

Homework #2

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2. Logit MLE

a) Derive the score of the logit-likelihood: First, noting that

$$\frac{\partial \Lambda(X'_i \beta)}{\partial \beta} = \frac{\exp(X'_i \beta) X'_i (1 + \exp(X'_i \beta)) - \exp(X'_i \beta) \exp(X'_i \beta) X'_i}{(1 + \exp(X'_i \beta))^2} = \frac{\exp(X'_i \beta) X'_i}{(1 + \exp(X'_i \beta))^2}$$

The score of the logit log-likelihood is therefore

$$\begin{aligned} s(\beta) &= \sum_i \frac{Y_i \exp(X'_i \beta) X'_i}{\Lambda(X'_i \beta) (1 + \exp(X'_i \beta))^2} - \frac{(1 - Y_i) \exp(X'_i \beta) X'_i}{(1 - \Lambda(X'_i \beta)) (1 + \exp(X'_i \beta))^2} \\ &= \sum_i \left(Y_i - (1 - Y_i) \exp(X'_i \beta) \right) \frac{X'_i}{1 + \exp(X'_i \beta)} \\ &= \sum_i \left(Y_i (1 + \exp(X'_i \beta)) - \exp(X'_i \beta) \right) \frac{X'_i}{1 + \exp(X'_i \beta)} \\ &= \sum_i \left(Y_i - \frac{\exp(X'_i \beta)}{1 + \exp(X'_i \beta)} \right) X'_i \end{aligned}$$

b) The moment condition identifying β_{ML} is

$$\begin{aligned} E \left[\left(Y_i - \frac{\exp(X'_i \beta)}{1 + \exp(X'_i \beta)} \right) X'_i \right] &= 0 \\ E \left[E \left[Y_i - \frac{\exp(X'_i \beta)}{1 + \exp(X'_i \beta)} \middle| X_i \right] X'_i \right] &= 0 \end{aligned}$$

c) The moment conditions identifying β_{NLLS} are

$$\begin{aligned} 0 &= E \left[-2(Y_i - \Lambda(X'_i \beta)) \frac{\exp(X'_i \beta)}{(1 + \exp(X'_i \beta))^2} X'_i \right] \\ &= E \left[E[(Y - \Lambda(X'_i \beta)) | X_i] \frac{\exp(X'_i \beta)}{(1 + \exp(X'_i \beta))^2} X'_i \right] \end{aligned}$$

d) Under which conditions does β_{NLLS} coincide with β_{ML} ? The moment conditions will coincide when $E[Y_i - \Lambda(X'_i \beta) | X'_i] = 0$ for all X_i ; that is, they coincide when the logit model is correctly specified.

3. Matlab Probit DGP

- a) (Matlab program attached)
- b) The ML point estimates of $\hat{\beta}^{con} = (\hat{\alpha}, \hat{b})$ are $(-0.0175, 0.9159)$. The standard errors are $(0.3030, 0.3607)$, respectively.
- c) Score test: Following Wooldridge (2010), page 570, the LM statistic is the ESS from the following regression

$$\frac{\hat{u}_i}{\sqrt{\hat{G}_i(1 - \hat{G}_i)}} = \alpha \frac{\hat{g}_i}{\sqrt{\hat{G}_i(1 - \hat{G}_i)}} x_i + \gamma \frac{\hat{g}_i}{\sqrt{\hat{G}_i(1 - \hat{G}_i)}} z_i$$

where x_i is the regressor matrix in the unconstrained regression (constant and X) and z_i is the vector of X_i^2 . The ESS from this regression was 0.2963, which has a p-value of 0.58621. So, one does not reject the null that $b_2 = 0$.

- d) The unrestricted model yields point estimates of $\hat{\beta}^{unc} = (\hat{\alpha}, \hat{b}, \hat{b}_2) = (-0.0435, 0.9175, 0.0428)$. The Wald test will test the null $g(\beta) = 0$ where

$$g(\beta) \equiv (0, 0, 1) \cdot \beta = b_2$$

$$G(\beta) \equiv \frac{\partial g(\beta)}{\partial \beta} = (0, 0, 1)$$

and the Wald statistic is given by

$$N \cdot \hat{b}_2 \cdot \left(G \cdot \frac{1}{\sqrt{N}} H^{-1} \cdot G' \right)^{-1} \cdot \hat{b}_2$$

where H is the average Hessian at the ML estimate. Because we know that the model is correctly specified, I use the inverse Hessian instead of the sandwich estimator. The Wald evaluates to 7.4592, which has a p-value of 0.0063 under the χ^2 distribution with d.f. 1, so one rejects the null that $b_2 = 0$. This is starkly different from the score test result, which did not reject the null. This makes sense, as the Wald tends to reject more than the LM test. If one bumps the number of observations up (say, to 5000), both tests reject the null.

4. Clustered DGP in Stata

- a) Regressing Y_{ic} on D_{ic} , we get the results in table 1.

The basic SE are 0.03237. The standard errors are slightly larger (0.03244) when we correct for heteroskedasticity in column 2 and significantly larger (0.09508) when we account for the intracluster correlation of observations from the same cluster.

- b) Changing σ_η^2 to 0. Regressing Y_{ic} on D_{ic} , we get the results in table :

The standard errors are fairly similar for columns 1 and 2 to what we had in A. They are 0.02732 for the regular SE, 0.027731 for robust SE. The clustered SE (0.02420) are similar to the magnitude of the non-clustered version.

Table 1: Clustered DGP

	(1) Y_ic, Regular SE	(2) Y_ic, Robust SE	(3) Y_ic, Clustered SE
d_ic	1.010*** (0.03237)	1.010*** (0.03244)	1.010*** (0.09508)
Constant	0.0394* (0.02278)	0.0394* (0.02037)	0.0394 (0.10761)
R^2	0.089	0.089	0.089
Observations	10000	10000	10000

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Table 2: Clustered DGP

	(1) Y_ic, Regular SE	(2) Y_ic, Robust SE	(3) Y_ic, Clustered SE
d_ic	0.992*** (0.02732)	0.992*** (0.02731)	0.992*** (0.02420)
Constant	-0.00105 (0.01911)	-0.00105 (0.01923)	-0.00105 (0.09650)
R^2	0.117	0.117	0.117
Observations	10000	10000	10000

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

- c) Comment on your differences in the answers to a) and b).
The SE are fairly similar in columns 1 and 2 for parts a) and b) though they are slightly smaller in part b as a result of the lower variance of y_{ic} in part b. In part b since there is no intra-cluster correlation in the coefficient on d_{ic} the clustered standard errors are of a similar magnitude to the non-clustered version (0.02420).
- d) In table 3, we see that the simulation rejects the null that the coefficient on $d_{ic}=1$ 57% of the time when we don't cluster our standard errors and about 5% (which we would expect) when we cluster. So the non-clustered version is rejecting too often and the clustered version is rejecting at about the rate of p which we set.

Table 3: Monte Carlo Simulations

(1)					
	count	mean	sd	min	max
reject	1000	.567	.4957386	0	1
reject_cluster	1000	.055	.2280943	0	1

- e) Here we have that the regression with non-clustered standard errors rejects the test that the coefficient equals the average of the cluster level η 's 3% of the time and the clustered standard errors are never rejecting, so in this case clustering is causing us not reject often enough.

Table 4: Monte Carlo Simulations

(1)					
	count	mean	sd	min	max
reject	1000	.03	.1706726	0	1
reject_cluster	1000	0	0	0	0

- f) TBD

5. CPS and WLS

a) See results in column 1 of table 5.

Table 5: Wage Regression (Weighted)

	(1) Log wage	(2) Log wage (weighted by cell size)	(3) Log wage (weighted by 1/cell var)
agecat==2	0.314*** (0.00746)	0.314*** (0.00073)	0.314*** (0.00047)
agecat==3	0.458*** (0.00731)	0.458*** (0.00071)	0.458*** (0.00046)
agecat==4	0.494*** (0.00771)	0.494*** (0.00075)	0.494*** (0.00048)
agecat==5	0.472*** (0.00966)	0.472*** (0.00094)	0.472*** (0.00061)
agecat==6	0.295*** (0.01651)	0.295*** (0.00161)	0.295*** (0.00104)
educat==2	0.265*** (0.00754)	0.265*** (0.00073)	0.265*** (0.00047)
educat==3	0.415*** (0.00764)	0.415*** (0.00074)	0.415*** (0.00048)
educat==4	0.771*** (0.00783)	0.771*** (0.00076)	0.771*** (0.00049)
Sex	-0.261*** (0.00458)	-0.261*** (0.00045)	-0.261*** (0.00029)
Constant	2.153*** (0.00995)	2.153*** (0.00097)	2.153*** (0.00063)
R^2	0.297	0.978	0.978
Observations	56182	56182	135056

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

b) Collapsed results used in analysis below.

c) See column 2 of table 5. The coefficients are identical (as we would expect) but the standard errors are much smaller when we weight by bin size.

d) If wages are *iid* within a cell, then the variance of each cell is σ^2/N_c where σ^2 is the population variance and N_c is the number of observations in the cell.

e) Formula above is used to calculate the variance of the cell means.

f) See results in column 3 of table 5.

- g) With an RSS of 0.133 and 38 degrees of freedom we are not able to reject this model at the 5% level. In other words we cannot reject our model explains the data up to a sampling error.
- h) See results in table 7.
- i) With an RSS of 0.009 and 15 degrees of freedom we cannot reject this goodness of fit of this model at the 5% level either.
- j) See fig. 1. The model does a very nice job of fitting the data. The place it has a bit of trouble fitting is for older more educated individuals. The model overestimates the wages of 65+ males with some college and underestimates the wages for males with a BA or more.
- k) TBD

6. Random Coefficient Binary Choice Model:

- a) As was derived in the lecture notes, the simulated log likelihood can be derived as follows. Denote the Logistic cdf with Λ and its pdf with λ .

First observe that

$$Pr(Y_i = 1|X_i, b_i) = Pr(\epsilon_i > -b_i x_i) = 1 - Pr(\epsilon_i < -b_i x_i) = 1 - \Lambda(-b_i x_i) = \Lambda(b_i x_i)$$

Thus, assuming that b_i and ϵ_i are independent:

$$Pr(Y_i = 1|X_i) = \int \Lambda(b x_i) \frac{1}{\sigma} \phi\left(\frac{b_i - \mu}{\sigma}\right) db$$

The likelihood of this model for a single observation is

$$L(Y_i, X_i; \mu, \sigma) = \left(\int \Lambda(b x_i) \frac{1}{\sigma} \phi\left(\frac{b_i - \mu}{\sigma}\right) db \right)^{Y_i} \left(1 - \int \Lambda(b x_i) \frac{1}{\sigma} \phi\left(\frac{b_i - \mu}{\sigma}\right) db \right)^{1-Y_i}$$

Thus, for **fixed** random draws $\{u_i\}_{m=1}^M$ from $\mathcal{N}(0, 1)$, the simulated likelihood is

$$\hat{L}_M(Y_i, X_i, \mu, \sigma) = \left(\frac{1}{M} \sum_{m=1}^M \Lambda(\underbrace{(\mu + \sigma u_{im})}_{\equiv b} X_i) \right)^{Y_i} \left(1 - \frac{1}{M} \sum_{m=1}^M \Lambda(\underbrace{(\mu + \sigma u_{im})}_{\equiv b} X_i) \right)^{1-Y_i}$$

Clearly, taking logs (of the product of all the likelihoods) allows us to recover the simulated log-likelihood of the full sample.

Let's differentiate with respect to μ and σ !

For μ :

$$\begin{aligned} \frac{\partial}{\partial \mu} \frac{1}{N} \sum_i Y_i \log \left(\frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i) \right) &+ (1 - Y_i) \log \left(1 - \frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i) \right) \\ &= \frac{1}{N} \sum_i \left[\frac{Y_i X_i \frac{1}{M} \sum_{m=1}^M \lambda((\mu + \sigma u_{im}) X_i)}{\frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i)} - \frac{(1 - Y_i) X_i \frac{1}{M} \sum_{m=1}^M \lambda((\mu + \sigma u_{im}) X_i)}{1 - \frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i)} \right] \end{aligned}$$

Table 6: Wage Regression (Weighted)

	(1) Log wage
agecat==2	0.301*** (0.00103)
agecat==3	0.414*** (0.00103)
agecat==4	0.424*** (0.00103)
agecat==5	0.407*** (0.00106)
agecat==6	0.355*** (0.00068)
educat==2	0.109*** (0.00076)
educat==3	0.238*** (0.00083)
educat==4	0.757*** (0.00081)
Sex	-0.192*** (0.00054)
<=25 × Dropout	0 (.)
<=25 × HS	0.0576*** (0.00078)
<=25 × Some College	0.00573*** (0.00084)
<=25 × BA+	-0.135*** (0.00087)
25-35 × Dropout	0.00900*** (0.00084)
25-35 × HS	0.135*** (0.00071)
25-35 × Some College	0.152*** (0.00078)
25-35 × BA+	0 (.)
35-45 × Dropout	-0.0677*** (0.00084)
35-45 × HS	0.0784*** (0.00070)
35-45 × Some College	0.154*** (0.00078)
35-45 × BA+	0 (.)
45-55 × Dropout	-0.0676*** (0.00086)
45-55 × HS	0.112*** (0.00071)
45-55 × Some College	0.158*** (0.00079)
45-55 × BA+	0 (.)
55-65 × Dropout	-0.00166* (0.00090)
55-65 × HS	0.0884*** (0.00076)
R^2	7 0.998
Observations	135056

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

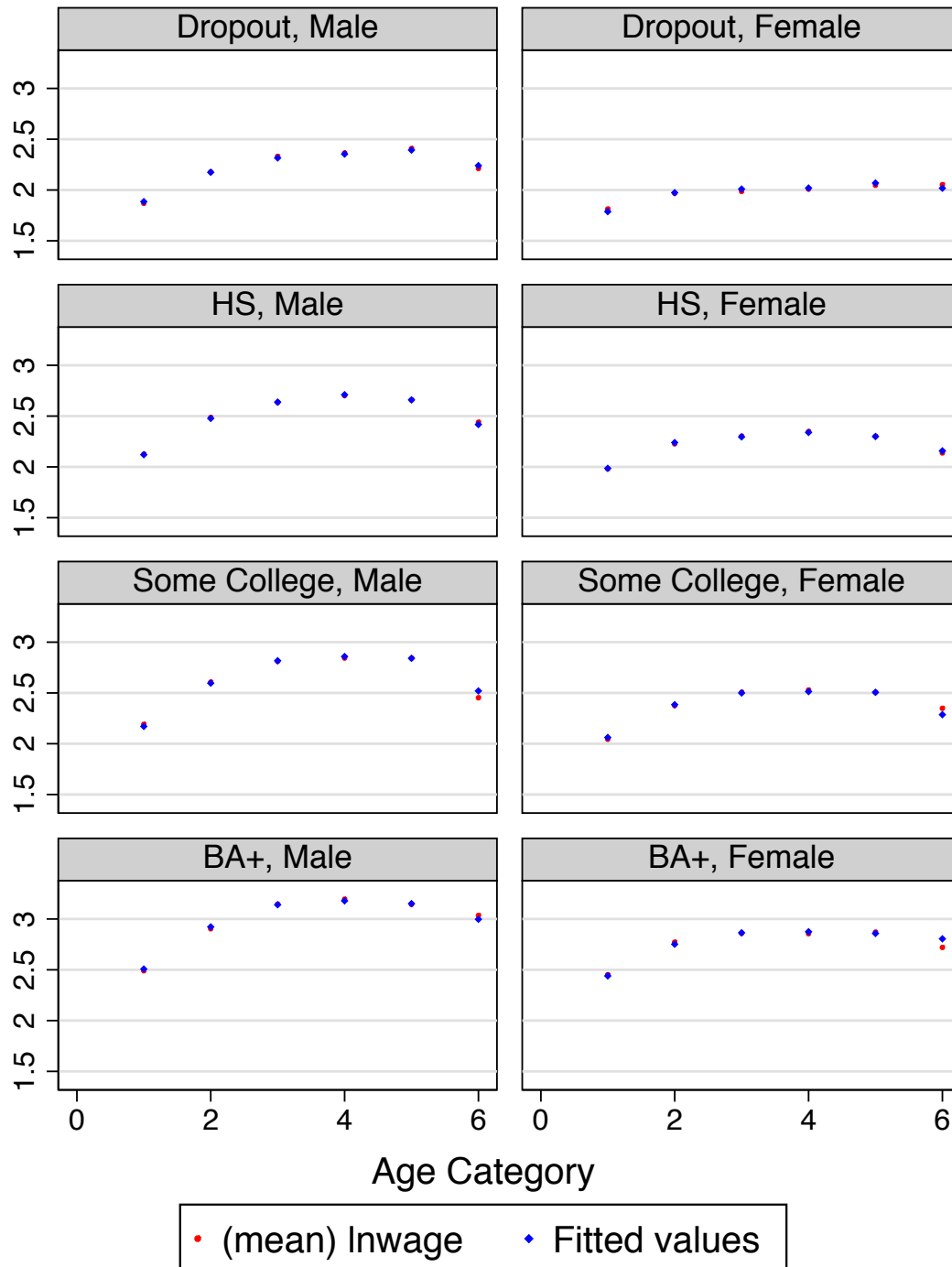
Table 7: Wage Regression (Weighted) cont

	(1) Log wage
55-65 \times Some College	0.167*** (0.00084)
55-65 \times BA+	0 (.)
>65 \times Dropout	0 (.)
>65 \times HS	0 (.)
>65 \times Some College	0 (.)
>65 \times BA+	0 (.)
$\leq 25 \times$ Male	0 (.)
$\leq 25 \times$ Female	0.124*** (0.00055)
25-35 \times Male	-0.0212*** (0.00054)
25-35 \times Female	0 (.)
35-45 \times Male	0.0836*** (0.00054)
35-45 \times Female	0 (.)
45-55 \times Male	0.112*** (0.00054)
45-55 \times Female	0 (.)
55-65 \times Male	0.101*** (0.00058)
55-65 \times Female	0 (.)
>65 \times Male	0 (.)
>65 \times Female	0 (.)
Dropout \times Male	0 (.)
Dropout \times Female	-0.0302*** (0.00026)
HS \times Male	0.0672*** (0.00020)
HS \times Female	0 (.)
Some College \times Male	0.0410*** (0.00020)
Some College \times Female	0 (.)
BA+ \times Male	0 (.)
BA+ \times Female	0 (.)
Constant	2.078*** (0.00057)
R^2	8 0.998
Observations	135056

Standard errors in parentheses

* p<0.10, ** p<0.05, *** p<0.01

Figure 1: Predicted and actual log wages by subgroup



Graphs by Education Category and Sex

For σ :

$$\begin{aligned} & \frac{\partial}{\partial \sigma} \frac{1}{N} \sum_i Y_i \log \left(\frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i) \right) + (1 - Y_i) \log \left(1 - \frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i) \right) \\ &= \frac{1}{N} \sum_i \left[\frac{Y_i X_i \frac{1}{M} \sum_{m=1}^M \lambda((\mu + \sigma u_{im}) X_i) u_{im}}{\frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i)} - \frac{(1 - Y_i) X_i \frac{1}{M} \sum_{m=1}^M \lambda((\mu + \sigma u_{im}) X_i) u_{im}}{1 - \frac{1}{M} \sum_{m=1}^M \Lambda((\mu + \sigma u_{im}) X_i)} \right] \end{aligned}$$

- b) See *max_sml.m* file.
- c) See *max_sml.m* file.
- d) These results are invariant to initial conditions:

$$\begin{bmatrix} \mu \\ \ln(\sigma) \end{bmatrix} = \begin{bmatrix} 1.6011 \\ 0.5139 \end{bmatrix}$$

- e) Let $\hat{\theta}_{MSL}$ denote the estimated parameters from the method of simulated likelihood and the true parameters $\theta = [\mu, \sigma]'$. Under proper specification, we know that

$$\sqrt{N}(\hat{\theta}_{MSL} - \theta) \rightarrow \mathcal{N}(\mathbf{0}, \mathbf{H}(\theta)^{-1})$$

By the delta-method, with $g(\theta) \equiv [\mu \ln(\sigma)]'$, we have that

$$\sqrt{N}(g(\hat{\theta}_{MSL}) - g(\theta)) \rightarrow \mathcal{N}(\mathbf{0}, \mathbf{G} \mathbf{H}(\theta)^{-1} \mathbf{G})$$

where

$$\mathbf{G} \equiv \frac{\partial g(\theta)}{\partial \theta} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sigma} \end{bmatrix}$$

When we calculate the square root of the diagonal elements of $\frac{1}{N} \mathbf{H}(\hat{\theta})^{-1}$, we derive the standard errors of our estimates of μ and $\ln(\sigma)$: (0.2633, 0.3884).

- f) To calculate the standard errors for the incorrect model, we replace $\mathbf{H}(\theta)^{-1}$ with $\mathbf{H}(\theta)^{-1} \mathbf{V}_s \mathbf{H}(\theta)^{-1}$, where \mathbf{V}_s is the population variance-covariance matrix of the scores—defined by the the criterion function we are maximizing.

Again, we use the analogy principle to replace the asymptotic variance terms with the simulated likelihood, sample averages. This yields standard errors of μ and $\ln(\sigma)$: (0.4839, 0.6417).