1 = b\sqrt{1-m^2} \quad z_1 = c\sqrt{1-m^2}$

$x_2 = na \quad y_2 = b\sqrt{1-n^2} \quad z_2 = c\sqrt{1-n^2}$

$x_3 = La \quad y_3 = b\sqrt{1-L^2} \quad z_3 = c\sqrt{1-L^2}$

$$V_{JEM} = 2\left(V_1' + V_2' + V_3' + V_4\right)$$

$$V_1' = \frac{1}{3} ma\left(\pi bc + \pi b\sqrt{1-m^2}\, c\sqrt{1-m^2} + \sqrt{\pi bc \cdot \pi b\sqrt{1-m^2}\, c\sqrt{1-m^2}}\right)$$

$$= \frac{\pi abc}{3} m\left(2-m^2+\sqrt{1-m^2}\right)$$

$$V_2' = \frac{\pi abc}{3}(n-m)\left(2-n^2-m^2+\sqrt{(1-n^2)(1-m^2)}\right)$$

$$V_3' = \frac{1}{3}(La-na)\left(\pi b\sqrt{1-L^2}\cdot c\sqrt{1-L^2} + \pi b\sqrt{1-n^2}\, c\sqrt{1-n^2} + \sqrt{\pi bc(1-L^2)\cdot \pi bc(1-n^2)}\right)$$

$$= \frac{\pi abc}{3}(L-n)\left(2-L^2-n^2+\sqrt{(1-L^2)(1-n^2)}\right)$$

$$V_4' = \frac{\pi abc}{3}(1-L)(1-L^2)$$

$$m=\frac{1}{5} < n=\frac{3}{5} < L=\frac{4}{5}$$

$$V_1' = \frac{\pi abc}{3} \cdot m\left(2-m^2+\sqrt{1-m^2}\right)$$

$$= \frac{\pi abc}{3} \cdot \frac{49+10\sqrt{6}}{125}$$

$$V_2' = \frac{\pi \cdot abc}{3} (n-m)\left(2-n^2-m^2+\sqrt{(1-n^2)(1-m^2)}\right)$$

$$= \frac{\pi \cdot abc}{3} \cdot \frac{80+16\sqrt{6}}{125}$$

$$V_3' = \frac{\pi abc}{3} (L-n)\left(2-L^2-n^2+\sqrt{(1-L^2)(1-n^2)}\right)$$

$$= \frac{\pi abc}{3} \left(\frac{4}{5}-\frac{3}{5}\right)\left(2-\frac{16}{25}-\frac{9}{25}+\sqrt{\frac{16}{25}\cdot\frac{9}{25}}\right)$$

$$= \frac{\pi abc}{3} \cdot \frac{37}{125}$$

$$V_4' = \frac{\pi abc}{3} (1-L)(1-L^2) = \frac{\pi \cdot abc}{3} \left(1-\frac{4}{5}\right)\left(1-\frac{16}{25}\right)$$

$$= \frac{\pi \cdot abc}{3} \cdot \frac{9}{125}$$

$$V_{JCM} / V_{real} = (175+26\sqrt{6}) / 250$$