# Lecture-1-6

## Course Plan / Outline

### A. Course Information

1. Course Title: Differential Calculus and Integral Calculus
2. Course Code: Math 1113
3. Prerequisite: None
4. Semester: 1st Year Odd
5. Credit: 3.00
6. Class Hours: Three Hours per Week

### B. Instructors' Details

1. Name of the Instructor: Dr. Md. Helal Uddin Molla and Md. Zahangir Alam
2. Designation: Professor and Assistant Professor
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7. Consultation Hours:

### C. Course Rationale

Differential and Integral Calculus has become one of the most important courses in which one can be gain knowledge about function, limit, continuity, differentiability, expansion of a function, successive differentiation, Partial differentiation, maxima and minima of a function at any point, tangent and normal, Asymptotes, curvature and curve tracing. In this course one can also gain knowledge about integration by various methods indefinite and definite integrals, beta and gamma function, area and volume of various curves, surface of revolution and Multiple Integration.

### D. Course Content

**Differential calculus**: Limit, Continuity and Differentiability. Differentiation of explicit and implicit function and parametric equations. Significance of derivatives, Differentials, Successive differentiation of various type of functions. Leibnitz's theorem, Rolle's theorem, mean value theorem. Taylor's theorem in finite and infinite forms. Maclaurin's theorem in finite and infinite forms. Lagrange's form of remainders. Cauchy's form of remainder. Expansion of functions by differentiation and integration. Partial differentiation, Euler's theorem. Tangent and Normal, maxima and minima, Points of inflection and their applications. Evaluation of indeterminate forms by L'Hospital rule, curvature, evaluate and inviolate. Asymptotes, Envelopes, Curve tracing.

**Integral Calculus:** Definitions of integration, integration by the method of substitutions, Integration by the method of successive reduction. Definite integrals. Beta function and gamma function. Area under a plane curve in Cartesian and Polar coordinates. Area of the region enclosed by two curves in Cartesian and Polar coordinates. Parametric and Pedal equations. Intrinsic equation. Volumes of solids of revolution. Volume of hollow solids of revolution by shell method. Area of surface of revolution, Multiple Integration.

### Text Books

1. Differential and Integral Calculus, B.C. Dass and B.N. Mukherjee
2. An Elementary Treatise on the Differential Calculus, Joseph Edwards
3. Integral Calculus with Application, A.K. Hazra

**Assessment and Marks Distribution**

Students will be assessed on the basis of their overall performance in all the exams (class tests, final exam, assignments, projects, and presentations. Final numeric reward will be the compilation of:

1. Continuous Assessment (Class Tests, Assignments, Projects, Presentations, etc.): (40%)
    a. Class Tests: (20%)
    b. Attendance (10%)
    c. Others i.e., Assignments/Projects/Presentations: (10%)
2. Final Exam: (60%)

# Function

**Q1. Define function, even, odd and monotone function**.

**Function:** A function is a relation between dependent and independent variables.

Example: (i) $f(x) = x^2 + 2$. (ii) $y = 2x + 1$

**Odd and Even functions:** A function $f(x)$ is said to be odd if it changes sign with the change of sign of the variable $x$ that is if $f(-x) = -f(x)$ and A function $f(x)$ is said to be even if it does not change sign with the change of sign of the variable $x$ that is if $f(-x) = f(x)$.

Examples: (i) $f(x) = x^3$is odd function and (ii) $f(x) = x^2$is even function.

**Home Work: Q2. Determine odd or even function of the following cases:**

**(i)** $\boldsymbol{f(x) = \frac{1}{2}(2^x + 2^{-x})}$**,**

**(ii)** $\boldsymbol{f(x) = \sqrt{(1 + 5x + 7x^2)} - \sqrt{(1 - 5x + 7x^2)}}$**,**

**(iii)** $\boldsymbol{f(x) = Log\left(\frac{2+x}{2-x}\right)}$**,**

**(iv)** $\boldsymbol{f(x) = Log(x + \sqrt{1 + x^2})}$**.**

**Monotone functions:** Let, $x_1,\ x_2$be any two points such that $x_1 <\ x_2$ in the interval of definition of a function$f(x)$, then $f(x)$ is said to be monotonically increasing if $f(x_1) < f(x_2)$ and monotonically decreasing if $f(x_1) > f(x_2)$.

**Q3: Show that** $\boldsymbol{f(x) = \frac{x}{x+1}}$ **is monotone ascending,** $\boldsymbol{x > 0}$.

**Solution:** Given $f(x) = \frac{x}{x+1}$, $x > 0$.

Now, for any two points $x_1$ and $x_2$ where, $x_1 > 0,\ x_2 > 0$.

$$f(x_1) = \frac{x_1}{x_1+1} \text{ and } f(x_2) = \frac{x_2}{x_2+1}$$

$$\therefore f(x_1) - f(x_2) = \frac{x_1}{x_1+1} - \frac{x_2}{x_2+1}$$

$$= \frac{x_1x_2+x_1-x_1x_2-x_2}{(x_1+1)(x_2+1)} = \frac{x_1-x_2}{(x_1+1)(x_2+1)}$$

If $x_1 > x_2$ then $x_1 - x_2\ > 0$

so, $f(x_1) - f(x_2) > 0$

or, $f(x_1) > f(x_2)$, Hence $f(x)$ is monotone ascending/increasing.

**Q4: Show that, $f(x)=\frac{1}{x+1}+\frac{1}{x+2}+\cdots\cdots\cdots+\frac{1}{x+n}$ is monotone descending, $x>0$.**

**Solution:** Given $f(x)=\frac{1}{x+1}+\frac{1}{x+2}+\cdots\cdots\cdots+\frac{1}{x+n}$, $x>0$.

Now, for any two points $x_1$ and $x_2$ were, $x_1>0,\ x_2>0$.

$$f(x_1)=\frac{1}{x_1+1}+\frac{1}{x_1+2}+\cdots\cdots\cdots+\frac{1}{x_1+n}$$

and $f(x_2)=\frac{1}{x_2+1}+\frac{1}{x_2+2}+\cdots\cdots\cdots+\frac{1}{x_2+n}$

$$\therefore f(x_1)-f(x_2)=\left(\frac{1}{x_1+1}-\frac{1}{x_2+1}\right)+\left(\frac{1}{x_1+2}-\frac{1}{x_2+2}\right)+\cdots\cdots\cdots+\left(\frac{1}{x_1+n}-\frac{1}{x_2+n}\right)$$

$$=\left(\frac{x_2-x_1}{(x_1+1)(x_2+1)}\right)+\left(\frac{x_2-x_1}{(x_1+2)(x_2+2)}\right)+\cdots\cdots\cdots+\left(\frac{x_2-x_1}{(x_1+n)(x_2+n)}\right)$$

$$=(x_2-x_1)\left(\frac{1}{(x_1+1)(x_2+1)}\right)+\left(\frac{1}{(x_1+2)(x_2+2)}\right)+\cdots\cdots\cdots+\left(\frac{1}{(x_1+n)(x_2+n)}\right)$$

If $x_1>x_2$ then $x_2-x_1<0$

so, $f(x_1)-f(x_2)<0$

or, $f(x_1)<f(x_2)$, Hence $f(x)$ is monotone descending/decreasing.

**Home Work: Q5. Show that, $f(x)=\left(1+\frac{1}{x}\right)^x$, $x>0$, is monotone ascending.**

# Limit

**Q1. Define limit. What are the differences between $\underset{x\to a}{Lt}\, f(x)$ and $f(a)$.**

**Limit:** A function $f(x)$ is said to have a limit $\ell$ at $x$ tends to $a$ if for every positive number $\varepsilon$, we can determine another small positive number $\delta$, such that $|f(x)-\ell|<\varepsilon$ for all values of $x$ which satisfying the inequality $|x-a|\le\delta$ and it is denoted by $\underset{x\to a}{lt}\, f(x)=\ell$.

**Right hand Limit:** A function $f(x)$ is said to have a right hand limit $\ell_1$ at $x$ tends to $a$ if for every positive number $\varepsilon$, we can determine another small positive number $\delta$, such that $|f(x)-\ell_1|<\varepsilon$for all values of $x$ which satisfying the inequality $x-a\le\delta$ and it is denoted by $\underset{x\to a^+}{lt}\, f(x)=\ell_1$ or, $\underset{h\to 0}{lt}\, f(a+h)=\ell_1$.

**Left hand Limit:** A function $f(x)$ is said to have a left hand limit $\ell_2$ at $x$ tends to $a$ if for every positive number $\varepsilon$, we can determine another small positive number $\delta$, such that $|f(x)-\ell_2|<\varepsilon$ for all values of $x$ which satisfying the inequality $a-x\le\delta$ and it is denoted by $\underset{x\to a^-}{lt}\, f(x)=\ell_2$ or, $\underset{h\to 0}{lt}\, f(a-h)=\ell_2$.

**2nd Part:** The statement $\underset{x\to a}{Lt}\, f(x)$ is a statement about the value of $f(x)$ when $x$ has any arbitrary value near to $a$ except $a$. But $f(a)$ stands for the value of $f(x)$ when $x$ is exactly equal to $a$ or else by substitution of $a$ for $x$ in the expression $f(x)$ when it exists.

**Q2. A function is defined as $f(x)=\begin{cases} x & when\ x>0 \\ 0 & when\ x=0 \\ -x & when\ x<0\end{cases}$, find the value of $\underset{x\to 0}{Lt}\, f(x)$.**

**Solution:** Right hand limit: $\underset{x\to 0^+}{Lt}\, f(x)=\underset{x\to 0^+}{Lt}\, x=0$

Left hand limit: $\underset{x\to 0^-}{Lt}\, f(x)=\underset{x\to 0^-}{Lt}\, (-x)=0$

Since R.H.L=L.H.L, So limit exist and $\underset{x\to 0}{Lt}\, f(x)=0$ Ans.

**H.W. Q3. A function is defined as $\varphi(x)=\begin{cases} x^2 & when\ x<1 \\ 2.5 & when\ x=1 \\ x^2+2 & when\ x>1\end{cases}$, Does $\underset{x\to 1}{Lt}\, \varphi(x)$ exist?**

**H.W. Q4. Evaluate $\underset{n\to\infty}{Lt}\left\{\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots\cdots+\frac{n}{n^2}\right\}$.**

# Continuity

**Q1. Define Continuity. A function is defined as $f(x)=\begin{cases} xSin(1/x) & for\ x\neq 0 \\ 0 & for\ x=0 \end{cases}$ Show that $f(x)$ is continuous at $x=0$.**

**Ans:** **Continuity:** A function $f(x)$ is said to be continuous at $x=a$ if its limit exists at that point and equal to its functional value at $x=a$ i.e, $\underset{x\to a^+}{lt} f(x) = \underset{x\to a^-}{lt} f(x) = f(a)$. or, $\underset{h\to 0}{lt} f(a+h) = \underset{h\to 0}{lt} f(a-h) = f(a)$.

**2nd Part:** Right hand limit (R.H.L): $\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt} xSin(1/x) = 0$

Left hand limit (L.H.L): $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt} xSin(1/x) = 0$

and functional value (F.v): when $x=0$ then $f(x)=0$ or $f(0)=0$.

Since R.H.L = L.H.L = F.V

Hence $f(x)$ is continuous at $x=0$.

**Q2. A function is defined as $\varphi(x)=\begin{cases} x^2 & when\ x<1 \\ 2.5 & when\ x=1 \\ x^2+2 & when\ x>1 \end{cases}$, Is $\varphi(x)$ continuous at $x=1$?**

**Solution:** R.H.L: $\underset{x\to 1^+}{Lt} \varphi(x) = \underset{x\to 1^+}{Lt} (x^2+2) = 3$

L.H.L: $\underset{x\to 1^-}{Lt} \varphi(x) = \underset{x\to 1^-}{Lt} x^2 = 1$

Since R.H.L≠L.H.L

Hence $\varphi(x)$ is discontinuous at $x=1$.

**Q3. Find the values of c that makes f$(x)=\begin{cases} \frac{x-c}{c+1} & if\ x\leq 0 \\ x^2+c & if\ x>0 \end{cases}$, continuous for every value of x.**

**Solution** : If a function is continuous at any point then

Right hand limit (R.H.L) = Left hand limit (L.H.L) = Functional value (F.V) at that point.

Here the existing point is x=0.

Now, Right hand limit (R.H.L): $\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt} x^2+c = c$

Left hand limit (L.H.L): $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt} \frac{x-c}{c+1} = \frac{-c}{c+1}$

Since R.H.L=L.H.L

So, $c=\frac{-c}{c+1}$

Or, $c^2+2c=0$

Or, $c(c+2)=0$

Or, $c=0,-2$ Ans.

**Q4. Find the points of discontinuity of the following functions: (i) $\frac{x^3+2x+5}{x^2-8x+12}$ H.W. (ii) $\frac{x^3+2x+5}{x^2-8x+16}$**

**Solution:** (i) The function is undefined at the point where $x^2-8x+12=0$

Or, $x^2-6x-2x+12=0$

Or, $x(x-6)-2(x-6)=0$

Or, $x=2,6$.

Hence the points of discontinuity are $x=2,6$.

**H.W. Q5. A function is defined as** $f(x) = \begin{cases} \frac{1}{2} - x & when\ 0 < x < \frac{1}{2} \\ \frac{1}{2} & when\ x = \frac{1}{2} \\ \frac{3}{2} - x & when\ \frac{1}{2} < x < 1 \end{cases}$ **Show that** $f(x)$ **is discontinuous at** $x = \frac{1}{2}$.

**H.W. Q6. A function is defined as** $f(x) = \begin{cases} -x & when\ x \le 0 \\ x & when\ 0 < x < 1 \\ 2 - x & when\ x \ge 1 \end{cases}$**, show that it is continuous at** $x = 0$ **and** $x = 1$.

**H.W. Q7. A function is defined as** $f(x) = \begin{cases} 3 + 2x & for\ -\frac{3}{2} \le x < 0 \\ 3 - 2x & for\ 0 \le x < \frac{3}{2} \\ -3 - 2x & for\ x \ge \frac{3}{2} \end{cases}$ **, Show that** $f(x)$ **is continuous at** $x = 0$ **and discontinuous at** $x = \frac{3}{2}$.

# Differentiation

**Q1. Define differentiability of a function at any point. Show that every differentiable function is continuous but converse is not always true. Or, Prove that if** $f'(a)$ **is finite, then** $f(x)$ **must be continuous at** $x = a$.

**Ans: Differentiable:** A function $f(x)$ is said to be differentiable at $x = a$ if

$$\underset{h\to 0^+}{Lt} \frac{f(a+h)-f(a)}{h} = \underset{h\to 0^-}{Lt} \frac{f(a-h)-f(a)}{-h}$$

i,e. R.H.L of $f'(x)$ = L.H.L of $f'(x)$.

**2nd Part:** From the definition of differentiation, we can write $f'(a) = \underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{h}$

Now, we can write $f(a+h) - f(a) = \frac{f(a+h)-f(a)}{h} \times h$

$$\text{Or, } \underset{h\to 0}{Lt} f(a+h) - f(a) = \underset{h\to 0}{Lt} \left\{\frac{f(a+h)-f(a)}{h} \times h\right\}$$

$$= \underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{h} \times \underset{h\to 0}{Lt}\, h$$

$$= f'(a) \times 0$$

$$= 0 \quad \text{since } f'(a) \text{ is finite.}$$

$$\underset{h\to 0}{Lt} f(a+h) = f(a).$$

$\therefore$ from the definition of continuity, it follows that $f(x)$ is continuous at $x = a$.

Hence, for the differential coefficient of $f(x)$ to exist finitely for any value of $x$, the function $f(x)$ must be continuous at the point. The converse is not always true.

**Q2. Find from first principles the differential coefficient of the following functions:**

**(i)** $sin\, x$ **(ii)** $Log\, cos\, x$ **(iii)** $x^x$ **(iv)** $tan^{-1} x$

**Solution: (i)** Let, $f(x) = sin\, x$

We know, $f'(x) = \frac{d}{dx} f(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$

$$\text{Or, } f'(x) = \underset{h\to 0}{Lt} \frac{sin(x+h) - sin\, x}{h}$$

$$= \lim_{h\to 0} \frac{2\sin(h/2)\cos(x+h/2)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \times \lim_{h\to 0} \cos(x+h/2)$$

$= \cos x$ Ans.

**(ii)** Let, $f(x) = Log\cos x$ and $u = \cos x$

$\therefore f(x+h) = Log\cos(x+h)$ $\quad$ $\therefore u + k = \cos(x+h)$

$\therefore k = \cos(x+h) - \cos x$ [by subtracting]

When $h \to 0$ then $k \to 0$

We know, $f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Or, $f'(x) = \lim_{h\to 0} \frac{Log\cos(x+h) - Log\cos x}{h}$

$$= \lim_{k\to 0} \frac{Log(u+k) - Log\,u}{k} \times \lim_{h\to 0} \frac{k}{h}$$

$$= \lim_{k\to 0} \frac{1}{k} Log\left(1 + {}^{k}/_{u}\right) \times \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{k/u\to 0} \frac{1}{k/u} Log\left(1 + {}^{k}/_{u}\right) \times \frac{1}{u} \times \lim_{h\to 0} \frac{-2\sin(x+h/2)\sin\ h/2}{h}$$

$=1 \times \frac{1}{\cos x} \times (-) \lim_{h\to 0} \lim_{h\to 0} \sin(x+h/2) \times \lim_{h\to 0} \frac{\sin\ h/2}{h/2}$ $\quad$ [as $k \to 0$ so, ${}^{k}/_{u} \to 0$]

Hence, $f'(x) = -\frac{\sin x}{\cos x} = -\tan x$ Ans.

**(iii)** $\boldsymbol{x^x}$

Let, $f(x) = x^x = e^{x\log x}$

$\therefore f(x+h) = e^{(x+h)\log(x+h)}$

$\therefore f(x+h) - f(x) = e^{(x+h)\log(x+h)} - e^{x\log x}$

$$= e^{x\log x}\left[e^{(x+h)\log(x+h) - x\log x} - 1\right]$$

$$= e^{x\log x}[e^z - 1]$$

Where, $z = (x+h)\log(x+h) - x\log x$

When $h \to 0$ then $z \to 0$

We know, $f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{e^{x\log x}[e^z - 1]}{h}$$

$$= e^{x\log x} \lim_{h\to 0} \left[\frac{e^z - 1}{z} \times \frac{z}{h}\right]$$

$$= e^{x\log x} \lim_{z\to 0} \frac{e^z - 1}{z} \times \lim_{h\to 0} \frac{z}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, $\lim_{z\to 0} \frac{e^z - 1}{z} = 1,$

and $\lim_{h\to 0} \frac{z}{h} = \lim_{h\to 0} \frac{(x+h)\log(x+h) - x\log}{h}$

$$= \lim_{h\to 0} \frac{x\log(x+h) + h\log(x+h) - x\log}{h}$$

$$= \lim_{h\to 0} \frac{x\{\log(x+h) - \log x\} + h\log(x+h)}{h}$$

$$= \lim_{h\to 0} \frac{x}{h}\log(1 + h/x) + \lim_{h\to 0} \log(x+h)$$

$$= \lim_{h/x\to 0} \frac{1}{h/x}\log(1 + h/x) + \lim_{h\to 0} \log(x+h)$$

$$= 1 + \log x$$

Hence, from Eq. (1) we get, $f'(x) = e^{x\log x} \lim_{z\to 0} \frac{e^z - 1}{z} \times \lim_{h\to 0} \frac{z}{h}$

$= x^x(1 + \log x)$ Ans.

**(iv) $\tan^{-1} x$**

Let, $f(x) = \tan^{-1} x = y$ (say)

Or, $\tan y = x$ ………… (1)

$\therefore f(x+h) = \tan^{-1}(x+h) = y + k$ (say)

Or, $\tan(y+k) = x + h$ ………… (2)

Subtracting (1) from (2) we get, $\tan(y+k) - \tan y = h$

When $h \to 0$ then $k \to 0$

We know, $f'(x) = \frac{d}{dx} f(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$

$$\text{Or, } f'(x) = \underset{h\to 0}{Lt} \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h}$$

$$= \underset{k\to 0}{Lt} \frac{y+k-y}{\tan(y+k) - \tan y}$$

$$= \underset{k\to 0}{Lt} \frac{k}{\tan(y+k) - \tan y}$$

$$= \underset{k\to 0}{Lt} \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}$$

$$= \underset{k\to 0}{Lt} \frac{k}{\sin(y+k)\cos y - \cos(y+k)\sin y} \times \cos(y+k)\cos y$$

$$= \underset{k\to 0}{Lt} \frac{k}{\sin k} \times \cos(y+k)\cos y$$

$$= \cos y \cos y$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \text{ Ans.}$$

**H.W Q3. Find from first principles the differential coefficient of the following functions:**

**(i) $\cos x$ (ii) $\tan x$ (iii) $Log_{10} x$ (iv) $e^{\sin x}$ (v) $\sin^{-1} x$ (vi) $\cos^{-1} x$**

**Q4. A function is defined as $f(x) = \begin{cases} x\sin(1/x) & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$, Show that $f'(0)$ does not exist.**

**Solution:** We know that $f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$

$$\text{Or, } f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

**R.H.D:** $f'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$

$$\text{Or,} f'(0) = \lim_{h\to 0^+} \frac{h\,Sin(1/h) - 0}{h}$$

$$\text{Or,} f'(0) = \lim_{h\to 0^+} \frac{h\,Sin(1/h)}{h}$$

Or, $f'(0) = \lim_{h\to 0^+} Sin(1/h)$ which is not defined.

Hence $f'(0)$ does not exist.

**Q5. A function is defined as $f(x) = \begin{cases} x^2 Sin(1/x) & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$, Does $f'(0)$ exist?**

**<u>Solution:</u>** We know that $f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$

Or, $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

**<u>R.H.D:</u>** $f'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$

Or, $f'(0) = \lim_{h\to 0^+} \frac{h^2\ Sin(1/h)-0}{h}$

Or, $f'(0) = \lim_{h\to 0^+} \frac{h^2 Sin(1/h)}{h}$

Or, $f'(0) = \lim_{h\to 0^+} h\ Sin(1/h) = 0$

**<u>L.H.D:</u>** $f'(0) = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} \frac{-h^2\ Sin(1/h)}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} \frac{h^2 Sin(1/h)}{h}$

Or, $f'(0) = \lim_{h\to 0^-} h\ Sin(1/h) = 0$

Since R.H.D = L.H.D

Hence $f'(0)$exist.

**Q6. If $f(x) = \begin{cases} 3+2x & for\ -\frac{3}{2} < x \leq 0 \\ 3-2x & for\ 0 < x < \frac{3}{2} \end{cases}$, then show that $f(x)$ is continuous at $x = 0$ but $f'(0)$ does not exist.**

**<u>Solution:</u> 1st Part:** <u>R.H.L:</u> $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3-2x = 3-0 = 3$

<u>L.H.L:</u> $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3+2x = 3+0 = 3$

<u>F.V:</u> when $x = 0$ then $f(x) = 3+2x$ or, $f(0) = 3$

Since R.H.L=L.H.L=F.V, Hence $f(x)$ is continuous at $x$=0. (Proved)

**2nd Part:** We know that $f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x)}{h}$

Or, $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

**<u>R.H.D:</u>** $f'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$

Or, $f'(0) = \lim_{h\to 0^+} \frac{3-2h-}{h}$

Or, $f'(0) = \lim_{h\to 0^+} \frac{-2h}{h}$

Or, $f'(0) = \lim_{h\to 0^+} -2 = -2$

**<u>L.H.D:</u>** $f'(x) = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} \frac{3-2h-3}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} \frac{-2h}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} 2 = 2$

Since R.H.D $\neq$ L.H.D, Hence $f'(0)$ does not exist. (Proved)

**Q7. If** $f(x) = \begin{cases} 1+x & for \;\; x<0 \\ 1 & for \; 0 \le x \le 1 \\ 2x^2+4x+5 & for \; x>1 \end{cases}$ **, find $f'(x)$ for all values of $x$ for which it exists.**

**Or, Does $f'(x)$ exist at those values of $x$ for which it exists?**

**Solution:** Given $f(x) = \begin{cases} 1+x & for \;\; x<0 \\ 1 & for \; 0 \le x \le 1 \\ 2x^2+4x+5 & for \; x>1 \end{cases}$

Here, existing points are $x = 0$ and $x = 1$

We know that $f'(x) = \underset{h\to 0}{\mathrm{Lt}} \frac{f(x+h)-f(x)}{h}$

**At $x = 0$:**

**R.H.D:** $f'(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$

Or, $f'(0) = \lim_{h\to 0^+} \frac{1-1}{h} = 0$

**L.H.D:** $f'(0) = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} \frac{1-h-1}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} \frac{-h}{-h}$

Or, $f'(0) = \lim_{h\to 0^-} 1 = 1$

Since R.H.D $\neq$ L.H.D. Hence $f'(0)$ does not exist.

**At x=1:**

**R.H.D:** $f'(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$

Or, $f'(1) = \lim_{h\to 0^+} \frac{2(1+h)^2+4(1+h)+5-1}{h}$

Or, $f'(1) = \lim_{h\to 0^+} \frac{2+4h+\quad {}^2+4+4h+}{h}$

Or, $f'(1) = \lim_{h\to 0^+} \frac{10}{h} + \lim_{h\to 0^+} 8 + \lim_{h\to 0^+} 2h = \infty$

**L.H.D:** $f'(1) = \lim_{h\to 0^-} \frac{f(1-h)-f(1)}{-h}$

Or, $f'(1) = \lim_{h\to 0^-} \frac{1-1}{-h} = 0$

Since R.H.D $\neq$ L.H.D. Hence $f'(1)$ does not exist.

**Home Works:**

**Q8. If** $f(x) = \begin{cases} 1 & for \quad x<0 \\ 1+\sin x & for \; 0 \le x < \frac{\pi}{2} \\ 2+(x-\frac{\pi}{2})^2 & for \; \frac{\pi}{2} \le x \end{cases}$**, then show that $f'(x)$is exist at $x = \frac{\pi}{2}$ but does not exist at $x = 0$.**

**Q9. If** $f(x) = \begin{cases} 5x-4 & for \; 0 < x \le 1 \\ 4x^2-3x & for \; 1<x<2 \\ 3x+4 & for \; x \ge 2 \end{cases}$**, then discuss the continuity of $f(x)$ for$x = 1$ and 2, and the existence of $f'(x)$ for these values.**

**Q10. If** $f(x)=\begin{cases} x & for\ 0< x<1 \\ 2-x & for\ 1\le x\le 2 \\ x-\frac{x^2}{2} & for\ x>2 \end{cases}$**, Is** $f(x)$ **continuous at** $x=1$ **and 2? Does** $f'(x)$ **exist for these values?**

**Q11. If** $f(x)=\begin{cases} -\frac{x^2}{2} & for\ \ x\le 0 \\ x^n\, sin(1/x) & for\ x>0 \end{cases}$**, find whether** $f'(0)$ **exists for** $n=1$ **and 2.**

**Q12. A function is defined as** $f(x)=\begin{cases} x\cos(1/x) & for\ x\ne 0 \\ 0 & for\ x=0 \end{cases}$**, Show that the function is continuous at** $x=0$ **but is not differentiable at that point.**

**Q13. Find the differential coefficients of the following with respect to x:**

**(i)** $x^n e^x$ **(ii)** $10^x x^{10}$ **(iii)** $\frac{x^4}{\sin x}$

**Q14. Find the differential coefficient of : (i)** $sin^2(\,log sec\, x)$ **(ii)** $x^x$ **(iii)** $(1+x)^x$ **(iv)** $x^{\log x}$ **(v)** $x^{1+x+x^2}$ **(vi)** $a^{a^x}$ **(vii)** $e^{e^x}$ **(viii)** $e^{x^x}$ **(ix)** $x^{e^x}$ **(x)** $(\sin x)^{\tan x}$ **(xi)** $x^{\cot^{-1}x}$ **(xii)** $(\sin x)^{\log x}$ **(xiii)** $x^{x^x}$ **(xiv)** $(\sin x)^{\cos x}+(\cos x)^{\sin x}$ **(xv)** $(\tan x)^{\cot x}+(\cot x)^{\tan x}$ **(xvi)** $\tan^{-1}\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)}$ **(xvii)** $\sin\left\{2\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right\}$ **(xviii)** $\log\frac{a+b\tan x}{a-b\tan x}$

**Q15. Find** $\frac{dy}{dx}$ **in the following cases: (i)** $y=x^y$ **(ii)** $x^y=y^x$ **(iii)** $(\cos x)^y=(\sin y)^x$

**Q16. If** $y=e^{\sin^{-1}x}$ **and** $z=e^{-\cos^{-1}x}$ **then show that** $dy/dz$ **is independent of** $x$.

**Q17. Differentiate (i)** $\sec x$ **with respect to** $\tan x$ **(ii)** $\tan^{-1}x$ **with respect to** $x^2$

**(iii)** $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ **with respect to** $\tan^{-1}x$ **(iv)** $x^{\sin^{-1}x}$ **with respect to** $\sin^{-1}x$

**(v)** $\cos^{-1}\frac{1-x^2}{1+x^2}$ **with respect to** $\tan^{-1}\frac{2x}{1-x^2}$ **(v)** $x^x$ **with respect to** $\sin^{-1}x$

# Successive Differentiation

**Definition:** If $y=f(x)$ be a general function of $x$, then its differential coefficient $y_1=dy/dx=f'(x)$ is called the first derivative of $f(x)$. The differential coefficient of $f'(x)$ is called the 2nd derivative of $f(x)$ and it is denoted by $f''(x)$. Similarly, the differential coefficient of $f''(x)$ is called the 3rd derivative of $f(x)$ and it is denoted by $f'''(x)$ and so on. If $f(x)$ is differentiated $n$ times with respect to $x$ then it is called the $n$th derivative of $f(x)$ and it is denoted by $f^n(x)$.So the successive derivatives of $y=f(x)$ are denoted by $f'(x), f''(x), \dots\dots\dots, f^n(x), \dots\dots\dots$

**Q 1. If** $y=e^{ax}$ **then find** $y_n$**.**

**Solution:** Given $y=e^{ax}$

$\therefore\ y_1=ae^{ax}$ (by differentiating with respect to $x$.)

$y_2=a^2e^{ax}$ (again by differentiating with respect to $x$.)

$y_3=a^3e^{ax}$ (again by differentiating with respect to $x$.)

Similarly, $y_4=a^4e^{ax}$

$\dots\dots$

$y_n=a^ne^{ax}$ Ans.

**Q 2. If $y = Log(x+a)$ then find $y_n$.**

Solution: Given $y = Log(x+a)$

$\therefore\ y_1 = \frac{1}{x+a}$ (by differentiating with respect to $x$.)

$y_2 = (-1)\frac{1}{(x+a)^2}$ (again by differentiating with respect to $x$.)

$y_3 = (-1)(-2)\frac{1}{(x+a)^3} = (-1)^{3-1}\frac{(3-1)!}{(x+a)^3}$ (again by differentiating with respect to $x$.)

Similarly, $y_4 = (-1)(-2)(-3)\frac{1}{(x+a)^4} = (-1)^{4-1}\frac{(4-1)!}{(x+a)^4}$

$\dots\dots\dots$

$y_n = (-1)^{n-1}\frac{(n-1)!}{(x+a)^n}$ Ans.

**Q 3. If $y = sin(ax+b)$ then find $y_n$.**

Solution: Given $y = sin(ax+b)$

$\therefore\ y_1 = a\ cos(ax+b)$ (by differentiating with respect to $x$.)

$y_1 = a\ sin(\frac{\pi}{2} + ax + b)$

$y_2 = a^2 cos(\frac{\pi}{2} + ax + b)$ (again by differentiating with respect to $x$.)

$y_2 = a^2 sin(\frac{2\pi}{2} + ax + b)$

$y_3 = a^3 cos(\frac{2\pi}{2} + ax + b)$ (again by differentiating with respect to $x$.)

$y_3 = a^3 sin(\frac{3\pi}{2} + ax + b)$

Similarly, $y_4 = a^4 sin(\frac{4\pi}{2} + ax + b)$

$\dots\dots$

$y_n = a^n sin(\frac{n\pi}{2} + ax + b)$ Ans.

H.W **Q 4. If $y = cos(ax+b)$ then find $y_n$.** Ans. $y_n = a^n cos(\frac{n\pi}{2} + ax + b)$

**Q 5. If $y = e^{ax} sin(bx+c)$ then find $y_n$.**

Solution: Given, $y = e^{ax} sin(bx+c)$

$\therefore y_1 = ae^{ax} sin(bx+c) + be^{ax} cos(bx+c)$ Let, $a = r\cos\theta$ and $b = r\sin\theta$

So, $y_1 = e^{ax}\{r\cos\theta\, sin(bx+c) + r\sin\theta\, cos(bx+c)\}$ $\therefore r^2 = a^2 + b^2$ and $\theta = tan^{-1}(b/a)$

$= re^{ax} sin(bx+c+\theta)$

$y_2 = r\{ae^{ax} sin(bx+c+\theta) + be^{ax} cos(bx+c+\theta)\}$

$= re^{ax}\{r\cos\theta\, sin(bx+c+\theta) + r\sin\theta\, cos(bx+c+\theta)\}$

$= r^2 e^{ax} sin(bx+c+\theta+\theta)$

$= r^2 e^{ax} sin(bx+c+2\theta)$

$\dots\dots$

Similarly, $y_n = r^n e^{ax} sin(bx+c+n\theta)$ Ans. here, $r = \sqrt{a^2+b^2}$ and $\theta = tan^{-1}(b/a)$

**Q 6. If $y = x^{n-1}Logx$ then prove that $y_n = \frac{(n-1)!}{x}$.**

Proof: Given $y = x^{n-1}Logx$

$\therefore\ y_n = \frac{d^n}{dx^n}(x^{n-1}Logx)$ (by differentiating $n$times with respect to $x$.)

or, $y_n = D^n(x^{n-1}Logx)$

or, $y_n = D^{n-1}.D(x^{n-1}Logx)$

or, $y_n = D^{n-1}.\{(n-1)x^{n-2}Logx + x^{n-2}\}$

or, $y_n = (n-1)D^{n-1}(x^{n-2}Logx) + D^{n-1}x^{n-2}$

or, $y_n = (n-1)D^{n-2}D(x^{n-2}Logx) + 0$

or, $y_n = (n-1)D^{n-2}.\{(n-2)x^{n-3}Logx + x^{n-3}\}$

or, $y_n = (n-1)(n-2)D^{n-2}.(x^{n-3}Logx) + (n-1)D^{n-2}x^{n-3}$

or, $y_n = (n-1)(n-2)D^{n-3}.D(x^{n-3}Logx) + 0$

or, $y_n = (n-1)(n-2)D^{n-3}\{(n-3)x^{n-4}Logx + x^{n-4}\}$

or, $y_n = (n-1)(n-2)(n-3)D^{n-3}(x^{n-4}Logx) + (n-1)(n-2)D^{n-3}x^{n-4}$

$\cdots\cdots$

Proceeding in this way, $y_n = (n-1)(n-2)(n-3)\cdots\cdots 2.1.D(x^0 Logx)$

or, $y_n = (n-1)(n-2)(n-3)\cdots\cdots 2.1.\frac{1}{x}$

or, $y_n = \frac{(n-1)!}{x}$ (Proved)

**Leibnitz's Theorem:**

**Statement:** If $u$ and $v$ are two functions of $x$, Then the $n$th derivative of their product is

$$(uv)_n = u_n v + n_{C_1}u_{n-1}v_1 + n_{C_2}u_{n-2}v_2 + \cdots\cdots + n_{C_r}u_{n-r}v_r + \cdots\cdots + uv_n,$$

where the suffixes in $u$and $v$ denote the order of derivations of $u$and $v$ with respect to $x$.

**Proof:** Let, $y = uv$

$\therefore y_1 = u_1 v + uv_1$

$\therefore y_2 = u_2 v + u_1 v_1 + u_1 v_1 + uv_2$

$= u_2 v + 2u_1 v_1 + uv_2$

$= u_2 v + 2_{C_1}u_{2-1}v_1 + uv_2$

Similarly,

$y_3 = u_3 v + u_2 v_1 + 2u_2 v_1 + 2u_1 v_2 + u_1 v_2 + uv_3$

$= u_3 v + 3u_2 v_1 + 3u_1 v_2 + uv_3$

$= u_3 v + 3_{C_1}u_{3-1}v_1 + 3_{C_2}u_{3-2}v_2 + uv_3$

$\cdots\cdots$

$y_n = u_n v + n_{C_1}u_{n-1}v_1 + n_{C_2}u_{n-2}v_2 + \cdots\cdots\cdots + n_{C_r}u_{n-r}v_r + \cdots\cdots\cdots + uv_n.$ (Proved).

**Q 7. If $y = tan^{-1}x$ then prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$.**

**Proof:** Given $y = tan^{-1}x$

$\therefore\ y_1 = \frac{1}{1+x^2}$ (by differentiating with respect to $x$.)

$(1+x^2)y_1 = 1 \cdots\cdots\cdots\cdots\cdots\cdots$ (i)

Differentiating Eq. (i), $n$times by Leibnitz's theorem, where, $u = y_1$and $v = 1+x^2$.

$y_{n+1}(1+x^2) + n_{C_1}y_n.2x + n_{C_2}y_{n-1}.2 = 0$

or, $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$. (proved).

**Q 8. If $y = (sin^{-1}x)^2$ then prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$.**

**Proof:** Given $y = (sin^{-1}x)^2$

$\therefore\ y_1 = 2\,sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ (by differentiating with respect to $x$.)

or, $(\sqrt{1-x^2})y_1 = 2\,sin^{-1}x$

or, $(1-x^2)y_1^2 = 4(sin^{-1}x)^2$

or, $(1-x^2)y_1^2 = 4y$

or, $2y_1y_2(1-x^2) + y_1^2(-2x) = 4y_1$ (again by differentiating with respect to $x$.)

or,$y_2(1-x^2) - y_1x - 2 = 0 \cdots\cdots\cdots\cdots\cdots\cdots$ (i) [Dividing by$2y_1$]

Differentiating Eq. (i), $n$times by Leibnitz's theorem

$y_{n+2}(1-x^2) + n_{C_1}y_{n+1}.(-2x) + n_{C_2}y_n.(-2) - y_{n+1}.x - n_{C_1}y_n = 0$

or, $(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n=0$

or, $(1-x^2)y_{n+2}-2nxy_{n+1}-n^2y_n+ny_n-xy_{n+1}-ny_n=0$

or, $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$. (proved).

**Q 9. If $y=a\cos(Logx)+b\sin(Logx)$ then prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$.**

**Proof:** Given $y=a\ \cos(Logx)+b\ \sin(Logx)$

$\therefore\ y_1=-a\ \sin(Logx).\frac{1}{x}+b\ \cos(Logx).\frac{1}{x}$ (by differentiating with respect to $x$)

or, $y_1=-\frac{1}{x}\{a\ \sin(Logx)-b\ \cos(Logx)\}$

or, $y_2=-\frac{1}{x}\left\{a\ \cos(Logx).\frac{1}{x}+b\ \sin(Logx).\frac{1}{x}\right\}+\{a\ \sin(Logx)-b\ \cos(Logx)\}.\frac{1}{x^2}$

or, $y_2=\frac{1}{x^2}[-\{a\ \cos(Logx)+b\ \sin(Logx)\}+\{a\ \sin(Logx)-b\ \cos(Logx)\}]$

or, $y_2=\frac{1}{x^2}[-y-xy_1]$

or, $x^2y_2+xy_1+y=0\cdots\cdots\cdots\cdots\cdots$ (i)

Differentiating Eq. (i), $n$times by Leibnitz's theorem

$y_{n+2}.x^2+n_{C_1}y_{n+1}.2x+n_{C_2}y_n.2+y_{n+1}.x+n_{C_1}y_n.1+y_n=0$

or, $x^2y_{n+2}+2nxy_{n+1}+n(n-1)y_n.+xy_{n+1}+ny_n+y_n=0$

or, $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2-n+n+1)y_n.=0$

or, $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$. (proved)

**Q 10. If $y=A\left(x+\sqrt{x^2+a^2}\right)^n+B\left(x+\sqrt{x^2+a^2}\right)^{-n}$ then prove that $(x^2+a^2)y_{m+2}+(2m+1)xy_{m+1}+(m^2-n^2)y_m=0$.**

**Proof:** Given $y=A\left(x+\sqrt{x^2+a^2}\right)^n+B\left(x+\sqrt{x^2+a^2}\right)^{-n}$

or, $y_1=An\left(x+\sqrt{x^2+a^2}\right)^{n-1}\left\{1+\frac{1}{2\sqrt{x^2+a^2}}.2x\right\}-Bn\left(x+\sqrt{x^2+a^2}\right)^{-n-1}\left\{1+\frac{1}{2\sqrt{x^2+a^2}}.2x\right\}$

or, $y_1=An\left(x+\sqrt{x^2+a^2}\right)^{n-1}\left\{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right\}-Bn\left(x+\sqrt{x^2+a^2}\right)^{-n-1}\left\{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right\}$

or, $y_1=\frac{An\left(x+\sqrt{x^2+a^2}\right)^n}{\sqrt{x^2+a^2}}-\frac{Bn\left(x+\sqrt{x^2+a^2}\right)^{-n}}{\sqrt{x^2+a^2}}$

or, $\left(\sqrt{x^2+a^2}\right)y_1=n\left\{A\left(x+\sqrt{x^2+a^2}\right)^n-B\left(x+\sqrt{x^2+a^2}\right)^{-n}\right\}$

or, $\left(\sqrt{x^2+a^2}\right)y_2+\frac{xy_1}{\sqrt{x^2+a^2}}=n\left\{\frac{An\left(x+\sqrt{x^2+a^2}\right)^n}{\sqrt{x^2+a^2}}+\frac{Bn\left(x+\sqrt{x^2+a^2}\right)^{-n}}{\sqrt{x^2+a^2}}\right\}$

or, $(x^2+a^2)y_2+xy_1=n^2\left\{A\left(x+\sqrt{x^2+a^2}\right)^n+B\left(x+\sqrt{x^2+a^2}\right)^{-n}\right\}$

or, $(x^2+a^2)y_2+xy_1=n^2y$

or, $(x^2+a^2)y_2+xy_1-n^2y=0\cdots\cdots\cdots\cdots\cdots$ (i)

Differentiating Eq. (i), $m$ times by Leibnitz's theorem

or, $(x^2+a^2)y_{m+2}+m_{C_1}2x.y_{m+1}+m_{C_2}2.y_m+xy_{m+1}+m_{C_1}1.y_m-n^2y_m=0$

or, $(x^2+a^2)y_{m+2}+2mx.y_{m+1}+m(m-1)y_m+xy_{m+1}+my_m-n^2y_m=0$

or, $(x^2+a^2)y_{m+2}+(2m+1)xy_{m+1}+(m^2-n^2)y_m=0$. (Proved).

**H.W. Q 11. If $y=e^{\tan^{-1}x}$ then prove that $(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0$.**

**Q 12. If $y=Sin(m\sin^{-1}x)$ then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$.**

**Q 13. If $y=\sin^{-1}x$ then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$.**

**Q 14. If $y=(x^2-1)^n$ then prove that $(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$.**

**Q 15. If $y=e^{a\sin^{-1}x}$ then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$.**

**Q 16. If $Logy=\tan^{-1}x$ then show that $(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0$.**