Chapter I
Indefinite integral

Problem 1: Integrate the following integrals

i) $\int \sin^4 x dx$ ii) $\int \sin^2 x \cos^2 x dx$ iii) $\int \sin^2 x \cos 2x dx$

Solution: i) $\int \sin^4 x dx = \frac{1}{4}\int (2\sin^2 x)^2 dx = \frac{1}{4}\int (1-\cos 2x)^2 dx$

$= \frac{1}{4}\int (1-2\cos 2x+\cos^2 2x)dx = \frac{1}{4}\int (1-2\cos 2x)dx + \frac{1}{8}\int 2\cos^2 2x dx$

$= \frac{1}{4}(x-\sin 2x) + \frac{1}{8}\int (1+\cos 4x)dx = \frac{1}{4}(x-\sin 2x) + \frac{1}{8}(x+\frac{\sin 4x}{4}) + c$

$= \frac{3x}{4} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$

ii) $\int \sin^2 x \cos^2 x dx = \frac{1}{4}\int 2\sin^2 x.2\cos^2 x dx = \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)dx$

$= \frac{1}{4}\int (1-\cos^2 2x)dx = \frac{x}{4} - \frac{1}{8}\int 2\cos^2 2x dx = \frac{x}{4} - \frac{1}{8}\int (1+\cos 4x)dx = \frac{x}{4} - \frac{x}{8} - \frac{\sin 4x}{32} + c$

$= \frac{x}{8} - \frac{\sin 4x}{32} + c$

iii) $\int \sin^2 x \cos 2x dx = \frac{1}{2}\int 2\sin^2 x \cos 2x dx = \frac{1}{2}\int (1-\cos 2x)\cos 2x dx = \frac{1}{2}\int (\cos 2x - \cos^2 2x)dx$

$= \frac{\sin 2x}{4} - \frac{1}{4}\int 2\cos^2 2x dx = \frac{\sin 2x}{4} - \frac{1}{4}\int (1+\cos 4x)dx = \frac{\sin 2x}{4} - \frac{x}{4} - \frac{\sin 4x}{16} + c$

Chapter IIA
Method of substitution

Problem 1: Integrate the following integrals

i) $\int \frac{\sin 2x dx}{a\sin^2 x + b\cos^2 x}$ ii) $\int \frac{\tan x dx}{a+b\tan^2 x}$ iii) $\int \frac{\sin 2x dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2}$ iv) $\int \sqrt{\frac{a+x}{a-x}}dx$ v) $\int \sqrt{\frac{x}{a-x}}dx$

vi) $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$ vii) $\int \frac{dx}{x\sqrt{x^4-1}}$

Solution: i) $\int \frac{\sin 2x dx}{a\sin^2 x + b\cos^2 x} = \int \frac{2\sin x \cos x dx}{a\sin^2 x + b(1-\sin^2 x)} = \int \frac{2\sin x\cos x dx}{(a-b)\sin^2 x + b}$

$= \frac{1}{(a-b)}\int \frac{dz}{z} = \frac{1}{(a-b)}\ln z + c = \frac{1}{(a-b)}\ln[(a-b)\sin^2 x + b] + c$

Put

$(a-b)\sin^2 x + b = z$
$\Rightarrow 2(a-b)\sin x \cos x dx = dz$

ii) $\int \frac{\tan x dx}{a+b\tan^2 x} = \int \frac{(\frac{\sin x}{\cos x})dx}{a+\frac{b\sin^2 x}{\cos^2 x}} = \int \frac{\sin x \cos x dx}{a\cos^2 x + b\sin^2 x}$

$= \frac{1}{2}\int \frac{2\sin x\cos x dx}{(b-a)\sin^2 x + a} = \frac{1}{2(b-a)}\ln[(b-a)\sin^2 x + a] + c$ similar as (i)

iii) $\int \frac{\sin 2x dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

$= \int \frac{2\sin x\cos x dx}{[a^2\sin^2 x + b^2(1-\sin^2 x)]^2} = \int \frac{2\sin x\cos x dx}{[(a^2-b^2)\sin^2 x + b^2]^2}$

$= \frac{1}{(a^2-b^2)}\int \frac{dz}{z^2} = \frac{1}{(a^2-b^2)}(-\frac{1}{z}) + c$

$= -\frac{1}{(a^2-b^2)}\frac{1}{(a^2-b^2)\sin^2 x + b^2} + c$

Put
$(a^2-b^2)\sin^2 x + b^2 = z$
$\Rightarrow 2(a^2-b^2)\sin x\cos x = dz$

iv) $\int \sqrt{\frac{a+x}{a-x}}dx = -\int \sqrt{\frac{a+a\cos 2t}{a-a\cos 2t}}2a\sin 2t dt$

$= -\int \sqrt{\frac{2a\cos^2 t}{2a\sin^2 t}}2a\sin 2t dt = -\int \frac{\cos t}{\sin t}2a.2\sin t\cos t dt$

$= -2a\int 2\cos^2 t dt = -2a\int (1+\cos 2t)dt = -2at - a\sin 2t + c$

$= -a\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + c$

Put
$x = a\cos 2t$
$\Rightarrow dx = -2a\sin 2t dt$

v) $\int \sqrt{\frac{x}{a-x}}dx = \int \sqrt{\frac{a\sin^2 t}{a-a\sin^2 t}}2a\sin t\cos t dt$

$= \int 2a\sin^2 t dt = a\int (1-\cos 2t)dt = at - \frac{a\sin 2t}{2} + c$

$= at - a\sin t\cos t + c = a\sin^{-1}\sqrt{\frac{x}{a}} - a\sqrt{\frac{x}{a}}\sqrt{\frac{a-x}{a}} + c$

$= at - a\sin t\cos t + c = a\sin^{-1}\sqrt{\frac{x}{a}} - \sqrt{x(a-x)} + c$

Put
$x = a\sin^2 t$
$\Rightarrow dx = 2a\sin t\cos dt$

vi) $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx = -\int \cos\left(2\cot^{-1}\sqrt{\frac{1-\cos t}{1+\cos t}}\right)\sin t dt$

$= -\int \cos\left(2\cot^{-1}\frac{\sin t/2}{\cos t/2}\right)\sin t dt = -\int \cos(2\cot^{-1}\tan t/2)\sin t dt$

$= -\int \cos(2\cot^{-1}\cot(\pi/2 - t/2))\sin t dt = -\int \cos(2(\pi/2 - t/2))\sin t dt$

$= -\int \cos(\pi - t)\sin t dt = \int \cos t\sin t dt = -\frac{\cos^2 t}{2} + c = -\frac{x^2}{2} + c$

Put
$x = \cos t$
$\Rightarrow dx = -\sin t dt$

vii) $\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{\sec t\tan t dt}{2\sec t\sqrt{\sec^2 - 1}} = \int \frac{\sec t\tan t dt}{2\sec t\tan t}$

$= \int \frac{dt}{2} = \frac{t}{2} + c = \frac{\sec^{-1} x^2}{2} + c$

Put
$x^2 = \sec t$
$\Rightarrow 2xdx = \sec t\tan t dt$
$\Rightarrow dx = \frac{\sec t\tan t dt}{2x}$

Chapter IIB
Method of substitution with some formulae

Problem 1: Integrate the following integrals

i) $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ ii) $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$ iii) $\int \sqrt{\frac{a+x}{x}}dx$ iv) $\int \frac{\sqrt{x-a}}{x}dx$

Solution: i) $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \int \frac{2zdz}{\sqrt{z^2(\beta-\alpha-z^2)}}$

Put
$x-\alpha = z^2$
$\Rightarrow dx = 2zdz$

$$= \int \frac{2dz}{\sqrt{\beta-\alpha-z^2}} = 2\int \frac{dz}{\sqrt{\left(\sqrt{\beta-\alpha}\right)^2 - z^2}} = 2\sin^{-1}\frac{z}{\sqrt{\beta-\alpha}} + c$$

$$= 2\sin^{-1}\frac{\sqrt{x-\alpha}}{\sqrt{\beta-\alpha}} + c$$

Put
$1+x = 1/z$
$\Rightarrow dx = -1/z^2 dz$

ii) $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} = \int \frac{-1/z^2 dz}{1/z\sqrt{1+2(1/z-1)-(1/z-1)^2}}$

$$= -\int \frac{dz}{z\sqrt{1+2/z-2-1/z^2+2/z-1}} = -\int \frac{dz}{z\sqrt{4/z-2-1/z^2}} = -\int \frac{dz}{z\sqrt{\frac{4z-2z^2-1}{z^2}}}$$

$$= -\int \frac{dz}{\sqrt{4z-2z^2-1}} = -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{2z-z^2-1/2}} = -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{-1+2z-z^2-1/2+1}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{-(1-2z+z^2)+1/2}} = -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{\left(\sqrt{1/2}\right)^2-(z-1)^2}}$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\frac{z-1}{1/\sqrt{2}} + c = -\frac{1}{\sqrt{2}}\sin^{-1}\frac{\frac{1}{1+x}-1}{1/\sqrt{2}} + c = -\frac{1}{\sqrt{2}}\sin^{-1}\frac{-\sqrt{2}x}{1+x} + c = \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}x}{1+x} + c$$

iii) $\int \sqrt{\frac{a+x}{x}}dx = \int \frac{\sqrt{a+x}\sqrt{a+x}}{\sqrt{x}\sqrt{a+x}}dx = \int \frac{a+x}{\sqrt{x^2+ax}}dx = \frac{1}{2}\int \frac{a+2x}{\sqrt{x^2+ax}}dx + \frac{1}{2}\int \frac{a}{\sqrt{x^2+ax}}dx$

$$= \frac{1}{2}.2\sqrt{x^2+ax} + \frac{1}{2}\int \frac{a}{\sqrt{x^2+ax+\frac{a^2}{4}-\frac{a^2}{4}}}dx = \sqrt{x^2+ax} + \frac{1}{2}\int \frac{adx}{\sqrt{\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}}$$

$$= \sqrt{x^2+ax} + \frac{a}{2}\ln\left\{x+\frac{a}{2}+\sqrt{\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}\right\} + c = \sqrt{x^2+ax} + \frac{a}{2}\ln\left\{x+\frac{a}{2}+\sqrt{x^2+ax}\right\} + c$$

$$= \sqrt{x^2+ax} + \frac{a}{2}\ln\left\{\frac{2x+a+2\sqrt{x}\sqrt{x+a}}{2}\right\} + c = \sqrt{x(x+a)} + \frac{a}{2}\ln\left\{\frac{x+x+a+2\sqrt{x}\sqrt{x+a}}{2}\right\} + c$$

$$= \sqrt{x(x+a)} + \frac{a}{2}\ln\left\{\frac{\left(\sqrt{x}+\sqrt{x+a}\right)^2}{2}\right\} + c = \sqrt{x(x+a)} + \frac{a}{2}\ln\left(\sqrt{x}+\sqrt{x+a}\right)^2 - \frac{a}{2}\ln 2 + c$$

$$= \sqrt{x(x+a)} + a\ln\left(\sqrt{x}+\sqrt{x+a}\right) + c_0$$

iv) $\int \frac{\sqrt{x-a}}{x}dx = \int \frac{\sqrt{x-a}\sqrt{x-a}}{x\sqrt{x-a}}dx = \int \frac{x-a}{x\sqrt{x-a}}dx$

$= \int \frac{x}{x\sqrt{x-a}}dx - \int \frac{a}{x\sqrt{x-a}}dx = \int \frac{dx}{\sqrt{x-a}} - \int \frac{a}{x\sqrt{x-a}}dx$

$= 2\sqrt{x-a} - \int \frac{2azdz}{(a+z^2)\sqrt{z^2}} = 2\sqrt{x-a} - \int \frac{2adz}{a+z^2}$

$= 2\sqrt{x-a} - 2a\int \frac{dz}{\left(\sqrt{a}\right)^2 + z^2} = 2\sqrt{x-a} - 2a.\frac{1}{\sqrt{a}}\tan^{-1}\frac{z}{\sqrt{a}} + c$

$= 2\sqrt{x-a} - 2\sqrt{a}\tan^{-1}\frac{\sqrt{x-a}}{\sqrt{a}} + c = 2\sqrt{x-a} - 2\sqrt{a}\tan^{-1}\sqrt{\frac{x-a}{a}} + c$

Put (2nd integral)
$x-a = z^2$
$\Rightarrow dx = 2zdz$

Chapter III

Integration by parts

Problem 1: Integrate the following integrals

i) $\int \sin^{-1}\sqrt{\frac{x}{x+a}}dx$ ii) $\int \frac{e^x}{x}(1+x\log x)dx$ iii) $\int e^x \frac{x^2+1}{(x+1)^2}dx$ iv) $\int e^x \frac{x-1}{(x+1)^3}dx$

v) $\int \sqrt{2ax-x^2}\,dx$ vi) $\int \sqrt{(x-\alpha)(\beta-x)}dx$ vii) $\int (x+2)\sqrt{x^2+2x+10}dx$

Solution: i) $\int \sin^{-1}\sqrt{\frac{x}{x+a}}dx = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}.2atan\theta sec^2\theta d\theta$

Put $x = atan^2\theta$
$dx = 2atan\theta sec^2\theta d\theta$

$= 2a\int \sin^{-1}\sin\theta.tan\theta sec^2\theta d\theta = 2a\int \theta tan\theta sec^2\theta d\theta$

$= 2a\left[\theta\frac{\tan^2\theta}{2} - \frac{1}{2}\int \tan^2\theta\, d\theta\right]$

$= 2a\left[\theta\frac{\tan^2\theta}{2} - \frac{1}{2}\int (\sec^2\theta - 1)d\theta\right]$

$= a[\theta\tan^2\theta - (\tan\theta - \theta)] = a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right]$

ii) $\int \frac{e^x}{x}(1+x\log x)dx = \int e^x\left(\frac{1}{x} + \log x\right)dx = e^x\log x$

iii) $\int e^x\frac{(x^2+1)}{(x+1)^2}dx = \int e^x\frac{(x^2-1+2)}{(x+1)^2}dx = \int e^x\left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right]dx$

$= e^x\frac{x-1}{x+1}$

iv) $\int e^x\frac{(x-1)}{(x+1)^3}dx = \int e^x\frac{(x+1-2)}{(x+1)^3}dx = \int e^x\left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right]dx$

$= e^x\frac{1}{(x+1)^2}$

v) $\int \sqrt{(2ax-x^2)}dx = \int \sqrt{\{a^2-(x-a)^2\}}\,dx$

$= \frac{1}{2}(x-a)\sqrt{\{a^2-(x-a)^2\}} + \frac{a^2}{2}\sin^{-1}\frac{(x-a)}{a}$

$$= \frac{1}{2}(x-a)\sqrt{(2ax-x^2)} + \frac{a^2}{2}\sin^{-1}\frac{(x-a)}{a}$$

vi) $\int \sqrt{(x-a)(b-x)}dx = \int \sqrt{bx - x^2 - ab + ax}dx = \int \sqrt{(a+b)x - x^2 - ab}dx$

$$= \int \sqrt{-\left(\frac{a+b}{2}\right)^2 + (a+b)x - x^2 - ab + \left(\frac{a+b}{2}\right)^2}\, dx$$

$$= \int \sqrt{\left(\frac{a+b}{2}\right)^2 - ab - \left\{x^2 - (a+b)x + \left(\frac{a+b}{2}\right)^2\right\}}\, dx$$

$$= \int \sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2}\, dx$$

$$= \frac{\left(x - \frac{a+b}{2}\right)\sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2}}{2} + \frac{\left(\frac{a-b}{2}\right)^2}{2}\sin^{-1}\frac{\left(x - \frac{a+b}{2}\right)}{\left(\frac{a-b}{2}\right)}$$

$$= \frac{(2x-a-b)\sqrt{(x-a)(b-x)}}{4} + \frac{(a-b)^2}{8}\sin^{-1}\frac{(2x-a-b)}{(a-b)}$$

vii) $\int (x+2)\sqrt{x^2+2x+10}dx = \frac{1}{2}\int (2x+2)\sqrt{x^2+2x+10}dx + \int \sqrt{x^2+2x+10}\, dx$

$$= \frac{1}{2}\int \sqrt{z}dz + \int \sqrt{(x+1)^2+9}dx = \frac{1}{2}\int z^{\frac{1}{2}}dz + \int \sqrt{3^2 + (x+1)^2}dx$$

$$= \frac{1}{2}z^{\frac{3}{2}} \times \frac{2}{3} + \frac{(x+1)\sqrt{x^2+2x+10}}{2} + \frac{9}{2}\log\left\{(x+1) + \sqrt{x^2+2x+10}\right\}$$

$$= \frac{1}{3}(x^2+2x+10)^{\frac{3}{2}} + \frac{(x+1)\sqrt{x^2+2x+10}}{2} + \frac{9}{2}\log\left\{(x+1) + \sqrt{x^2+2x+10}\right\}$$

Chapter IV

Integration of trigonometric function

Problem 1: $i) \int \frac{dx}{1+\tan x}$ $ii) \int \frac{\cos x}{2\sin x + 3\cos x}dx$ $iii) \int \frac{dx}{a+b\tan x}$ $iv) \int \frac{dx}{5+4\sin x}$ $v) \int \frac{dx}{5+4\cos x}$

Solution:

$$i) \int \frac{dx}{1+\tan x} = \int \frac{\cos x\, dx}{\sin x + \cos x} = \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{(\cos x - \sin x)}{\sin x + \cos x}dx = \frac{x}{2} + \frac{1}{2}\log(\sin x + \cos x)$$

$$ii) \int \frac{\cos x}{2\sin x + 3\cos x}dx = \frac{3}{13}\int \frac{(3\cos x + 2\sin x) + \frac{2}{3}(2\cos x - 3\sin x)}{2\sin x + 3\cos x}dx$$

$$= \frac{3}{13}\int dx + \frac{2}{13}\int \frac{(2\cos x - 3\sin x)}{2\sin x + 3\cos x}dx = \frac{3x}{13} + \frac{2}{13}\log(2\sin x + 3\cos x)$$

$$iii)\int \frac{dx}{a+b\tan x}=\int \frac{\cos x dx}{a\cos x+b\sin x}$$

$$=\frac{a}{a^2+b^2}\int \frac{(a\cos x+b\sin x)+\frac{b}{a}(b\cos x-a\sin x)}{a\cos x+b\sin x}dx$$

$$=\frac{a}{a^2+b^2}\int dx+\frac{b}{a^2+b^2}\int \frac{(b\cos x-a\sin x)}{a\cos x+b\sin x}dx$$

$$=\frac{ax}{a^2+b^2}+\frac{b}{a^2+b^2}\log(a\cos x+b\sin x)$$

$$iv)\int \frac{dx}{5+4\sin x}=\int \frac{dx}{5\cos^2\frac{x}{2}+5\sin^2\frac{x}{2}+8\sin\frac{x}{2}\cos\frac{x}{2}}=\int \frac{\sec^2\frac{x}{2}dx}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}}$$

Put $\tan\frac{x}{2}=z$

$\sec^2\frac{x}{2}dx=2dz$

$$=2\int \frac{dz}{5+5z^2+8z}=\frac{2}{5}\int \frac{dz}{z^2+\frac{8}{5}z+1}=\frac{2}{5}\int \frac{dz}{(z+\frac{4}{5})^2+1-\frac{16}{25}}$$

$$=\frac{2}{5}\int \frac{dz}{(z+\frac{4}{5})^2+\frac{9}{25}}=\frac{2}{5}\int \frac{dz}{\left(z+\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}$$

$$=\frac{2}{5}\times\frac{5}{3}\tan^{-1}\frac{\left(z+\frac{4}{5}\right)}{\frac{3}{5}}=\frac{2}{3}\tan^{-1}\frac{5z+4}{3}=\frac{2}{3}\tan^{-1}\frac{5\tan\frac{x}{2}+4}{3}$$

$$v)\int \frac{dx}{5+4\cos x}=\int \frac{dx}{5\cos^2\frac{x}{2}+5\sin^2\frac{x}{2}+4\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\int \frac{\sec^2\frac{x}{2}dx}{5+5\tan^2\frac{x}{2}+4-4\tan^2\frac{x}{2}}=\int \frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2}+9}$$

Put $\tan\frac{x}{2}=z$

$\sec^2\frac{x}{2}dx=2dz$

$$=2\int \frac{dz}{z^2+3^2}=\frac{2}{3}\tan^{-1}\frac{z}{3}$$

$$=\frac{2}{3}\tan^{-1}\frac{\tan\frac{x}{2}}{3}$$

Chapter V

Rational function

Problem 1: $i)\int \frac{x-1}{(x-2)(x-3)}dx$ $ii)\int \frac{dx}{(x-1)^2(x+1)}$ $iii)\int \frac{x}{x^4-1}dx$ $iv)\int \frac{x}{x^4-1}dx$ $v)\int \frac{dx}{x^4-1}$

Solution:

$$i)\ \int \frac{x-1}{(x-2)(x-3)}dx = \int \left\{\frac{2}{(x-3)} - \frac{1}{(x-2)}\right\}dx = 2\log(x-3) - \log(x-2)$$

$$ii)\ \int \frac{dx}{(x-1)^2(x+1)}$$

Now,

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Put $x = 1$

$$B = \frac{1}{2}$$

Put $x = -1$

$$C = \frac{1}{4}$$

Equating the coefficient of x^2

$$A + C = 0 \therefore\ A = -\frac{1}{4}$$

$$\int \frac{dx}{(x-1)^2(x+1)} = \int \left\{-\frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}\right\}dx$$

$$= -\frac{1}{4}\log(x-1) - \frac{1}{2(x-1)} + \frac{1}{4}\log(x+1)$$

$$iii)\ \int \frac{x}{x^4-1}dx = \int \frac{x}{(x-1)(x+1)(x^2+1)}dx = \int \left\{\frac{1}{4(x-1)} + \frac{1}{4(x+1)} - \frac{x}{2(x^2+1)}\right\}$$

$$= \frac{1}{4}\log(x-1) + \frac{1}{4}\log(x+1) - \frac{1}{4}\log(x^2+1)$$

$$iv)\ \int \frac{x}{x^4-1}dx = \frac{1}{4}\int \frac{4x^3}{x^4(x^4-1)}dx$$

Put $x^4 = z,\ 4x^3dx = dz$

$$= \frac{1}{4}\int \frac{dz}{z(z-1)} = \frac{1}{4}\int \left\{\frac{1}{z-1} - \frac{1}{z}\right\}dz = \frac{1}{4}[\log(z-1) - \log z] = \frac{1}{4}[\log(x^4-1) - \log x^4]$$

$$v)\ \int \frac{dx}{x^4-1} = \int \frac{dx}{(x^2+1)(x^2-1)} = \frac{1}{2}\int \left\{\frac{1}{(x^2-1)} - \frac{1}{(x^2+1)}\right\}dx = \frac{1}{2}\left[\frac{1}{2}\log\frac{1+x}{1-x} - \tan^{-1}x\right]$$

CHAPTER VI

Definite integral

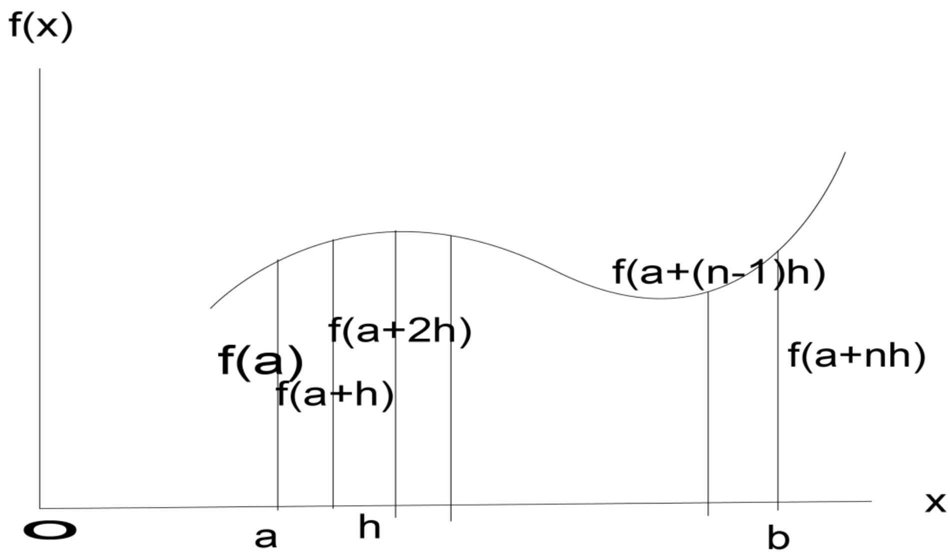
Problem 1: $i)\ \int_0^1 x(\tan^{-1}x)^2\,dx\ \ ii)\ \int_0^a \sin^{-1}\frac{2x}{1+x^2}\,dx\ \ iii)\ \int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

Solution:

$$i)\ \int_0^1 x(\tan^{-1}x)^2\,dx = \left[\frac{x^2}{2}(\tan^{-1}x)^2\right]_0^1 - \frac{1}{2}\int_0^1 x^2 . 2\frac{\tan^{-1}x}{1+x^2}dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \frac{x^2 + 1 - 1}{1 + x^2} \tan^{-1} x \, dx$$
$$= \frac{\pi^2}{32} - \int_0^1 \tan^{-1} x \, dx + \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$$
$$= \frac{\pi^2}{32} - [x \tan^{-1} x]_0^1 + \int_0^1 x \frac{1}{1 + x^2} dx + \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$$
$$= \frac{\pi^2}{32} - [x \tan^{-1} x]_0^1 + \left[\frac{1}{2} \log(1 + x^2\right]_0^1 + \frac{1}{2} [(\tan^{-1} x)^2]_0^1$$
$$= \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2$$
$$ii) \int_0^a \sin^{-1} \frac{2x}{1 + x^2} dx = \int_0^a 2 \tan^{-1} x \, dx = [2x \tan^{-1} x]_0^a - \int_0^a \frac{2x}{1 + x^2} dx$$
$$= 2a \tan^{-1} a - [\log(1 + x^2)]_0^a = 2a \sin^{-1} a - \log(1 + a^2)$$
$$iii) \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \int_0^{\pi/2} \frac{\sec^4 x \, dx}{(a^2 + b^2 \tan^2 x)^2}$$
Put $b \tan x = a \tan \theta, b \sec^2 x dx = \ a \sec^2 \theta d\theta$
$$= \int_0^{\pi/2} \frac{\left(1 + \frac{a^2}{b^2} \tan^2 \theta\right) \frac{a}{b} \sec^2 \theta d\theta}{(a^2 + a^2 \tan^2 \theta)^2} = \frac{1}{a^3 b^3} \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2 \theta) \frac{a}{b} \sec^2 \theta d\theta}{(1 + \tan^2 \theta)^2}$$
$$= \frac{1}{a^3 b^3} \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2 \theta) \sec^2 \theta d\theta}{\sec^4 \theta} = \frac{1}{a^3 b^3} \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2 \theta) d\theta}{\sec^2 \theta}$$
$$= \frac{1}{a^3 b^3} \int_0^{\pi/2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta) d\theta = \frac{1}{2a^3 b^3} \int_0^{\pi/2} (b^2 2 \cos^2 \theta + a^2 \, 2\sin^2 \theta) d\theta$$
$$= \frac{1}{2a^3 b^3} \int_0^{\pi/2} (b^2(1 + \cos 2\theta) + a^2(1 - \cos 2\theta)) d\theta$$
$$= \frac{1}{2a^3 b^3} \left[b^2 \left(\theta + \frac{\sin 2\theta}{2}\right) + a^2 \left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} = \frac{(a^2 + b^2)\pi}{4a^3 b^3}$$

First principle of integral calculus:



b-a=nh

Area of region by the curve f(x) with the x-axis bounded by x=a and x=b is

$$\lim_{h\to 0}[hf(a) + hf(a+h) + hf(a+2h) + \cdots + hf(a+(n-1)h)]$$

$$\lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh)$$

Symbolically this area is represented as $\int_a^b f(x)dx$ implies that

$$\int_a^b f(x)dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=0}^{n-1} f(a + r\frac{b-a}{n})$$

If $a = 0$ and $b = 1$ then

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f(\frac{r}{n})$$

Problem 2: Evaluate from first principle $\int_a^b e^x dx$

Solution:

$$\int_a^b e^x dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} e^{a+rh}$$

$$= \lim_{h\to 0} h\left[e^a + e^{a+h} + e^{a+2h} + \cdots + e^{a+(n-1)h}\right]$$

$$= \lim_{h\to 0} he^a\left[1 + e^h + e^{2h} + \cdots + e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} he^a \frac{(e^h)^n - 1}{e^h - 1} = \lim_{h\to 0} he^a \frac{e^{nh} - 1}{e^h - 1} = \lim_{h\to 0} he^a \frac{e^{b-a} - 1}{e^h - 1}$$

$$= e^a(e^{b-a} - 1)\lim_{h\to 0} \frac{h}{e^h - 1} = e^a(e^{b-a} - 1) = e^b - e^a$$

Problem 3: Evaluate from first principle $\int_0^1 x^2dx$

Solution:

$$\int_0^1 x^2dx = \lim_{h\to 0} h\sum_{r=1}^{n}(rh)^2$$

$$= \lim_{h\to 0} h[1^2h^2 + 2^2h^2 + 3^2h^2 + \cdots + n^2h^2]$$

$$= \lim_{h\to 0} h^3[1^2 + 2^2 + 3^2 + \cdots + n^2]$$

$$= \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\lim_{h\to 0}(2n^3h^3 + 3n^2h^2.h + nh.h^2$$;here nh=b-a=1-0=1

$$= \frac{1}{6}\lim_{h\to 0}(2 + 3h + h^2 = \frac{2}{6} = \frac{1}{3}$$

Problem 4: Evaluate

i) $\lim\limits_{n\to\infty}\left[\frac{1}{n+m} + \frac{1}{n+2m} + \frac{1}{n+3m} + \cdots + \frac{1}{n+nm}\right]$

ii) $\lim\limits_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{n}{n^2+n^2}\right]$

iii) $\lim\limits_{n\to\infty}\left[\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \frac{3^2}{n^3+3^3} + \cdots + \frac{n^2}{n^3+n^3}\right]$

iv) $\lim\limits_{n\to\infty}\left[\left(1+\frac{1^2}{n^2}\right)^{\frac{2}{n^2}} + \left(1+\frac{2^2}{n^2}\right)^{\frac{4}{n^2}} + \left(1+\frac{3^2}{n^2}\right)^{\frac{6}{n^2}} + \cdots + \left(1+\frac{n^2}{n^2}\right)^{\frac{2n}{n^2}}\right]$

Solution:

i) $\lim\limits_{n\to\infty}\left[\frac{1}{n+m} + \frac{1}{n+2m} + \frac{1}{n+3m} + \cdots + \frac{1}{n+nm}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{m}{n}} + \frac{1}{1+\frac{2m}{n}} + \frac{1}{1+\frac{3m}{n}} + \cdots + \frac{1}{1+\frac{nm}{n}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{rm}{n}} = \int_0^1\frac{1}{1+mx}dx = \left[\frac{1}{m}\log(1+mx)\right]_0^1$$

$$= \frac{1}{m}\log(1+m)$$

ii) $\lim\limits_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\frac{n}{n^2}\left[\frac{1}{1+\frac{1^2}{n^2}} + \frac{1}{1+\frac{2^2}{n^2}} + \frac{1}{1+\frac{3^2}{n^2}} + \cdots + \frac{1}{1+\frac{n^2}{n^2}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{1+\frac{r^2}{n^2}} = \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

iii) $\lim\limits_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\cdots+\frac{n^2}{n^3+n^3}\right]$

$$= \lim_{n\to\infty}\frac{n^2}{n^3}\left[\frac{\frac{1^2}{n^2}}{1+\frac{1^3}{n^3}}+\frac{\frac{2^2}{n^2}}{1+\frac{2^3}{n^3}}+\frac{\frac{3^2}{n^2}}{1+\frac{3^3}{n^3}}+\cdots+\frac{\frac{n^2}{n^2}}{1+\frac{n^3}{n^3}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\frac{r^2}{n^2}}{1+\frac{r^3}{n^3}} = \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}[\log(1+x^3)]_0^1$$

$$= \frac{1}{3}\log 2$$

iv) $\lim\limits_{n\to\infty}\left[\left(1+\frac{1^2}{n^2}\right)^{\frac{2}{n^2}}+\left(1+\frac{2^2}{n^2}\right)^{\frac{4}{n^2}}+\left(1+\frac{3^2}{n^2}\right)^{\frac{6}{n^2}}+\cdots+\left(1+\frac{n^2}{n^2}\right)^{\frac{2n}{n^2}}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(1+\frac{r^2}{n^2}\right)^{\frac{2r}{n^2}}$$

Let

$$A = \lim_{n\to\infty}\sum_{r=1}^{n}\left(1+\frac{r^2}{n^2}\right)^{\frac{2r}{n^2}}$$

$$\log A = \log \lim_{n\to\infty}\sum_{r=1}^{n}\left(1+\frac{r^2}{n^2}\right)^{\frac{2r}{n^2}} = \sum_{r=1}^{n}\lim_{n\to\infty}\frac{2r}{n^2}\log\left(1+\frac{r^2}{n^2}\right) = \frac{1}{n}\sum_{r=1}^{n}\lim_{n\to\infty}\frac{2r}{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \int_0^1 2x\log(1+x^2)\,dx$$

Put $1+x^2 = z,\ 2xdx = dz,\ x\to 0, z\to 1; x\to 1, z\to 2$

$$= \int_1^2 \log z dz = [z\log z - z]_1^2 = 2\log 2 - 2 + 1$$

$$= \log 2^2 - 1 = \log 4 - \log e = \log\frac{4}{e}$$

$$\therefore A = \frac{4}{e}$$

Chapter VII

General properties and reduction fomula

Some properties of definite integral:

i) $\int_a^b f(x)dx = \int_a^b f(z)dz$

ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\,, if\ a < c < b$

iv) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

v) $\int_0^{na} f(x)dx = n\int_0^a f(x)dx,\ if\ f(a+x) = f(x)$

vi) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx,\ if\ f(-x) = f(x)$

$\qquad = 0, if\ f(-x) = -f(x)$

Problem 1: Integrate $i)\ \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ $ii)$ show that $\int_0^{\frac{\pi}{2}} \log\sin x dx = \int_0^{\frac{\pi}{2}} \log\cos x dx = \frac{\pi}{2}\log\frac{1}{2}$ $iii)$ show that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2$

Solution:

$$\mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\therefore\ 2\mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$2\mathrm{I} = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\mathrm{I} = \frac{\pi}{4}$$

$$ii)\ I = \int_0^{\frac{\pi}{2}} \log\sin x dx = \int_0^{\frac{\pi}{2}} \log\cos x dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\sin x dx + \int_0^{\frac{\pi}{2}} \log\cos x dx = \int_0^{\frac{\pi}{2}} \log\left(\sin x\cos x\right)dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\sin 2x\,dx - \int_0^{\frac{\pi}{2}} \log 2\,dx$$

Put $2x = z, dx = \frac{dz}{2}, x \to 0, z \to 0; x \to \frac{\pi}{2}, z \to \pi$

$$= \frac{1}{2}\int_0^{\pi} \log\sin z\,dz - [\mathrm{x}\log 2]_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{2}} \log\sin z\,dz - \frac{\pi}{2}\log 2$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2 = I - \frac{\pi}{2} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}$$

$$iii)\ I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

Put $x = \tan\theta,\ dx = \sec^2\theta d\theta,\ x \to 0, \theta \to 0; x \to 1, \theta \to \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta = \int_0^{\frac{\pi}{4}} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right] d\theta = \int_0^{\frac{\pi}{4}} \log\left[1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[\frac{1 + \tan\theta + 1 - \tan\theta}{1 + \tan\theta}\right] d\theta = \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1 + \tan\theta}\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2 d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta = \int_0^{\frac{\pi}{4}} \log 2 d\theta - I$$

$$2I = [\theta \log 2]_0^{\frac{\pi}{4}} = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

Theorem: state and prove walli’s formula for definite integral or show that

If n be positive integer

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Or

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

According as n is even or odd.

Proof:

$$I_n = \int \sin^n x dx = \int \sin^{n-1} x \sin x \, dx = \sin^{n-1} x\, (-\cos x) + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

$$= \sin^{n-1} x\, (-\cos x) + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) I_n$$

$$n I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx$$

$$I_n = \left[\frac{-\sin^{n-1} x \cos x}{n}\right]_0^{\frac{\pi}{2}} + \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

Similarly if we proceed we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot I_0$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$$

according as n is even or odd.

$$I_0 = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Or

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

according as n is even or odd.

Theorem: If both m and n are even

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\cdots 3 \cdot 1 (n-1)(n-3)\cdots 3 \cdot 1}{(m+n)(m+n-2)\cdots 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

If m is even and n is odd

$$= \frac{(m-1)(m-3)\cdots 3 \cdot 1 (n-1)(n-3)\cdots 4 \cdot 2}{(m+n)(m+n-2)\cdots 5 \cdot 3 \cdot 1} \cdot 1$$

If both m and n are odd

$$= \frac{(m-1)(m-3)\cdots 4 \cdot 2 (n-1)(n-3)\cdots 4 \cdot 2}{(m+n)(m+n-2)\cdots 6 \cdot 4 \cdot 2} \cdot \frac{1}{2}$$

Problem 2: Evaluate $\int_0^1 x^6 \sqrt{1-x^2} dx$

Solution: Put $x = \sin\theta, dx = \cos\theta d\theta, x \to 0, \theta \to 0, x \to 1, \theta \to \pi/2$

$$\int_0^1 x^6 \sqrt{1-x^2} dx = \int_0^{\frac{\pi}{2}} sin^6\theta \sqrt{1 - sin^2\theta} \cos\theta \, d\theta = \int_0^{\frac{\pi}{2}} sin^6\theta \cos^2\theta \, d\theta$$

$$= \frac{5 \cdot 3 \cdot 1 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5\pi}{256}$$

Problem 3: Evaluate $\int_0^1 x^2 (1-x)^{3/2} dx$

Solution: Put $x = \sin^2\theta, dx = 2\sin\theta\cos\theta d\theta, x \to 0, \theta \to 0, x \to 1, \theta \to \pi/2$

$$\int_0^1 x^2(1-x)^{3/2}dx = \int_0^{\frac{\pi}{2}} sin^4\theta(1-sin^2\theta)^{\frac{3}{2}} 2\sin\theta\cos\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} sin^5\theta\cos^4\theta\, d\theta$$

$$= 2\frac{4\cdot 2\cdot 3\cdot 1}{9\cdot 7\cdot 5\cdot 3\cdot 1} = \frac{16}{315}$$

Problem 4: Evaluate $i)\ \int_0^{\frac{\pi}{2}} \sin^9 x dx\ ii)\ \int_0^{\frac{\pi}{2}} \sin^{10} x dx\ iii)\ \int_0^{\frac{\pi}{2}} \sin^5 x\cos^7 x\, dx$

Solution:

$$i)\ \int_0^{\frac{\pi}{2}} \sin^9 x dx = \frac{8\cdot 6\cdot 4\cdot 2}{9\cdot 7\cdot 5\cdot 3\cdot 1} = \frac{128}{315}$$

$$ii)\ \int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{63\pi}{512}$$

$$iii)\ \int_0^{\frac{\pi}{2}} \sin^5 x\cos^7 x\, dx = \frac{4\cdot 2\cdot 6\cdot 4\cdot 2}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}\cdot\frac{1}{2} = \frac{1}{240}$$

Problem 5: show that $\int_0^\pi \frac{xdx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{(a^2+b^2)\pi^2}{4a^3b^3}$

Solution:

$$I = \int_0^\pi \frac{xdx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \int_0^\pi \frac{(\pi - x)dx}{(a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x))^2}$$

$$= \int_0^\pi \frac{(\pi - x)dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \int_0^\pi \frac{\pi dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} - \int_0^\pi \frac{xdx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$$

$$\int_0^\pi \frac{\pi dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} - I$$

$$\therefore I = \frac{1}{2}\int_0^\pi \frac{\pi dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \int_0^{\frac{\pi}{2}} \frac{\pi dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{(a^2+b^2)\pi^2}{4a^3b^3}$$

(Solution See problem 1 (iii) chapter Vi)

Problem 6: Obtain reduction formulae for $i)\ \int \tan^n x dx\ ii)\ \int_0^{\pi/4} \tan^n x dx$ hence deduce $iii)\ \int \tan^5 x dx\ iv)\ \int \tan^6 x dx$

Solution:

$$i)\ I_n = \int \tan^n x dx = \int \tan^{n-2} x\tan^2 x dx = \int \tan^{n-2} x(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x\sec^2 x dx - \int \tan^{n-2} x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\therefore I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$ii)\ J_n = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x dx$$

$$\therefore J_n = \frac{1}{n-1} - J_{n-2}$$

$$iii)\ I_5 = \int \tan^5 x dx = \frac{\tan^4 x}{4} - I_3$$

$$I_3 = \int \tan^3 x dx = \frac{\tan^2 x}{2} - I_1$$

$$I_1 = \int \tan \mathrm{x}\, dx = \log \sec x$$

$$\therefore\ I_5 = \int \tan^5 x dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x$$

$$iv)\ I_6 = \int \tan^6 x dx = \frac{\tan^5 x}{5} - I_4$$

$$I_4 = \int \tan^4 x dx = \frac{\tan^3 x}{3} - I_2$$

$$I_2 = \int \tan^2 \mathrm{x}\, dx = \int (\sec^2 \mathrm{x} - 1) dx = \tan x - x$$

$$\therefore\ I_6 = \int \tan^6 x dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x$$

Problem 7: Obtain reduction formulae for $i) \int \sec^n x dx$ hence deduce $ii)\ \int \sec^6 x dx$
$iii)\ \int \sec^7 x dx$

Solution:

$$i)\ I_n = \int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x\, dx$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x\, (\sec^2 x - 1) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$(1 + n - 2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$ii)\ I_6 = \int \sec^6 x dx = \frac{\sec^4 x tan\ x}{5} + \frac{4}{5} I_4$$

$$I_4 = \int \sec^4 x dx = \frac{\sec^2 x tan\ x}{3} + \frac{2}{3} I_2$$

$$I_2 = \int \sec^2 x dx = tan\ x$$

$$I_6 = \int \sec^6 x dx = \frac{\sec^4 x tan\ x}{5} + \frac{4}{5} \frac{\sec^2 x tan\ x}{3} + \frac{4}{5}\frac{2}{3} tan\ x$$

$$iii)\ I_7 = \int \sec^7 x dx = \frac{\sec^5 x tan\ x}{6} + \frac{5}{6} I_5$$

$$I_5 = \int \sec^5 x dx = \frac{\sec^3 x tan\ x}{4} + \frac{3}{4} I_3$$

$$I_3 = \frac{\sec x \tan x}{2} + \frac{1}{2} I_1$$

$$I_1 = \int \sec x \, dx = \log(\sec x + \tan x)$$

$$I_7 = \int \sec^7 x dx = \frac{\sec^5 x \tan x}{6} + \frac{5}{6}\frac{\sec^3 x \tan x}{4} + \frac{5}{6}\frac{3}{4}\frac{\sec x \tan x}{2} + \frac{5}{6}\frac{3}{4}\frac{1}{2}\log(\sec x + \tan x)$$

Problem 8: Obtain reduction formulae for $\int e^{ax}\cos^n x dx$ hence deduce $\int_0^\infty e^{-4x}\cos^5 x \, dx$

Solution:

$$I_n = \int e^{ax}\cos^n x dx = \frac{e^{ax}\cos^n x}{a} + \frac{n}{a}\int e^{ax}\cos^{n-1} x \sin x \, dx$$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{n}{a}\Big[\frac{e^{ax}}{a}\cos^{n-1} x \sin x$$
$$- \frac{1}{a}\int e^{ax}\{(n-1)\cos^{n-2} x(-\sin x)\sin x + \cos^{n-1} x \cos x\}dx\Big]$$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{ne^{ax}}{a^2}\cos^{n-1} x \sin x + \frac{n}{a^2}\int e^{ax}\{(n-1)\cos^{n-2} x(1-\cos^2 x) + \cos^n x\} \, dx$$

$$= \frac{e^{ax}\cos^{n-1} x\,(a\cos x + n\sin x)}{a^2} + \frac{n(n-1)}{a^2}\int e^{ax}\cos^{n-2} x dx - \frac{n}{a^2}\int e^{ax}\{(n-1)\cos^n x$$
$$+ \cos^n x\}dx$$

$$= \frac{e^{ax}\cos^{n-1} x\,(a\cos x + n\sin x)}{a^2} + \frac{n(n-1)}{a^2}\int e^{ax}\cos^{n-2} x dx - \frac{n^2}{a^2}\int e^{ax}\cos^n x dx$$

$$\left(1 + \frac{n^2}{a^2}\right) I_n = \frac{e^{ax}\cos^{n-1} x\,(a\cos x + n\sin x)}{a^2} + \frac{n(n-1)}{a^2} I_{n-2}$$

$$I_n = \frac{e^{ax}\cos^{n-1} x\,(a\cos x + n\sin x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

Now the reduction formula for $\int_0^\infty e^{-ax}\cos^n x dx$ is

$$I_n = \left[\frac{e^{-ax}\cos^{n-1} x\,(-a\cos x + n\sin x)}{a^2 + n^2}\right]_0^\infty + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

$$I_n = \frac{a}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

$$\therefore\ I_5 = \frac{4}{41} + \frac{20}{41} I_3$$

$$I_3 = \frac{4}{25} + \frac{6}{25} I_1$$

$$I_1 = \int_0^\infty e^{-4x}\cos x \, dx = \frac{4}{4^2 + 1^2} = \frac{4}{17}$$

$$\therefore\ I_5 = \frac{708}{3485}$$

Chapter VIII

Gama beta function

Gama function: The second Eulerian integral is called gama function and is defined as

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$, and $n > 0$ but need not be integer.

Beta function: The first Eulerian integral is called beta function and is defined as

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$, and $m, n > 0$ but need not be integer.

Problem 1: Show that $i)$ $\beta(m,n) = \beta(n,m)$ $ii)$ $\Gamma(1) = 1$ $iii)$ $\Gamma(n+1) = n\Gamma(n)$

$iv)$ $\int_0^\infty e^{-kx}\, x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

Solution:

$$i)\ \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \int_0^1 (1-x)^{m-1}(1-1+x)^{n-1}\, dx$$

$$= \int_0^1 x^{n-1}\,(1-x)^{m-1} dx = \beta(n,m)$$

$$ii)\ \Gamma(1) = \int_0^\infty e^{-x}\, dx = 1$$

$$iii)\ \Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1}\, dx = \int_0^\infty e^{-x} x^n\, dx$$

$$= [-x^n e^{-x}]_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\, dx = 0 + n\int_0^\infty e^{-x} x^{n-1}\, dx = n\Gamma(n)$$

When n is positive integer $\Gamma(n+1) = n!$

$$iv)\ \Gamma(n) = \int_0^\infty e^{-y}\, y^{n-1} dy$$

Put $y = kx, dy = kdx, y \to 0, x \to 0; y \to \infty, x \to \infty$

$$\Gamma(n) = \int_0^\infty e^{-kx}\,(kx)^{n-1} k dx = \int_0^\infty e^{-kx}\, x^{n-1} k^n dx$$

$$\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx}\, x^{n-1} dx$$

Problem 2: show that $i)$ $\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2n-1} dx$ $ii)$ $\Gamma(n+1) = \int_0^\infty e^{-x^{\frac{1}{n}}}\, dx$

$iii)$ $\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$ hence integrate $iv)$ $\int_0^\infty e^{-x^2}\, dx$ $v)$ $\int_0^\infty e^{-x^{\frac{1}{4}}}\, dx$ $vi)$ $\int_0^1 \left(\log\frac{1}{x}\right)^5 dx$

Solution:

$i)$ we know

$\Gamma(n) = \int_0^\infty e^{-y}\, y^{n-1} dy$

Put $y = x^2, dy = 2xdx; y \to 0, x \to 0, y \to \infty, x \to \infty$

$\Gamma(n) = \int_0^\infty e^{-x^2}\, x^{2n-2} 2x dx = 2\int_0^\infty e^{-x^2}\, x^{2n-1} dx$

$ii)$ we know

$\Gamma(n) = \int_0^\infty e^{-y}\, y^{n-1} dy$

Put $y^n = x, ny^{n-1} dy = dx, y^{n-1} dy = \frac{dx}{n}; y \to 0, x \to 0, y \to \infty, x \to \infty$

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{\frac{1}{n}}}\, dx \Rightarrow n\Gamma(n) = \int_0^\infty e^{-x^{\frac{1}{n}}}\, dx$$

$$\Rightarrow \Gamma(n+1) = \int_0^\infty e^{-x^{\frac{1}{n}}}\, dx$$

$iii)$ we know

$\Gamma(n) = \int_0^\infty e^{-y}\, y^{n-1} dy$

Put $e^{-y} = x, e^y = \frac{1}{x}, y = \log\frac{1}{x}, -e^{-y}dy = dx; y \to 0, x \to 1, y \to \infty, x \to 0$

$\Gamma(n) = -\int_1^0 \left(\log\frac{1}{x}\right)^{n-1} dx = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$

$iv)$ we know

$\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2n-1} dx$

Put $n = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dx \Rightarrow \int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$v)$ we know

$$\Gamma(n+1) = \int_0^\infty e^{-x^{\frac{1}{n}}}\, dx$$

Put $n = 4$

$$\Gamma(5) = \int_0^\infty e^{-x^{\frac{1}{4}}}\, dx \Rightarrow \int_0^\infty e^{-x^{\frac{1}{4}}}\, dx = \Gamma(5) = 24$$

$vi)$ we know

$\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$

Put $n = 6$

$\Gamma(6) = \int_0^1 \left(\log\frac{1}{x}\right)^5 dx \Rightarrow \int_0^1 \left(\log\frac{1}{x}\right)^5 dx = \Gamma(6) = 120$

Problem 3: Show that $\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$ hence integrate $\int_0^\infty \frac{y^7}{(1+y)^{14}} dy$

Solution: we know

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Put $x = \frac{1}{1+y}, 1+y = \frac{1}{x}, y = \frac{1}{x} - 1, dx = -\frac{1}{(1+y)^2} dy, x \to 0, y \to \infty, x \to 1, y \to 0$

$$\beta(m,n) = -\int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1 - \frac{1}{1+y}\right)^{n-1} \frac{1}{(1+y)^2} dy$$

$$= \int_0^\infty \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1} \frac{1}{(1+y)^2} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Similarly

$$\beta(n,m) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Since $\beta(m,n) = \beta(n,m)$

$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$ (proved)

We know

$$\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Put $m=7, n=8$

$$\int_0^\infty \frac{y^7}{(1+y)^{15}}\,dy = \beta(7{,}8) = \frac{\Gamma(7)\Gamma(8)}{\Gamma(7+8)} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14}$$

$$= \frac{1}{2\cdot 3\cdot 2\cdot 11\cdot 13\cdot 14}$$

Problem 4: Show that

$i)\ \beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$ii)\ \displaystyle\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

Solution:

$i)$ we know

$$\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}\,dx$$

And

$$\frac{\Gamma(n)}{x^n} = \int_0^\infty e^{-xy}y^{n-1}\,dy$$

$$\Gamma(n) = \int_0^\infty e^{-xy}x^n y^{n-1}\,dy$$

$$\therefore \Gamma(m)\Gamma(n) = \int_0^\infty \left\{\int_0^\infty e^{-x(1+y)}x^{m+n-1}\,dx\right\} y^{n-1}\,dy = \int_0^\infty \frac{\Gamma(m+n)}{(1+y)^{m+n}}y^{n-1}\,dy$$

$$= \Gamma(m+n)\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \Gamma(m+n)\beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$ii)$ we know

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put $x=\sin^2\theta\,, dx = 2\sin\theta\cos\theta\,d\theta, x\to 0, \theta\to 0, x\to 1, \theta\to\frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2}\,2\sin\theta\cos\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin\theta^{2m-1}\cos\theta^{2n-1}\,d\theta$$

Or

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{\beta(m,n)}{2} = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Put $2m-1=p$ and $2n-1=q \therefore m=\frac{p+1}{2}, n=\frac{q+1}{2}$

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)} = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Example:

$$\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^4\theta\,d\theta = \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{5+4+2}{2}\right)} = \frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{2\cdot 1\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{8}{315}$$

Problem 5: Show that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Solution: We know

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Let $m+n=1$

$$\Gamma(n)\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}dx = \int_0^1 \frac{x^{n-1}}{1+x}dx + \int_1^\infty \frac{x^{n-1}}{1+x}dx$$

$$= \int_0^1 \frac{x^{n-1}}{1+x}dx + I$$

Here

$$I = \int_1^\infty \frac{x^{n-1}}{1+x}dx$$

Put $x = \frac{1}{y}, dx = -\frac{1}{y^2}dy, x\to 1, y\to 1; x\to\infty, y\to 0$

$$I = -\int_1^0 \frac{\left(\frac{1}{y}\right)^{n-1}}{1+\frac{1}{y}}\frac{1}{y^2}dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^{n+1}}{\frac{1+y}{y}}dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^{n+1} y}{1+y}dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^{n}}{1+y}dy$$

$$I = \int_0^1 \frac{y^{-n}}{1+y}dy = \int_0^1 \frac{x^{-n}}{1+x}dx$$

$$\therefore\ \Gamma(n)\Gamma(1-n) = \int_0^1 \frac{x^{n-1}}{1+x}dx + \int_0^1 \frac{x^{-n}}{1+x}dx = \int_0^1 \frac{x^{n-1}+x^{-n}}{1+x}dx$$

$$= \int_0^1 (x^{n-1}+x^{-n})\,(1-x+x^2-x^3+\cdots)dx$$

$$= \int_0^1 (x^{n-1}-x^n+x^{n+1}-x^{n+2}+\cdots)\,dx + \int_0^1 (x^{-n}-x^{-n+1}+x^{-n+2}-x^{-n+3}+\cdots)\,dx$$

$$= \left[\frac{x^n}{n}-\frac{x^{n+1}}{n+1}+\frac{x^{n+2}}{n+2}-\frac{x^{n+3}}{n+3}+\cdots\right]_0^1 + \left[\frac{x^{-n+1}}{-n+1}-\frac{x^{-n+2}}{-n+2}+\frac{x^{-n+3}}{-n+3}-\frac{x^{-n+4}}{-n+4}+\cdots\right]_0^1$$

$$= \left[\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}+\cdots\right] - \left[\frac{1}{n-1}-\frac{1}{n-2}+\frac{1}{n-3}-\frac{1}{n-4}+\cdots\right]$$

We know

$$cosec\ \theta = \left(\frac{1}{\theta}-\frac{1}{\theta+\pi}+\frac{1}{\theta+2\pi}-\frac{1}{\theta+3\pi}+\cdots\right) - \left(\frac{1}{\theta-\pi}-\frac{1}{\theta-2\pi}+\frac{1}{\theta-3\pi}-\frac{1}{\theta-4\pi}+\cdots\right)$$

If $\theta = n\pi$ then

$$cosec\ n\pi = \frac{1}{\pi}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots\right) - \frac{1}{\pi}\left(\frac{1}{n-1} - \frac{1}{n-2} + \frac{1}{n-3} - \frac{1}{n-4} + \cdots\right)$$

$$\Rightarrow \pi\ cosec\ n\pi = \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots\right) - \left(\frac{1}{n-1} - \frac{1}{n-2} + \frac{1}{n-3} - \frac{1}{n-4} + \cdots\right)$$

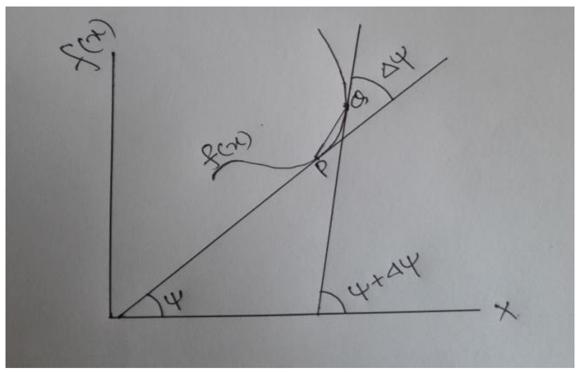
$$\Rightarrow \frac{\pi}{\sin n\pi} = \Gamma(n)\Gamma(1-n)$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Chapter Ix

Length of curves

Length of curves for Cartesian equation:



$$\frac{ds}{dx} = \sec\psi = \sqrt{1+\tan^2\psi} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

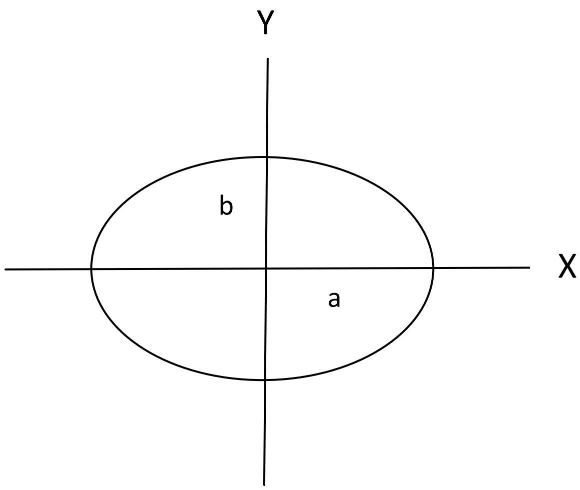
$$S = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Chapter x

Area and volume of curves and surface of revolution

Problem 1: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ between the major and minor axes

Solution:



Area of the ellipse

$$A = 4\int_0^a y dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2} dx$$

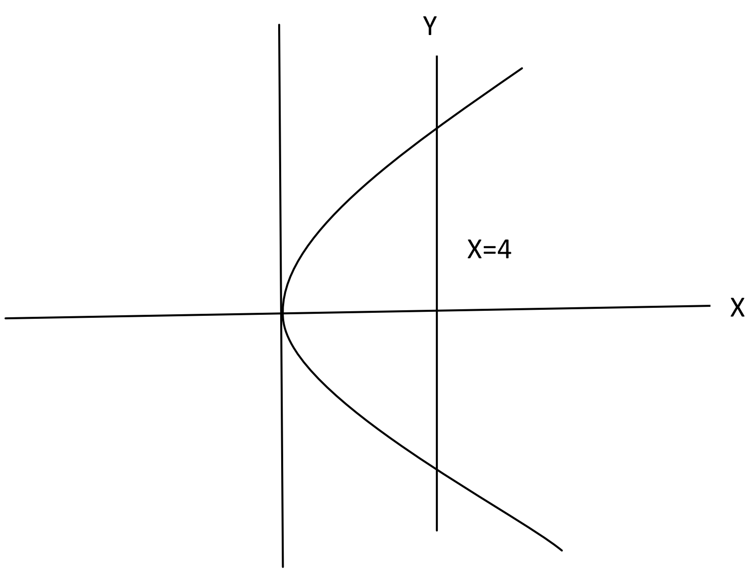
Put $x = a\sin\theta,\ dx = a\cos\theta d\theta, x \to 0, \theta \to 0; x \to a, \theta \to \frac{\pi}{2}$

$$A = 4\int_0^{\frac{\pi}{2}} \frac{b}{a}\sqrt{a^2 - a^2\sin^2\theta}\, a\cos\theta d\theta = 4\int_0^{\frac{\pi}{2}} ab\cos^2\theta d\theta = 2ab\int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta$$

$$= 2ab\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 2ab \cdot \frac{\pi}{2} = \pi ab$$

Problem 2: Find the area of the parabola $y^2 = 4ax$ bounded by $x = 4$

Solution:

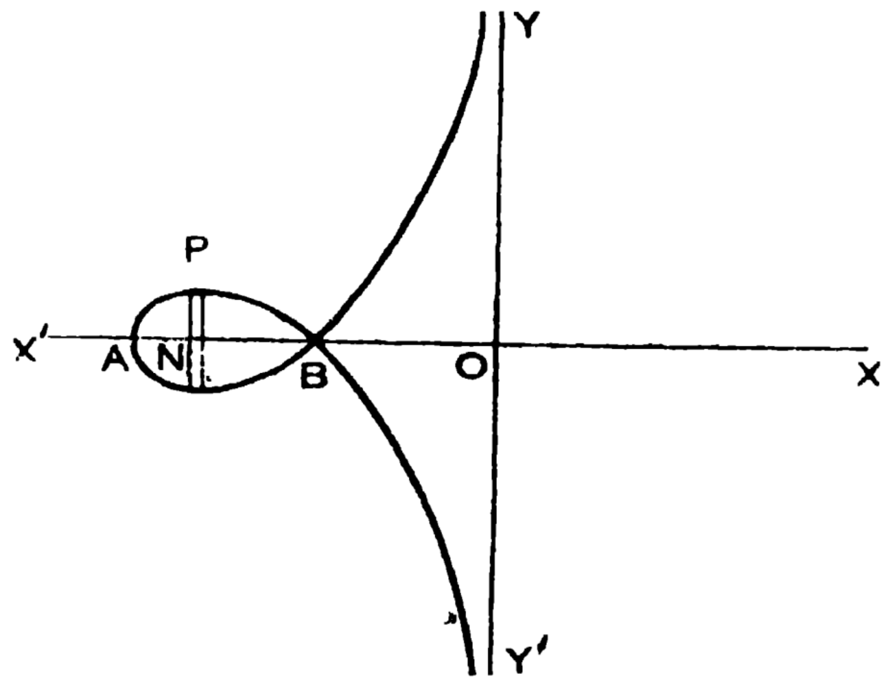


Area of the bounded region

$$A = 2\int_0^4 \sqrt{4ax}\, dx = 4\sqrt{a}\int_0^4 x^{1/2} dx = 4\sqrt{a} \cdot \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4 = \frac{8}{3}\sqrt{a} \cdot 4^{\frac{3}{2}} = \frac{64}{3}\sqrt{a}$$

Problem 3: Find the area of the loop of the curve $xy^2 + (x+a)^2(x+2a) = 0$

Solution:



If $y = 0, x = -a, -2a$ implies that the curve cut x axis at points $x = -a, -2a$

The area of the loop is

$$A = 2\int_{-2a}^{-a} y\,dx = 2\int_{-2a}^{-a} \sqrt{-\frac{(x+a)^2(x+2a)}{x}}\,dx$$

Put $x + 2a = z, dx = dz; x \to -2a, z \to 0; x \to -a, z \to a$

$$A = 2\int_0^a (a-z)\sqrt{\frac{z}{2a-z}}\,dz$$

Putting $z = 2a\sin^2\frac{\theta}{2}, dz = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta; z \to 0, \theta \to 0; z \to a, \theta \to \frac{\pi}{2}$

$$A = 2\int_0^{\frac{\pi}{2}} a\cos\theta \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta = 2a^2\int_0^{\frac{\pi}{2}} \cos\theta\,(1 - cos\theta)d\theta$$

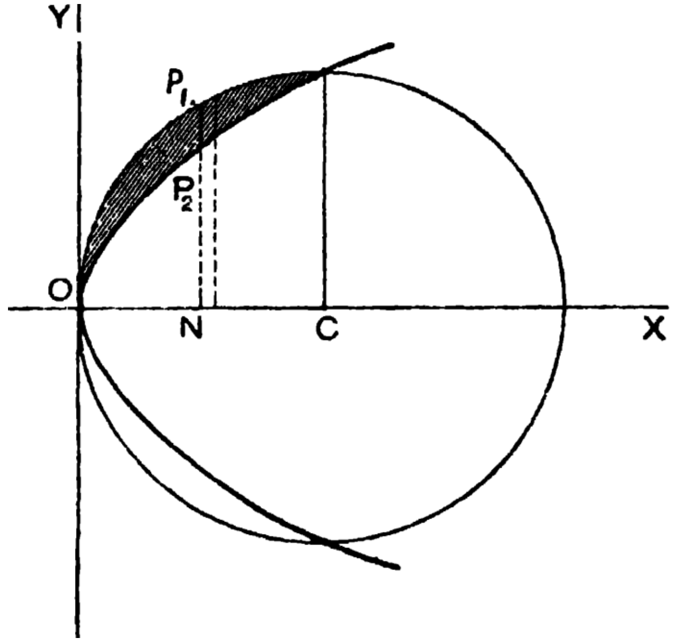
$$= 2a^2\int_0^{\frac{\pi}{2}} (\cos\theta - \cos^2\theta)d\theta = 2a^2\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta - 2a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 2a^2[\sin\theta]_0^{\frac{\pi}{2}} - a^2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta = 2a^2 - a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2a^2 - a^2\frac{\pi}{2} = \frac{a^2}{2}(4-\pi)$$

Problem 4: Find the area above the x axis included between the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 2ax$

Solution:



The points of intersection of the curves are (0,0) and $(a, \pm a)$

The required area is

$$A = \int_0^a (y_1 - y_2)dx = \int_0^a (\sqrt{2ax - x^2} - \sqrt{ax})dx$$

Putting $x = 2a\sin^2\theta, dx = 4a\sin\theta\cos\theta\, d\theta; x \to 0, \theta \to 0; x \to a, \theta \to \frac{\pi}{4}$

$$A = \int_0^{\frac{\pi}{4}} (\sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}\, 4a\sin\theta\cos\theta\, d\theta - \int_0^a \sqrt{a}\, x^{\frac{1}{2}} dx$$

$$= 2a^2 \int_0^{\frac{\pi}{4}} 2\sin\theta\cos\theta\, 2\sin\theta\cos\theta\, d\theta - \sqrt{a}\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^a$$

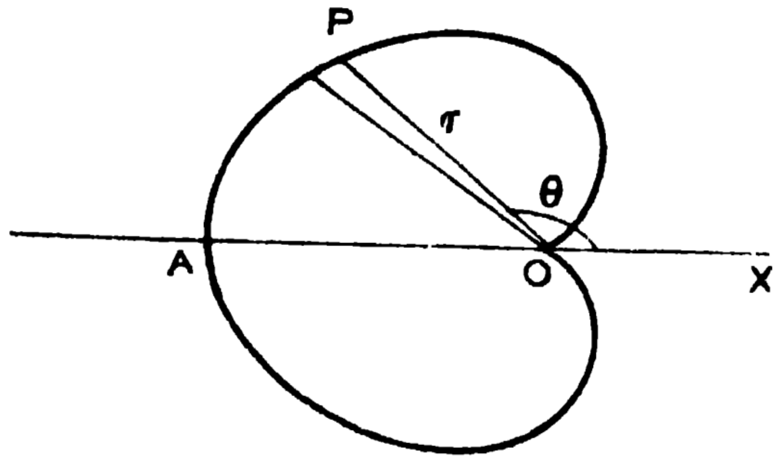
$$= a^2 \int_0^{\frac{\pi}{4}} 2\sin^2 2\theta\, d\theta - \frac{2}{3}\sqrt{a}a^{\frac{3}{2}}$$

$$= a^2 \int_0^{\frac{\pi}{4}} (1 - \cos 4\theta)\, d\theta - \frac{2}{3}a^2 = a^2\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}} - \frac{2}{3}a^2$$

$$= a^2\frac{\pi}{4} - \frac{2}{3}a^2 = a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

Problem 5: Find the area bounded by the cardioide $r = a(1 - \cos\theta)$

Solution:



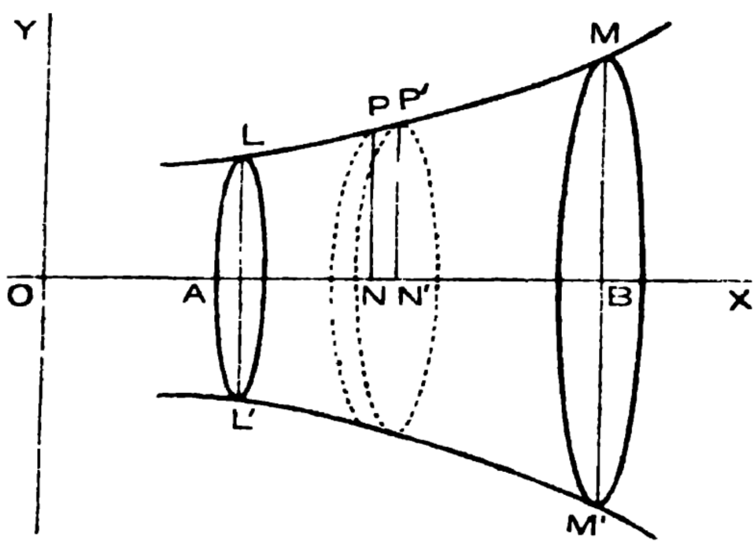
The curve is symmetric about the initial line then the required area is

$$A = 2\int_0^{\pi} \frac{1}{2} r^2 d\theta = \int_0^{\pi} a^2 \,(1 - \cos\theta)^2 \, d\theta = a^2 \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= a^2[\theta - 2\sin\theta]_0^{\pi} + \frac{a^2}{2}\int_0^{\pi} (1 + \cos 2\theta)\, d\theta = a^2\pi + \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= a^2\pi + \frac{a^2\pi}{2} = \frac{3a^2\pi}{2}$$

The solid of revolution, the axis of revolution is x axis:



Let a curve LM whose Cartesian equation is given $y = f(x)$ be rotated about the x axis and form a solid of revolution, let us consider the portion $LL'M'M$ of this solid bounded by $x = x_1$ and $x = x_2$. Consider a circular slices PP' with coordinates $P(x, y)$ and $P'(x + \Delta x, y + \Delta y)$.
The volume of the slices with thickness Δx is $\pi y^2 \Delta x$
Hence total volume of the solid bounded by $x = x_1$ and $x = x_2$ is

$$V = \lim_{\Delta x \to 0} \sum \pi y^2 \Delta x = \pi \int_{x_1}^{x_2} y^2 \, dx$$

Again the area of the slices is $2\pi y \Delta s$

Hence total surface area of the solid bounded by $x = x_1$ and $x = x_2$ is

$$S = \lim_{\Delta s \to 0} \sum 2\pi y \Delta s = 2\pi \int_{s_1}^{s_2} y \, ds = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

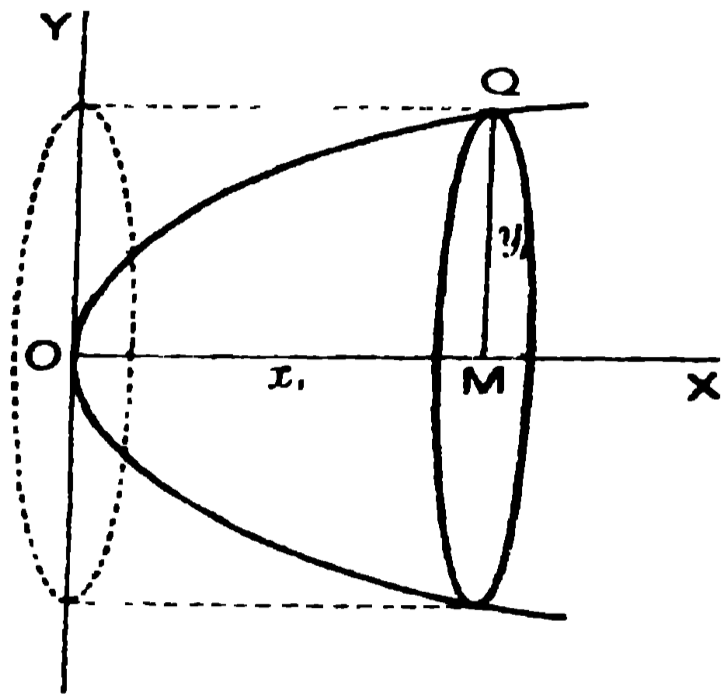
Similarly if the axis of revolution is y axis:

$$V = \pi \int_{y_1}^{y_2} x^2 \, dy$$

$$S = 2\pi \int_{y_1}^{y_2} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Problem 6: Find the volume and area of the curved surface of a paraboloid of revolution form by revolving the parabola $y^2 = 4ax$ about the x axis bounded by $x = 4$

Solution:



Here,

$$y = 2\sqrt{ax}, \frac{dy}{dx} = \sqrt{\frac{a}{x}}$$

The required volume is

$$V = \pi \int_0^4 y^2 dx = \pi \int_0^4 4axdx = [2ax^2]_0^4 = 32a$$

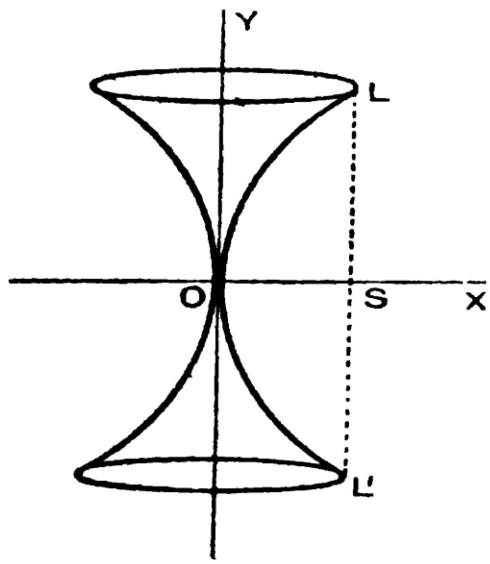
Also the required surface area is

$$S = 2\pi \int_0^4 y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^4 \sqrt{4ax}\sqrt{1+\frac{a}{x}} dx = 4\pi\sqrt{a}\int_0^4 \sqrt{a+x}\, dx$$

$$= 4\pi\sqrt{a}\frac{2}{3}\left[(a+x)^{\frac{3}{2}}\right]_0^4 = \frac{8\pi}{3}\sqrt{a}\left\{(a+4)^{\frac{3}{2}} - a^{\frac{3}{2}}\right\}$$

Problem 7: Find the volume and area of the curved surface of the reel generated by the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at vertex.

Solution:



Here the axis of revolution being y axis and the extreme values of y being $\pm 2a$

The volume is

$$V = \pi \int_{-2a}^{2a} x^2\, dy = \pi \int_{-2a}^{2a} \frac{y^4}{16a^2} dy = \frac{\pi}{16a^2}\left[\frac{y^5}{5}\right]_{-2a}^{2a} = \frac{\pi}{16a^2}\left(\frac{32a^5}{5} + \frac{32a^5}{5}\right)$$

$$= \frac{4\pi a^3}{5}$$

The surface is

$$S = 2\pi \int_{-2a}^{2a} x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_{-2a}^{2a} \frac{y^2}{4a}\sqrt{1+\frac{y^2}{4a^2}} dy$$

Putting $y = 2a\tan\theta, dy = 2a\sec^2\theta d\theta; y \to -2a, \theta \to -\frac{\pi}{4}, y \to 2a, \theta \to \frac{\pi}{4}$

$$S = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4a^2 tan\,^2\theta}{4a}\sqrt{1+\frac{4a^2 tan\,^2\theta}{4a^2}} 2a\sec^2\theta d\theta = 4\pi a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\,^2\theta \sec^3\theta d\theta$$

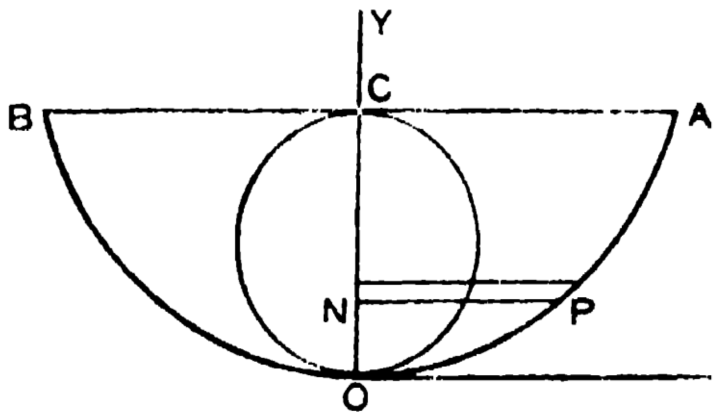
$$= 4\pi a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^5\theta - \sec^3\theta) d\theta$$

$$= 4\pi a^2\left[\frac{\tan\theta\sec^2\theta}{4} - \frac{\tan\theta\sec\theta}{8} - \frac{1}{8}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 4\pi a^2 \left[\frac{3}{4}\sqrt{2} - \frac{1}{4}\log \cot\frac{\pi}{8}\right]$$

Problem 8: Find the volume and surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ about its base.

Solution:



The extreme values of x are given by $\theta = \pm\pi$ *i.e* $x = \pm a\pi$

The required volume

$$V = \int_{-a\pi}^{a\pi} y^2\, dx = \pi a^3 \int_{-\pi}^{\pi} (1 + \cos\theta)^3\, d\theta = \pi a^3 \int_{-\pi}^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$= 8\pi a^3 \int_{-\pi}^{\pi} \cos^6\frac{\theta}{2} d\theta = 8\pi a^3 \cdot 2\int_0^{\pi} \cos^6\frac{\theta}{2} d\theta = 16\pi a^3 \int_0^{\frac{\pi}{2}} \cos^6 u \cdot 2du$$

$$= 32\pi a^3 \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = 5\pi^2 a^3$$

The required surface area

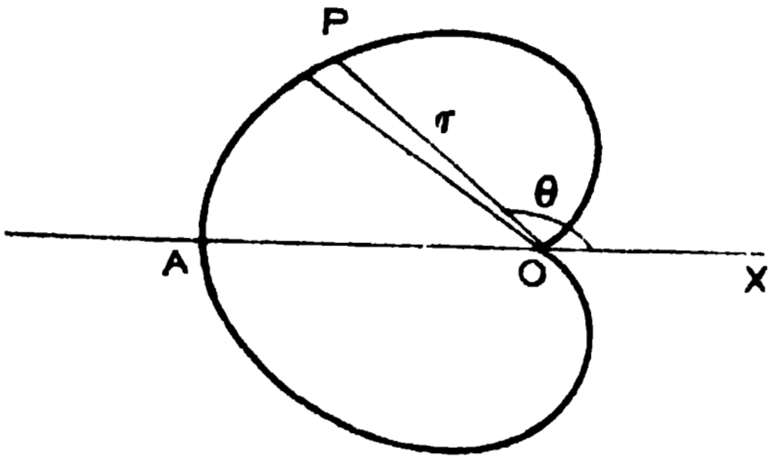
$$S = 2\pi \int y\sqrt{dx^2 + dy^2} = 2\pi \int_{-\pi}^{\pi} a(1 + \cos\theta)\sqrt{\{a(1 + \cos\theta)d\theta\}^2 + \{-a\sin\theta d\theta\}^2}$$

$$= 2\pi a^2 \int_{-\pi}^{\pi} (1 + \cos\theta)\sqrt{2(1 + \cos\theta)}\, d\theta = 8\pi a^2 \int_{-\pi}^{\pi} \cos^3\frac{\theta}{2} d\theta$$

$$= 8\pi a^2 \cdot 2\int_0^{\frac{\pi}{2}} \cos^3 u\, 2du = 32\pi a^2 \int_0^{\frac{\pi}{2}} \cos^3 u\, du = 32\pi a^2 \frac{2}{3\cdot 1}\cdot 1 = \frac{64}{3}\pi a^2$$

Problem 9: Find the volume and surface area of the solid generated by revolving the cardioide $r = a(1 - \cos\theta)$ about the initial line.

Solution:



The extreme points of curve are given by $\theta = 0, \pi$

The required volume

$$V = \pi \int y^2\, dx = \pi \int_0^{\pi} r^2 \sin^2\theta\, d(r\cos\theta)$$

$$= \pi a^3 \int_0^{\pi} (1-\cos\theta)^2 \sin^2\theta\, d((1-\cos\theta)\cos\theta)$$

$$= \pi a^3 \int_0^{\pi} (1-\cos\theta)^2 \sin^2\theta\, (-\sin\theta + 2\cos\theta\sin\theta)\, d\theta$$

Putting $z = \cos\theta, dz = -\sin\theta\, d\theta;\ \theta \to 0, z \to 1, \theta \to \pi, z \to -1$

$$= \pi a^3 \int_1^{-1} (1-z)^2(1-z^2)\,(1-2z)dz$$

$$= \pi a^3 \int_1^{-1} (1 - 4z + 4z^2 + 2z^3 - 5z^4 + 2z^5)\, dz$$

$$= \pi a^3 \left[z - 2z^2 + \frac{4}{3}z^3 + \frac{1}{2}z^4 - z^5 + \frac{1}{3}z^6\right]_1^{-1}$$

$$= \pi a^3 \left[\left(-1 - 2 - \frac{4}{3} + \frac{1}{2} + 1 + \frac{1}{3}\right) - \left(1 - 2 + \frac{4}{3} + \frac{1}{2} - 1 + \frac{1}{3}\right)\right]$$

$$= \pi a^3 \left(-2 - \frac{8}{3} + 2\right) = -\frac{8}{3}\pi a^3 = \frac{8}{3}\pi a^3$$

The required surface area

$$S = 2\pi \int r\sin\theta \sqrt{dr^2 + r^2 d\theta^2}$$

$$= 2\pi \int_0^{\pi} a(1-\cos\theta)\sin\theta \sqrt{(a\sin\theta d\theta)^2 + a^2(1-\cos\theta)^2\, d\theta^2}$$

$$= 2\pi a^2 \int_0^{\pi} (1-\cos\theta)\sin\theta \sqrt{2(1-\cos\theta)}\, d\theta$$

Putting $z = 1 - \cos\theta, dz = \sin\theta d\theta; \theta \to 0, z \to 0, \theta \to \pi, z \to 0$

$$= 2\pi a^2\sqrt{2} \int_0^{2} z\sqrt{z}dz = 2\sqrt{2}\pi a^2 \frac{2}{5}\left[z^{\frac{5}{2}}\right]_0^2$$

$$= \frac{32}{5}\pi a^2$$