Homework 08

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Work in groups of two. Include both of your names as a comment in the first line of each file. Then submit those files on canvas (once per group!).

1. You are given the matrices:

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -9 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & -2 \\ -1 & -3 & -1 \end{pmatrix}.$$

   (a) Test `PowerMethod.m` on the matrix A (pick a tolerance of 1e-8). You can choose the starting vector. Use the `eig` function to verify that your answer is correct.

   (b) Now test the matrix B with starting value (1,1,1). Why does the power method fail (check with `eig`)? Modify the code to print the vector x in each iteration (only print the first 15, show them as row vectors to save space). What happens and why does this happen?

   (c) With the same B, try running with the starting vector $x = [2; -2; 2]$. What happens now? Can you explain why?

   Note: This question requires you to include text as your interpretation. You can print the MATLAB output and write your conclusions on paper, or include them typed inside your submission.

2. Compute (by hand) the eigenvalues, the corresponding set of all eigenvectors for each eigenvalue, and give one example eigenvector for each eigenvalue (pick it to have small whole numbers as entries) of the matrices

$$A = \begin{pmatrix} 0 & -8 & 0 \\ -3 & 2 & 3 \\ 0 & 0 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}.$$

3. Consider the population example (see page 2). Suppose you have birth rates $b_1 = 0.3$, $b_2 = 0.3$, $b_3 = 0.3$, $b_4 = 0.1$ and death rates $d_1 = 0.1$, $d_2 = 0.2$, $d_3 = 0.5$, $d_4 = 0.9$.

   (a) What is the biggest eigenvalue of the resulting matrix? What do you expect to happen with the population over a long time period?

   (b) Suppose you have $P_1 = 100$, $P_2 = 200$, $P_3 = 150$, $P_4 = 75$ in year 0. What will the population be in year 1000? Does this agree with a)?

   (c) Suppose we change the problem by reducing the death rate $d_4 = 0.01$ of P_4 (from 0.9). What is the largest eigenvalue now? What do you expect to happen after a long time period? What is the population after 1000 years?

   Note: This question requires you to include text as your interpretation. You can print the MATLAB output and write your conclusions on paper, or include them typed inside your submission.

In population systems with approximately constant birth and death rates, we can develop matrices that show how the population changes from year to year. As we will show, the largest eigenvalue of this system determines the stability of the system.

We consider the following example. Suppose we have a population of animals, and each individual is classified as being in group P_1 if less than 1 year old, P_2 if between 1 and 2, P_3 if between 2 and 3, and P_4 if older than 3. The death rates, i.e. the chance that an individual will die in the next year, for each of these groups are d_1, d_2. d_3 and d_4, respectively. The birth rates for each group are b_1, b_2. b_3 and b_4. Thus if we know the size of the groups at year n, then we will know the size of the groups at year $n+1$ from the equations

$$\begin{aligned} P_1^{n+1} &= b_1 P_1^n + b_2 P_2^n + b_3 P_3^n + b_4 P_4^n \\ P_2^{n+1} &= (1-d_1) P_1^n \\ P_3^{n+1} &= (1-d_2) P_2^n \\ P_4^{n+1} &= (1-d_3) P_3^n + (1-d_4) P_4^n \end{aligned}$$

Writing this as matrices and vectors,

$$\begin{pmatrix} P_1^{n+1} \\ P_2^{n+1} \\ P_3^{n+1} \\ P_4^{n+1} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ (1-d_1) & 0 & 0 & 0 \\ 0 & (1-d_2) & 0 & 0 \\ 0 & 0 & (1-d_3) & (1-d_4) \end{pmatrix} \begin{pmatrix} P_1^n \\ P_2^n \\ P_3^n \\ P_4^n \end{pmatrix}$$

Using this relationship, we can determine the long time behavior of the population with repeated multiplication, since the year n population is related to year 0 population by

$$\begin{pmatrix} P_1^n \\ P_2^n \\ P_3^n \\ P_4^n \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ (1-d_1) & 0 & 0 & 0 \\ 0 & (1-d_2) & 0 & 0 \\ 0 & 0 & (1-d_3) & (1-d_4) \end{pmatrix}^n \begin{pmatrix} P_1^0 \\ P_2^0 \\ P_3^0 \\ P_4^0 \end{pmatrix}$$

Due to the repeated multiplication is by the same matrix, this process is exactly the (non-normalized) power method! Thus if the biggest eigenvalue is bigger than 1, then the population will grow to infinity as $n \to \infty$, and if the biggest eigenvalue is less than 1, then the population will eventually become extinct. A stable population will have a biggest eigenvalue of 1.