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Econ-240a: Pset 3

2 Linear Regression (theory)

1. Let $Z = \left(X', W'\right)'$. Then,

$$\mathbb{E}^*[Y|X,W] = \mathbb{E}^*[Y|Z] = \mathbb{E}[Y] + \sigma(Y,Z)\mathbb{V}(Z)^{-1}(Z - \mathbb{E}[Z])$$

Because W and X are independent, $\mathbb{V}(Z) = \begin{pmatrix} \mathbb{V}(X) & 0 \\ 0 & \mathbb{V}(W) \end{pmatrix}$. So,

$$\begin{aligned}
\mathbb{E}^*[Y|X,W] &= \mathbb{E}[Y] + \begin{pmatrix} \sigma(Y,X) & \sigma(Y,W) \end{pmatrix} \begin{pmatrix} \mathbb{V}(X)^{-1} & 0 \\ 0 & \mathbb{V}(W)^{-1} \end{pmatrix} \begin{pmatrix} X - \mathbb{E}[X] \\ W - \mathbb{E}[W] \end{pmatrix} \\
&= \mathbb{E}[Y] + \begin{pmatrix} \sigma(Y,X) & \sigma(Y,W) \end{pmatrix} \begin{pmatrix} \mathbb{V}(X)^{-1}(X - \mathbb{E}[X]) \\ \mathbb{V}(W)^{-1}(W - \mathbb{E}[W]) \end{pmatrix} \\
&= \mathbb{E}[Y] + \sigma(Y,X)\mathbb{V}(X)^{-1}(X - \mathbb{E}[X]) + \sigma(Y,W)\mathbb{V}(W)^{-1}(W - \mathbb{E}[W]) \\
&= \mathbb{E}[Y] + \mathbb{E}^*[Y|X] - \mathbb{E}[Y] + \mathbb{E}^*[Y|W] - \mathbb{E}[Y] \\
&= \mathbb{E}^*[Y|X] + \mathbb{E}^*[Y|W] - \mathbb{E}[Y]
\end{aligned}$$

2. Using the formula for the best linear predictor β_0

$$\begin{aligned}
|\beta_0| &= \frac{|\sigma(X,Y)|}{\mathbb{V}(X)} \\
&\leq \frac{\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}}{\mathbb{V}(X)} \\
&= \frac{\sqrt{\mathbb{V}(Y)}}{\sqrt{\mathbb{V}(X)}} = 1
\end{aligned}$$

3. From the formula of variance and applying the Law of Iterated Expectations,

$$\begin{aligned}
\mathbb{V}(Y) &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[\mathbb{E}[Y^2|X]] - \mathbb{E}[\mathbb{E}[Y|X]]^2 \\
&= \mathbb{E}[\mathbb{V}(Y|X) + \mathbb{E}[Y|X]^2] - \mathbb{E}[\mathbb{E}[Y|X]]^2 \\
&= \mathbb{E}[\mathbb{V}(Y|X)] + \mathbb{E}[\mathbb{E}[Y|X]^2] - [\mathbb{E}[Y|X]]^2 \\
&= \mathbb{E}[\mathbb{V}(Y|X)] + \mathbb{V}(\mathbb{E}[Y|X])
\end{aligned}$$

4.