Homework #1

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Identification I (OLS)

Identification II (A Structural Labor Supply Model)

Identification III (Mixture of Normals)

Quantile Treatment Effects

Iterated Projections

Prove the law of iterated projections

$$E^*[Y_i|X_i] = E^*[E^*[Y_i|X_i, Z_i]|X_i]$$

Proof.
Define $W_i' = \begin{bmatrix} X_i' & Z_i' \end{bmatrix}$. Recall that the linear projection of W_i onto Y_i requires that, for $\beta = E[W_i W_i']^{-1} E[W_i Y_i]$, the following must hold:

$$E[W_i(Y_i - W_i'\beta)] = 0.$$

This further implies that

$$E[X_i \underbrace{(Y_i - W_i'\beta)}_{\equiv u_i}] = 0$$

Thus,

$$\begin{aligned}
E^*[Y_i|X_i] &= X_i' E[X_i X_i'] E[X_i Y_i] \\
&= X_i' E[X_i X_i']^{-1} E[X_i(W_i'\beta + u_i)] \\
&= X_i' E[X_i X_i']^{-1} E[X_i W_i'\beta] + X_i' E[X_i X_i']^{-1} E[X_i u_i] \\
&= X_i' E[X_i X_i']^{-1} E[X_i E^*[Y_i|X_i, Z_i]] \\
&= E^*[E^*[Y_i|X_i, Z_i]|X_i]
\end{aligned}$$

The third equality follows from the linearity of the expectation operator. The second term in the third equality is equal to zero by the observation made above. The fourth follows from the definition $E^*[Y_i|X_i, Z_i] = W_i\beta$. $\square$

FWL Theorem

Suppose the population projection $E^*[Y_i|X_i, Z_i] = X_i'\beta + Z_i'\gamma$. Let $\tilde{Z}_i = Z_i - E^*[Z_i|X_i]$ and $\tilde{Y}_i = Y_i - E^*[Y_i|X_i]$:

(a) Show that $\gamma = E\left[\tilde{Z}_i\tilde{Z}_i'\right]^{-1} E\left[\tilde{Z}_i\tilde{Y}_i\right]$

Proof.
Observe that we can write $Y_i = X_i'\beta + Z_i'\gamma + \epsilon_i$ where ϵ_i is the conditional expectation error term from the projection of X_i and Z_i onto Y_i. Recall that by definition, $E[X_i\epsilon_i] = E[Z_i\epsilon_i] = 0$. Thus, we can write:

$$\begin{aligned}
Y_i &= X_i'\beta + Z_i'\gamma + \epsilon_i \\
X_iY_i &= X_iX_i'\beta + X_iZ_i'\gamma + X_i\epsilon_i \\
E[X_iY_i] &= E[X_iX_i'\beta] + E[X_iZ_i'\gamma] + E[X_i\epsilon_i] \\
X_i'E[X_iX_i']^{-1}E[X_iY_i] &= X_i'E[X_iX_i']^{-1}E[X_iX_i']\beta + X_i'E[X_iX_i']^{-1}E[X_iZ_i']\gamma \\
E^*[Y_i|X_i] &= X_i'\beta + E^*[Z_i|X_i]'\gamma
\end{aligned}$$

The third line follows from the first order condition of minimization. The fourth line premultiplies everything by $X_i'E[X_iX_i']^{-1}$. The final line simply replaces the expressions with their definition.

Now, subtract the final line from the first line:

$$\begin{aligned}
\underbrace{Y_i - E^*[Y_i|X_i]}_{\tilde{Y}_i} &= X_i'\beta - X_i'\beta + \underbrace{Z_i'\gamma - E^*[Z_i|X_i]'\gamma}_{\tilde{Z}_i'\gamma} + \epsilon_i \\
\tilde{Z}_i\tilde{Y}_i &= \tilde{Z}_i\tilde{Z}_i'\gamma + \tilde{Z}_i\epsilon_i \\
E[\tilde{Z}_i\tilde{Y}_i] &= E[\tilde{Z}_i\tilde{Z}_i']\gamma + E[\tilde{Z}_i\epsilon_i]
\end{aligned}$$

Observe that $E[Z_i\epsilon_i] = 0 \implies E[\epsilon|Z_i] = 0 \implies E[g(Z_i)\epsilon_i] = 0 \implies E[\tilde{Z}_i\epsilon_i] = 0$.[1] Thus, we have

$$E[\tilde{Z}_i\tilde{Y}_i] = E[\tilde{Z}_i\tilde{Z}_i']\gamma,$$

which after rearranging becomes $\gamma = E[\tilde{Z}_i\tilde{Z}_i']^{-1}E[\tilde{Z}_i\tilde{Y}_i]$, the desired result. □

(b) Show that $Y_i - E^*[Y_i|X_i, Z_i] = (Y_i - E^*[Y_i|X_i]) - (Z_i - E^*[Z_i|X_i]'\gamma)$

Proof.
Observe

$$\begin{aligned}
E^*[Y_i|X_i, Z_i] &= X_i'\beta + Z_i'\gamma \\
E^*[E^*[Y_i|X_i, Z_i]|X_i] &= X_i'\beta + E^*[Z_i|X_i]'\gamma \\
E^*[Y_i|X_i] &= X_i'\beta + E^*[Z_i|X_i]'\gamma
\end{aligned}$$

[1] Suppose the first implication did not hold, so that $E[\epsilon|Z_i] = a$ for some $a \in \mathbb{R}$. Then $E[Z_i\epsilon_i] = E[Z_iE[\epsilon|Z_i]] = aE[Z_i]$. This final term is not equal to zero except for the special case when $E[Z_i] = 0$.

Now, we have, after solving the above equation for $X_i'\beta$, the desired result:

$$\begin{aligned} Y_i - X_i'\beta - Z_i'\gamma &= Y_i - E^*[Y_i|X_i] + E^*[Z_i|X_i]'\gamma - Z_i'\gamma \\ &= (Y_i - E^*[Y_i|X_i]) - (Z_i - E^*[Z_i|X_i])'\gamma \end{aligned}$$

$\square$

Weighted Average Derivative Properties