

## Chapter I

### Indefinite integral

**Problem 1:** Integrate the following integrals

i)  $\int \sin^4 x dx$  ii)  $\int \sin^2 x \cos^2 x dx$  iii)  $\int \sin^2 x \cos 2x dx$

**Solution:** i)  $\int \sin^4 x dx = \frac{1}{4} \int (2 \sin^2 x)^2 dx = \frac{1}{4} \int (1 - \cos 2x)^2 dx$   
 $= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) dx = \frac{1}{4} \int (1 - 2 \cos 2x) dx + \frac{1}{8} \int 2 \cos^2 2x dx$   
 $= \frac{1}{4} (x - \sin 2x) + \frac{1}{8} \int (1 + \cos 4x) dx = \frac{1}{4} (x - \sin 2x) + \frac{1}{8} (x + \frac{\sin 4x}{4}) + c$   
 $= \frac{3x}{4} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$

ii)  $\int \sin^2 x \cos^2 x dx = \frac{1}{4} \int 2 \sin^2 x \cdot 2 \cos^2 x dx = \frac{1}{4} \int (1 - \cos 2x)(1 + \cos 2x) dx$   
 $= \frac{1}{4} \int (1 - \cos^2 2x) dx = \frac{x}{4} - \frac{1}{8} \int 2 \cos^2 2x dx = \frac{x}{4} - \frac{1}{8} \int (1 + \cos 4x) dx = \frac{x}{4} - \frac{x}{8} - \frac{\sin 4x}{32} + c$   
 $= \frac{x}{8} - \frac{\sin 4x}{32} + c$

iii)  $\int \sin^2 x \cos 2x dx = \frac{1}{2} \int 2 \sin^2 x \cos 2x dx = \frac{1}{2} \int (1 - \cos 2x) \cos 2x dx = \frac{1}{2} \int (\cos 2x - \cos^2 2x) dx$   
 $= \frac{\sin 2x}{4} - \frac{1}{4} \int 2 \cos^2 2x dx = \frac{\sin 2x}{4} - \frac{1}{4} \int (1 + \cos 4x) dx = \frac{\sin 2x}{4} - \frac{x}{4} - \frac{\sin 4x}{16} + c$

## Chapter IIA

### Method of substitution

**Problem 1:** Integrate the following integrals

i)  $\int \frac{\sin 2x dx}{a \sin^2 x + b \cos^2 x}$  ii)  $\int \frac{\tan x dx}{a + b \tan^2 x}$  iii)  $\int \frac{\sin 2x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$  iv)  $\int \sqrt{\frac{a+x}{a-x}} dx$  v)  $\int \sqrt{\frac{x}{a-x}} dx$   
vi)  $\int \cos \left( 2 \cot^{-1} \sqrt{\frac{1-x}{1+x}} \right) dx$  vii)  $\int \frac{dx}{x \sqrt{x^4 - 1}}$

**Solution:** i)  $\int \frac{\sin 2x dx}{a \sin^2 x + b \cos^2 x} = \int \frac{2 \sin x \cos x dx}{a \sin^2 x + b(1 - \sin^2 x)} = \int \frac{2 \sin x \cos x dx}{(a-b) \sin^2 x + b}$   
 $= \frac{1}{(a-b)} \int \frac{dz}{z} = \frac{1}{(a-b)} \ln z + c = \frac{1}{(a-b)} \ln[(a-b) \sin^2 x + b] + c$

ii)  $\int \frac{\tan x dx}{a + b \tan^2 x} = \int \frac{\left( \frac{\sin x}{\cos x} \right) dx}{a + \frac{b \sin^2 x}{\cos^2 x}} = \int \frac{\sin x \cos x dx}{a \cos^2 x + b \sin^2 x}$

$= \frac{1}{2} \int \frac{2 \sin x \cos x dx}{(b-a) \sin^2 x + a} = \frac{1}{2(b-a)} \ln[(b-a) \sin^2 x + a] + c$  similar as (i)

Put

$$(a-b) \sin^2 x + b = z$$

$$\Rightarrow 2(a-b) \sin x \cos x dx = dz$$

$$\begin{aligned}
 \text{iii) } & \int \frac{\sin 2x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \\
 &= \int \frac{2 \sin x \cos x dx}{[a^2 \sin^2 x + b^2 (1 - \sin^2 x)]^2} = \int \frac{2 \sin x \cos x dx}{[(a^2 - b^2) \sin^2 x + b^2]^2} \\
 &= \frac{1}{(a^2 - b^2)} \int \frac{dz}{z^2} = \frac{1}{(a^2 - b^2)} \left(-\frac{1}{z}\right) + c \\
 &= -\frac{1}{(a^2 - b^2)} \frac{1}{(a^2 - b^2) \sin^2 x + b^2} + c
 \end{aligned}$$

Put

$$\begin{aligned}
 (a^2 - b^2) \sin^2 x + b^2 &= z \\
 \Rightarrow 2(a^2 - b^2) \sin x \cos x &= dz
 \end{aligned}$$

$$\begin{aligned}
 \text{iv) } & \int \sqrt{\frac{a+x}{a-x}} dx = - \int \sqrt{\frac{a+a \cos 2t}{a-a \cos 2t}} 2a \sin 2t dt \\
 &= - \int \sqrt{\frac{2a \cos^2 t}{2a \sin^2 t}} 2a \sin 2t dt = - \int \frac{\cos t}{\sin t} 2a \cdot 2 \sin t \cos t dt \\
 &= -2a \int 2 \cos^2 t dt = -2a \int (1 + \cos 2t) dt = -2at - a \sin 2t + c \\
 &= -a \cos^{-1} \frac{x}{a} - \sqrt{a^2 - x^2} + c
 \end{aligned}$$

Put

$$\begin{aligned}
 x &= a \cos 2t \\
 \Rightarrow dx &= -2a \sin 2t dt
 \end{aligned}$$

$$\begin{aligned}
 \text{v) } & \int \sqrt{\frac{x}{a-x}} dx = \int \sqrt{\frac{a \sin^2 t}{a-a \sin^2 t}} 2a \sin t \cos t dt \\
 &= \int 2a \sin^2 t dt = a \int (1 - \cos 2t) dt = at - \frac{a \sin 2t}{2} + c \\
 &= at - a \sin t \cos t + c = a \sin^{-1} \sqrt{\frac{x}{a}} - a \sqrt{\frac{x}{a}} \sqrt{\frac{a-x}{a}} + c \\
 &= at - a \sin t \cos t + c = a \sin^{-1} \sqrt{\frac{x}{a}} - \sqrt{x(a-x)} + c
 \end{aligned}$$

Put

$$\begin{aligned}
 x &= a \sin^2 t \\
 \Rightarrow dx &= 2a \sin t \cos t dt
 \end{aligned}$$

$$\begin{aligned}
 \text{vi) } & \int \cos \left( 2 \cot^{-1} \sqrt{\frac{1-x}{1+x}} \right) dx = - \int \cos \left( 2 \cot^{-1} \sqrt{\frac{1-\cos t}{1+\cos t}} \right) \sin t dt \\
 &= - \int \cos \left( 2 \cot^{-1} \frac{\sin t/2}{\cos t/2} \right) \sin t dt = - \int \cos(2 \cot^{-1} \tan t/2) \sin t dt \\
 &= - \int \cos(2 \cot^{-1} \cot(\pi/2 - t/2)) \sin t dt = - \int \cos(2(\pi/2 - t/2)) \sin t dt \\
 &= - \int \cos(\pi - t) \sin t dt = \int \cos t \sin t dt = -\frac{\cos^2 t}{2} + c = -\frac{x^2}{2} + c
 \end{aligned}$$

Put

$$\begin{aligned}
 x &= \cos t \\
 \Rightarrow dx &= -\sin t dt
 \end{aligned}$$

$$\begin{aligned}
 \text{vii) } & \int \frac{dx}{x \sqrt{x^4 - 1}} = \int \frac{\sec t \tan t dt}{2 \sec t \sqrt{\sec^2 - 1}} = \int \frac{\sec t \tan t dt}{2 \sec t \tan t} \\
 &= \int \frac{dt}{2} = \frac{t}{2} + c = \frac{\sec^{-1} x^2}{2} + c
 \end{aligned}$$

Put

$$\begin{aligned}
 x^2 &= \sec t \\
 \Rightarrow 2x dx &= \sec t \tan t dt \\
 \Rightarrow dx &= \frac{\sec t \tan t dt}{2x}
 \end{aligned}$$

## Chapter IIB

### Method of substitution with some formulae

**Problem 1:** Integrate the following integrals

$$\text{i) } \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} \quad \text{ii) } \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} \quad \text{iii) } \int \sqrt{\frac{a+x}{x}} dx \quad \text{iv) } \int \frac{\sqrt{x-a}}{x} dx$$

**Solution:** i)  $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \int \frac{2zdz}{\sqrt{z^2(\beta-\alpha-z^2)}}$

$$= \int \frac{2dz}{\sqrt{\beta-\alpha-z^2}} = 2 \int \frac{dz}{\sqrt{(\sqrt{\beta-\alpha})^2 - z^2}} = 2 \sin^{-1} \frac{z}{\sqrt{\beta-\alpha}} + c$$

$$= 2 \sin^{-1} \frac{\sqrt{x-\alpha}}{\sqrt{\beta-\alpha}} + c$$

Put

$$x - \alpha = z^2 \\ \Rightarrow dx = 2zdz$$

ii)  $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} = \int \frac{-1/z^2 dz}{1/z \sqrt{1+2(1/z-1)-(1/z-1)^2}}$

$$= - \int \frac{dz}{z \sqrt{1+2/z-2-1/z^2+2/z-1}} = - \int \frac{dz}{z \sqrt{4/z-2-1/z^2}} = - \int \frac{dz}{z \sqrt{\frac{4z-2z^2-1}{z^2}}}$$

$$= - \int \frac{dz}{\sqrt{4z-2z^2-1}} = - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{2z-z^2-1/2}} = - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{-1+2z-z^2-1/2+1}}$$

$$= - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{-(1-2z+z^2)+1/2}} = - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(\sqrt{1/2})^2 - (z-1)^2}}$$

$$= - \frac{1}{\sqrt{2}} \sin^{-1} \frac{z-1}{1/\sqrt{2}} + c = - \frac{1}{\sqrt{2}} \sin^{-1} \frac{1+x}{1/\sqrt{2}} + c = - \frac{1}{\sqrt{2}} \sin^{-1} \frac{-\sqrt{2}x}{1+x} + c = \frac{1}{\sqrt{2}} \sin^{-1} \frac{\sqrt{2}x}{1+x} + c$$

Put

$$1+x = 1/z \\ \Rightarrow dx = -1/z^2 dz$$

iii)  $\int \sqrt{\frac{a+x}{x}} dx = \int \frac{\sqrt{a+x}\sqrt{a+x}}{\sqrt{x}\sqrt{a+x}} dx = \int \frac{a+x}{\sqrt{x^2+ax}} dx = \frac{1}{2} \int \frac{a+2x}{\sqrt{x^2+ax}} dx + \frac{1}{2} \int \frac{a}{\sqrt{x^2+ax}} dx$

$$= \frac{1}{2} \cdot 2\sqrt{x^2+ax} + \frac{1}{2} \int \frac{a}{\sqrt{x^2+ax+\frac{a^2}{4}-\frac{a^2}{4}}} dx = \sqrt{x^2+ax} + \frac{1}{2} \int \frac{adx}{\sqrt{\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2}}$$

$$= \sqrt{x^2+ax} + \frac{a}{2} \ln \left\{ x + \frac{a}{2} + \sqrt{\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} \right\} + c = \sqrt{x^2+ax} + \frac{a}{2} \ln \left\{ x + \frac{a}{2} + \sqrt{x^2+ax} \right\} + c$$

$$= \sqrt{x^2+ax} + \frac{a}{2} \ln \left\{ \frac{2x+a+2\sqrt{x}\sqrt{x+a}}{2} \right\} + c = \sqrt{x(x+a)} + \frac{a}{2} \ln \left\{ \frac{x+x+a+2\sqrt{x}\sqrt{x+a}}{2} \right\} + c$$

$$= \sqrt{x(x+a)} + \frac{a}{2} \ln \left\{ \frac{(\sqrt{x}+\sqrt{x+a})^2}{2} \right\} + c = \sqrt{x(x+a)} + \frac{a}{2} \ln (\sqrt{x}+\sqrt{x+a})^2 - \frac{a}{2} \ln 2 + c$$

$$= \sqrt{x(x+a)} + a \ln (\sqrt{x}+\sqrt{x+a}) + c_0$$

$$\begin{aligned}
\text{iv) } \int \frac{\sqrt{x-a}}{x} dx &= \int \frac{\sqrt{x-a} \sqrt{x-a}}{x\sqrt{x-a}} dx = \int \frac{x-a}{x\sqrt{x-a}} dx \\
&= \int \frac{x}{x\sqrt{x-a}} dx - \int \frac{a}{x\sqrt{x-a}} dx = \int \frac{dx}{\sqrt{x-a}} - \int \frac{a}{x\sqrt{x-a}} dx \\
&= 2\sqrt{x-a} - \int \frac{2azdz}{(a+z^2)\sqrt{z^2}} = 2\sqrt{x-a} - \int \frac{2adz}{a+z^2} \\
&= 2\sqrt{x-a} - 2a \int \frac{dz}{(\sqrt{a})^2 + z^2} = 2\sqrt{x-a} - 2a \cdot \frac{1}{\sqrt{a}} \tan^{-1} \frac{z}{\sqrt{a}} + c \\
&= 2\sqrt{x-a} - 2\sqrt{a} \tan^{-1} \frac{\sqrt{x-a}}{\sqrt{a}} + c = 2\sqrt{x-a} - 2\sqrt{a} \tan^{-1} \sqrt{\frac{x-a}{a}} + c
\end{aligned}$$

Put (2<sup>nd</sup> integral)

$$\begin{aligned}
x-a &= z^2 \\
\Rightarrow dx &= 2zdz
\end{aligned}$$

## Chapter III

### Integration by parts

**Problem 1:** Integrate the following integrals

$$\begin{aligned}
\text{i) } \int \sin^{-1} \sqrt{\frac{x}{x+a}} dx &\quad \text{ii) } \int \frac{e^x}{x} (1+x \log x) dx \quad \text{iii) } \int e^x \frac{x^2+1}{(x+1)^2} dx \quad \text{iv) } \int e^x \frac{x-1}{(x+1)^3} dx \\
\text{v) } \int \sqrt{2ax-x^2} dx &\quad \text{vi) } \int \sqrt{(x-\alpha)(\beta-x)} dx \quad \text{vii) } \int (x+2)\sqrt{x^2+2x+10} dx
\end{aligned}$$

$$\begin{aligned}
\text{Solution: i) } \int \sin^{-1} \sqrt{\frac{x}{x+a}} dx &= \int \sin^{-1} \sqrt{\frac{a \tan^2 \theta}{a+a \tan^2 \theta}} \cdot 2a \tan \theta \sec^2 \theta d\theta \\
&= 2a \int \sin^{-1} \sin \theta \cdot \tan \theta \sec^2 \theta d\theta = 2a \int \theta \tan \theta \sec^2 \theta d\theta \\
&= 2a \left[ \theta \frac{\tan^2 \theta}{2} - \frac{1}{2} \int \tan^2 \theta d\theta \right] \\
&= 2a \left[ \theta \frac{\tan^2 \theta}{2} - \frac{1}{2} \int (\sec^2 \theta - 1) d\theta \right] \\
&= a[\theta \tan^2 \theta - (\tan \theta - \theta)] = a \left[ \frac{x}{a} \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1} \sqrt{\frac{x}{a}} \right]
\end{aligned}$$

$$\begin{aligned}
\text{Put } x &= a \tan^2 \theta \\
dx &= 2a \tan \theta \sec^2 \theta d\theta
\end{aligned}$$

$$\begin{aligned}
\text{ii) } \int \frac{e^x}{x} (1+x \log x) dx &= \int e^x \left( \frac{1}{x} + \log x \right) dx = e^x \log x \\
\text{iii) } \int e^x \frac{(x^2+1)}{(x+1)^2} dx &= \int e^x \frac{(x^2-1+2)}{(x+1)^2} dx = \int e^x \left[ \frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx \\
&= e^x \frac{x-1}{x+1} \\
\text{iv) } \int e^x \frac{(x-1)}{(x+1)^3} dx &= \int e^x \frac{(x+1-2)}{(x+1)^3} dx = \int e^x \left[ \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} \right] dx \\
&= e^x \frac{1}{(x+1)^2} \\
\text{v) } \int \sqrt{(2ax-x^2)} dx &= \int \sqrt{a^2 - (x-a)^2} dx \\
&= \frac{1}{2} (x-a) \sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1} \frac{(x-a)}{a}
\end{aligned}$$

$$= \frac{1}{2}(x-a)\sqrt{(2ax-x^2)} + \frac{a^2}{2}\sin^{-1}\frac{(x-a)}{a}$$

$$\text{vi) } \int \sqrt{(x-a)(b-x)}dx = \int \sqrt{bx-x^2-ab+ax}dx = \int \sqrt{(a+b)x-x^2-ab}dx$$

$$= \int \sqrt{-\left(\frac{a+b}{2}\right)^2 + (a+b)x - x^2 - ab + \left(\frac{a+b}{2}\right)^2} dx$$

$$= \int \sqrt{\left(\frac{a+b}{2}\right)^2 - ab - \left\{x^2 - (a+b)x + \left(\frac{a+b}{2}\right)^2\right\}} dx$$

$$= \int \sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2} dx$$

$$= \frac{\left(x - \frac{a+b}{2}\right)\sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2}}{2} + \frac{\left(\frac{a-b}{2}\right)^2}{2}\sin^{-1}\frac{\left(x - \frac{a+b}{2}\right)}{\left(\frac{a-b}{2}\right)}$$

$$= \frac{(2x-a-b)\sqrt{(x-a)(b-x)}}{4} + \frac{(a-b)^2}{8}\sin^{-1}\frac{(2x-a-b)}{(a-b)}$$

$$\text{vii) } \int (x+2)\sqrt{x^2+2x+10}dx = \frac{1}{2}\int (2x+2)\sqrt{x^2+2x+10}dx + \int \sqrt{x^2+2x+10} dx$$

$$= \frac{1}{2}\int \sqrt{z}dz + \int \sqrt{(x+1)^2+9}dx = \frac{1}{2}\int z^{\frac{1}{2}}dz + \int \sqrt{3^2+(x+1)^2}dx$$

$$= \frac{1}{2}z^{\frac{3}{2}} \times \frac{2}{3} + \frac{(x+1)\sqrt{x^2+2x+10}}{2} + \frac{9}{2}\log\{(x+1)+\sqrt{x^2+2x+10}\}$$

$$= \frac{1}{3}(x^2+2x+10)^{\frac{3}{2}} + \frac{(x+1)\sqrt{x^2+2x+10}}{2} + \frac{9}{2}\log\{(x+1)+\sqrt{x^2+2x+10}\}$$

## Chapter IV

### Integration of trigonometric function

**Problem 1:** i)  $\int \frac{dx}{1+\tan x}$  ii)  $\int \frac{\cos x}{2\sin x+3\cos x}dx$  iii)  $\int \frac{dx}{a+b\tan x}$  iv)  $\int \frac{dx}{5+4\sin x}$  v)  $\int \frac{dx}{5+4\cos x}$

**Solution:**

$$\begin{aligned} \text{i) } \int \frac{dx}{1+\tan x} &= \int \frac{\cos x dx}{\sin x + \cos x} = \frac{1}{2} \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x} dx \\ &= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{(\cos x - \sin x)}{\sin x + \cos x} dx = \frac{x}{2} + \frac{1}{2} \log(\sin x + \cos x) \end{aligned}$$

$$\begin{aligned} \text{ii) } \int \frac{\cos x}{2\sin x + 3\cos x} dx &= \frac{3}{13} \int \frac{(3\cos x + 2\sin x) + \frac{2}{3}(2\cos x - 3\sin x)}{2\sin x + 3\cos x} dx \\ &= \frac{3}{13} \int dx + \frac{2}{13} \int \frac{(2\cos x - 3\sin x)}{2\sin x + 3\cos x} dx = \frac{3x}{13} + \frac{2}{13} \log(2\sin x + 3\cos x) \end{aligned}$$

$$\begin{aligned}
 \text{iii) } \int \frac{dx}{a + b \tan x} &= \int \frac{\cos x dx}{a \cos x + b \sin x} \\
 &= \frac{a}{a^2 + b^2} \int \frac{(a \cos x + b \sin x) + \frac{b}{a}(b \cos x - a \sin x)}{a \cos x + b \sin x} dx \\
 &= \frac{a}{a^2 + b^2} \int dx + \frac{b}{a^2 + b^2} \int \frac{(b \cos x - a \sin x)}{a \cos x + b \sin x} dx \\
 &= \frac{ax}{a^2 + b^2} + \frac{b}{a^2 + b^2} \log(a \cos x + b \sin x)
 \end{aligned}$$

$$\text{iv) } \int \frac{dx}{5 + 4 \sin x} = \int \frac{dx}{5 \cos^2 \frac{x}{2} + 5 \sin^2 \frac{x}{2} + 8 \sin \frac{x}{2} \cos \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2} dx}{5 + 5 \tan^2 \frac{x}{2} + 8 \tan \frac{x}{2}}$$

$$\text{Put } \tan \frac{x}{2} = z$$

$$\sec^2 \frac{x}{2} dx = 2dz$$

$$\begin{aligned}
 &= 2 \int \frac{dz}{5 + 5z^2 + 8z} = \frac{2}{5} \int \frac{dz}{z^2 + \frac{8}{5}z + 1} = \frac{2}{5} \int \frac{dz}{(z + \frac{4}{5})^2 + 1 - \frac{16}{25}} \\
 &= \frac{2}{5} \int \frac{dz}{(z + \frac{4}{5})^2 + \frac{9}{25}} = \frac{2}{5} \int \frac{dz}{\left(z + \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} \\
 &= \frac{2}{5} \times \frac{5}{3} \tan^{-1} \frac{\left(z + \frac{4}{5}\right)}{\frac{3}{5}} = \frac{2}{3} \tan^{-1} \frac{5z + 4}{3} = \frac{2}{3} \tan^{-1} \frac{5 \tan \frac{x}{2} + 4}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{v) } \int \frac{dx}{5 + 4 \cos x} &= \int \frac{dx}{5 \cos^2 \frac{x}{2} + 5 \sin^2 \frac{x}{2} + 4 \left( \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right)} \\
 &= \int \frac{\sec^2 \frac{x}{2} dx}{5 + 5 \tan^2 \frac{x}{2} + 4 - 4 \tan^2 \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2} dx}{\tan^2 \frac{x}{2} + 9}
 \end{aligned}$$

$$\text{Put } \tan \frac{x}{2} = z$$

$$\sec^2 \frac{x}{2} dx = 2dz$$

$$\begin{aligned}
 &= 2 \int \frac{dz}{z^2 + 3^2} = \frac{2}{3} \tan^{-1} \frac{z}{3} \\
 &= \frac{2}{3} \tan^{-1} \frac{\tan \frac{x}{2}}{3}
 \end{aligned}$$

## Chapter V

### Rational function

$$\text{Problem 1: i) } \int \frac{x-1}{(x-2)(x-3)} dx \text{ ii) } \int \frac{dx}{(x-1)^2(x+1)} \text{ iii) } \int \frac{x}{x^4-1} dx \text{ iv) } \int \frac{x}{x^4-1} dx \text{ v) } \int \frac{dx}{x^4-1}$$

**Solution:**

$$i) \int \frac{x-1}{(x-2)(x-3)} dx = \int \left\{ \frac{2}{(x-3)} - \frac{1}{(x-2)} \right\} dx = 2 \log(x-3) - \log(x-2)$$

$$ii) \int \frac{dx}{(x-1)^2(x+1)}$$

Now,

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Put  $x = 1$

$$B = \frac{1}{2}$$

Put  $x = -1$

$$C = \frac{1}{4}$$

Equating the coefficient of  $x^2$

$$A + C = 0 \therefore A = -\frac{1}{4}$$

$$\int \frac{dx}{(x-1)^2(x+1)} = \int \left\{ -\frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)} \right\} dx$$

$$= -\frac{1}{4} \log(x-1) - \frac{1}{2(x-1)} + \frac{1}{4} \log(x+1)$$

$$iii) \int \frac{x}{x^4-1} dx = \int \frac{x}{(x-1)(x+1)(x^2+1)} dx = \int \left\{ \frac{1}{4(x-1)} + \frac{1}{4(x+1)} - \frac{x}{2(x^2+1)} \right\}$$

$$= \frac{1}{4} \log(x-1) + \frac{1}{4} \log(x+1) - \frac{1}{4} \log(x^2+1)$$

$$iv) \int \frac{x}{x^4-1} dx = \frac{1}{4} \int \frac{4x^3}{x^4(x^4-1)} dx$$

Put  $x^4 = z$ ,  $4x^3 dx = dz$

$$= \frac{1}{4} \int \frac{dz}{z(z-1)} = \frac{1}{4} \int \left\{ \frac{1}{z-1} - \frac{1}{z} \right\} dz = \frac{1}{4} [\log(z-1) - \log z] = \frac{1}{4} [\log(x^4-1) - \log x^4]$$

$$v) \int \frac{dx}{x^4-1} = \int \frac{dx}{(x^2+1)(x^2-1)} = \frac{1}{2} \int \left\{ \frac{1}{(x^2-1)} - \frac{1}{(x^2+1)} \right\} dx = \frac{1}{2} \left[ \frac{1}{2} \log \frac{1+x}{1-x} - \tan^{-1} x \right]$$

## CHAPTER VI

### Definite integral

**Problem 1:** i)  $\int_0^1 x(\tan^{-1} x)^2 dx$  ii)  $\int_0^a \sin^{-1} \frac{2x}{1+x^2} dx$  iii)  $\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

**Solution:**

$$i) \int_0^1 x(\tan^{-1} x)^2 dx = \left[ \frac{x^2}{2} (\tan^{-1} x)^2 \right]_0^1 - \frac{1}{2} \int_0^1 x^2 \cdot 2 \frac{\tan^{-1} x}{1+x^2} dx$$

$$\begin{aligned}
&= \frac{\pi^2}{32} - \int_0^1 \frac{x^2 + 1 - 1}{1 + x^2} \tan^{-1} x \, dx \\
&= \frac{\pi^2}{32} - \int_0^1 \tan^{-1} x \, dx + \int_0^1 \frac{\tan^{-1} x}{1 + x^2} \, dx \\
&= \frac{\pi^2}{32} - [x \tan^{-1} x]_0^1 + \int_0^1 x \frac{1}{1 + x^2} \, dx + \int_0^1 \frac{\tan^{-1} x}{1 + x^2} \, dx \\
&= \frac{\pi^2}{32} - [x \tan^{-1} x]_0^1 + \left[ \frac{1}{2} \log(1 + x^2) \right]_0^1 + \frac{1}{2} [(\tan^{-1} x)^2]_0^1 \\
&= \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2
\end{aligned}$$

$$\text{ii) } \int_0^a \sin^{-1} \frac{2x}{1 + x^2} \, dx = \int_0^a 2 \tan^{-1} x \, dx = [2x \tan^{-1} x]_0^a - \int_0^a \frac{2x}{1 + x^2} \, dx$$

$$= 2a \tan^{-1} a - [\log(1 + x^2)]_0^a = 2a \sin^{-1} a - \log(1 + a^2)$$

$$\text{iii) } \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \int_0^{\pi/2} \frac{\sec^4 x \, dx}{(a^2 + b^2 \tan^2 x)^2} = ?$$

Put  $b \tan x = a \tan \theta$ ,  $b \sec^2 x \, dx = a \sec^2 \theta \, d\theta$

$$= \int_0^{\pi/2} \frac{\left(1 + \frac{a^2}{b^2} \tan^2 \theta\right) \frac{a}{b} \sec^2 \theta \, d\theta}{(a^2 + a^2 \tan^2 \theta)^2} = \frac{1}{a^3 b^3} \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2 \theta) \sec^2 \theta \, d\theta}{(1 + \tan^2 \theta)^2}$$

$$= \frac{1}{a^3 b^3} \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2 \theta) \sec^2 \theta \, d\theta}{\sec^4 \theta} = \frac{1}{a^3 b^3} \int_0^{\pi/2} \frac{(b^2 + a^2 \tan^2 \theta) \, d\theta}{\sec^2 \theta}$$

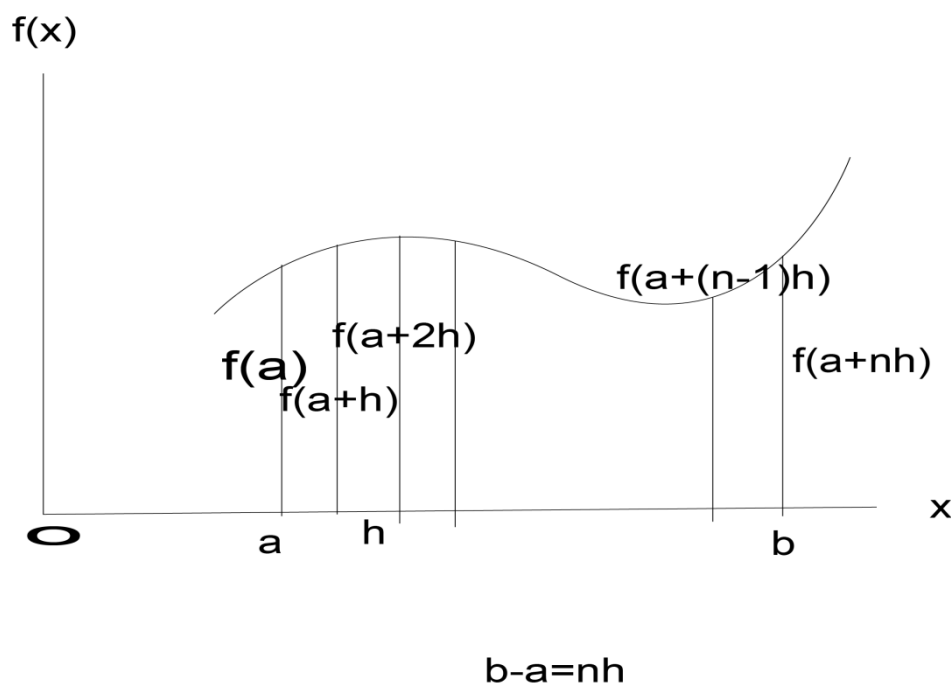
$$= \frac{1}{a^3 b^3} \int_0^{\pi/2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta) \, d\theta = \frac{1}{2a^3 b^3} \int_0^{\pi/2} (b^2 2 \cos^2 \theta + a^2 2 \sin^2 \theta) \, d\theta$$

$$= \frac{1}{2a^3 b^3} \int_0^{\pi/2} (b^2(1 + \cos 2\theta) + a^2(1 - \cos 2\theta)) \, d\theta$$

$$= \frac{1}{2a^3 b^3} \left[ b^2 \left( \theta + \frac{\sin 2\theta}{2} \right) + a^2 \left( \theta - \frac{\sin 2\theta}{2} \right) \right]_0^{\pi/2} = \frac{(a^2 + b^2)\pi}{4a^3 b^3}$$

**First principle of integral calculus:**





Area of region by the curve  $f(x)$  with the  $x$ -axis bounded by  $x=a$  and  $x=b$  is

$$\lim_{h \rightarrow 0} [hf(a) + hf(a+h) + hf(a+2h) + \dots + hf(a+(n-1)h)]$$

$$\lim_{h \rightarrow 0} \sum_{r=0}^{n-1} hf(a+rh)$$

Symbolically this area is represented as  $\int_a^b f(x)dx$  implies that

$$\int_a^b f(x)dx = \lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} f(a+rh) = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{r=0}^{n-1} f\left(a+r \frac{b-a}{n}\right)$$

If  $a=0$  and  $b=1$  then

$$\int_0^1 f(x)dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$$

**Problem 2:** Evaluate from first principle  $\int_a^b e^x dx$

**Solution:**

$$\begin{aligned} \int_a^b e^x dx &= \lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} e^{a+rh} \\ &= \lim_{h \rightarrow 0} h [e^a + e^{a+h} + e^{a+2h} + \dots + e^{a+(n-1)h}] \\ &= \lim_{h \rightarrow 0} h e^a [1 + e^h + e^{2h} + \dots + e^{(n-1)h}] \\ &= \lim_{h \rightarrow 0} h e^a \frac{(e^h)^n - 1}{e^h - 1} = \lim_{h \rightarrow 0} h e^a \frac{e^{nh} - 1}{e^h - 1} = \lim_{h \rightarrow 0} h e^a \frac{e^{b-a} - 1}{e^h - 1} \\ &= e^a (e^{b-a} - 1) \lim_{h \rightarrow 0} \frac{h}{e^h - 1} = e^a (e^{b-a} - 1) = e^b - e^a \end{aligned}$$

**Problem 3:** Evaluate from first principle  $\int_0^1 x^2 dx$

**Solution:**

$$\begin{aligned}
 \int_0^1 x^2 dx &= \lim_{h \rightarrow 0} h \sum_{r=1}^n (rh)^2 \\
 &= \lim_{h \rightarrow 0} h [1^2 h^2 + 2^2 h^2 + 3^2 h^2 + \dots + n^2 h^2] \\
 &= \lim_{h \rightarrow 0} h^3 [1^2 + 2^2 + 3^2 + \dots + n^2] \\
 &= \lim_{h \rightarrow 0} h^3 \frac{n(n+1)(2n+1)}{6} = \frac{1}{6} \lim_{h \rightarrow 0} (2n^3 h^3 + 3n^2 h^2 \cdot h + nh \cdot h^2) \quad nh = 1 \\
 &= \frac{1}{6} \lim_{h \rightarrow 0} (2 + 3h + h^2) = \frac{2}{6} = \frac{1}{3}
 \end{aligned}$$

**Problem 4:** Evaluate

i)  $\lim_{n \rightarrow \infty} \left[ \frac{1}{n+m} + \frac{1}{n+2m} + \frac{1}{n+3m} + \dots + \frac{1}{n+nm} \right]$

ii)  $\lim_{n \rightarrow \infty} \left[ \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{n}{n^2+n^2} \right]$

iii)  $\lim_{n \rightarrow \infty} \left[ \frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \frac{3^2}{n^3+3^3} + \dots + \frac{n^2}{n^3+n^3} \right]$

iv)  $\lim_{n \rightarrow \infty} \left[ \left(1 + \frac{1^2}{n^2}\right)^{\frac{2}{n^2}} + \left(1 + \frac{2^2}{n^2}\right)^{\frac{4}{n^2}} + \left(1 + \frac{3^2}{n^2}\right)^{\frac{6}{n^2}} + \dots + \left(1 + \frac{n^2}{n^2}\right)^{\frac{2n}{n^2}} \right]$

**Solution:**

$$\begin{aligned}
 \text{i) } \lim_{n \rightarrow \infty} &\left[ \frac{1}{n+m} + \frac{1}{n+2m} + \frac{1}{n+3m} + \dots + \frac{1}{n+nm} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{1}{1+\frac{m}{n}} + \frac{1}{1+\frac{2m}{n}} + \frac{1}{1+\frac{3m}{n}} + \dots + \frac{1}{1+\frac{nm}{n}} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{1}{1+\frac{rm}{n}} = \int_0^1 \frac{1}{1+mx} dx = \left[ \frac{1}{m} \log(1+mx) \right]_0^1 \\
 &= \frac{1}{m} \log(1+m)
 \end{aligned}$$

$$\begin{aligned}
 \text{ii) } \lim_{n \rightarrow \infty} &\left[ \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{n}{n^2+n^2} \right] \\
 &= \lim_{n \rightarrow \infty} \frac{n}{n^2} \left[ \frac{1}{1+\frac{1^2}{n^2}} + \frac{1}{1+\frac{2^2}{n^2}} + \frac{1}{1+\frac{3^2}{n^2}} + \dots + \frac{1}{1+\frac{n^2}{n^2}} \right]
 \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{1}{1 + \frac{r^2}{n^2}} = \int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

$$\text{iii) } \lim_{n \rightarrow \infty} \left[ \frac{1^2}{n^3 + 1^3} + \frac{2^2}{n^3 + 2^3} + \frac{3^2}{n^3 + 3^3} + \dots + \frac{n^2}{n^3 + n^3} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{n^3} \left[ \frac{\frac{1^2}{n^2}}{1 + \frac{1^3}{n^3}} + \frac{\frac{2^2}{n^2}}{1 + \frac{2^3}{n^3}} + \frac{\frac{3^2}{n^2}}{1 + \frac{3^3}{n^3}} + \dots + \frac{\frac{n^2}{n^2}}{1 + \frac{n^3}{n^3}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{\frac{r^2}{n^2}}{1 + \frac{r^3}{n^3}} = \int_0^1 \frac{x^2}{1+x^3} dx = \left[ \frac{1}{3} \log(1+x^3) \right]_0^1$$

$$= \frac{1}{3} \log 2$$

$$\text{iv) } \lim_{n \rightarrow \infty} \left[ \left(1 + \frac{1^2}{n^2}\right)^{\frac{2}{n^2}} + \left(1 + \frac{2^2}{n^2}\right)^{\frac{4}{n^2}} + \left(1 + \frac{3^2}{n^2}\right)^{\frac{6}{n^2}} + \dots + \left(1 + \frac{n^2}{n^2}\right)^{\frac{2n}{n^2}} \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(1 + \frac{r^2}{n^2}\right)^{\frac{2r}{n^2}}$$

Let

$$A = \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(1 + \frac{r^2}{n^2}\right)^{\frac{2r}{n^2}}$$

$$\log A = \log \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(1 + \frac{r^2}{n^2}\right)^{\frac{2r}{n^2}} = \sum_{r=1}^n \lim_{n \rightarrow \infty} \frac{2r}{n^2} \log \left(1 + \frac{r^2}{n^2}\right) = \frac{1}{n} \sum_{r=1}^n \lim_{n \rightarrow \infty} \frac{2r}{n} \log \left(1 + \frac{r^2}{n^2}\right)$$

$$= \int_0^1 2x \log(1+x^2) dx$$

Put  $1+x^2 = z$ ,  $2x dx = dz$ ,  $x \rightarrow 0, z \rightarrow 1$ ;  $x \rightarrow 1, z \rightarrow 2$

$$= \int_1^2 \log z dz = [z \log z - z]_1^2 = 2 \log 2 - 2 + 1$$

$$= \log 2^2 - 1 = \log 4 - \log e = \log \frac{4}{e}$$

$$\therefore A = \frac{4}{e}$$

$$\int \log x dx = x \log x - x$$

## Chapter VII

### General properties and reduction formula

Some properties of definite integral:

$$i) \int_a^b f(x) dx = \int_a^b f(z) dz$$

$$ii) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$iii) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, \text{ if } a < c < b$$

$$iv) \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$v) \int_0^{na} f(x) dx = n \int_0^a f(x) dx, \text{ if } f(a+x) = f(x)$$

$$vi) \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(-x) = f(x) \\ = 0, \text{ if } f(-x) = -f(x)$$

**Problem 1: Integrate** i)  $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$  ii) show that  $\int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx =$

$\frac{\pi}{2} \log \frac{1}{2}$  iii) show that  $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2$

**Solution:**

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}} dx \\ = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \\ \therefore 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \\ 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$ii) I = \int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx$$

$$2I = \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx = \int_0^{\pi/2} \log(\sin x \cos x) dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log \sin 2x dx - \int_0^{\pi/2} \log 2 dx$$

$$\text{Put } 2x = z, dx = \frac{dz}{2}, x \rightarrow 0, z \rightarrow 0; x \rightarrow \frac{\pi}{2}, z \rightarrow \pi$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin z dz - [x \log 2]_0^{\pi/2} = \int_0^{\pi/2} \log \sin z dz - \frac{\pi}{2} \log 2$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2 = I - \frac{\pi}{2} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}$$

$$ii) I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

Put  $x = \tan \theta$ ,  $dx = \sec^2 \theta d\theta$ ,  $x \rightarrow 0, \theta \rightarrow 0$ ;  $x \rightarrow 1, \theta \rightarrow \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) d\theta = \int_0^{\frac{\pi}{4}} \log \left[ 1 + \tan \left( \frac{\pi}{4} - \theta \right) \right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log \left[ 1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right] d\theta = \int_0^{\frac{\pi}{4}} \log \left[ 1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log \left[ \frac{1 + \tan \theta + 1 - \tan \theta}{1 + \tan \theta} \right] d\theta = \int_0^{\frac{\pi}{4}} \log \left[ \frac{2}{1 + \tan \theta} \right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log 2 d\theta - \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) d\theta = \int_0^{\frac{\pi}{4}} \log 2 d\theta - I$$

$$2I = [\theta \log 2]_0^{\frac{\pi}{4}} = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

**Theorem:** state and prove walli's formula for definite integral or show that

If  $n$  be positive integer

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Or

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

According as  $n$  is even or odd.

**Proof:**

$$I_n = \int \sin^n x dx = \int \underbrace{\sin^{n-1} x}_{\sqrt{\quad}} \underbrace{\sin x}_{\underbrace{\quad}} dx = \sin^{n-1} x (-\cos x) + (n-1) \int \sin^{n-2} x \cos^2 x dx$$

$$= \sin^{n-1} x (-\cos x) + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) I_n$$

$$n I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx$$

$$I_n = \left[ \frac{-\sin^{n-1} x \cos x}{n} \right]_0^{\frac{\pi}{2}} + \frac{(n-1)}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx$$

$$I_n = \frac{(n-1)}{n} I_{n-2} \quad \checkmark$$

Similarly if we proceed we get

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot I_0$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$$

according as  $n$  is even or odd.

$$I_0 = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Or

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

according as  $n$  is even or odd.

→ 1st hit  $\frac{\pi}{2}$

**Theorem:** If both  $m$  and  $n$  are even

2 tai even hole  $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{(m-1)(m-3) \cdots 3 \cdot 1 (n-1)(n-3) \cdots 3 \cdot 1}{(m+n)(m+n-2) \cdots 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

If  $m$  is even and  $n$  is odd

$$= \frac{(m-1)(m-3) \cdots 3 \cdot 1 (n-1)(n-3) \cdots 4 \cdot 2}{(m+n)(m+n-2) \cdots 5 \cdot 3 \cdot 1} \cdot 1$$

If both  $m$  and  $n$  are odd

$$= \frac{(m-1)(m-3) \cdots 4 \cdot 2 (n-1)(n-3) \cdots 4 \cdot 2}{(m+n)(m+n-2) \cdots 6 \cdot 4 \cdot 2} \cdot \frac{1}{2}$$

2 ta odd hole  $1/2$

**Problem 2:** Evaluate  $\int_0^1 x^6 \sqrt{1-x^2} dx$

**Solution:** Put  $x = \sin \theta$ ,  $dx = \cos \theta d\theta$ ,  $x \rightarrow 0, \theta \rightarrow 0, x \rightarrow 1, \theta \rightarrow \pi/2$

$$\int_0^1 x^6 \sqrt{1-x^2} dx = \int_0^{\frac{\pi}{2}} \sin^6 \theta \sqrt{1-\sin^2 \theta} \cos \theta d\theta = \int_0^{\frac{\pi}{2}} \sin^6 \theta \cos^2 \theta d\theta$$

$$= \frac{5 \cdot 3 \cdot 1 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{5\pi}{256}$$

**Problem 3:** Evaluate  $\int_0^1 x^2 (1-x)^{3/2} dx$

**Solution:** Put  $x = \sin^2 \theta$ ,  $dx = 2 \sin \theta \cos \theta d\theta$ ,  $x \rightarrow 0, \theta \rightarrow 0, x \rightarrow 1, \theta \rightarrow \pi/2$

$$\int_0^1 x^2(1-x)^{3/2} dx = \int_0^{\pi/2} \sin^4 \theta (1 - \sin^2 \theta)^{3/2} 2 \sin \theta \cos \theta d\theta = 2 \int_0^{\pi/2} \sin^5 \theta \cos^4 \theta d\theta$$

$$= 2 \frac{4 \cdot 2 \cdot 3 \cdot 1}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = \frac{16}{315}$$

**Problem 4:** Evaluate i)  $\int_0^{\pi/2} \sin^9 x dx$  ii)  $\int_0^{\pi/2} \sin^{10} x dx$  iii)  $\int_0^{\pi/2} \sin^5 x \cos^7 x dx$

**Solution:**

i)  $\int_0^{\pi/2} \sin^9 x dx = \frac{8 \cdot 6 \cdot 4 \cdot 2}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = \frac{128}{315}$

ii)  $\int_0^{\pi/2} \sin^{10} x dx = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{63\pi}{512}$

iii)  $\int_0^{\pi/2} \sin^5 x \cos^7 x dx = \frac{4 \cdot 2 \cdot 6 \cdot 4 \cdot 2}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{1}{2} = \frac{1}{240}$

**Problem 5:** show that  $\int_0^{\pi} \frac{x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{(a^2 + b^2)\pi^2}{4a^3b^3}$

**Solution:**

$$I = \int_0^{\pi} \frac{x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \int_0^{\pi} \frac{(\pi - x) dx}{(a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x))^2}$$

$$= \int_0^{\pi} \frac{(\pi - x) dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \int_0^{\pi} \frac{\pi dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} - \int_0^{\pi} \frac{x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$$

$$\int_0^{\pi} \frac{\pi dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} - I$$

$$\therefore I = \frac{1}{2} \int_0^{\pi} \frac{\pi dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \int_0^{\pi/2} \frac{\pi dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{(a^2 + b^2)\pi^2}{4a^3b^3}$$

(Solution See problem 1 (iii) chapter Vi)

**Problem 6:** Obtain reduction formulae for i)  $\int \tan^n x dx$  ii)  $\int_0^{\pi/4} \tan^n x dx$  hence deduce iii)  $\int \tan^5 x dx$  iv)  $\int \tan^6 x dx$

**Solution:**

i)  $I_n = \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\therefore I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

ii)  $J_n = \left[ \frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x dx$

$$\therefore J_n = \frac{1}{n-1} - J_{n-2}$$

$$iii) I_5 = \int \tan^5 x dx = \frac{\tan^4 x}{4} - I_3$$

$$I_3 = \int \tan^3 x dx = \frac{\tan^2 x}{2} - I_1$$

$$I_1 = \int \tan x dx = \log \sec x$$

$$\therefore I_5 = \int \tan^5 x dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log \sec x$$

$$iv) I_6 = \int \tan^6 x dx = \frac{\tan^5 x}{5} - I_4$$

$$I_4 = \int \tan^4 x dx = \frac{\tan^3 x}{3} - I_2$$

$$I_2 = \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x$$

$$\therefore I_6 = \int \tan^6 x dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x$$

**Problem 7:** Obtain reduction formulae for i)  $\int \sec^n x dx$  hence deduce ii)  $\int \sec^6 x dx$   
iii)  $\int \sec^7 x dx$

**Solution:**

$$i) I_n = \int \sec^n x dx = \int \underbrace{\sec^{n-2} x}_{u} \underbrace{\sec^2 x dx}_{v}$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x dx$$

$$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x (\sec^2 x - 1) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$(1+n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$ii) I_6 = \int \sec^6 x dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} I_4$$

$$I_4 = \int \sec^4 x dx = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2$$

$$I_2 = \int \sec^2 x dx = \tan x$$

$$I_6 = \int \sec^6 x dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \frac{\sec^2 x \tan x}{3} + \frac{4}{5} \frac{2}{3} \tan x$$

$$iii) I_7 = \int \sec^7 x dx = \frac{\sec^5 x \tan x}{6} + \frac{5}{6} I_5$$

$$I_5 = \int \sec^5 x dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3$$



$$I_3 = \frac{\sec x \tan x}{2} + \frac{1}{2} I_1$$

$$I_1 = \int \sec x \, dx = \log(\sec x + \tan x)$$

$$I_7 = \int \sec^7 x \, dx = \frac{\sec^5 x \tan x}{6} + \frac{5}{6} \frac{\sec^3 x \tan x}{4} + \frac{5}{6} \frac{3 \sec x \tan x}{2} + \frac{5}{6} \frac{1}{2} \log(\sec x + \tan x)$$

**Problem 8:** Obtain reduction formulae for  $\int e^{ax} \cos^n x \, dx$  hence deduce  $\int_0^\infty e^{-4x} \cos^5 x \, dx$

**Solution:**

$$\begin{aligned} I_n &= \int e^{ax} \cos^n x \, dx = \frac{e^{ax} \cos^n x}{a} + \frac{n}{a} \int e^{ax} \cos^{n-1} x \sin x \, dx \\ &= \frac{e^{ax} \cos^n x}{a} + \frac{n}{a} \left[ \frac{e^{ax}}{a} \cos^{n-1} x \sin x \right. \\ &\quad \left. - \frac{1}{a} \int e^{ax} \{(n-1) \cos^{n-2} x (-\sin x) \sin x + \cos^{n-1} x \cos x\} dx \right] \\ &= \frac{e^{ax} \cos^n x}{a} + \frac{ne^{ax}}{a^2} \cos^{n-1} x \sin x + \frac{n}{a^2} \int e^{ax} \{(n-1) \cos^{n-2} x (1 - \cos^2 x) + \cos^n x\} dx \\ &= \frac{e^{ax} \cos^{n-1} x (a \cos x + n \sin x)}{a^2} + \frac{n(n-1)}{a^2} \int e^{ax} \cos^{n-2} x \, dx - \frac{n}{a^2} \int e^{ax} \{(n-1) \cos^n x \\ &\quad + \cos^n x\} dx \\ &= \frac{e^{ax} \cos^{n-1} x (a \cos x + n \sin x)}{a^2} + \frac{n(n-1)}{a^2} \int e^{ax} \cos^{n-2} x \, dx - \frac{n^2}{a^2} \int e^{ax} \cos^n x \, dx \\ \left(1 + \frac{n^2}{a^2}\right) I_n &= \frac{e^{ax} \cos^{n-1} x (a \cos x + n \sin x)}{a^2} + \frac{n(n-1)}{a^2} I_{n-2} \\ I_n &= \frac{e^{ax} \cos^{n-1} x (a \cos x + n \sin x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2} \end{aligned}$$

Now the reduction formula for  $\int_0^\infty e^{-ax} \cos^n x \, dx$  is

$$I_n = \left[ \frac{e^{-ax} \cos^{n-1} x (-a \cos x + n \sin x)}{a^2 + n^2} \right]_0^\infty + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

$$I_n = \frac{a}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

$$\therefore I_5 = \frac{4}{41} + \frac{20}{41} I_3$$

$$I_3 = \frac{4}{25} + \frac{6}{25} I_1$$

$$I_1 = \int_0^\infty e^{-4x} \cos x \, dx = \frac{4}{4^2 + 1^2} = \frac{4}{17}$$

$$\therefore I_5 = \frac{708}{3485}$$

## Chapter VIII\*

### Gama beta function

**Gama function:** The second Eulerian integral is called gama function and is defined as

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ , and  $n > 0$  but need not be integer.

**Beta function:** The first Eulerian integral is called beta function and is defined as

$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ , and  $m, n > 0$  but need not be integer.

**Problem 1:** Show that i)  $\beta(m, n) = \beta(n, m)$  ii)  $\Gamma(1) = 1$  iii)  $\Gamma(n+1) = n\Gamma(n)$

iv)  $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$

**Solution:**

i)  $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^1 (1-x)^{m-1} (1-1+x)^{n-1} dx$   
 $= \int_0^1 x^{n-1} (1-x)^{m-1} dx = \beta(n, m)$

ii)  $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$

iii)  $\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1} dx = \int_0^\infty e^{-x} x^n dx$   
 $= [-x^n e^{-x}]_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx = 0 + n \int_0^\infty e^{-x} x^{n-1} dx = n\Gamma(n)$

When  $n$  is positive integer  $\Gamma(n+1) = n!$

iv)  $\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$

Put  $y = kx$ ,  $dy = kdx$ ,  $y \rightarrow 0, x \rightarrow 0; y \rightarrow \infty, x \rightarrow \infty$

$\Gamma(n) = \int_0^\infty e^{-kx} (kx)^{n-1} kdx = \int_0^\infty e^{-kx} x^{n-1} k^n dx$

$\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx} x^{n-1} dx$

**Problem 2:** show that i)  $\Gamma(n) = 2 \int_0^\infty e^{-x^2} x^{2n-1} dx$  ii)  $\Gamma(n+1) = \int_0^\infty e^{-x^{\frac{1}{n}}} dx$

iii)  $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$  hence integrate iv)  $\int_0^\infty e^{-x^2} dx$  v)  $\int_0^\infty e^{-x^{\frac{1}{4}}} dx$  vi)  $\int_0^1 \left(\log \frac{1}{x}\right)^5 dx$

**Solution:**

i) we know

$\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$

Put  $y = x^2$ ,  $dy = 2x dx$ ;  $y \rightarrow 0, x \rightarrow 0, y \rightarrow \infty, x \rightarrow \infty$

$\Gamma(n) = \int_0^\infty e^{-x^2} x^{2n-2} 2x dx = 2 \int_0^\infty e^{-x^2} x^{2n-1} dx$

ii) we know

$\Gamma(n) = \int_0^\infty e^{-y} y^{n-1} dy$

Put  $y^n = x$ ,  $ny^{n-1} dy = dx$ ,  $y^{n-1} dy = \frac{dx}{n}$ ;  $y \rightarrow 0, x \rightarrow 0, y \rightarrow \infty, x \rightarrow \infty$

$\Gamma(n) = \frac{1}{n} \int_0^\infty e^{-x^{\frac{1}{n}}} dx \Rightarrow n\Gamma(n) = \int_0^\infty e^{-x^{\frac{1}{n}}} dx$

$\Rightarrow \Gamma(n+1) = \int_0^\infty e^{-x^{\frac{1}{n}}} dx$

iii) we know

$$\Gamma(n) = \int_0^{\infty} e^{-y} y^{n-1} dy$$

Put  $e^{-y} = x, e^y = \frac{1}{x}, y = \log \frac{1}{x}, -e^{-y} dy = dx; y \rightarrow 0, x \rightarrow 1, y \rightarrow \infty, x \rightarrow 0$

$$\Gamma(n) = -\int_1^0 \left(\log \frac{1}{x}\right)^{n-1} dx = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$$

iv) we know

$$\Gamma(n) = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

Put  $n = \frac{1}{2}$

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-x^2} dx \Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}$$

v) we know

$$\Gamma(n+1) = \int_0^{\infty} e^{-x^{\frac{1}{n}}} dx$$

Put  $n = 4$

$$\Gamma(5) = \int_0^{\infty} e^{-x^{\frac{1}{4}}} dx \Rightarrow \int_0^{\infty} e^{-x^{\frac{1}{4}}} dx = \Gamma(5) = 24$$

vi) we know

$$\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$$

Put  $n = 6$

$$\Gamma(6) = \int_0^1 \left(\log \frac{1}{x}\right)^5 dx \Rightarrow \int_0^1 \left(\log \frac{1}{x}\right)^5 dx = \Gamma(6) = 120$$

**Problem 3:** Show that  $\beta(m, n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$  hence integrate  $\int_0^{\infty} \frac{y^7}{(1+y)^{10}} dy$

**Solution:** we know

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Put  $x = \frac{1}{1+y}, 1+y = \frac{1}{x}, y = \frac{1}{x} - 1, dx = -\frac{1}{(1+y)^2} dy, x \rightarrow 0, y \rightarrow \infty, x \rightarrow 1, y \rightarrow 0$

$$\begin{aligned} \beta(m, n) &= -\int_{\infty}^0 \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \frac{1}{(1+y)^2} dy \\ &= \int_0^{\infty} \left(\frac{1}{1+y}\right)^{m-1} \left(\frac{y}{1+y}\right)^{n-1} \frac{1}{(1+y)^2} dy = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy \end{aligned}$$

Similarly

$$\beta(n, m) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Since  $\beta(m, n) = \beta(n, m)$

$$\beta(m, n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy \text{ (proved)}$$

We know

$$\beta(m, n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

Put  $m = 7, n = 8$

$$\int_0^{\infty} \frac{y^7}{(1+y)^{15}} dy = \beta(7,8) = \frac{\Gamma(7)\Gamma(8)}{\Gamma(7+8)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14}$$

$$= \frac{1}{2 \cdot 3 \cdot 2 \cdot 11 \cdot 13 \cdot 14}$$

**Problem 4:** Show that

i)  $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

ii)  $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

**Solution:**

i) we know

$$\Gamma(m) = \int_0^{\infty} e^{-x} x^{m-1} dx$$

And

$$\frac{\Gamma(n)}{x^n} = \int_0^{\infty} e^{-xy} y^{n-1} dy$$

$$\Gamma(n) = \int_0^{\infty} e^{-xy} x^n y^{n-1} dy$$

$$\therefore \Gamma(m)\Gamma(n) = \int_0^{\infty} \left\{ \int_0^{\infty} e^{-x(1+y)} x^{m+n-1} dx \right\} y^{n-1} dy = \int_0^{\infty} \frac{\Gamma(m+n)}{(1+y)^{m+n}} y^{n-1} dy$$

$$= \Gamma(m+n) \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy = \Gamma(m+n)\beta(m, n)$$

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

ii) we know

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Put  $x = \sin^2 \theta$ ,  $dx = 2 \sin \theta \cos \theta d\theta$ ,  $x \rightarrow 0, \theta \rightarrow 0, x \rightarrow 1, \theta \rightarrow \frac{\pi}{2}$

$$\beta(m, n) = \int_0^{\frac{\pi}{2}} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{n-1} 2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin \theta)^{2m-2} (\cos \theta)^{2n-2} 2 \sin \theta \cos \theta d\theta = 2 \int_0^{\frac{\pi}{2}} \sin \theta^{2m-1} \cos \theta^{2n-1} d\theta$$

Or

$$\int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{\beta(m, n)}{2} = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Put  $2m-1 = p$  and  $2n-1 = q \therefore m = \frac{p+1}{2}, n = \frac{q+1}{2}$

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)} = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Example:

$$\int_0^{\frac{\pi}{2}} \sin^5 \theta \cos^4 \theta d\theta = \frac{\Gamma\left(\frac{5+1}{2}\right) \Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{5+4+2}{2}\right)} = \frac{\Gamma(3) \Gamma\left(\frac{5}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{2 \cdot 1 \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{2 \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)} = \frac{8}{315}$$

**Problem 5:** Show that  $\Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

**Solution:** We know

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Let  $m+n=1$

$$\begin{aligned} \Gamma(n) \Gamma(1-n) &= \int_0^1 \frac{x^{n-1}}{1+x} dx = \int_0^1 \frac{x^{n-1}}{1+x} dx + \int_1^\infty \frac{x^{n-1}}{1+x} dx \\ &= \int_0^1 \frac{x^{n-1}}{1+x} dx + I \end{aligned}$$

Here

$$I = \int_1^\infty \frac{x^{n-1}}{1+x} dx$$

Put  $x = \frac{1}{y}$ ,  $dx = -\frac{1}{y^2} dy$ ,  $x \rightarrow 1, y \rightarrow 1$ ;  $x \rightarrow \infty, y \rightarrow 0$

$$I = - \int_1^0 \frac{\left(\frac{1}{y}\right)^{n-1}}{1+\frac{1}{y}} \frac{1}{y^2} dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^{n+1}}{\frac{1+y}{y}} dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^{n+1} y}{1+y} dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^n}{1+y} dy$$

$$I = \int_0^1 \frac{y^{-n}}{1+y} dy = \int_0^1 \frac{x^{-n}}{1+x} dx$$

$$\therefore \Gamma(n) \Gamma(1-n) = \int_0^1 \frac{x^{n-1}}{1+x} dx + \int_0^1 \frac{x^{-n}}{1+x} dx = \int_0^1 \frac{x^{n-1} + x^{-n}}{1+x} dx$$

$$= \int_0^1 (x^{n-1} + x^{-n}) (1-x+x^2-x^3+\dots) dx$$

$$= \int_0^1 (x^{n-1} - x^n + x^{n+1} - x^{n+2} + \dots) dx + \int_0^1 (x^{-n} - x^{-n+1} + x^{-n+2} - x^{-n+3} + \dots) dx$$

$$= \left[ \frac{x^n}{n} - \frac{x^{n+1}}{n+1} + \frac{x^{n+2}}{n+2} - \frac{x^{n+3}}{n+3} + \dots \right]_0^1 + \left[ \frac{x^{-n+1}}{-n+1} - \frac{x^{-n+2}}{-n+2} + \frac{x^{-n+3}}{-n+3} - \frac{x^{-n+4}}{-n+4} + \dots \right]_0^1$$

$$= \left[ \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \dots \right] - \left[ \frac{1}{n-1} - \frac{1}{n-2} + \frac{1}{n-3} - \frac{1}{n-4} + \dots \right]$$

We know

$$\operatorname{cosec} \theta = \left( \frac{1}{\theta} - \frac{1}{\theta+\pi} + \frac{1}{\theta+2\pi} - \frac{1}{\theta+3\pi} + \dots \right) - \left( \frac{1}{\theta-\pi} - \frac{1}{\theta-2\pi} + \frac{1}{\theta-3\pi} - \frac{1}{\theta-4\pi} + \dots \right)$$

If  $\theta = n\pi$  then

$$\operatorname{cosec} n\pi = \frac{1}{\pi} \left( \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \dots \right) - \frac{1}{\pi} \left( \frac{1}{n-1} - \frac{1}{n-2} + \frac{1}{n-3} - \frac{1}{n-4} + \dots \right)$$

$$\Rightarrow \pi \operatorname{cosec} n\pi = \left( \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \dots \right) - \left( \frac{1}{n-1} - \frac{1}{n-2} + \frac{1}{n-3} - \frac{1}{n-4} + \dots \right)$$

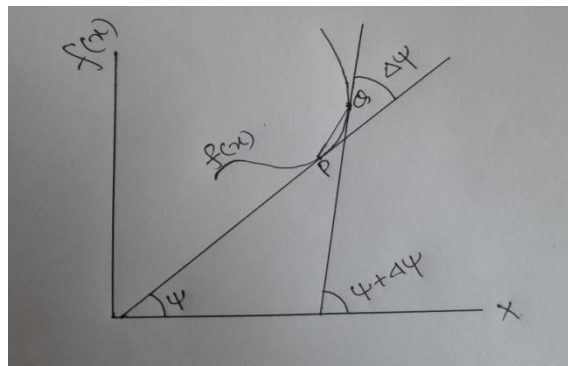
$$\Rightarrow \frac{\pi}{\sin n\pi} = \Gamma(n)\Gamma(1-n)$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

## Chapter IX

### Length of curves

Length of curves for Cartesian equation:



$$\frac{ds}{dx} = \sec \psi = \sqrt{1 + \tan^2 \psi} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2}$$

$$ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

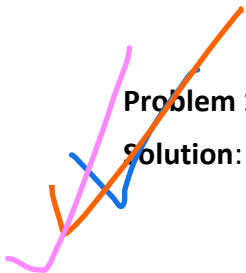
$$S = \int \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

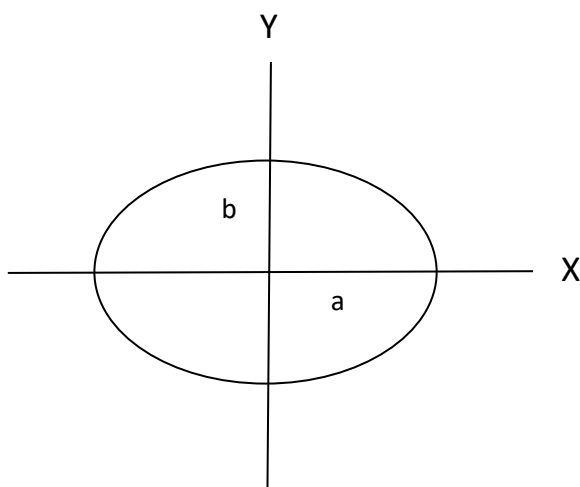
## Chapter X

### Area and volume of curves and surface of revolution

**Problem 1:** Find the area of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  between the major and minor axes

**Solution:**





Area of the ellipse

$$A = 4 \int_0^a y dx = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

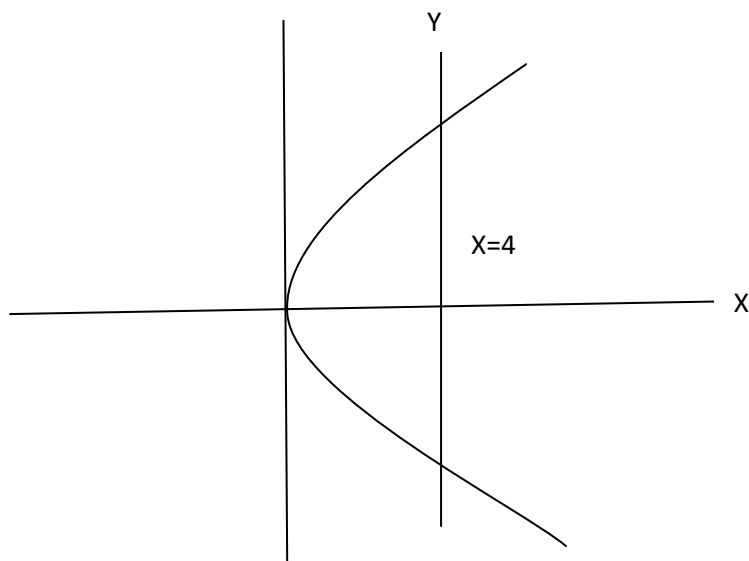
Put  $x = a \sin \theta$ ,  $dx = a \cos \theta d\theta$ ,  $x \rightarrow 0, \theta \rightarrow 0$ ;  $x \rightarrow a, \theta \rightarrow \frac{\pi}{2}$

$$A = 4 \int_0^{\frac{\pi}{2}} \frac{b}{a} \sqrt{a^2 - a^2 \sin^2 \theta} a \cos \theta d\theta = 4 \int_0^{\frac{\pi}{2}} ab \cos^2 \theta d\theta = 2ab \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta$$

$$= 2ab \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 2ab \cdot \frac{\pi}{2} = \pi ab$$

**Problem 2:** Find the area of the parabola  $y^2 = 4ax$  bounded by  $x = 4$

**Solution:**

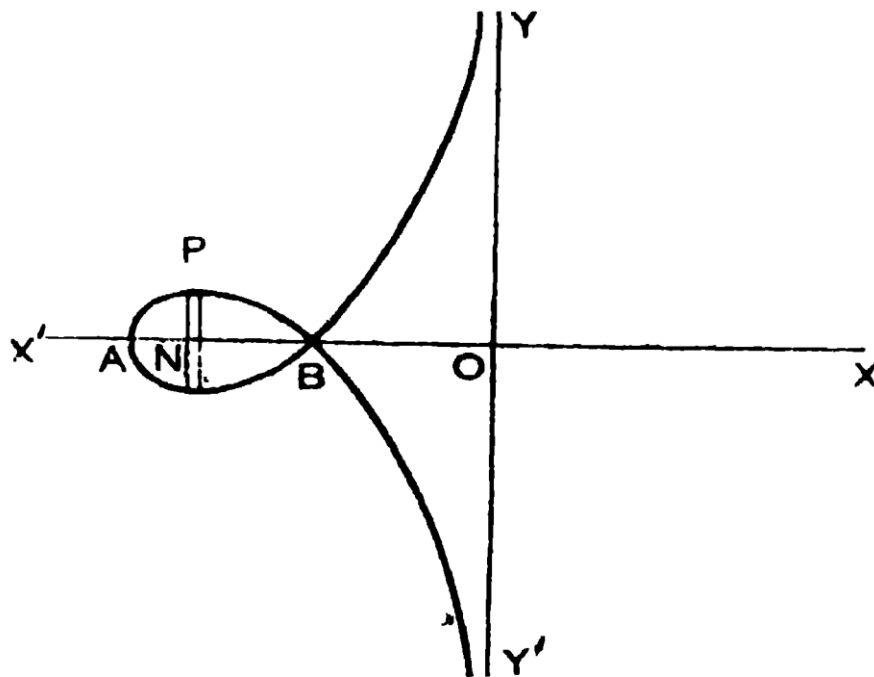


Area of the bounded region

$$A = 2 \int_0^4 \sqrt{4ax} dx = 4\sqrt{a} \int_0^4 x^{1/2} dx = 4\sqrt{a} \cdot \frac{2}{3} \left[ x^{3/2} \right]_0^4 = \frac{8}{3} \sqrt{a} \cdot 4^{3/2} = \frac{64}{3} \sqrt{a}$$

**Problem 3:** Find the area of the loop of the curve  $xy^2 + (x+a)^2(x+2a) = 0$

**Solution:**



If  $y = 0, x = -a, -2a$  implies that the curve cut x axis at points  $x = -a, -2a$

The area of the loop is

$$A = 2 \int_{-2a}^{-a} y dx = 2 \int_{-2a}^{-a} \sqrt{-\frac{(x+a)^2(x+2a)}{x}} dx$$

Put  $x + 2a = z, dx = dz; x \rightarrow -2a, z \rightarrow 0; x \rightarrow -a, z \rightarrow a$

$$A = 2 \int_0^a (a-z) \sqrt{\frac{z}{2a-z}} dz \quad (z-a) \text{ ke } (a-z) \text{ e convert... and } (z-2a) \text{ ke } (2a-z) \text{ e convert for the neg sign}$$

Putting  $z = 2a \sin^2 \frac{\theta}{2}, dz = 2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta; z \rightarrow 0, \theta \rightarrow 0; z \rightarrow a, \theta \rightarrow \frac{\pi}{2}$

$$A = 2 \int_0^{\frac{\pi}{2}} a \cos \theta \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} 2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta = 2a^2 \int_0^{\frac{\pi}{2}} \cos \theta (1 - \cos \theta) d\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{2}} (\cos \theta - \cos^2 \theta) d\theta = 2a^2 \int_0^{\frac{\pi}{2}} \cos \theta d\theta - 2a^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$= 2a^2 [\sin \theta]_0^{\frac{\pi}{2}} - a^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta = 2a^2 - a^2 \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}}$$

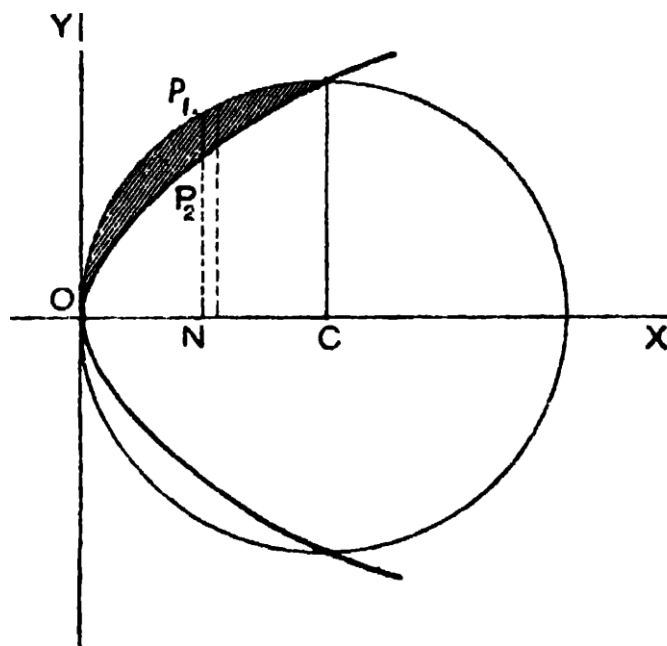
$$= 2a^2 - a^2 \frac{\pi}{2} = \frac{a^2}{2} (4 - \pi)$$

**Problem 4:** Find the area above the x axis included between the parabola  $y^2 = ax$  and the circle  $x^2 + y^2 = 2ax$

only the above part



**Solution:**



The points of intersection of the curves are  $(0,0)$  and  $(a, \pm a)$

The required area is

$$A = \int_0^a (y_1 - y_2) dx = \int_0^a (\sqrt{2ax - x^2} - \sqrt{ax}) dx$$

Putting  $x = 2a \sin^2 \theta$ ,  $dx = 4a \sin \theta \cos \theta d\theta$ ;  $x \rightarrow 0, \theta \rightarrow 0$ ;  $x \rightarrow a, \theta \rightarrow \frac{\pi}{4}$

$$A = \int_0^{\frac{\pi}{4}} (\sqrt{4a^2 \sin^2 \theta - 4a^2 \sin^4 \theta} 4a \sin \theta \cos \theta d\theta - \int_0^a \sqrt{a} x^{\frac{1}{2}} dx$$

$$= 2a^2 \int_0^{\frac{\pi}{4}} 2 \sin \theta \cos \theta 2 \sin \theta \cos \theta d\theta - \sqrt{a} \frac{2}{3} \left[ x^{\frac{3}{2}} \right]_0^a$$

$$= a^2 \int_0^{\frac{\pi}{4}} 2 \sin^2 2\theta d\theta - \frac{2}{3} \sqrt{a} a^{\frac{3}{2}}$$

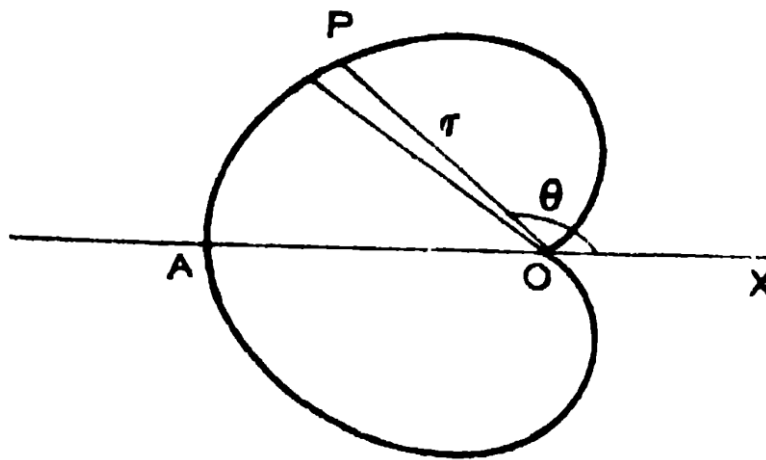
$$= a^2 \int_0^{\frac{\pi}{4}} (1 - \cos 4\theta) d\theta - \frac{2}{3} a^2 = a^2 \left[ \theta - \frac{\sin 4\theta}{4} \right]_0^{\frac{\pi}{4}} - \frac{2}{3} a^2$$

$$= a^2 \frac{\pi}{4} - \frac{2}{3} a^2 = a^2 \left( \frac{\pi}{4} - \frac{2}{3} \right)$$

**Problem 5:** Find the area bounded by the cardioid  $r = a(1 - \cos \theta)$

**Solution:**

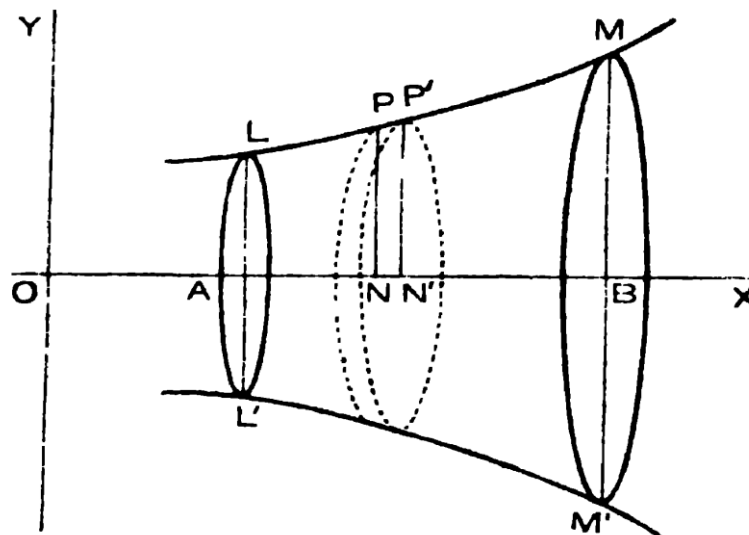




The curve is symmetric about the initial line then the required area is

$$\begin{aligned}
 A &= 2 \int_0^{\pi} \frac{1}{2} r^2 d\theta = \int_0^{\pi} a^2 (1 - \cos \theta)^2 d\theta = a^2 \int_0^{\pi} (1 - 2\cos \theta + \cos^2 \theta) d\theta \\
 &= a^2 [\theta - 2 \sin \theta]_0^{\pi} + \frac{a^2}{2} \int_0^{\pi} (1 + \cos 2\theta) d\theta = a^2 \pi + \frac{a^2}{2} \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\pi} \\
 &= a^2 \pi + \frac{a^2 \pi}{2} = \frac{3a^2 \pi}{2}
 \end{aligned}$$

The solid of revolution, the axis of revolution is x axis:



Let a curve  $LM$  whose Cartesian equation is given  $y = f(x)$  be rotated about the  $x$  axis and form a solid of revolution, let us consider the portion  $LL'M'M$  of this solid bounded by  $x = x_1$  and  $x = x_2$ . Consider a circular slices  $PP'$  with coordinates  $P(x, y)$  and  $P'(x + \Delta x, y + \Delta y)$ .

The volume of the slices with thickness  $\Delta x$  is  $\pi y^2 \Delta x$

Hence total volume of the solid bounded by  $x = x_1$  and  $x = x_2$  is

$$V = \lim_{\Delta x \rightarrow 0} \sum \pi y^2 \Delta x = \pi \int_{x_1}^{x_2} y^2 dx$$

Again the area of the slices is  $2\pi y \Delta s$

Hence total surface area of the solid bounded by  $x = x_1$  and  $x = x_2$  is

$$S = \lim_{\Delta s \rightarrow 0} \sum 2\pi y \Delta s = 2\pi \int_{s_1}^{s_2} y ds = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

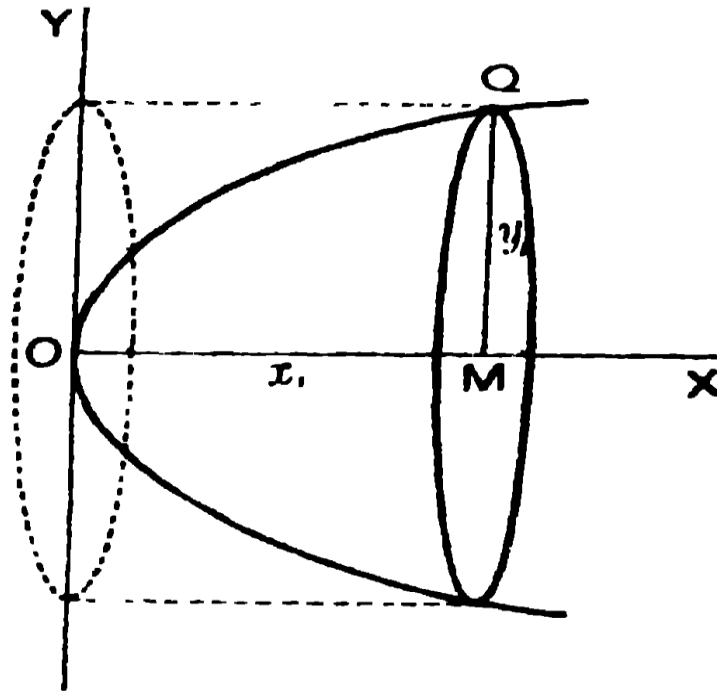
Similarly if the axis of revolution is y axis:

$$V = \pi \int_{y_1}^{y_2} x^2 dy$$

$$S = 2\pi \int_{y_1}^{y_2} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

**Problem 6:** Find the volume and area of the curved surface of a paraboloid of revolution form by revolving the parabola  $y^2 = 4ax$  about the  $x$  axis bounded by  $x = 4$

**Solution:**



Here,

$$y = 2\sqrt{ax}, \frac{dy}{dx} = \sqrt{\frac{a}{x}}$$

The required volume is

$$V = \pi \int_0^4 y^2 dx = \pi \int_0^4 4ax dx = [2ax^2]_0^4 = 32a$$

Also the required surface area is

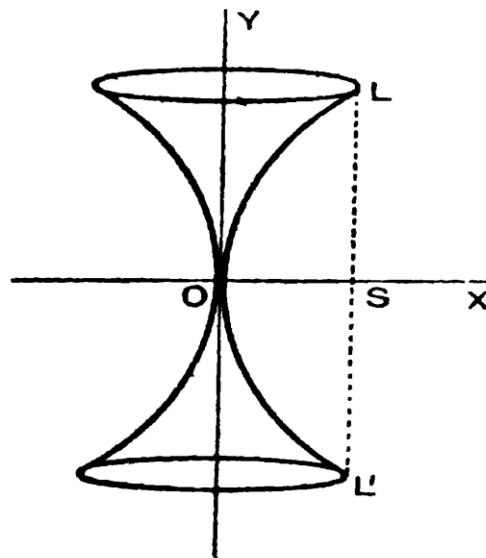
$$S = 2\pi \int_0^4 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_0^4 \sqrt{4ax} \sqrt{1 + \frac{a}{x}} dx = 4\pi\sqrt{a} \int_0^4 \sqrt{a+x} dx$$

$$= 4\pi\sqrt{a} \frac{2}{3} \left[ (a+x)^{\frac{3}{2}} \right]_0^4 = \frac{8\pi}{3} \sqrt{a} \left\{ (a+4)^{\frac{3}{2}} - a^{\frac{3}{2}} \right\}$$

**Problem 7:** Find the volume and area of the curved surface of the reel generated by the parabola  $y^2 = 4ax$  bounded by the latus rectum revolves about the tangent at vertex.

**Solution:**

$\pm 2a$



focus diye jeta jai

Here the axis of revolution being y axis and the extreme values of y being  $\pm 2a$

The volume is

$$V = \pi \int_{-2a}^{2a} x^2 dy = \pi \int_{-2a}^{2a} \frac{y^4}{16a^2} dy = \frac{\pi}{16a^2} \left[ \frac{y^5}{5} \right]_{-2a}^{2a} = \frac{\pi}{16a^2} \left( \frac{32a^5}{5} + \frac{32a^5}{5} \right)$$

$$= \frac{4\pi a^3}{5}$$

The surface is

$$S = 2\pi \int_{-2a}^{2a} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = 2\pi \int_{-2a}^{2a} \frac{y^2}{4a} \sqrt{1 + \frac{y^2}{4a^2}} dy$$

Putting  $y = 2a \tan \theta$ ,  $dy = 2a \sec^2 \theta d\theta$ ;  $y \rightarrow -2a, \theta \rightarrow -\frac{\pi}{4}, y \rightarrow 2a, \theta \rightarrow \frac{\pi}{4}$

$$S = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4a^2 \tan^2 \theta}{4a} \sqrt{1 + \frac{4a^2 \tan^2 \theta}{4a^2}} 2a \sec^2 \theta d\theta = 4\pi a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 \theta \sec^3 \theta d\theta$$

$$= 4\pi a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^5 \theta - \sec^3 \theta) d\theta$$

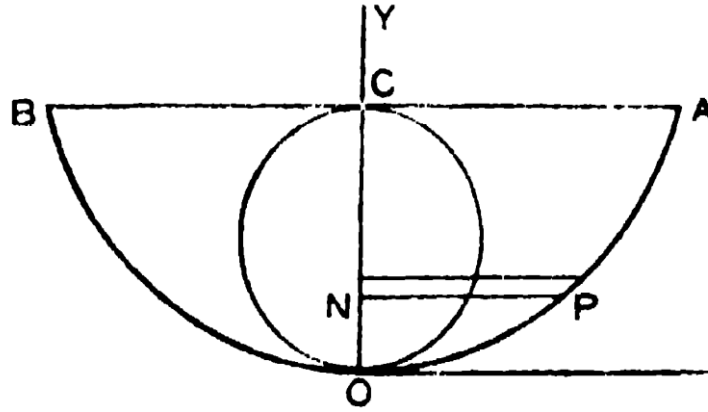
$$= 4\pi a^2 \left[ \frac{\tan \theta \sec^2 \theta}{4} - \frac{\tan \theta \sec \theta}{8} - \frac{1}{8} \log \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

reduction formula

$$= 4\pi a^2 \left[ \frac{3}{4}\sqrt{2} - \frac{1}{4}\log \cot \frac{\pi}{8} \right]$$

**Problem 8:** Find the volume and surface area of the solid generated by revolving the cycloid  $x = a(\theta + \sin \theta)$ ,  $y = a(1 + \cos \theta)$  about its base.

**Solution:**



The extreme values of  $x$  are given by  $\theta = \pm\pi$  i.e.  $x = \pm a\pi$

The required volume

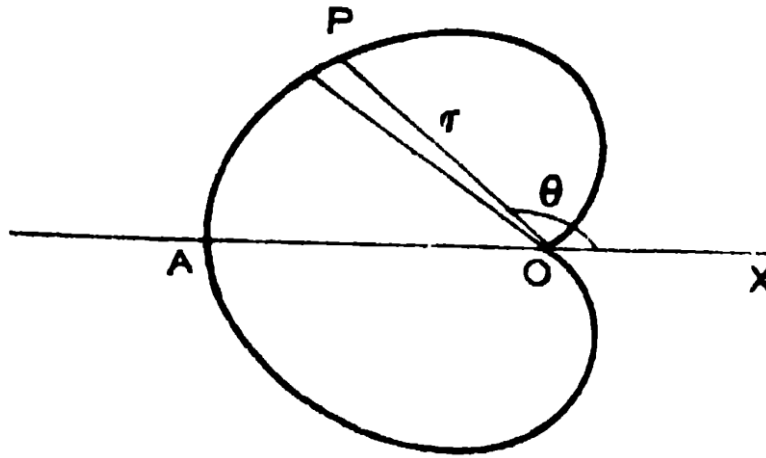
$$\begin{aligned} V &= \int_{-a\pi}^{a\pi} y^2 dx = \pi a^3 \int_{-\pi}^{\pi} (1 + \cos \theta)^3 d\theta = \pi a^3 \int_{-\pi}^{\pi} \left(2 \cos^2 \frac{\theta}{2}\right)^3 d\theta \\ &= 8\pi a^3 \int_{-\pi}^{\pi} \cos^6 \frac{\theta}{2} d\theta = 8\pi a^3 \cdot 2 \int_0^{\pi} \cos^6 \frac{\theta}{2} d\theta = 16\pi a^3 \int_0^{\frac{\pi}{2}} \cos^6 u \cdot 2 du \\ &= 32\pi a^3 \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = 5\pi^2 a^3 \end{aligned}$$

The required surface area

$$\begin{aligned} S &= 2\pi \int y \sqrt{dx^2 + dy^2} = 2\pi \int_{-\pi}^{\pi} a(1 + \cos \theta) \sqrt{\{a(1 + \cos \theta)d\theta\}^2 + \{-a \sin \theta d\theta\}^2} \\ &= 2\pi a^2 \int_{-\pi}^{\pi} (1 + \cos \theta) \sqrt{2(1 + \cos \theta)} d\theta = 8\pi a^2 \int_{-\pi}^{\pi} \cos^3 \frac{\theta}{2} d\theta \\ &= 8\pi a^2 \cdot 2 \int_0^{\frac{\pi}{2}} \cos^3 u \cdot 2 du = 32\pi a^2 \int_0^{\frac{\pi}{2}} \cos^3 u du = 32\pi a^2 \frac{2}{3 \cdot 1} \cdot 1 = \frac{64}{3} \pi a^2 \end{aligned}$$

**Problem 9:** Find the volume and surface area of the solid generated by revolving the cardioid  $r = a(1 - \cos \theta)$  about the initial line.

**Solution:**



The extreme points of curve are given by  $\theta = 0, \pi$

The required volume

$$\begin{aligned} V &= \pi \int y^2 dx = \pi \int_0^\pi r^2 \sin^2 \theta d(r \cos \theta) \\ &= \pi a^3 \int_0^\pi (1 - \cos \theta)^2 \sin^2 \theta d((1 - \cos \theta) \cos \theta) \\ &= \pi a^3 \int_0^\pi (1 - \cos \theta)^2 \sin^2 \theta (-\sin \theta + 2 \cos \theta \sin \theta) d\theta \end{aligned}$$

Putting  $z = \cos \theta, dz = -\sin \theta d\theta; \theta \rightarrow 0, z \rightarrow 1, \theta \rightarrow \pi, z \rightarrow -1$

$$\begin{aligned} &= \pi a^3 \int_1^{-1} (1 - z)^2 (1 - z^2) (1 - 2z) dz \\ &= \pi a^3 \int_1^{-1} (1 - 4z + 4z^2 + 2z^3 - 5z^4 + 2z^5) dz \\ &= \pi a^3 \left[ z - 2z^2 + \frac{4}{3}z^3 + \frac{1}{2}z^4 - z^5 + \frac{1}{3}z^6 \right]_1^{-1} \\ &= \pi a^3 \left[ \left( -1 - 2 - \frac{4}{3} + \frac{1}{2} + 1 + \frac{1}{3} \right) - \left( 1 - 2 + \frac{4}{3} + \frac{1}{2} - 1 + \frac{1}{3} \right) \right] \\ &= \pi a^3 \left( -2 - \frac{8}{3} + 2 \right) = -\frac{8}{3} \pi a^3 = \frac{8}{3} \pi a^3 \end{aligned}$$

The required surface area

$$\begin{aligned} S &= 2\pi \int r \sin \theta \sqrt{dr^2 + r^2 d\theta^2} \\ &= 2\pi \int_0^\pi a(1 - \cos \theta) \sin \theta \sqrt{(a \sin \theta d\theta)^2 + a^2(1 - \cos \theta)^2 d\theta^2} \\ &= 2\pi a^2 \int_0^\pi (1 - \cos \theta) \sin \theta \sqrt{2(1 - \cos \theta)} d\theta \end{aligned}$$

Putting  $z = 1 - \cos \theta, dz = \sin \theta d\theta; \theta \rightarrow 0, z \rightarrow 0, \theta \rightarrow \pi, z \rightarrow 2$

$$\begin{aligned} &= 2\pi a^2 \sqrt{2} \int_0^2 z \sqrt{z} dz = 2\sqrt{2} \pi a^2 \frac{2}{5} \left[ z^{\frac{5}{2}} \right]_0^2 \\ &= \frac{32}{5} \pi a^2 \end{aligned}$$

