Introduction and theoretical background

August 29, 2025

Contents

1 TODO

- Better understand the underlying theory
- Create a function which allows to input an arbitrary reflection function and get the associated geometry

2 Theory

2.1 Basis facts from electromagnetism

This section is baed on chaptor 3 from the book [1]

2.1.1 Maxwell' equations

Maxwell's equation describe the classic propagation of electromagnetic waves. These equations are given by

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{2}$$

$$\nabla \cdot \boldsymbol{D} = \rho_v \tag{3}$$

$$\nabla \cdot \boldsymbol{B} = 0 \ , \tag{4}$$

where $\boldsymbol{E}/\boldsymbol{B}$ are the electric/magnetic field intensity, $\boldsymbol{B}/\boldsymbol{H}$ are the electric/magnetic flux density, $\boldsymbol{J}$ is the electric current density and ρ_v is the volume charge density. Furthermore the following relations complement the relations between the vectorfields

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \tag{5}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{6}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E} \ , \tag{7}$$

where $\epsilon = \epsilon_0 \epsilon_r$ is the dielectric permittivity, $\mu = \mu_0 \mu_r$ is the permeability and σ is the electric conductivity.

The Maxwell's equatios can also be represented by an integral formulation. In this formualtions Gauss's and Stokes's theorem can be applied. By analyzing how the integral formulation of the Maxwell's equations behave at the boundary between two materials, one obtains the boundary conditions described in table 1. A detailed derivation of the formulas can be found in [1] chapter 3.2.

We can see from table 1 that the tangential component of E and the normal component of B change continuously, whereas the normal component of D and the tangential component of H do not chamge continuously. An image describing the situation at the boundary can be seen in figure 1.

Field components	General form	Specific form
Tangential E	$\boldsymbol{n}_2 \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = 0$	$E_{1t} = E_{2t}$
Normal D	$\boldsymbol{n}_2 \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$
Tangential H	$\boldsymbol{n}_2 \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_s$	$H_{1t} - H_{2t} = J_s$
Normal B	$\boldsymbol{n}_2 \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0$	$B_{1n} = B_{2n}$

Table 1: Table to test captions and labels.

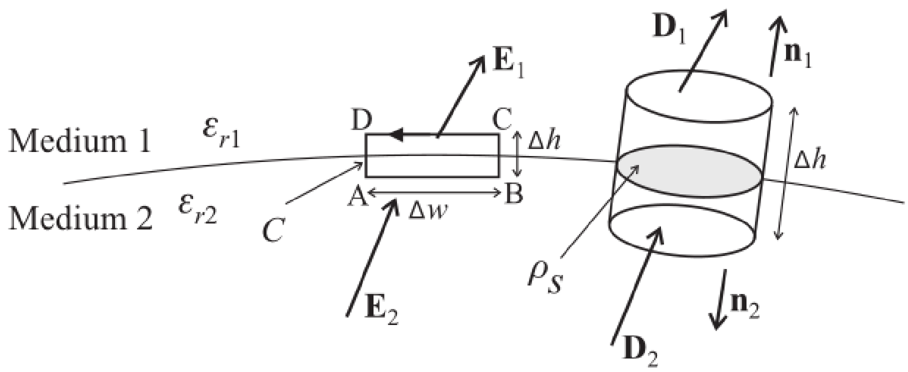

Figure 1: An interface between two media. Contour and volume used to derive boundary conditions for fields between two different dielectrics are shown. Image taken from [1].

2.1.2 Wave equation

A wave equation can be derived from a source free medium where $\rho_V = 0 = \boldsymbol{J}$ by applying $\nabla\times$ on both sides of the equation. The result is the follwoing wave equation

$$\nabla^2 \boldsymbol{E} = \frac{n^2 \partial^2}{c^2 \partial t^2} \boldsymbol{E} \ , \tag{8}$$

where $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is the speed of light and n is the refractive index of the medium. In many practical situations the solution to the wave equation are time-harmonics, i.e. the solution can be written as

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r}) e^{i\omega t}\} \ , \tag{9}$$

where ω is the angular frequency. Often the fields can also be written as plane waves, i.e. in the form

$$\boldsymbol{E} \propto e^{i\omega t - \boldsymbol{r}\boldsymbol{k}} \ , \tag{10}$$

where $\boldsymbol{k}$ is the wave vektor. If both electric and magnetic fields can be written in this form, then the following relation between them holds

$$\boldsymbol{k} \times \boldsymbol{E} = \omega \mu_0 \boldsymbol{H}. \tag{11}$$

This relation is visualized in figure 2.

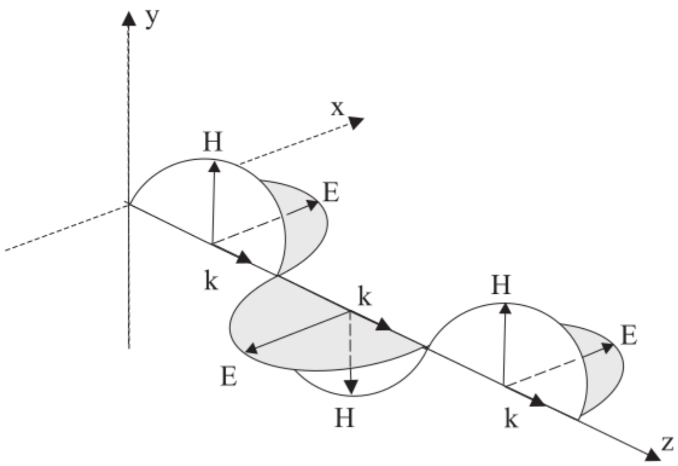

Figure 2: Visualization of the electric and magnetic fields from [1]

2.1.3 Polarized waves

Polarization characterizes the curve which the E vector makes (in the plane orthogonal to the direction of propagation) at a given point in space as a function of time. In the most general case, the curve produced is an ellipse and, accordingly, the wave is called elliptically polarized. Under certain conditions, the ellipse may be reduced to a circle or a segment of a straight line. In those cases it is said that the wave's polarization is circular or linear, respectively. Since the magnetic field vector is related to the electric field vector, it does not need separate discussion. The different types of polarization can be seen in figure 3.

Furthermore, we distinguish between TE (or s) polarization and TM (or p) polarization. If a wave is TE polarized, then the electric field is normal to the plane of incidence (**s**enkrecht in german) or equivalent it is parallel to the the interface between both media. This configuration can be seen in figure 4.

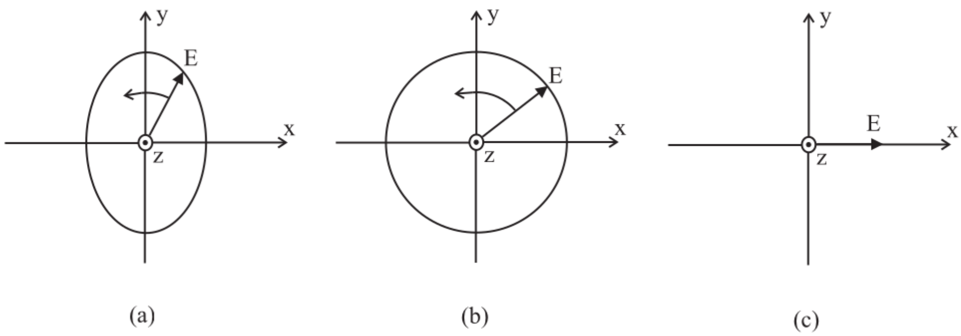

Figure 3: Typical states of polarization: (a) elliptic, (b) circular and (c) linear from [1]

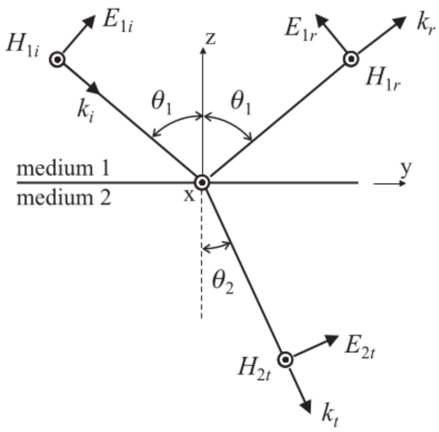

Figure 4: Directions of vectors for a TM polarized wave from [1]

On the other hand, if a wave is TM polariced, then the electric field is **p**arallel to the to the plane of incidence. That implies that the magnetic field is normal to the plance of incidence.

2.1.4 Snell's Law

Snell's law describes how light behaves bewtween two isotropic materials. The setup can be seen in figure 5.

Snell's law is described by the equation

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} = \frac{v_1}{v_2} \ , \tag{12}$$

where θ_1 is the angle of incident, θ_2 is the angle of refraction, $n_{1/2}$ are the refrection indices associated to the isotropic materials and $v_{1/2}$ are the phase velocities in the two media.

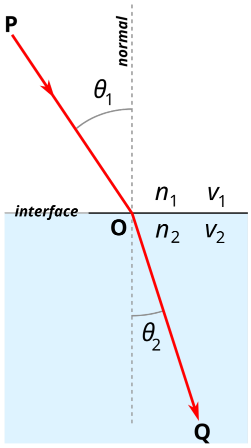

Figure 5: Refraction of light at the interface between two media of different refractive indices, with $n_2 > n_1$. Since the velocity is lower in the second medium ($v_2 < v_1$), the angle of refraction θ_2 is less than the angle of incidence θ_1.

Snell's Law in 12 can be derived from *Fermat's principle of least time*, which can be derived from the propagation of light waves. Fermat's theorem states that a path taken by a light ray between two given points is the one which minimizes the time.

2.1.5 Fresnel coefficients and phases

The Frensel coefficient describes the portion of the electric field, which gets reflected/transmitted at the interface between two dielectrica. The Fresnel coefficients are functions of the angle of incidence and material properties of the two dielectrics. Further, Fressnel phases are determined from the Fressnel coefficients using the following relation

$$r = e^{-2i\phi} \ . \tag{13}$$

We begin by analyzing the TE polarized waves. Referring to figure 6 the E_{1i} , E_{1r} , E_{2t} are complex values of incident, reflected and transmitted electric fields in medium 1 and 2.

A full derivation of the equation can be found in [1] Chapter 3.6.1. The Fressnel coefficient for TE polarized waves is given by

$$r_{TE} = \frac{n_1 \cos\theta_1 - \sqrt{n_2^2 - n_1^2 \cos\theta_1}}{n_1 \cos\theta_1 - \sqrt{n_2^2 + n_1^2 \cos\theta_1}} \ . \tag{14}$$

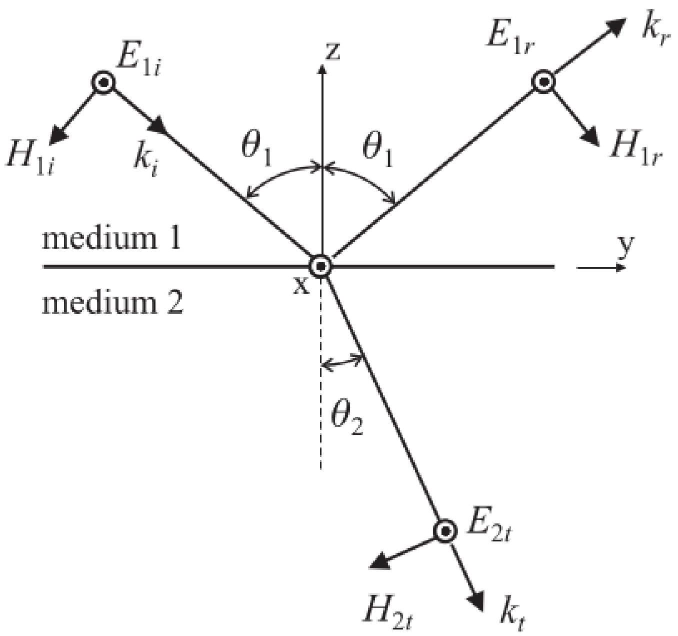

Figure 6: Fresnel reflection for TE polarization. Directions of vectors for TE polarized wave. From [1].

Note, that the square root becomes negative for angles θ_1 such that $n_2^2 - n_1^2 \cos\theta_1 < 0$ and there the coefficient becomes complex. A complex coefficient implies absorption of the material. The Fressnel phase is given by the equation

$$\tan\phi_{TE} = \frac{\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}}{n_1 \cos\theta_1} \ . \tag{15}$$

The above equation represents phase shift during reflection for TE polarized electromagnetic wave.

The same calculation can be done for a TM polarizied wave. Here the coefficient is given by

$$r_{Tm} = \frac{n_2 \cos\theta_1 - \sqrt{n_1^2 - n_2^2 \cos\theta_1}}{n_2 \cos\theta_1 - \sqrt{n_1^2 + n_2^2 \cos\theta_1}} \ , \tag{16}$$

and the Fressnel phase is given by

$$\tan \phi_{TM} = \frac{n_1^2}{n_2^2} \frac{\sqrt{n_1^2 \sin^2 \theta_1 - n_2^2}}{n_1 \cos \theta_1} \ . \tag{17}$$

An example of the Fressnel coefficients for the air-glass reflection with $n = 1.5$ is given in figure 7.

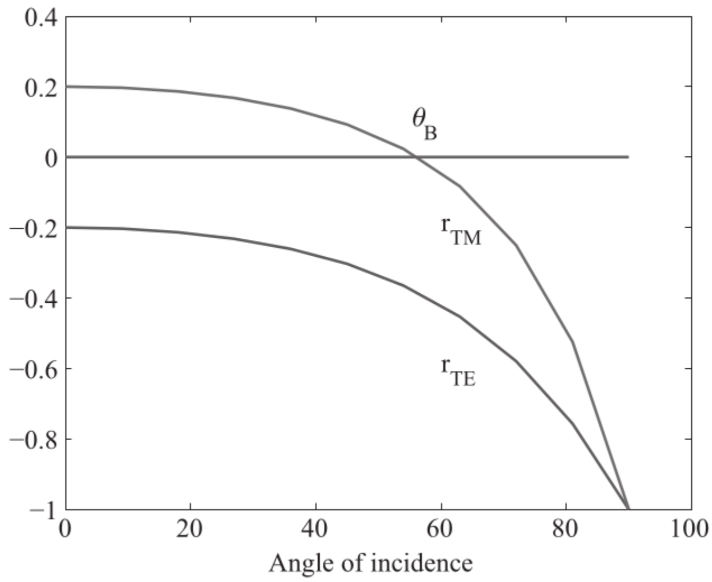

Figure 7: Reflections of TE and TM modes for external reflection with $n = 1.5$. Brewster's angle θ_B is also shown. From [1].

Note that there is an angle, where the TM polarized wave gets not reflected. This angle is called Brewster's angle θ_B and can be used to polarize light.

2.1.6 Bragg Equation

TODO

References

[1] Marek S Wartak. *Computational photonics: an introduction with MATLAB*. Cambridge University Press, 2013.