Homework #2

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2. Logit MLE

a) Derive the score of the logit-likelihood: First, noting that

$$\frac{\partial \Lambda(X_i'\beta)}{\partial \beta} = \frac{\exp(X_i'\beta)X_i'(1 + X_i'\beta) - \exp(X_i'\beta)\exp(X_i'\beta)X_i'}{(1 + \exp(X_i'\beta))^2} = \frac{\exp(X_i'\beta)X_i'}{(1 + \exp(X_i'\beta))^2}$$

The score of the logit log-likelihood is therefore

$$\begin{aligned} s(\beta) &= \sum_i \frac{Y_i \exp(X_i'\beta)X_i'}{\Lambda(X_i'\beta)(1 + \exp(X_i'\beta))^2} - \frac{(1 - Y_i)\exp(X_i'\beta)X_i'}{(1 - \Lambda(X_i'\beta))(1 + \exp(X_i'\beta))^2} \\ &= \sum_i \Big(Y_i - (1 - Y_i)\exp(X_i'\beta)\Big)\frac{X_i'}{1 + \exp(X_i'\beta)} \\ &= \sum_i \Big(Y_i(1 + \exp(X_i'\beta)) - \exp(X_i'\beta)\Big)\frac{X_i'}{1 + \exp(X_i'\beta)} \\ &= \sum_i \Big(Y_i - \frac{\exp(X_i'\beta)}{1 + \exp(X_i'\beta)}\Big)X_i' \end{aligned}$$

b) The moment condition identifying β_{ML} is

$$E\Bigg[\Big(Y_i - \frac{\exp(X_i'\beta)}{1 + \exp(X_i'\beta)}\Big)X_i'\Bigg] = 0$$
$$E\Bigg[E[Y_i - \frac{\exp(X_i'\beta)}{1 + \exp(X_i'\beta)}|X_i]X_i'\Bigg] = 0$$

c) The moment conditions identifying β_{NLLS} are

$$\begin{aligned} 0 &= E\Bigg[-2(Y_i - \Lambda(X_i'\beta))\frac{\exp(X_i'\beta)}{(1 + \exp(X_i'\beta))^2}X_i'\Bigg] \\ &= E\Bigg[E[(Y - \Lambda(X_i'\beta))|X_i]\frac{\exp(X_i'\beta)}{(1 + \exp(X_i'\beta))^2}X_i'\Bigg] \end{aligned}$$

d) Under which conditions does β_{NLLS} coincide with β_{ML}? The moment conditions will coincide when $E[Y_i - \Lambda(X_i'\beta)|X_i'] = 0$ for all X_i; that is, they coincide when the logit model is correctly specified.

e)

3. Matlab Probit DGP

a) (Matlab program attached)

b) The ML point estimates of $\hat{\beta}^{con} = (\hat{\alpha}, \hat{b})$ are $(-0.0175, 0.9159)$. The standard errors are $(0.0918, 0.1301)$.

c) Score test: Regressing the restricted model's generalized residuals, given by

$$gres_i = \frac{Y_i \phi(X_i' \hat{\beta}^{con})}{\Phi(X_i' \hat{\beta}^{con})} - \frac{(1 - Y_i)\phi(X_i' \hat{\beta}^{con})}{1 - \Phi(X_i' \hat{\beta}^{con})}$$

on X_i^2 yields a coefficient of 0.0039 with a p-value of effectively 0, rejecting the null of 0.

d) The unrestricted model yields point estimates of $\hat{\beta}^{unc} = (\hat{\alpha}, \hat{b}, \hat{b_2}) = (-0.0435, 0.9175, 0.0428)$. The wald test, which tests $g(\beta) = 0$, where

$$g(\beta) \equiv (0, 0, 1) \cdot \beta = b_2$$
$$G(\beta) \equiv \frac{\partial g(\beta)}{\partial \beta} = (0, 0, 1)$$

and the Wald statistic is given by

$$N \cdot \hat{b_2} \cdot \left(G \cdot H^{-1} \cdot G' \right)^{-1} \cdot \hat{b_2}$$

where H is the average Hessian at the ML estimate. Here, the Wald evaluates to 0.3336, which has a p-value of 0.5636 under the χ^2 distribution with d.f. 1, much higher than the 0 obtained in part b.