Homework #1

Professor: Pat Kline

Students: Christina Brown, Sam Leone, Peter McCrory, Preston Mui

Identification I (OLS)

Identification II (A Structural Labor Supply Model)

1. Derive the payoffs to working 20 and 40 hours:

$$u_{20}(w_i; d_i, a) = 20w_i - \frac{d_i}{1+a}20^{1+a}$$

$$u_{40}(w_i; d_i, a) = 40w_i - \frac{d_i}{1+a}40^{1+a}$$

2. Derive the probability of working 40 hours a week given wages:

   Given wages, the worker will work 40 hours if $u_{40}(w_i; d_i, a) > u_{20}(w_i; d_i, a)$, which holds when

$$40w_i - \frac{d_i}{1+a}40^{1+a} > 20w_i - \frac{d_i}{1+a}20^{1+a}$$

$$2w_i - \frac{2d_i}{1+a}40^a > w_i - \frac{d_i}{1+a}20^a$$

$$w_i(1+a) > 2d_i 40^a - d_i 20^a$$

$$\implies d_i < \frac{w_i(1+a)}{2 \cdot 40^a - 20^a}$$

$$\implies \log d_i < \log w_i + \log(1+a) - \log(2 \cdot 40^a - 20^a)$$

   The probability of which is $\Phi\left(\frac{\log w_i + \log(1+a) - \log(2 \cdot 40^a - 20^a) - \mu}{\sigma}\right)$, where Φ is the c.d.f. of the standard normal.

3. Derive an individual's contribution to the likelihood of the observed data: Given an observation (h_i, w_i) under a given parameter a and parameters on the distribution of $\log d_i$, the likelihood of the observed data is $\Phi\left(\frac{\log w_i + \log(1+a) - \log(2 \cdot 40^a - 20^a) - \mu}{\sigma}\right)$ if $h_i = 40$ and $1 - \Phi\left(\frac{\log w_i + \log(1+a) - \log(2 \cdot 40^a - 20^a) - \mu}{\sigma}\right)$ if $h_i = 20$.

4. These likelihoods are, naturally, used in the maximum likelihood estimator.

5. The parameters (a, μ, σ) are set-identified (any two will imply the third).

6. Derive an expression for the probability of working 20 and 40 hours a week when the choice set for hours is $\{10, 20, 40\}$: The utility from working 20 and 40 hours is as

above, and the utility of working 10 hours is

$$u_{10}(w_i; d_i, a) = 10w_i - \frac{d_i}{1+a}10^{1+a}$$

A worker works 40 hours when $u_{40} > u_{20}$ (as above) and $u_{40} > u_{10}$. The latter happens when

$$40w_i - \frac{d_i}{1+a}40^{1+a} > 10w_i - \frac{d_i}{1+a}10^{1+a}$$

$$4w_i - \frac{4d_i}{1+a}40^a > w_i - \frac{d_i}{1+a}10^a$$

$$3w_i(1+a) - 4d_i40^a > -d_i10^a$$

$$\implies d_i < \frac{3w_i(1+a)}{4 \cdot 40^a - 10^a}$$

However, this condition is met when $u_{40} > u_{20}$, unless $a < 0$ which is outside of the parameter space. So, the probability that the worker chooses 40 hours the same as in the $\{20, 40\}$ choice set.

Using a similar derivation to the above, the worker chooses 20 hours over 10 hours when $d_i < \frac{w_i(1+a)}{2 \cdot 20^a - 10^a}$. So, the probability that the worker chooses 20 hours is

$$\Phi\left(\frac{\log w_i + \log(1+a) - \log(2 \cdot 20^a - 10^a) - \mu}{\sigma}\right)$$

$$-\Phi\left(\frac{\log w_i + \log(1+a) - \log(2 \cdot 40^a - 20^a) - \mu}{\sigma}\right)$$

7. With the addition of the option to work 10 hours, the parameters are still set-identified, in the sense that choosing one value would imply the others.

Identification III (Mixture of Normals)

Quantile Treatment Effects

a) Let $Z_i = g(Y_i)$, where $g(\cdot)$ is a monotonically increasing function. Prove that $Q_\tau(Z_i) = g(Q_\tau(Y_i))$:

$$F_Z(g(Q_\tau(Y_i))) = Pr(Z_i < g(Q_\tau(Y_i)))$$
$$= Pr(Y_i < Q_\tau(Y_i)) \qquad \text{(Since } g(\cdot) \text{ is monotonically increasing)}$$
$$= F_Y(Q_\tau(Y_i)) = \tau$$
$$\implies g(Q_\tau(Y_i)) = F_Z^{-1}(\tau) = Q_\tau(Z_i) \qquad \square$$

Iterated Projections

Prove the law of iterated projections

$$E^*[Y_i|X_i] = E^*[E^*[Y_i|X_i, Z_i]|X_i]$$

Proof.
Define $W_i' = \begin{bmatrix} X_i' & Z_i' \end{bmatrix}$. Recall that the linear projection of W_i onto Y_i requires that, for $\beta = E[W_iW_i']^{-1}E[W_iY_i]$, the following must hold:

$$E[W_i(Y_i - W_i'\beta)] = 0.$$

This further implies that

$$E[X_i \underbrace{(Y_i - W_i'\beta)}_{\equiv u_i}] = 0$$

Thus,

$$
\begin{aligned}
E^*[Y_i|X_i] &= X_i'E[X_iX_i']E[X_iY_i] \\
&= X_i'E[X_iX_i']^{-1}E[X_i(W_i'\beta + u_i)] \\
&= X_i'E[X_iX_i']^{-1}E[X_iW_i'\beta] + X_i'E[X_iX_i']^{-1}E[X_iu_i] \\
&= X_i'E[X_iX_i']^{-1}E[X_iE^*[Y_i|X_i, Z_i]] \\
&= E^*[E^*[Y_i|X_i, Z_i]|X_i]
\end{aligned}
$$

The third equality follows from the linearity of the expectation operator. The second term in the third equality is equal to zero by the observation made above. The fourth follows from the definition $E^*[Y_i|X_i, Z_i] = W_i\beta$. $\qquad\square$

FWL Theorem

Suppose the population projection $E^*[Y_i|X_i, Z_i] = X_i'\beta + Z_i'\gamma$. Let $\tilde{Z}_i = Z_i - E^*[Z_i|X_i]$ and $\tilde{Y}_i = Y_i - E^*[Y_i|X_i]$:

(a) Show that $\gamma = E\left[\tilde{Z}_i\tilde{Z}_i'\right]^{-1} E\left[\tilde{Z}_i\tilde{Y}_i\right]$

Proof.
Observe that we can write $Y_i = X_i'\beta + Z_i'\gamma + \epsilon_i$ where ϵ_i is the conditional expectation error term from the projection of X_i and Z_i onto Y_i. Recall that by definition, $E[X_i\epsilon_i] = E[Z_i\epsilon_i] = 0$. Thus, we can write:

$$
\begin{aligned}
Y_i &= X_i'\beta + Z_i'\gamma + \epsilon_i \\
X_iY_i &= X_iX_i'\beta + X_iZ_i'\gamma + X_i\epsilon_i \\
E[X_iY_i] &= E[X_iX_i'\beta] + E[X_iZ_i'\gamma] + E[X_i\epsilon_i] \\
X_i'E[X_iX_i']^{-1}E[X_iY_i] &= X_i'E[X_iX_i']^{-1}E[X_iX_i']\beta + X_i'E[X_iX_i']^{-1}E[X_iZ_i']\gamma \\
E^*[Y_i|X_i] &= X_i'\beta + E^*[Z_i|X_i]'\gamma
\end{aligned}
$$

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The third line follows from the first order condition of minimization. The fourth line premultiplies everything by $X_i' E[X_i X_i']^{-1}$. The final line simply replaces the expressions with their definition.

Now, subtract the final line from the first line:

$$\underbrace{Y_i - E^*[Y_i|X_i]}_{\tilde{Y}_i} = X_i'\beta - X_i'\beta + \underbrace{Z_i'\gamma - E^*[Z_i|X_i]'\gamma}_{\tilde{Z}_i'\gamma} + \epsilon_i$$

$$\tilde{Z}_i \tilde{Y}_i = \tilde{Z}_i \tilde{Z}_i'\gamma + \tilde{Z}_i \epsilon_i$$

$$E[\tilde{Z}_i \tilde{Y}_i] = E[\tilde{Z}_i \tilde{Z}_i']\gamma + E[\tilde{Z}_i \epsilon_i]$$

Observe that $E[Z_i \epsilon_i] = 0 \implies E[\epsilon|Z_i] = 0 \implies E[g(Z_i)\epsilon_i] = 0 \implies E[\tilde{Z}_i \epsilon_i] = 0.$[1] Thus, we have

$$E[\tilde{Z}_i \tilde{Y}_i] = E[\tilde{Z}_i \tilde{Z}_i']\gamma,$$

which after rearranging becomes $\gamma = E[\tilde{Z}_i \tilde{Z}_i']^{-1} E[\tilde{Z}_i \tilde{Y}_i]$, the desired result. $\square$

(b) Show that $Y_i - E^*[Y_i|X_i, Z_i] = (Y_i - E^*[Y_i|X_i]) - (Z_i - E^*[Z_i|X_i]'\gamma)$

Proof.
Observe

$$E^*[Y_i|X_i, Z_i] = X_i'\beta + Z_i'\gamma$$
$$E^*[E^*[Y_i|X_i, Z_i]|X_i] = X_i'\beta + E^*[Z_i|X_i]'\gamma$$
$$E^*[Y_i|X_i] = X_i'\beta + E^*[Z_i|X_i]'\gamma$$

Now, we have, after solving the above equation for $X_i'\beta$, the desired result:

$$Y_i - X_i'\beta - Z_i'\gamma = Y_i - E^*[Y_i|X_i] + E^*[Z_i|X_i]'\gamma - Z_i'\gamma$$
$$= (Y_i - E^*[Y_i|X_i]) - (Z_i - E^*[Z_i|X_i])'\gamma$$

$\square$

Weighted Average Derivative Properties

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[1] Suppose the first implication did not hold, so that $E[\epsilon|Z_i] = a$ for some $a \in \mathbb{R}$. Then $E[Z_i \epsilon_i] = E[Z_i E[\epsilon|Z_i]] = a E[Z_i]$. This final term is not equal to zero except for the special case when $E[Z_i] = 0$.