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Article *in* IEEE Transactions on Fuzzy Systems · May 2018

DOI: 10.1109/TFUZZ.2018.2838560

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it's a framework about the belief reliability, the main works can referent the contribution in the end of this paper. Generally speaking this paper apply the belief reliability to replace the reliability, while the belief reliability can be measured by chance.And the big difference between chance and probability is that chance theory regards the reliability of system as the probability of the system at the feasible state condition,and define a new formula to calculate the chance value. Besides, if the state variable ξ degenerate to a random variable, the belief reliability metric will be a probability. And if the state variable is a uncertain random variable, it will obey the uncertain distributation. Hence, the uncertainty theory tell us that the product uncertain measure equal the minimum value of all the uncertain measures, which will alleviate the problem of vanishing probability when calculate the product of probability. What's more, an uncertain random system can be simplified to be composed of two types of subsystems – a random subsystem only including random components and an uncertain subsystem only containing uncertain components, and the two types of subsystems will be connected in either series or parallel. The random subsystem obeys the probability framework, while the uncertain subsystem obeys brief reliability.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2018.2838560, IEEE Transactions on Fuzzy Systems

Belief reliability for uncertain random systems

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***Abstract*—Measuring system reliability by a reasonable metric is a basic problem in reliability engineering. Since the real systems are usually uncertain random systems which is affected by both aleatory and epistemic uncertainties, the existed reliability metrics may not work well. This paper aims to develop a general reliability metric, called belief reliability metric, to cope with the problem. In this paper, the belief reliability is defined as the chance that a system state is within a feasible domain. Mathematically, the metric can degenerate to either probability theory-based reliability, which mainly copes with aleatory uncertainty, or uncertainty theory-based reliability, which mainly considers the effect of epistemic uncertainty. Based on the proposed metric, some commonly used belief reliability indexes, such as belief reliability distribution, mean time to failure and belief reliable life, are introduced. We also develop some system belief reliability formulas for different systems configurations. To further illustrate the formulas, a real case study is finally performed in this paper.**

***Index Terms*—Belief reliability, Chance theory, Uncertain random system, Reliability metric.**

definition of reliability

I. Introduction

RELIABILITY is one of the most important properties of systems. It refers to the capability that a component or system can perform a required function for a given period of time under stated operating conditions [1]. In reliability engineering, quantifying reliability by a quantitative metric is a fundamental problem. Only on the basis of a reasonable reliability metric can we better carry out reliability design, reliability analysis and reliability assessments. The key problem for determining a reliability metric is how to cope with uncertainties affecting products. In general, there are two types of uncertainties: aleatory uncertainty caused by inherent randomness of the physical world, and epistemic uncertainty coming from our lack of knowledge about a system [2], [3].

Traditional reliability metrics are based on probability theory. At the very beginning, the reliability of a product is calculated using statistical methods to analyze the product's failure time data [4], [5]. Based on the law of large numbers, the acquisition of this reliability metric requires large samples of failure time data. Since our knowledge about the system is included in the data, reflected as the system-to-system variations of failure times, there is no need to strictly distinguish the two types of uncertainties. However, in the product development process, it is often difficult to collect enough statistical data of the system failure time. This promots the physical model-based reliability metric, where the failure of a system is regarded to be determined by physics-of-failure (PoF) models [6], [7]. Through PoF models, the reliability of a system can be improved by eliminating weak points. In this method, the parameters in the PoF models are usually described as probability distributions, which reflects the effect of aleatory uncertainty. The system reliability can be, then, obtained by propagating the aleatory uncertainty through the PoF models [8]. However, because of our limited information of the products, the process of selecting or establishing a PoF model is often influenced by epistemic uncertainty, and this is especially true in the design of innovative products. Without accounting for the effect of epistemic uncertainty, this metric may overestimate the reliability of a system [9]. Therefore, to better design and improve system reliability, people tend to consider the two types of uncertainties separately.

Considering the effect of epistemic uncertainty, many reliability metrics are proposed based on different mathematical theories, such as evidence theory-based reliability metric [10], [11], interval analysis-based reliability metric [12], [13], fuzzy interval analysis-based reliability metric [14] and posbist reliability metric [15]. Among the four reliability metrics, the first three are all given as reliability intervals, and the last one is defined as a possibility measure. As pointed out by Kang et al. [16], the reliability interval-based metrics may cause interval extension problems when calculating system reliability and posbist reliability does not satisfy duality property which may lead to counter-intuitive results.

For this reason, a new mathematical theory called uncertainty theory is utilized to measure system reliability. Uncertainty theory was founded by Liu [17] in 2007 and refined by Liu [18] in 2010. By introducing the uncertain measure, the uncertain variable, the uncertainty distribution and other concepts, uncertainty theory is viewed as an appropriate mathematical system to model epistemic uncertainty [19], [16]. To simulate the evolution of uncertain phenomena over time, researchers also proposed the tools of uncertain process [20] and uncertain differential equation [21]. After these years of development, uncertainty theory has been applied in various areas, including uncertain finance [22], [23], decision making [24], [25], uncertain control [26], maintenance optimization [27], etc.

In 2010, Liu [28] first described system reliability as an uncertain measure mathematically and proposed a reliability index theorem to calculate the reliability of boolean systems. Later in 2013, Zeng et al. [29] name this reliability metric as *belief reliability*, and interpreted the metric as the belief degree of the system to be reliable. They also clarified the strong need for a new metric in reliability engineering, and proposed a significant belief reliability analysis method for applications. Since the theoretical basis of this reliability metric is uncertainty theory [30], in this paper, we call this metric uncertainty theory-based reliability for convenience.

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Manuscript received December 13, 2017; revised March 26, 2018.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2018.2838560, IEEE Transactions on Fuzzy Systems

Because of the axioms of uncertainty theory, this metric satisfy duality property and will not cause interval extension problems. The uncertainty theory-based reliability metric seems to be more proper to measure system reliability considering epistemic uncertainty. Nowadays, however, real engineering systems are usually consist of different types of components. Some components may suffer great epistemic uncertainty and their reliabilities are modeled by uncertainty theory, while others may be mainly affected by aleatory uncertainty and their reliabilities are measured based on probability theory. We call these kind of real systems, in this paper, as uncertain random systems. Obviously, the reliability of an uncertain random system cannot be analyzed based only on probability theory or only on uncertainty theory.

To address this problem, a chance theory proposed by Liu [31] in 2013 is introduced in this paper. Chance theory can be regarded as a mixture of probability theory and uncertainty theory, and can be utilized to describe systems with both randomness and uncertainty. In chance theory, the uncertain random variable and the chance distribution are two fundamental concepts. To describe uncertain random events over time, uncertain random process was proposed [32], [33]. In recent years, chance theory has developed steadily and applied widely in many fields such as uncertain random risk analysis [34], portfolio optimization [35] and project scheduling [36].

The reliability metric based on chance theory is first introduced by Wen and Kang [37] to measure the reliability of an uncertain random Boolean system. However, the metric has the following shortcomings. First, it does not consider the effect of time, which is a significant factor in reliability engineering. Second, only the reliability of systems with two states is defined, and the multi-state system reliability is not given. Thirdly, they only discuss the system reliability mathematically, and some physical meanings are still insufficient.

To avoid the above shortcomings, in this paper, we aim to expand the connotation of *belief reliability* and develop a general definition of belief reliability metric. The general metric, of course, can degenerate to both traditional probability theory-based reliability metric and uncertainty theory-based belief reliability metric. In addition, for the need of engineering applications, the new reliability metric can be estimated by means of either failure time data, performance margin or system function level. Some system belief reliability formulas are also discussed in this paper.

The remainder of this paper are structured as follows. Section II introduces some mathematical basis of uncertainty theory and chance theory. In Section III, the belief reliability metric is defined and discussed based on chance theory. Some important belief reliability indexes in reliability engineering are defined in Section IV. In Section V, formulas for system belief reliability are given for simple and complex systems, respectively. A real case study about the reliability analysis of an apogee engine is performed to illustrate the formulas. Finally, some conclusions are made in Section VI.

II. Preliminary

In this section, some basic concepts and results of uncertainty theory and chance theory are introduced.

A. Uncertainty theory

Uncertainty theory is a new branch of axiomatic mathematics built on four axioms, i.e., Normality, Duality, Subadditivity and Product Axioms. Founded by Liu [17] in 2007 and refined by Liu [18] in 2010, uncertainty theory has been widely applied as a new tool for modeling subjective (especially human) uncertainties. In uncertainty theory, belief degrees of events are quantified by defining uncertain measures:

Definition II.1 (Uncertain measure [17])**.** Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms,

Axiom 1 (Normality Axiom)**.** *$\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*

Axiom 2 (Duality Axiom)**.** *$\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\} = 1$ for any event $\Lambda \in \mathcal{L}$.*

Axiom 3 (Subadditivity Axiom)**.** *For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

Uncertain measures of product events are calculated following the product axiom [38]:

Axiom 4 (Product Axiom)**.** *Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

the symbol of \Lambda means min{}

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\left\{\Lambda_k\right\},$$

where $\mathcal{L}_k$ are σ-algebras over Γ_k, and Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition II.2 (Uncertain variable [17])**.** An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in \mathcal{B}\}$ is an event for any Borel set $\mathcal{B}$ of real numbers.

Definition II.3 (Independence [38])**.** The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Definition II.4 (Uncertainty distribution [17])**.** The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number x.

An uncertainty distribution Φ is said to be regular if it is a continuous and strictly increasing with respect to x, with $0 < \Phi(x) < 1$, and $\lim_{x\to-\infty} \Phi(x) = 0$, $\lim_{x\to+\infty} \Phi(x) = 1$. A regular uncertainty distribution has an inverse function, which is defined as the inverse uncertainty distribution, denoted by $\Phi^{-1}(\alpha)$, $\alpha \in (0, 1)$. Inverse uncertainty distributions play a central role in uncertainty theory, since the uncertainty distribution of a function of uncertain variables is calculated using the inverse uncertainty distributions:

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Theorem II.1 (Operational law [18]). *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution*

$$\begin{aligned}\Psi^{-1}(\alpha) = & f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \\ & \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).\end{aligned} \tag{II.1}$$

Definition II.5 (Expected value [17]). Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}dx. \tag{II.2}$$

The expected value of an uncertain variable can be also calculated using its uncertainty distribution or inverse uncertainty distribution.

Theorem II.2. [17] *Let ξ be an uncertain variable with an uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx.$$

Theorem II.3. [18] *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha.$$

B. Chance theory

Chance theory is founded by Liu [31], [39] as a mixture of uncertainty theory and probability theory, to deal with problems affected by both aleatory uncertainty (randomness) and epistemic uncertainty. The basic concept in chance theory is the chance measure of an event in a chance space.

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a chance space.

Definition II.6 (chance measure [31]). Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an event. Then the chance measure of Θ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\} \, dx. \tag{II.3}$$

Theorem II.4. [31] *Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space. Then*

$$\mathrm{Ch}\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times \Pr\{A\} \tag{II.4}$$

for any $\Lambda \in \mathcal{L}$ and any $A \in \mathcal{A}$. Especially, we have

$$\mathrm{Ch}\{\varnothing\} = 0, \quad \mathrm{Ch}\{\Gamma \times \Omega\} = 1. \tag{II.5}$$

Definition II.7 (Uncertain random variable [31]). An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B of real numbers.

Random variables and uncertain variables are two special cases of uncertain random variables. If an uncertain random variable $\xi(\gamma, \omega)$ does not vary with γ, it degenerates to a random variable. If an uncertain random variable $\xi(\gamma, \omega)$ does not vary with ω, it degenerates to an uncertain variable.

Example II.1. Let $\eta_1, \eta_2, \cdots, \eta_m$ be random variables and $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. If f is a measurable function, then

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

is an uncertain random variable determined by

$$\begin{aligned}\xi(\gamma, \omega) = & f(\eta_1(\omega), \eta_2(\omega), \cdots, \eta_m(\omega), \\ & \tau_1(\gamma), \tau_2(\gamma), \cdots, \tau_n(\gamma))\end{aligned}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Definition II.8. Let ξ be an uncertain random variable. Then its chance distribution is defined by

$$\Phi(x) = \mathrm{Ch}\{\xi \leq x\} \tag{II.6}$$

for any $x \in \Re$.

Example II.2. As a special uncertain random variable, the chance distribution of a random variable η is just its probability distribution, that is,

$$\Phi(x) = \mathrm{Ch}\{\eta \leq x\} = \Pr\{\eta \leq x\}.$$

Example II.3. As a special uncertain random variable, the chance distribution of an uncertain variable τ is just its uncertainty distribution, that is,

$$\Phi(x) = \mathrm{Ch}\{\tau \leq x\} = \mathcal{M}\{\tau \leq x\}.$$

Theorem II.5. [39] *Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. Assume f is a measurable function. Then the uncertain random variable*

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$$

has a chance distribution

$$\begin{aligned}\Phi(x) = \int_{\Re^m} & F(x; y_1, y_2, \cdots, y_m) \\ & d\Psi_1(y_1)d\Psi_2(y_2)\cdots d\Psi_m(y_m),\end{aligned} \tag{II.7}$$

where $F(x; y_1, y_2, \cdots, y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$.

Definition II.9. [31] Let ξ be an uncertain random variable. Then its expected value is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \geq x\}dx - \int_{-\infty}^0 \mathrm{Ch}\{\xi \leq x\}dx, \tag{II.8}$$

provided that at least one of the two integrals is finite.

Theorem II.6. [31] *Let ξ be an uncertain random variable with chance distribution Φ. Then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx. \tag{II.9}$$

If $\Phi(x)$ is a regular chance distribution, we can calculate the expected value by means of the inverse distribution [31]:

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \tag{II.10}$$

Definition II.10. Let ξ be an uncertain random variable with finite expected value e. Then the variance of ξ is

$$V[\xi] = E[(\xi - e)^2]. \tag{II.11}$$

Since $(\xi - e)^2$ is a nonnegative uncertain random variable, we also have

$$V[\xi] = \int_0^{+\infty} \text{Ch}\{(\xi - e)^2 \geq x\}dx. \tag{II.12}$$

III. Belief reliability metric

In this section, we will introduce the belief reliability metric to measure reliability of uncertain random systems. Usually, we are interested in the state of a system under given conditions. Therefore, we define a state variable which is able to describe the system function or failure behaviors. In an uncertain random system, the states of some components are modeled as random variables and those of other components are described as uncertain variables. Therefore, the state variable of the system will be an uncertain random variable. When the system state variable is within a feasible domain, which reflects our tolerance degree of failure, the system is regarded to be reliable under this state. Based on this, the belief reliability is defined as follows.

Definition III.1 (Belief reliability)**.** Let a system state variable ξ be an uncertain random variable, and Ξ be the feasible domain of the system state. Then the belief reliability is defined as the chance that the system state is within the feasible domain, i.e.,

$$R_B = \text{Ch}\{\xi \in \Xi\}. \tag{III.1}$$

Remark III.1. If the state variable ξ degenerate to a random variable, the belief reliability metric will be a probability. Let $R_B^{(P)}$ denotes the belief reliability under probability theory. Then

$$R_B = R_B^{(P)} = \Pr\{\xi \in \Xi\}. \tag{III.2}$$

This means the system is mainly influenced by aleatory uncertainty, and the belief reliability degenerate to the probability theory-based reliability metric.

Remark III.2. If the state variable ξ degenerate to an uncertain variable, the belief reliability metric will be a belief degree. Let $R_B^{(U)}$ denotes the belief reliability under uncertainty theory. Then

$$R_B = R_B^{(U)} = \mathcal{M}\{\xi \in \Xi\}. \tag{III.3}$$

This means the system is mainly influenced by epistemic uncertainty, and the belief reliability degenerate to the uncertainty theory-based reliability metric.

In Definition III.1, the state variable ξ describes the system behavior, while the feasible domain Ξ is a reflection of failure criteria. In reliability engineering, ξ is a physical quantity that can be measured or predicted through tests, physical models or online monitoring. Ξ is usually described mathematically as a subset of real numbers (eg., an interval) which include the acceptable values of ξ. When the value of ξ falls in Ξ, we say the system is working well, otherwise we say the system fails. For example, ξ may represent the system performance margin and $\xi > 0$ means the system is working, then correspondingly, Ξ will be interval $(0, +\infty)$. To better demonstrate belief reliability metric and the physical meaning of ξ and Ξ, we will offer 3 examples in the following parts. Another point that should be emphasized is that both ξ and Ξ can be relevant to time t, since the system behavior and the failure criteria usually vary with time in practice. Therefore, the belief reliability is usually a function of t, denoted as $R_B(t)$, which is regarded as belief reliability function in this paper.

Example III.1. The state variable can be the system failure time T which describes system failure behaviors. The system is regarded to be reliable at time t if the failure time is greater than t. Thus, the belief reliability of the system at time t can be obtained by letting the feasible domain of T to be $(t, +\infty)$. In this case, the belief reliability will be written as the form with respect to the failure time:

$$R_B(t) = \text{Ch}\{T > t\}. \tag{III.4}$$

If the system is mainly affected by aleatory uncertainty, the system failure time will be modeled as a random variable $T^{(P)}$. The system belief reliability becomes

$$R_B(t) = R_B^{(P)}(t) = \Pr\{T^{(P)} > t\}.$$

Similarly, if the system contains great epistemic uncertainty, the system failure time will be described as an uncertain variable $T^{(U)}$. The system belief reliability becomes

$$R_B(t) = R_B^{(U)}(t) = \mathcal{M}\{T^{(U)} > t\}.$$

Example III.2. The state variable can represent the performance margin m (defined by Zeng et al. [40]) of a system, which describes system function behaviors. m indicates the distance between a performance parameter and the associated failure threshold. A failure will occur if $m < 0$ and $m = 0$ indicates an unstable critical state. Therefore, the system feasible domain, in this case, should be $(0, +\infty)$ and the system belief reliability can be written as the form with respect to the performance margin:

$$R_B = \text{Ch}\{m > 0\}. \tag{III.5}$$

If we consider the degradation process of the performance margin, i.e., the state variable is relevant to t, $R_B(t)$ will be

$$R_B(t) = \text{Ch}\{m(t) > 0\}. \tag{III.6}$$

The failure time T and the performance margin m are the most commonly used system state variable in reliability engineering. Considering the effect of time, the function behavior represented by m will finally convert to the failure behavior described by T. Therefore, the meanings expressed by the two forms of belief reliability metric (III.4) and (III.6) are consistent. Actually, $m(t)$ is an uncertain random process with a threshold level of 0. The first hitting time of $m(t)$ will be

$$t_0 = \inf\{t \geq 0 | m(t) = 0\} \tag{III.7}$$

with a chance distribution of $\Upsilon(t)$. Since t_0 is just the failure time of the product, we have

$$R_B(t) = \text{Ch}\{m(t) > 0\} = 1 - \Upsilon(t) = \text{Ch}\{T > t\}. \tag{III.8}$$

Example III.3. If we consider a multi-state system, the state variable should be the system function level, denoted as G, which describes the behavioral status of a system as it performs its specified function. Assume the system has k different function levels $G = i, i = 0, 1, \cdots, k$ with a lowest acceptable level of $G = s$. Let $G = k$ represent the system functions perfectly, and $G = i, i = s, s+1, \cdots, k-1$ reflect the different degraded working states, then the system belief reliability can be obtained by letting the system feasible domain to be $\{s, s+1, \cdots, k\}$, i.e.,

$$R_B = \text{Ch}\left\{G \in \{s, s+1, \cdots, k\}\right\}. \tag{III.9}$$

The effect of time, of course, can be also considered by assuming the system function level vary with time, that is

$$R_B(t) = \text{Ch}\left\{G(t) \in \{s, s+1, \cdots, k\}\right\}. \tag{III.10}$$

For Eq. (III.9), if the system only has two function levels, namely, complete failure with $G = 0$ and perfectly function with $G = 1$, the system belief reliability will just be the metric proposed by Wen and Kang [37], i.e.,

$$R_B = \text{Ch}\left\{G = 1\right\}. \tag{III.11}$$

Therefore, the reliability metric they developed is exactly a special case of the belief reliability defined in this paper.

IV. Some Belief Reliability Indexes

In this section, some commonly used belief reliability indexes, including belief reliability distribution, belief reliable life, mean time to failure and variance of failure time, are defined based on the belief reliability metric.

Definition IV.1 (Belief Reliability Distribution)**.** Assume that a system state variable ξ is an uncertain random variable, then the chance distribution of ξ, i.e.,

$$\Phi(x) = Ch\{\xi \leq x\}, \tag{IV.1}$$

is defined as the belief reliability distribution.

Example IV.1. If ξ represents the product failure time T, the belief reliability distribution will be the chance distribution of T, denoted as $\Phi(t) = \text{Ch}\{T \leq t\}$. In this case, the sum of $\Phi(t)$ and $R_B(t)$ equals 1, i.e.,

$$\Phi(t) + R_B(t) = 1. \tag{IV.2}$$

Example IV.2. If ξ represents the system performance margin m, the belief reliability distribution will be the chance distribution of m, denoted as $\Phi(x) = \text{Ch}\{m \leq x\}$.

Definition IV.2 (Belief Reliable Life)**.** Assume the system failure time T is an uncertain random variable with a belief reliability function $R_B(t)$. Let α be a real number from $(0, 1)$. The system belief reliable life $T(\alpha)$ is defined as

$$T(\alpha) = \sup\{t | R_B(t) \geq \alpha\}. \tag{IV.3}$$

Example IV.3. BL1 life is defined as

$$t_{BL1} = T(0.99) = \sup\{t | R_B(t) \geq 0.99\},$$

which is one of the most commonly used belief reliable life. It means that the systems have a chance of 0.99 to survive till this time.

Example IV.4. The median time to failure is also commonly used in reliability engineering, which is defined as

$$t_{\text{med}} = T(0.5) = \sup\{t | R_B(t) \geq 0.5\}.$$

Apparently, to identify whether a system has the chance to work till t_{med} is the most difficult.

Definition IV.3 (Mean Time to Failure, MTTF)**.** Assume the system failure time T is an uncertain random variable with a belief reliability function $R_B(t)$. The mean time to failure (MTTF) is defined as

$$\text{MTTF} = E[T] = \int_0^\infty \text{Ch}\{T > t\} dt = \int_0^\infty R_B(t) dt. \tag{IV.4}$$

Theorem IV.1. *Let $R_B(t)$ be a continuous and strictly decreasing function with respect to t at which $0 < R_B(t) < R_B(0) \leq 1$ and $\lim_{t \to +\infty} R_B(t) = 0$. If $T(\alpha)$ is defined by (IV.3), then we have*

$$MTTF = \int_0^1 T(\alpha) d\alpha. \tag{IV.5}$$

Proof. Assume the belief reliability distribution of T is $\Phi(t)$, then we have $R_B(t) = \text{Ch}\{T > t\} = 1 - \text{Ch}\{T < t\} = 1 - \Phi(t)$. Since $R_B(t)$ has inverse function, $\Phi(t)$ has an inverse distribution $\Phi^{-1}(\alpha)$. It follows from (IV.3) that

$$T(\alpha) = \sup\{t | \Phi(t) \leq 1 - \alpha\} = \Phi^{-1}(1 - \alpha). \tag{IV.6}$$

Thus, MTTF can be written as

$$\begin{aligned} \text{MTTF} = E[T] &= \int_0^1 \Phi^{-1}(\alpha) d\alpha \\ &= \int_0^1 \Phi^{-1}(1-\alpha) d\alpha = \int_0^1 T(\alpha) d\alpha. \end{aligned} \tag{IV.7}$$

Definition IV.4 (Variance of failure time, VFT)**.** Assume the system failure time T is an uncertain random variable and the mean time to failure is MTTF. The variance of failure time (VFT) is defined as

$$\text{VFT} = V[T] = E\left[(T - \text{MTTF})^2\right]. \tag{IV.8}$$

Theorem IV.2. *Let the belief reliability function be $R_B(t)$, then the VFT can be calculated by*

$$\begin{aligned} VFT = \int_0^{+\infty} &(R_B(MTTF + \sqrt{t}) + \\ &1 - R_B(MTTF - \sqrt{t})) dt. \end{aligned} \tag{IV.9}$$

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Proof. Since $(T-\text{MTTF})^2$ is a nonnegative uncertain random variable, we have

$$\begin{aligned}\text{VFT} &= \int_0^{+\infty} \text{Ch}\{(T-MTTF)^2 \geq t\}dt \\ &= \int_0^{+\infty} \text{Ch}\{(T \geq MTTF+\sqrt{t}) \cup \\ &\qquad (T \leq MTTF-\sqrt{t})\}dt \\ &\leq \int_0^{+\infty} (\text{Ch}\{T \geq MTTF+\sqrt{t}\}+ \\ &\qquad \text{Ch}\{T \leq MTTF-\sqrt{t}\})dt \\ &= \int_0^{+\infty} (R_B(MTTF+\sqrt{t})+ \\ &\qquad 1-R_B(MTTF-\sqrt{t}))dt.\end{aligned} \tag{IV.10}$$

In this case, we stipulate that VFT takes the maximum value in (IV.10), i.e.,

$$\begin{aligned}\text{VFT} = \int_0^{+\infty} &(R_B(MTTF+\sqrt{t})+ \\ &1-R_B(MTTF-\sqrt{t}))dt.\end{aligned} \tag{IV.11}$$

V. System belief reliability fomulas

This section will propose some system belief reliability formulas. In this paper, we only discuss the situation that the system state variable ξ is failure time. The circumstances that ξ stands for performance margin or function level can be studied similarly.

Here, in an uncertain random system, the components mainly affected by aleatory uncertainty are regarded as random components whose failure times are described as random variables, while those mainly influenced by epistemic uncertainty are called uncertain components with failure times represented by uncertain variables.

A. Belief reliability formula for simple systems

Sometimes, an uncertain random system can be simplified to be composed of two types of subsystems — a random subsystem only including random components and an uncertain subsystem only containing uncertain components, and the two types of subsystems will be connected in either series or parallel. We do not strictly require the configurations inside the two subsystems and they can be very complex. The belief reliability of this kind of systems can be also calculated based on the following theorems.

Theorem V.1. *Assume an uncertain random system is simplified to be composed of a random subsystem with belief reliability $R_{B,R}^{(P)}(t)$ and an uncertain subsystem with belief reliability $R_{B,U}^{(U)}(t)$. If the two subsystems are connected in series, the system belief reliability $R_{B,S}(t)$ will be*

$$R_{B,S}(t) = R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t). \tag{V.1}$$

Proof. Assume the failure times of the random components and uncertain components in the two types of subsystems are $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively. According to the configurations of the subsystems, the failure times of the random subsystem $T_R^{(P)}$ and the uncertain subsystem $T_U^{(U)}$ are determined by $T_R^{(P)} = f(\eta_1, \eta_2, \cdots, \eta_m)$ and $T_U^{(U)} = g(\tau_1, \tau_2, \cdots, \tau_n)$, respectively, where f and g are two measurable functions. Therefore, $T_R^{(P)}$ is a random variable and $T_U^{(U)}$ is an uncertain variable. Since the two subsystems are connected in series, the system failure time can be written as:

$$T = T_R^{(P)} \wedge T_U^{(U)}.$$

Then we have

$$\begin{aligned}R_{B,S}(t) &= \text{Ch}\{T > t\} \\ &= \text{Ch}\{T_R^{(P)} \wedge T_U^{(U)} > t\} \\ &= \text{Ch}\left\{\left(T_R^{(P)} > t\right) \cap \left(T_U^{(U)} > t\right)\right\} \\ &= \text{Pr}\left\{T_R^{(P)} > t\right\} \times \mathcal{M}\left\{T_U^{(U)} > t\right\} \\ &= R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t).\end{aligned} \tag{V.2}$$

Example V.1 (Series system)**.** Consider an uncertain random series system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Then the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with series configurations, and the two subsystems are connected in series. Assume the failure times of random and uncertain components are $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively. Then the belief reliability of the system $R_{B,S}(t)$ can be calculated according to Theorem V.1:

$$\begin{aligned}R_{B,S}(t) &= R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t) \\ &= \text{Pr}\left\{\bigwedge_{i=1}^{m} \eta_i > t\right\} \times \mathcal{M}\left\{\bigwedge_{j=1}^{n} \tau_j > t\right\} \\ &= \text{Pr}\left\{\bigcap_{i=1}^{m} (\eta_i > t)\right\} \times \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j > t)\right\} \\ &= \prod_{i=1}^{m} R_{B,i}^{(P)}(t) \cdot \bigwedge_{j=1}^{n} R_{B,j}^{(U)}(t).\end{aligned} \tag{V.3}$$

Example V.2 (Parallel series system)**.** Consider an uncertain random parallel series system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Suppose the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with parallel configurations, and the two subsystems are connected in series. By assuming the failure times and random and uncertain components to be $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively, the belief reliability of the system $R_{B,S}(t)$ can

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be calculated according to Theorem V.1:

$$\begin{aligned} R_{B,S}(t) &= R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t) \\ &= \Pr\left\{\bigvee_{i=1}^{m} \eta_i > t\right\} \times \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j > t\right\} \\ &= \left(1 - \Pr\left\{\bigvee_{i=1}^{m} \eta_i \le t\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j \le t\right\}\right) \\ &= \left(1 - \Pr\left\{\bigcap_{i=1}^{m} (\eta_i \le t)\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j \le t)\right\}\right) \\ &= \left(1 - \prod_{i=1}^{m} (1 - R_{B,i}^{(P)}(t))\right) \cdot \bigvee_{j=1}^{n} R_{B,j}^{(U)}(t). \end{aligned} \tag{V.4}$$

Theorem V.2. *Assume an uncertain random system is simplified to be composed of a random subsystem with belief reliability $R_{B,R}^{(P)}$ and an uncertain subsystem with belief reliability $R_{B,U}^{(U)}$. If the two subsystems are connected in parallel, the system belief reliability will be*

$$R_{B,S}(t) = 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right). \tag{V.5}$$

Proof. Similar to the proof of Theorem V.1, the failure times of random subsystem $T_R^{(P)}$ and uncertain subsystem $T_U^{(U)}$ are random variable and uncertain variable, respectively. Since the two subsystems are connected in parallel, the system failure time can be written as:

$$T = T_R^{(P)} \vee T_U^{(U)}.$$

Then we have

$$\begin{aligned} R_{B,S}(t) &= \mathrm{Ch}\{T > t\} \\ &= \mathrm{Ch}\{T_R^{(P)} \vee T_U^{(U)} > t\} \\ &= 1 - \mathrm{Ch}\left\{T_R^{(P)} \vee T_U^{(U)} \le t\right\} \\ &= 1 - \mathrm{Ch}\left\{\left(T_R^{(P)} \le t\right) \cap \left(T_U^{(U)} \le t\right)\right\} \\ &= 1 - \Pr\left\{T_R^{(P)} \le t\right\} \times \mathcal{M}\left\{T_U^{(U)} \le t\right\} \\ &= 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right). \end{aligned} \tag{V.6}$$

Example V.3 (Parallel system)**.** Consider an uncertain random parallel system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Then the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with parallel configurations, and the two subsystems are connected in parallel. Assume the failure times of random and uncertain components are $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively. Then the belief reliability of the system $R_{B,S}(t)$ can be calculated according to Theorem V.2:

$$\begin{aligned} R_{B,S}(t) &= 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right) \\ &= 1 - \left(1 - \Pr\left\{\bigvee_{i=1}^{m} \eta_i > t\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j > t\right\}\right) \\ &= 1 - \Pr\left\{\bigvee_{i=1}^{m} \eta_i \le t\right\} \times \mathcal{M}\left\{\bigvee_{j=1}^{n} \tau_j \le t\right\} \\ &= 1 - \Pr\left\{\bigcap_{i=1}^{m} (\eta_i \le t)\right\} \times \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j \le t)\right\} \\ &= 1 - \left(\prod_{i=1}^{m} (1 - R_{B,i}^{(P)}(t))\right) \cdot \left(1 - \bigvee_{j=1}^{n} R_{B,j}^{(U)}(t)\right). \end{aligned} \tag{V.7}$$

Example V.4 (Series parallel system)**.** Consider an uncertain random series parallel system comprising m random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \cdots, m$, and n uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \cdots, n$. Suppose the system can be simplified to be consist of a random subsystem and an uncertain subsystem both with series configurations, and the two subsystems are connected in parallel. By assuming the failure times and random and uncertain components to be $\eta_1, \eta_2, \cdots, \eta_m$ and $\tau_1, \tau_2, \cdots, \tau_n$, respectively, the belief reliability of the system $R_{B,S}(t)$ can be calculated according to Theorem V.2:

$$\begin{aligned} R_{B,S}(t) &= 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right) \\ &= 1 - \left(1 - \Pr\left\{\bigwedge_{i=1}^{m} \eta_i > t\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigwedge_{j=1}^{n} \tau_j > t\right\}\right) \\ &= 1 - \left(1 - \Pr\left\{\bigcap_{i=1}^{m} (\eta_i > t)\right\}\right) \cdot \\ &\qquad \left(1 - \mathcal{M}\left\{\bigcap_{j=1}^{n} (\tau_j > t)\right\}\right) \\ &= 1 - \left(1 - \prod_{i=1}^{m} R_{B,i}^{(P)}(t)\right) \cdot \left(1 - \bigwedge_{j=1}^{n} R_{B,j}^{(U)}(t)\right). \end{aligned} \tag{V.8}$$

B. Belief reliability formula for complex systems

For more complex system, such as an uncertain random k-out-of-n system, it is much harder to obtain the system belief reliability functions directly. In this paper, we assume the system only have two states. Therefore, at first, we do not consider the effect of time, then the uncertain random system

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can be regarded as a Boolean system. In this case, the system reliability formula proposed by Wen and Kang [37] is first adopted. It is noted that the formula can be easily extended to a time-variant situation by performing the formula at each time of the system lifetime.

Theorem V.3. (Wen and Kang[37]) *Assume that a Boolean system has a structure function f and contains random components with belief reliabilities $R^{(P)}_{B,i}(t), i = 1, 2, \cdots, m$ and uncertain components with belief reliabilities $R^{(U)}_{B,j}(t), j = 1, 2, \cdots, n$. Then the belief reliability of the system is*

$$R_{B,S}(t) = \sum_{(y_1,\cdots,y_m)\in\{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(y_i,t)\right) \cdot Z(y_1, y_2, \cdots, y_m, t), \tag{V.9}$$

where

$$Z(y_1, y_2, \cdots, y_m, t) = \begin{cases} \sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=1} \min\limits_{1\le j\le n} \nu_j(z_j,t), \\ \quad \text{if} \sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=1} \min\limits_{1\le j\le n} \nu_j(z_j,t) < 0.5, \\ 1 - \sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=0} \min\limits_{1\le j\le n} \nu_j(z_j,t), \\ \quad \text{if} \sup\limits_{f(y_1,\cdots,y_m,z_1,\cdots,z_n)=1} \min\limits_{1\le j\le n} \nu_j(z_j,t) \ge 0.5, \end{cases}$$

$$\mu_j(y_i,t) = \begin{cases} R^{(P)}_{B,i}(t), & \text{if } y_i = 1, \\ 1 - R^{(P)}_{B,i}(t), & \text{if } y_i = 0, \end{cases} (i = 1, 2, \cdots, m),$$

$$\nu_j(z_j,t) = \begin{cases} R^{(U)}_{B,i}(t), & \text{if } z_j = 1, \\ 1 - R^{(U)}_{B,i}(t), & \text{if } z_j = 0, \end{cases} (j = 1, 2, \cdots, n).$$

C. Case study

Consider an apogee engine of a satellite proposed in [41]. The mission of the engine is to start at apogee and send the satellite into the synchronous orbit. To ensure the the success of the mission, the engine should not fail within the first 2 working hours. The engine is mainly consist of four parts: an ignition structure, an engine shell, a propellant grain and a nozzle. The ignition structure can be further decomposed into three kinds of components, namely, an igniter, two spark plugs (including one back-up spark plug) and some ignition composition. Among the components, the ignition composition and the nozzle are innovative products with few failure data, so they are modeled as uncertain components in this paper. The failure time distributions of the two components are obtained based on experts' empirical data. Since other components are mature product with a lot of failure time data, we model them as random components, whose failure time distributions are obtained through statistical method based on field or experimental failure data.

The working process of the apogee engine can be summarized as follows. First the igniter receives command and generates a pulse, then it ignites the ignition composition through the spark plugs. Later, the propellant grain is burned, generating a lot of gas. The gas will ejected from the nozzle to outside, thereby propel the satellite. The whole process takes place in the engine shell. It can be easily noticed that the engine will fail whenever a sort of component fail. Therefore, the reliability block diagram of the apogee engine can be represented by Fig. 1. The failure time distributions of these components are listed in Table I.

TABLE I
FAILURE TIME DISTRIBUTIONS OF COMPONENTS

No.	Component type	Failure time distribution
1-1,1-2,1-2'	Random	Exponential$(\lambda = 10^{-2.5}h^{-1})$
1-3	Uncertain	$\mathcal{L}(150h, 400h)$
2,3	Random	Exponential$(\lambda = 5 \times 10^{-3.5}h^{-1})$
4	Uncertain	$\mathcal{L}(100h, 500h)$

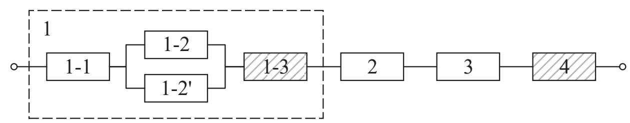

Fig. 1. Reliability block diagram of the apogee engine (1: ignition structure, 2: engine shell, 3: propellant grain, 4: nozzle, a: igniter, b: spark plug, b: back-up plug, c: ignition composition)

By merging the same type of components, we obtain two subsystems: a random subsystem containing all random components 1-1, 1-2, 1-2', 2 and 3, and an uncertain subsystem containing all uncertain components 1-3 and 4. We first calculate the belief reliability of the two subsystems.

$$\begin{aligned} R^{(P)}_{B,R}(t) &= \left[1 - \left(1 - R^{(P)}_{B,1-2}(t)\right)\left(1 - R^{(P)}_{B,1-2'}(t)\right)\right] \cdot \\ &\quad R^{(P)}_{B,1-1}(t) \cdot R^{(P)}_{B,2}(t) \cdot R^{(P)}_{B,3}(t) \\ &= e^{-2\times10^{-2.5}t} \cdot \left(2e^{-10^{-2.5}t} - e^{-2\times10^{-2.5}t}\right) \\ &= 2e^{-3\times10^{-2.5}t} - e^{-4\times10^{-2.5}t}, \end{aligned} \tag{V.10}$$

$$\begin{aligned} R^{(U)}_{B,U}(t) &= R^{(U)}_{B,1-3}(t) \wedge R^{(U)}_{B,4}(t) \\ &= \begin{cases} 1, & \text{if } t \le 100h, \\ \frac{600-t}{500}, & \text{if } 100h < t \le 200h, \\ \frac{400-t}{250}, & \text{if } 200h < t \le 400h, \\ 0, & \text{if } t > 400h. \end{cases} \end{aligned} \tag{V.11}$$

Then the system can be regarded as a series system consisting of a random subsystem and an uncertain subsystem. According to Theorem V.1, we have

$$R_{B,S}(t) = R^{(P)}_{B,R}(t) \cdot R^{(U)}_{B,U}(t). \tag{V.12}$$

The system belief reliability function is illustrated in Fig.2. We can get the system reliability at $2h$ is $R_{B,S}(2h) = 0.9874$. Using numerical methods, the MTTF of the system can be also calculated to be MTTF $= 114.25h$. This means that we can have a good faith that the mission will be successful.

VI. CONCLUSION

In this paper, belief reliability metric is defined based on chance theory to measure reliability of uncertain random

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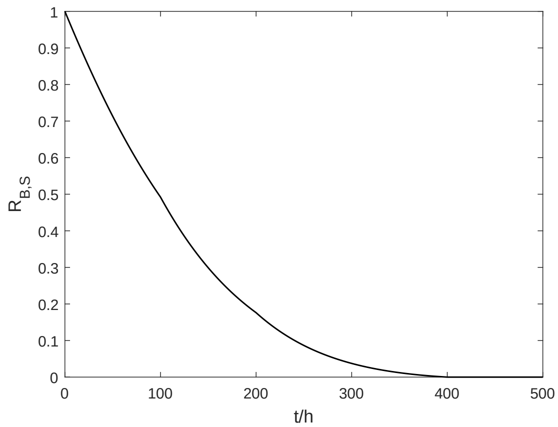

Fig. 2. System belief reliability of the apogee engine

systems affected by both aleatory and epistemic uncertainties. The developed metric can degenerate to the probability theory-based reliability metric or the uncertainty theory-based belief reliability metric. Some belief reliability indexes, including belief reliability distribution, MTTF, belief reliable life, are proposed on the basis of the belief reliability metric. In addition, this paper managed to propose some system belief reliability formulas. For different system configurations, different belief reliability formula can be used. A simple case study is given to further illustrated the proposed metric and formulas.

In conclusion, the contributions of this paper are as follows:

(1) The definition and concept of *belief reliability* is expanded based on chance theory considering both aleatory and epistemic uncertainties.
(2) The belief reliability metric and its physical meaning is first interpreted in detail from the view of failure time, performance margin and function level.
(3) Some new belief reliability indexes are proposed based on the new definition of belief reliability.
(4) Several new system belief reliability formulas are developed to analyze the belief reliability of uncertain random systems.

The future works may focus on the reliability modeling, analysis methods, etc. One of the most interesting issues is to obtain the belief reliability of a product through its physical model of performance margin, as briefly elaborated in this paper. Another important and interesting problem is to give a belief reliability evaluation method of muti-state systems.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (No.61573043 and No.71671009). Besides, we would like to express our gratitude to the associate editor and the anonymous referees for their constructive comments and suggestions, which have significantly improved this paper.

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