

Mathematical Proofs

CHAPTER 2 – EXERCISE SOLUTIONS AND NOTES

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Section 1: Describing a Set

Exercises

1) Which of the following are sets?

- a) $1, 2, 3$ Not a set
- b) $\{1, 2\}, 3$ Not a set
- c) $\{\{1\}, 2\}, 3$ Not a set
- d) $\{1, \{2\}, 3\}$ Set
- e) $\{1, 2, a, b\}$ Set

2) Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x .

- a) $A = \{1, 2, 3\} = \{x \in S : x > 0\}$
- b) $B = \{0, 1, 2, 3\} = \{x \in S : x \geq 0\}$
- c) $C = \{-2, -1\} = \{x \in S : x < 0\}$
- d) $D = \{-2, 2, 3\} = \{x \in S : |x| \geq 2\}$

3) Determine the cardinality of each of the following sets:

- a) $A = \{1, 2, 3, 4, 5\}$ $|A| = 5$
- b) $B = \{0, 2, 4, \dots, 20\}$ $|B| = 11$
- c) $C = \{25, 26, 27, \dots, 75\}$ $|C| = 51$
- d) $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$ $|D| = 2$
- e) $E = \{\emptyset\}$ $|E| = 1$
- f) $F = \{2, \{2, 3, 4\}\}$ $|F| = 2$

4) Write each of the following sets by listing its elements within braces.

- a) $A = \{n \in \mathbb{Z} : -4 < n \leq 4\} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
- b) $B = \{n \in \mathbb{Z} : n^2 < 5\} = \{-2, -1, 0, 1, 2\}$
- c) $C = \{n \in \mathbb{N} : n^3 < 100\} = \{1, 2, 3, 4\}$
- d) $D = \{x \in \mathbb{R} : x^2 - x = 0\} = \{0, 1\}$
- e) $E = \{x \in \mathbb{R} : x^2 + x = 0\} = \{-1, 0\}$

5) Write each of the following sets in the form $\{x \in \mathbb{Z} : p(x)\}$, where $p(x)$ is a property concerning x .

- a) $A = \{-1, -2, -3, \dots\} = \{x \in \mathbb{Z} : x < 0\}$
- b) $B = \{-3, -2, \dots, 3\} = \{x \in \mathbb{Z} : |x| \leq 3\}$
- c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbb{Z} : 0 < |x| \leq 2\}$

6) The set $E = \{2x : x \in \mathbb{Z}\}$ can be described by listing its elements, namely $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$. List the elements of the following sets in a similar manner.

- a) $A = \{2x + 1 : x \in \mathbb{Z}\} = \{\dots, -3, -1, 1, 3, 5, \dots\}$
- b) $B = \{4n : n \in \mathbb{Z}\} = \{\dots, -8, -4, 0, 4, 8, \dots\}$
- c) $C = \{3q + 1 : q \in \mathbb{Z}\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$

- 7) The set $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbb{Z}\} = \{2x : x \in \mathbb{Z}\}$. Describe the following sets in a similar manner.
- $A = \{\dots, -4, -1, 2, 5, 8, \dots\} = \{3x - 1 : x \in \mathbb{Z}\}$
 - $B = \{\dots, -10, -5, 0, 5, 10, \dots\} = \{5x : x \in \mathbb{Z}\}$
 - $C = \{1, 8, 27, 64, 125, \dots\} = \{x^3 : x \in \mathbb{N}\}$
- 8) Let $A = \{n \in \mathbb{Z} : 2 \leq |n| < 4\}$, $B = \{x \in \mathbb{Q} : 2 < x \leq 4\}$, $C = \{x \in \mathbb{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$ and $D = \{x \in \mathbb{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$.
- Describe the set A by listing its elements.
 - $A = \{-2, -3, 2, 3\}$
 - Give an example of three elements that belong to B but do not belong to A.
 - $\frac{5}{2}, \frac{7}{3}, \frac{9}{4}$
 - Describe the set C by listing its elements.
 - $C = \{\sqrt{2}, 2\}$
 - Describe the set D in another manner.
 - $D = \{2\}$
 - Determine the cardinality of the sets A, C and D.
 - $|A| = 4; |C| = 2; |D| = 1$
- 9) For $A = \{2, 3, 5, 7, 8, 10, 13\}$, let $B = \{x \in A : x = y + z, \text{ where } y, z \in A\}$ and $C = \{r \in B : r + s \in B \text{ for some } s \in B\}$. Determine C.
- $B = \{5, 7, 8, 10, 13\}$
 - $C = \{5, 8\}$ (because $5 + 8 = 13$ and $8 + 5 = 13$)

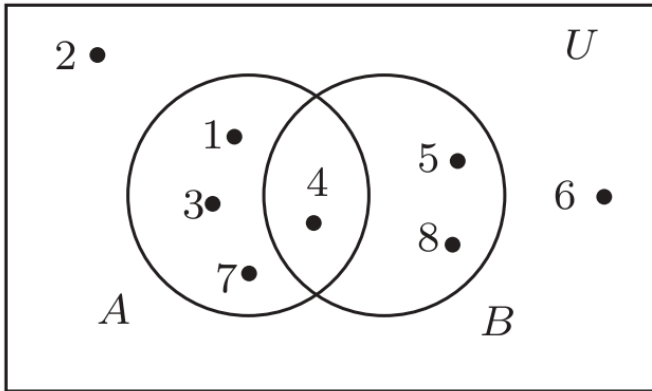
Section 2: Subsets

Exercises

- 10) Give examples of three sets A, B and C such that
- $A \subseteq B \subset C$
 - $A = \{1\}; B = \{1\}; C = \{1, 2\}$
 - $A \in B, B \in C$ and $A \notin C$
 - $A = \{1\}; B = \{\{1\}\}; C = \{\{\{1\}\}\}$
 - $A \in B$ and $A \subset C$
 - $A = \{1\}; B = \{\{1\}\}; C = \{1, 2\}$
- 11) Let (a, b) be an open interval of real numbers and let $c \in (a, b)$. Describe an open interval I centered at c such that $I \subseteq (a, b)$.
- Let $r = \min(c - a, b - c)$, then $I = (c - r, c + r)$
- 12) Which of the following sets are equal?
- $A = \{n \in \mathbb{Z} : |n| < 2\} = \{-1, 0, 1\}$
 - $B = \{n \in \mathbb{Z} : n^3 = n\} = \{-1, 0, 1\}$
 - $C = \{n \in \mathbb{Z} : n^2 \leq n\} = \{0, 1\}$

- d) $D = \{n \in \mathbb{Z} : n^2 \leq 1\} = \{-1, 0, 1\}$
 e) $E = \{-1, 0, 1\}$
 f) Conclusion: The elements in $\{A, B, D, E\}$ are equal and C is on its own.

13) For a universal set $U = \{1, 2, \dots, 8\}$ and two sets $A = \{1, 3, 4, 7\}$ and $B = \{4, 5, 8\}$, draw a Venn diagram that represents these sets



14) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for

- a) $A = \{1, 2\}$;
 i) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$;
 ii) $|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$
 b) $A = \{\emptyset, 1, \{a\}\}$;
 i) $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{1\}, \{\{a\}\}, \{\emptyset, 1\}, \{\emptyset, \{a\}\}, \{1, \{a\}\}, \{\emptyset, 1, \{a\}\}\}$
 ii) $|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8$

15) Find $\mathcal{P}(A)$ for $A = \{0, \{0\}\}$.

- a) $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, \{0, \{0\}\}\} = \{\emptyset, \{0\}, \{\{0\}\}, A\}$

16) Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

- a) $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$
 b) $\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
 c) $|\mathcal{P}(\mathcal{P}(\{1\}))| = 2^{|\mathcal{P}(\{1\})|} = 2^2 = 4$

17) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{0, \emptyset, \{\emptyset\}\}$.

- a) $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$
 b) $|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8$

18) For $A = \{x : x = 0 \text{ or } x \in \mathcal{P}(\{0\})\}$, determine $\mathcal{P}(A)$.

- a) $\mathcal{P}(\{0\}) = \{\emptyset, \{0\}\}$
 b) $A = \{\emptyset, 0, \{0\}\}$
 c) $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{0\}, \{\{0\}\}, \{\emptyset, 0\}, \{\emptyset, \{0\}\}, \{0, \{0\}\}, A\}$

19) Give an example of a set S such that

- a) $S \subseteq \mathcal{P}(\mathbb{N})$
 - i) $S = \emptyset$
- b) $S \in \mathcal{P}(\mathbb{N})$
 - i) $S = \{1\}$
- c) $S \subseteq \mathcal{P}(\mathbb{N})$ and $|S| = 5$
 - i) $S = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$
- d) $S \in \mathcal{P}(\mathbb{N})$ and $|S| = 5$
 - i) $S = \{1, 2, 3, 4, 5\}$

20) Determine whether the following statements are true or false.

- a) If $\{1\} \in \mathcal{P}(A)$, then $1 \in A$ but $\{1\} \notin A$
 - i) **False**, e.g. $A = \{1, \{1\}\}$
- b) If A, B and C are sets such that $A \subset \mathcal{P}(B) \subset C$ and $|A| = 2$, then $|C|$ can be 5 but $|C|$ cannot be 4.
 - i) **True**. If $|A| = 2$, then the cardinality of $\mathcal{P}(B) = 2^2 = 4$. Since $\mathcal{P}(B)$ is a proper subset of C , C must at least have a cardinality of 5.
- c) If a set B has one more element than a set A , then $\mathcal{P}(B)$ has at least two more elements than $\mathcal{P}(A)$.
 - i) **False**, if $A = \emptyset$ then $|\mathcal{P}(A)| = 2^{|\emptyset|} = 2^0 = 1$ and $|\mathcal{P}(B)| = 2^{|\emptyset|+1} = 2^1 = 2$. (It is true if $A \neq \emptyset$)
- d) If four sets A, B, C and D are subsets of $\{1, 2, 3\}$ such that $|A| = |B| = |C| = |D| = 2$, then at least two of these sets are equal.
 - i) **True**. Different combinations of $\{1, 2, 3\}$ with cardinality 2: $\frac{3!}{(3-2)! \cdot 2!} = \frac{3!}{2!} = \frac{3 \cdot 2}{2} = 3$.
Namely $\{1, 2\}, \{1, 3\}$ and $\{2, 3\}$.

21) Three subsets A, B and C of $\{1, 2, 3, 4, 5\}$ have the same cardinality. Furthermore,

- a) 1 belongs to A and B but not to C .
- b) 2 belongs to A and C but not to B .
- c) 3 belongs to A and exactly one of B and C .
- d) 4 belongs to an even number of A, B and C .
- e) 5 belongs to an odd number of A, B and C .
- f) The sums of the elements in two of the sets A, B and C differ by 1.
- g) What is B ?
 - i) $A = \{1, 2, 3\}$
 - ii) $B = \{1, 4, 5\}$
 - iii) $C = \{2, 3, 4\}$