Practice Exam 2

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1. You are given the following nonlinear system:

$$\begin{aligned} 1 &= xy \\ 2 &= x^2 \end{aligned}$$

Which method would use use to solve the system numerically? State the equation to compute the approximation x^{n+1} and execute one step with $x^0 = (1, 0)$.

Solution:

only Newton works for higher dimensions.

1) set up function f where root is solution to nonlinar system (note different variables):

$$f(\vec{x}) = f(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} 1 - x_1 x_2 \\ 2 - x_1^2 \end{pmatrix}$$

2) Jacobian:

$$\nabla f(\vec{x}) = \begin{pmatrix} -x_2 & -x_1 \\ -2x_1 & 0 \end{pmatrix}$$

3) iteration:

$$x^{k+1} = x^k - (\nabla f(x^k))^{-1} f(x^k)$$

For one step we first need to compute:

$$f(x^0) = f(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \nabla f(x^0) = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$$

so:

$$x^1 = x^0 - \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

solve:

$$\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

gives $a_1 = -1/2$, $a_2 = -1$. Finally:

$$x^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1/2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$$

2. You are given the fixed point iteration $g(x) = x^2$. Find an interval I in which g has a unique fixed point and where the fixed point iteration converges for any starting value. Is the convergence linear or quadratic?

Solution:
For $I = [0, 1/4]$ you have the $g(I) = [0, 1/16] \subset I$ and

$$|g(x) - g(y)| = \|x^2 - y^2\| \leq \|x + y\| \cdot \|x - y\| \leq 1/2 \|x - y\|$$

We also have $g'(x^*) = 0$, so we can expect quadratic convergence.

3. State the pros/cons of Bisection, Newton, and Secant method.

   Solution:

   See overview table in lecture notes (need interval with sign change for Bisection, derivative for Newton. Convergence orders: 1, 2, 1.6, etc.).

4. a) Compute the eigenvalues and one eigenvector for each eigenvalue for the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}.$$

   b) State a numerical algorithm to compute eigenvalue(s) for a matrix. Apply two iterations of the algorithm to the matrix A above. Use the infinity norm and start with $x_0 = (1, 1)$.

   c) What value will the method in b) return for the eigenvalue of the matrix A in a)?

   Solution:

   a) eigenvalues are -2 and 1. example for vectors: (-1,1) for -2 and (-1,-2) for 1.

   b) PowerMethod:

$$x_{n+1} = \frac{Ax_n}{\|Ax_n\|_\infty}$$

   two iterations:

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_1 = \frac{Ax_0}{\|Ax_0\|_\infty} = \frac{1}{\|Ax_0\|_\infty} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_2 = \frac{Ax_1}{\|Ax_1\|_\infty} = \frac{1}{\|Ax_1\|_\infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   c) will return -2.

5. You are given the quadrature rule

$$\int_a^b f(x)dx \approx Q(f) = \frac{b-a}{3}\left(f(a) + f\left(\frac{a+b}{2}\right) + f(b)\right).$$

   (a) Determine the order m of this quadrature rule. For this, prove that $Q(f)$ integrates polynomials up to degree m exactly and show that it is not exact for all polynomials of degree $m+1$ (find a counterexample).

   (b) Would you use this quadrature over the Simpson rule? Why/why not?

   Solution:

   a)

   1. $m \geq 0$?

   check $f(x) = 1$:

$$Q(f, a, b) = \frac{b-a}{3}(1 + 1 + 1) = b - a$$

   is equal to

$$\int_a^b f(x)dx = x \mid_a^b = b - a$$

so Q is exact for all constant functions.

2. $m \geq 1$?

check $f(x) = x$:

$$Q(f,a,b) = \frac{b-a}{3}(a + \frac{a+b}{2} + b) = \frac{b-a}{3}\frac{3(a+b)}{2} = \frac{(b-a)(a+b)}{2}$$

is equal to

$$\int_a^b f(x)dx = \frac{1}{2}x^2 \mid_a^b = \frac{b^2-a^2}{2}$$

so Q is exact for all linear functions.

3. $m \geq 2$?

check $f(x) = x^2$:

$$Q(f,a,b) = \frac{b-a}{3}(a^2 + \frac{(a+b)^2}{4} + b^2) = \frac{b-a}{3}\frac{4a^2+4b^2+2ab}{4}$$

is not equal to

$$\int_a^b f(x)dx = \frac{1}{3}x^3 \mid_a^b = \frac{b^3-a^3}{3}$$

so Q is not exact for all quadratic functions.

4. So $m = 1$.

b) no, because Simpson is more accurate with the same number of evaluations.

6. a) State the formula for a composite quadrature rule $Q^c(f,a,b)$ to approximate the integral $\int_a^b f(x)\,dx$ given a quadrature $Q(f,a,b)$ that approximates f on $[a,b]$. Draw a picture. Make sure you define x_i and specify the correct loop bounds.

   b) Derive a bound for the error in a)

   c) What is the reason for using composite quadrature rules?

   Solution:

   a)

$$Q^c(f,a,b) = \sum_{i=1}^{N} Q(f,x_i,x_{i+1})$$

   with N points $x_1,\ldots,x_N$ and $a = x_1$ and $b = x_N$.

   b) See lecture notes.

   c) You want a more accurate answer than a simple quadrature formula gives, higher order quadratures are hard to set up and need f to be $m+1$ times differentiable to give optimal convergence rates.

7. a) Given a set of points (x_i, y_i), what is the definition of an interpolating function?

   b) You are given the following data points: $(1,-1),(2,3),(3,0),(4,1)$. Construct (don't solve) a linear system that determines the coefficients of the interpolating polynomial of degree three.

   c) You are given a large number of points (say $n > 10$) that contain noisy data. What kind of interpolating function from lecture would you use/not use and why?

   Solution:

a) a function $f(x)$ with $f(x_i) = y_i$ (goes through all points)

b) you get $Ac = y$ with:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

to get the polynomial $p(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$ (see lecture notes for how the matrix is set up).

c) Not a single polynomial, so piece-wise polynomials (either linears or cubic splines).