2016

# Mathematical Proofs

## CHAPTER 3 – SETS (EXERCISE SOLUTIONS)

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## Section 1: Describing a Set

### Exercises

1) Which of the following are sets?
   a) 1, 2, 3 Not a set
   b) {1, 2}, 3 Not a set
   c) {{1}, 2}, 3 Not a set
   d) {1, {2,}, 3} Set
   e) {1, 2, a, b} Set

2) Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S: p(x)\}$, where p(x) is some condition on x.
   a) $A = \{1, 2, 3\} = \{x \in S: x > 0\}$
   b) $B = \{0, 1, 2, 3\} = \{x \in S: x \geq 0\}$
   c) $C = \{-2, -1\} = \{x \in S: x < 0\}$
   d) $D = \{-2, 2, 3\} = \{x \in S: |x| \geq 2\}$

3) Determine the cardinality of each of the following sets:
   a) $A = \{1, 2, 3, 4, 5\}$ $|A| = 5$
   b) $B = \{0, 2, 4, \dots, 20\}$ $|B| = 11$
   c) $C = \{25, 26, 27, \dots, 75\}$ $|C| = 51$
   d) $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$ $|D| = 2$
   e) $E = \{\emptyset\}$ $|E| = 1$
   f) $F = \{2, \{2, 3, 4\}\}$ $|F| = 2$

4) Write each of the following sets by listing its elements within braces.
   a) $A = \{n \in \mathbb{Z}: -4 < n \leq 4\} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
   b) $B = \{n \in \mathbb{Z}: n^2 < 5\} = \{-2, -1, 0, 1, 2\}$
   c) $C = \{n \in \mathbb{N}: n^3 < 100\} = \{1, 2, 3, 4\}$
   d) $D = \{x \in \mathbb{R}: x^2 - x = 0\} = \{0, 1\}$
   e) $E = \{x \in \mathbb{R}: x^2 + x = 0\} = \{-1, 0\}$

5) Write each of the following sets in the form $\{x \in \mathbb{Z}: p(x)\}$, where p(x) is a property concerning x.
   a) $A = \{-1, -2, -3, \dots\} = \{x \in \mathbb{Z}: x < 0\}$
   b) $B = \{-3, -2, \dots, 3\} = \{x \in \mathbb{Z}: |x| \leq 3\}$
   c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbb{Z}: 0 < |x| \leq 2\}$

6) The set $E = \{2x : x \in \mathbb{Z}\}$ can be described by listing its elements, namely $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$. List the elements of the following sets in a similar manner.
   a) $A = \{2x + 1 : x \in \mathbb{Z}\} = \{\dots, -3, -1, 1, 3, 5, \dots\}$
   b) $B = \{4n : n \in \mathbb{Z}\} = \{\dots, -8, -4, 0, 4, 8, \dots\}$
   c) $C = \{3q + 1 : q \in \mathbb{Z}\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$

7) The set $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbb{Z}\} = \{2x : x \in \mathbb{Z}\}$. Describe the following sets in a similar manner.
   a) $A = \{\dots, -4, -1, 2, 5, 8, \dots\} = \{3x - 1 : x \in \mathbb{Z}\}$
   b) $B = \{\dots, -10, -5, 0, 5, 10, \dots\} = \{5x : x \in \mathbb{Z}\}$
   c) $C = \{1, 8, 27, 64, 125, \dots\} = (x^3 : x \in \mathbb{N}\}$

8) Let $A = \{n \in \mathbb{Z} : 2 \leq |n| < 4\}, B = \{x \in \mathbb{Q} : 2 < x \leq 4\}, C = \{x \in \mathbb{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$ $and$ $D = \{x \in \mathbb{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0.$
   a) Describe the set A by listing its elements.
      i) $A = \{-2, -3, 2, 3\}$
   b) Give an example of three elements that belong to B but do not belong to A.
      i) $\frac{5}{2}, \frac{7}{3}, \frac{9}{4}$
   c) Describe the set C by listing its elements.
      i) $C = \{\sqrt{2}, 2\}$
   d) Describe the set D in another manner.
      i) $D = \{2\}$
   e) Determine the cardinality of the sets A, C and D.
      i) $|A| = 4; |C| = 2; |D| = 1$

9) For $A = \{2, 3, 5, 7, 8, 10, 13\}$, let $B = \{x \in A : x = y + z, where\ y, z \in A\}$ and $C = \{r \in B : r + s \in B\ for\ some\ s \in B\}$. Determine C.
   a) $B = \{5, 7, 8, 10, 13\}$
   b) $C = \{5, 8\}$ (because $5 + 8 = 13$ $and$ $8 + 5 = 13$)

## Section 2: Subsets

### Exercises

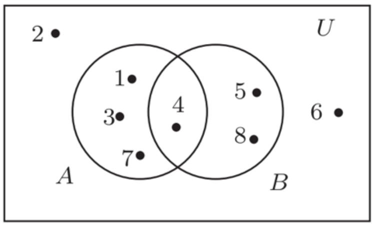
10) Give examples of three sets A, B and C such that
   a) $A \subseteq B \subset C$
      i) $A = \{1\}; B = \{1\}; C = \{1, 2\}$
   b) $A \in B, B \in C$ $and$ $A \notin C$
      i) $A = \{1\};\ B = \{\{1\}\}; C = \{\{\{1\}\}\}$
   c) $A \in B$ $and$ $A \subset C$
      i) $A = \{1\}; B = \{\{1\}\}; C = \{1, 2\}$

11) Let (a, b) be an open interval of real numbers and let $c \in (a, b)$. Describe an open interval I centered at c such that $I \subseteq (a, b)$.
   a) Let $r = \min(c - a, b - c)$, then $I = (c - r, c + r)$

12) Which of the following sets are equal?
   a) $A = \{n \in \mathbb{Z} : |n| < 2\} = \{-1, 0, 1\}$
   b) $B = \{n \in \mathbb{Z} :\ n^3 = n\} = \{-1, 0, 1\}$
   c) $C = \{n \in \mathbb{Z} :\ n^2 \leq n\} = \{0, 1\}$

d) $D = \{n \in \mathbb{Z} : n^2 \leq 1\} = \{-1, 0, 1\}$
e) $E = \{-1, 0, 1\}$
f) Conclusion: The elements in $\{A, B, D, E\}$ are equal and C is on its own.

13) For a universal set $U = \{1, 2, \dots, 8\}$ and two sets $A = \{1, 3, 4, 7\}$ and $B = \{4, 5, 8\}$, draw a Venn diagram that represents these sets




14) Find $\mathcal{P}(A)$ $and$ $|\mathcal{P}(A)|$ for
a) $A = \{1, 2\}$;
   i) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$;
   ii) $|\mathcal{P}(a)| = 2^{|A|} = 2^2 = 4$
b) $A = \{\emptyset, 1, \{a\}\}$;
   i) $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{1\}, \{\{a\}\}, \{\emptyset, 1\}, \{\emptyset, \{a\}\}, \{1, \{a\}\}, \{\emptyset, 1, \{a\}\}\}$
   ii) $|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8$

15) Find $\mathcal{P}(A)$ for $A = \{0, \{0\}\}$.
a) $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, \{0, \{0\}\}\} = \{\emptyset, \{0\}, \{\{0\}\}, A\}$

16) Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.
a) $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$
b) $\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
c) $|\mathcal{P}(\mathcal{P}(\{1\}))| = 2^{|P(\{1\})|} = 2^2 = 4$

17) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{0, \emptyset, \{\emptyset\}\}$.
a) $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$
b) $|\mathcal{P}(a)| = 2^{|A|} = 2^3 = 8$

18) For $A = \{x : x = 0 \; or \; x \in \mathcal{P}(\{0\})\}$, determine $\mathcal{P}(A)$.
a) $\mathcal{P}(\{0\}) = \{\emptyset, \{0\}\}$
b) $A = \{\emptyset, 0, \{0\}\}$
c) $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{0\}, \{\{0\}\}, \{\emptyset, 0\}, \{\emptyset, \{0\}\}, \{0, \{0\}\}, A\}$

19) Give an example of a set S such that
a) $S \subseteq \mathcal{P}(\mathbb{N})$
   i) $S = \emptyset$
b) $S \in \mathcal{P}(\mathbb{N})$
   i) $S = \{1\}$

c) $S \subseteq \mathcal{P}(\mathbb{N})\ and\ |S| = 5$
   i) $S = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$
d) $S \in \mathcal{P}(\mathbb{N})\ and\ |S| = 5$
   i) $S = \{1, 2, 3, 4, 5\}$

20) Determine whether the following statements are true or false.
a) If $\{1\} \in \mathcal{P}(A), then\ 1 \in A\ but\ \{1\} \notin A$
   i) **False**, e.g. $A = \{1, \{1\}\}$
b) If A, B and C are sets such that $A \subset \mathcal{P}(B) \subset C\ and\ |A| = 2$, then $|C|$ can be $5\ but\ |C|\ cannot\ be\ 4$.
   i) **True**. If $|A| = 2$, then the cardinality of $\mathcal{P}(B) = 2^2 = 4$. Since $\mathcal{P}(B)$ is a proper subset of C, C must at least have a cardinality of 5.
c) If a set B has one more element than a set A, then $\mathcal{P}(B)$ has at least two more elements than $\mathcal{P}(A)$.
   i) **False**, if $A = \emptyset$ then $|\mathcal{P}(A)| = 2^{|\emptyset|} = 2^0 = 1$ and $|\mathcal{P}(B)| = 2^{|\emptyset|+1} = 2^1 = 2$. (It is true if $A \neq \emptyset$)
d) If four sets A, B, C and D are subsets of {1, 2, 3} such that $|A| = |B| = |C| = |D| = 2$, then at least two of these sets are equal.
   i) **True**. Different combinations of {1, 2, 3} with cardinality 2: $\frac{3!}{(3-2)!*2!} = \frac{3!}{2!} = \frac{3*2}{2} = 3$. Namely {1, 2}, {1, 3} and {2, 3}.

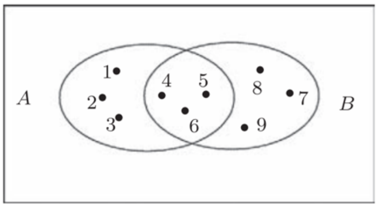
21) Three subsets A, B and C of $\{1, 2, 3, 4, 5\}$ have the same cardinality. Furthermore,
a) 1 belongs to A and B but not to C.
b) 2 belongs to A and C but not to B.
c) 3 belongs to A and exactly one of B and C.
d) 4 belongs to an even number of A, B and C.
e) 5 belongs to an odd number of A, B and C.
f) The sums of the elements in two of the sets A, B and C differ by 1.
g) What is B?
   i) $A = \{1, 2, 3\}$
   ii) $B = \{1, 4, 5\}$
   iii) $C = \{2, 3, 4\}$

## Section 3: Set Operations

### Exercises

22) Let $U = \{1, 3, \dots, 15\}$ be the universal set, $A = \{1, 5, 9, 13\}$, and $B = \{3, 9, 15\}$. Determine the following.
a) $A \cup B = \{1, 3, 5, 9, 13, 15\}$
b) $A \cap B = \{9\}$
c) $A - B = \{1, 5, 13\}$
d) $B - A = \{3, 15\}$
e) $\bar{A} = U - A = \{3, 7, 11, 15\}$
f) $A \cap \bar{B} = A - B = \{1, 5, 13\}$

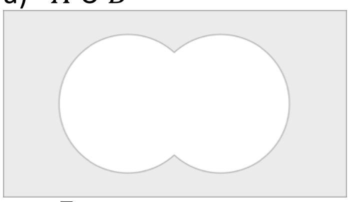
23) Give examples of two sets A and B such that $|A - B| = |A \cap B| = |B - A| = 3$. Draw the accompanying Venn diagram.
   a) $A = \{1, 2, 3, 4, 5, 6\}; B = \{4, 5, 6, 7, 8, 9\}$



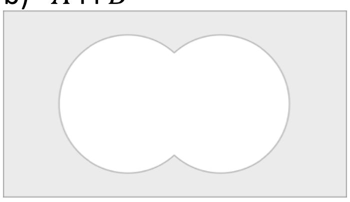

24) Give examples of three sets A, B and C such that $B \neq C$ but $B - A = C - A$
   a) $A = \{1, 2\}$
   b) $B = \{1, 2, 3\}$
   c) $C = \{1, 3\}$

25) Give examples of three sets A, B and C such that
   a) $A \in B, A \subseteq C\ and\ B \nsubseteq C$
      i) $A = \{1\}$
      ii) $B = \{\{1\}\}$
      iii) $C = \{1\}$
   b) $B \in A, B \subset C\ and\ A \cap C \neq \emptyset$
      i) $A = \{\{1\}\}$
      ii) $B = \{1\}$
      iii) $C = \{1, \{1\}\}$
   c) $A \in B, B \subseteq C\ and\ A \nsubseteq C$
      i) $A = \{1\}$
      ii) $B = \{\{1\}\}$
      iii) $C = \{\{1\}\}$

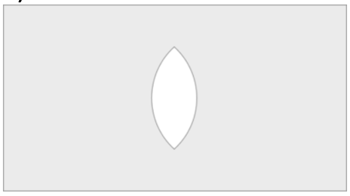
26) Let U be a universal set and let A and B be two subsets of U. Draw a Venn diagram for each of the following sets.
   a) $\overline{A \cup B}$



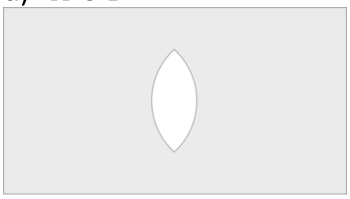
   b) $\bar{A} \cap \bar{B}$



c) $\overline{A \cap B}$

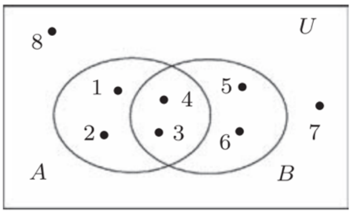


d) $\bar{A} \cup \bar{B}$



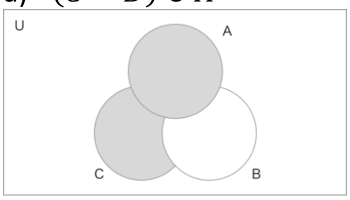
27) Give an example of a universal set U, two sets A and B and accompanying Venn diagram such that $|A \cap B| = |A - B| = |B - A| = \overline{|A \cup B|} = 2$

a) $U = \{1, 2, \ldots, 8\};\ A = \{1, 2, 3, 4\};\ B = \{3, 4, 5, 6\}$

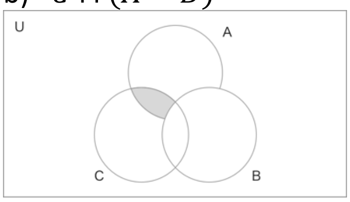



28) Let A, B and C be nonempty subsets of a universal set U. Draw a Venn diagram for each of the following set operations.

a) $(C - B) \cup A$




b) $C \cap (A - B)$




29) Let $A = \left\{\emptyset, \{\emptyset\}, \left\{\{\emptyset\}\right\}\right\}$.

a) Determine which of the following are elements of A: $\emptyset, \{\emptyset\}, \left\{\emptyset, \{\emptyset\}\right\}$
   i) $\emptyset\ and\ \{\emptyset\}$ are elements of A
b) Determine $|A| = 3$
c) Determine which of the following are subsets of A: $\emptyset, \{\emptyset\}, \left\{\emptyset, \{\emptyset\}\right\}$
   i) $\emptyset, \{\emptyset\}\ and\ \left\{\emptyset, \{\emptyset\}\right\}$ are subsets of A

For (d)-(i), determine the indicated sets.

d) $\emptyset \cap A = \emptyset$
e) $\{\emptyset\} \cap A = \{\emptyset\}$
f) $\{\emptyset, \{\emptyset\}\} \cap A = \{\emptyset, \{\emptyset\}\}$
g) $\emptyset \cup A = A$
h) $\{\emptyset\} \cup A = A$
i) $\{\emptyset, \{\emptyset\}\} \cup A = A$

30) Let $A = \{x \in \mathbb{R} : |x - 1| \leq 2\}, B = \{x \in \mathbb{R} : |x| \geq 1\}\ and\ C = \{x \in \mathbb{R} : |x + 2| \leq 3\}$.
a) Express A, B and C using interval notation.
i) $A = \{x \in \mathbb{R} : -1 \leq x \leq 3\} = [-1, 3]$
ii) $B = (-\infty, -1] \cup [1, \infty)$
iii) $C = \{x \in \mathbb{R} : -5 \leq x \leq 1\} = [-5, 1]$

31) Give an example of four different sets A, B, C and D such that (1) $A \cup B = \{1, 2\}$ and $C \cap D = \{2, 3\}$ and (2) if B and C are interchanged and $\cup\ and\ \cap$ are interchanged, then we get the same result ($A \cap C = \{1, 2\}\ and\ B \cup D = \{2, 3\}$.
a) $A = \{1, 2\}$
b) $B = \{2\}$
c) $C = \{1, 2, 3\}$
d) $D = \{2, 3\}$

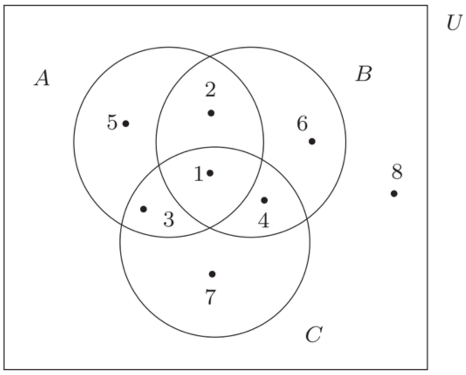
32) Give an example of four different subsets A, B, C and D of {1, 2, 3, 4} such that all intersections of two subsets are different.
a) $A = \{1, 2, 3\}$
b) $B = \{2, 4\}$
c) $C = \{2, 3, 4\}$
d) $D = \{1, 3, 4\}$
i) $A \cap B = B \cap A = \{2\}$
ii) $A \cap C = C \cap A = \{2, 3\}$
iii) $A \cap D = D \cap A = \{1, 3\}$
iv) $B \cap C = C \cap B = \{2, 4\}$
v) $B \cap D = D \cap B = \{4\}$
vi) $C \cap D = D \cap C = \{3, 4\}$

33) Give an example of two nonempty sets A and B such that $\{A \cup B, A \cap B, A - B, B - A\}$ is the power set of some set.
a) $A = \{1\}$
b) $B = \{2\}$
c) $\mathcal{P}(\{\{1,2\}\}) = \{\{1, 2\}, \emptyset, \{1\}, \{2\}\}$

34) Give examples of two subsets A and B of {1, 2, 3} such that all of the following sets are different: $A \cup B, A \cup \bar{B}, \bar{A} \cup B, \bar{A} \cup \bar{B}, A \cap B, A \cap \bar{B}, \bar{A} \cap B, \bar{A} \cap \bar{B}$.
a) $A = \{1, 2\}$
b) $B = \{2, 3\}$
c) Then the different results are: $\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{2\}, \{1\}, \{3\}, \emptyset$

35) Give examples of a universal set U and sets A, B and C such that each of the following sets contains exactly one element: $A \cap B \cap C, (A \cap B) - C, (A \cap C) - B, (B \cap C) - A, A - (B \cup C), B - (A \cup C), C - (A \cup B), \overline{A \cup B \cup C}$. Draw the accompanying Venn diagram.

a) $U = \{1, 2, \dots, 8\}$
b) $A = \{1, 2, 3, 5\}$
c) $B = \{1, 2, 4, 6\}$
d) $C = \{1, 3, 4, 7\}$
e) Then the different results are: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}$.




## Section 4: Indexed Collections of Sets

### Exercises

36) For a real number r, define $S_r$ to be the interval $[r-1, r+2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$.

a) $\bigcup_{\alpha \in A} S_\alpha = [0, 3] \cup [2, 5] \cup [3, 6] = [0, 6]$
b) $\bigcap_{\alpha \in A} S_\alpha = [0, 3] \cap [2, 5] \cap [3, 6] = \{3\}$

37) Let $A = \{1, 2, 5\}, B = \{0, 2, 4\}, C = \{2, 3, 4\}$ $and$ $S = \{A, B, C\}$. Determine $\bigcup_{X \in S} X$ $and$ $\bigcap_{X \in S} X$.

a) $\bigcup_{X \in S} X = \{1, 2, 5\} \cup \{0, 2, 4\} \cup \{2, 3, 4\} = \{0, 1, 2, \dots, 5\}$
b) $\bigcap_{X \in S} X = \{1, 2, 5\} \cap \{0, 2, 4\} \cap \{2, 3, 4\} = \{2\}$

38) For a real number r, define $A_r = \{r^2\}, B_r$ as the closed interval $[r-1, r+1]$ and $C_r$ as the interval $(r, \infty)$. For $S = \{1, 2, 4\}$, determine

a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$
   i) $\bigcup_{\alpha \in S} A_\alpha = \{1^2\} \cup \{2^2\} \cup \{4^2\} = \{1, 4, 16\}$
   ii) $\bigcap_{\alpha \in S} A_\alpha = \emptyset$
b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$
   i) $\bigcup_{\alpha \in S} B_\alpha = [0, 2] \cup [1, 3] \cup [3, 5] = [0, 5]$
   ii) $\bigcap_{\alpha \in S} B_\alpha = \emptyset$
c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$
   i) $\bigcup_{\alpha \in S} C_\alpha = (1, \infty) \cup (2, \infty) \cup (4, \infty) = (1, \infty)$
   ii) $\bigcap_{\alpha \in S} C_\alpha = (4, \infty)$

39) Let $A = \{a, b, \dots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let $A_\alpha$ consist of $\alpha$ and the two letters that follow it, where $A_y = \{y, z, a\}$ and $A_z = \{z, a, b\}$. Find a

set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set S has the required properties.

a) $S = \{a, d, g, j, m, p, s, v, y\}; |S| = 9$

b) $26\ letters\ in\ the\ alphabet\ divided\ by\ 3\ (cardinality\ og\ any\ A_\alpha) = \frac{26}{3} = 8{,}6\ \approx 9$

40) For $i \in \mathbb{Z}, let\ A_i = \{i - 1, i + 1\}$. Determine the following:

a) $\bigcup_{i=1}^{5} A_{2i} = \{1, 3\} \cup \{3, 5\} \cup \ldots \cup \{9, 11\} = \{1, 3, \ldots, 11\}$

b) $\bigcup_{i=1}^{5} (A_i \cap A_{i+1}) = (\{0, 2\} \cap \{1, 3\}) \cup (\{1, 3\} \cap \{2, 4\}) \cup \ldots \cup (\{4, 6\} \cap \{5, 7\}) = \emptyset$

c) $\bigcup_{i=1}^{5} (A_{2i-1} \cap A_{2i+1}) = (\{0, 2\} \cap \{2, 4\}) \cup (\{2, 4\} \cap \{4, 6\}) \cup \ldots \cup (\{8, 10\} \cap \{10, 12\}) = \{2, 4, \ldots, 10\}$

41) For each of the following, find an indexed collection $\{A_n\}_{n \in \mathbb{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.

a) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup_{n=1}^{\infty} A_n = [0,1]$

i) $\{A_n\}_{n \in \mathbb{N}}, where\ A_n = \left\{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{n}\right\} = \left[0, \frac{1}{n}\right]$

b) $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ and $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$

i) $\{A_n\}_{n \in \mathbb{N}}, where\ A_n = \{a \in \mathbb{Z} : |a| \leq n\}$

42) For each of the following collections of sets, define a set $A_n$ for each $n \in \mathbb{N}$ such that the indexed collection $\{A_n\}_{n \in \mathbb{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collections of sets.

a) $\left\{[1, 2 + 1), \left[1, 2 + \frac{1}{2}\right), \left[1, 2 + \frac{1}{3}\right), \ldots\right\}$

i) $\{A_n\}_{n \in \mathbb{N}}$ where $A_n = \left\{x \in \mathbb{R} : 1 \leq x < 2 + \frac{1}{n}\right\} = \left[1, 2 + \frac{1}{n}\right)$

b) $\left\{(-1, 2), \left(-\frac{3}{2}, 4\right), \left(-\frac{5}{3}, 6\right), \left(-\frac{7}{4}, 8\right), \ldots\right\}$

i) $\{A_n\}_{n \in \mathbb{N}}$ where $A_n = \left\{x \in \mathbb{R} : -\frac{2n-1}{n} < x < 2n\right\} = \left(-\frac{2n-1}{n}, 2n\right)$

43) For $r \in \mathbb{R}^+$, let $A_r = \{x \in \mathbb{R} : |x| < r\}$. Determine $\bigcup_{r \in \mathbb{R}^+} A_r$ and $\bigcap_{r \in \mathbb{R}^+} A_r$.

a) $\bigcup_{r \in \mathbb{R}^+} A_r = \mathbb{R}$

b) $\bigcap_{r \in \mathbb{R}^+} A_r = \{0\}$

44) Each of the following sets is a subset of $A = \{1, 2, \ldots, 10\}$: $A_1 = \{1, 5, 7, 9, 10\}, A_2 = \{1, 2, 3, 8, 9\}, A_3 = \{2, 4, 6, 8, 9\}, A_4 = \{2, 4, 8\}, A_5 = \{3, 6, 7\}, A_6 = \{3, 8, 10\}, A_7 = \{4, 5, 7, 9\}, A_8 = \{4, 5, 10\}, A_9 = \{4, 6, 8\}, A_{10} = \{5, 6, 10\}, A_{11} = \{5, 8, 9\}, A_{12} = \{6, 7, 10\}, A_{13} = \{6, 8, 9\}$.

a) Find a set $I \subseteq \{1, 2, \ldots, 13\}$ such that for every two distinct elements $j, k \in I, A_j \cap A_k = \emptyset$ and $|\bigcup_{i \in I} A_i|$ is maximum.

i) $I = \{1, 4\}$

45) For $n \in \mathbb{N}$, let $A_n = \left(-\frac{1}{n}, 2 - \frac{1}{n}\right)$. Determine $\bigcup_{n \in \mathbb{N}} A_n\ and\ \bigcap_{n \in \mathbb{N}} A_n$.

a) $\bigcup_{n \in \mathbb{N}} A_n = (-1, 2)$

b) $\bigcap_{n \in \mathbb{N}} A_n = [0, 1]$

## Section 5: Partitions of Sets

### Exercises

46) Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.
   a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$; **This is a partition of A**
   b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$; This is not a partition, because $\forall s \in S_2, g \notin s$
   c) $S_3 = \{A\}$; **This is a partition of A**
   d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$; This is not a partition because $\emptyset \in S_4$
   e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$; This is not a partition because b is in $\{b, f\}$ and in $\{b, g\}$.

47) Which of the following sets are partitions of $A = \{1, 2, 3, 4, 5\}$?
   a) $S_1 = \{\{1, 3\}, \{2, 5\}\}$; This is not a partition since the element 4 belongs to no element of $S_1$.
   b) $S_2 = \{\{1, 2\}, \{3, 4, 5\}\}$; **This is a partition**
   c) $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$; This is not a partition, since 2, 3 and 4 appear in multiple elements of $S_3$.
   d) $S_4 = A$; This is not a partition, since it isn't a collection of subsets of A, but the set A itself.

48) Let $A = \{1, 2, 3, 4, 5, 6\}$. Give an example of a partition S of A such that $|S| = 3$
   a) $S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$

49) Give an example of a set $A\ with\ |A| = 4$ and two disjoint partitions $S_1$ and $S_2$ of A with $|S_1| = |S_2| = 3$
   a) $A = \{1, 2, 3, 4\}$
   b) $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$
   c) $S_2 = \{\{1, 2\}, \{3\}, \{4\}\}$

50) Give an example of a partition of ℕ into three subsets.
   a) $S = \{\,\{1\}, \{x \in \mathbb{N} : x > 1\ and\ x\ is\ even\}, \{x \in \mathbb{N}: x > 1\ and\ x\ is\ odd\}\}$

51) Give an example of a partition of ℚ into three subsets.
   a) $S = \{\{x \in \mathbb{Q} : x < 0\}, \{0\}, \{x \in \mathbb{Q} : x > 0\}\}$

52) Give an example of three sets $A, S_1, S_2$ such that $S_1$ is a partition of A, $S_2$ is a partition of $S_1$ and $|S_2| < |S_1| < |A|$.
   a) $A = \{1, 2, 3, 4\}$
   b) $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$
   c) $S_2 = \{\{\{1\}, \{2\}\}, \{\{3, 4\}\}\}$
   d) Then $|A| = 4; |S_1| = 3; |S_2 = 2|; and\ |S_2| < |S_1| < |A|$.

53) Give an example of a partition of ℤ into four subsets.
   a) $S = \{A_1, A_2, A_3, A_4\}$
      i) $A_1 = \{x \in \mathbb{Z} : x < 0\ and\ x\ is\ odd\}$
      ii) $A_2 = \{x \in \mathbb{Z} : x < 0\ and\ x\ is\ even\}$

iii) $A_3 = \{x \in \mathbb{Z} : x \geq 0 \; and \; x \; is \; odd\}$
iv) $A_4 = \{x \in \mathbb{Z} : x \geq 0 \; and \; x \; is \; even\}$

54) Let $A = \{1, 2, \dots, 12\}$. Give an example of a partition S of A satisfying the following requirements: (i) $|S| = 5$, (ii) there is a subset T of S such that $|T| = 4 \; and \; |\bigcup_{X \in T} X| = 10$ and (iii) there is no element $B \in S$ such that $|B| = 3$.
a) $S = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$
b) $T = \{\{1, 2, 3, 4\}, \{5,6\}, \{7, 8\}, \{9, 10\}\}$
c) $\bigcup_{X \in T} X = \{1, 2, 3, \dots, 10\}$

55) A set S is partitioned into two subsets $S_1 \; and \; S_2$. This produces a partition $\mathcal{P}_1$ of S where $\mathcal{P}_1 = \{S_1, S_2\}$ and so $|\mathcal{P}_1| = 2$. One of the sets in $\mathcal{P}_1$ is then partitioned into two subsets, producing a partition $\mathcal{P}_2$ of S with $|\mathcal{P}_2| = 3$. A total of $|\mathcal{P}_1|$ sets in $\mathcal{P}_2$ are partitioned into two new subsets each, producing a partition $\mathcal{P}_3$ of S. Next, a total of $|\mathcal{P}_2|$ sets in $\mathcal{P}_3$ are partitioned into two new subsets, each producing a partition $\mathcal{P}_4$ of S. This is continued until partition $\mathcal{P}_6$ of S. What is $|\mathcal{P}_6|$?
a) $|\mathcal{P}_1| = 2$
b) $|\mathcal{P}_2| = 3$
c) $|\mathcal{P}_3| = 5$ (since $|\mathcal{P}_1|$ subsets are partitioned into two new subsets each)
d) $|\mathcal{P}_4| = 8$ (since $|\mathcal{P}_2|$ subsets are partitioned into two new subsets each)
e) $|\mathcal{P}_5| = 13$ (since $|\mathcal{P}_3|$ subsets are partitioned into two new subsets each)
f) $\underline{|\mathcal{P}_6| = 21}$ (since $|\mathcal{P}_4|$ subsets are partitioned into two new subsets each)

56) We mentioned that there are three ways that a collection S of subsets of a nonempty set A is defined to be a partition of A. **Definition 1**: The collection S consists of pairwise disjoint nonempty subsets of A and every element of A belongs to a subset in S**. Definition 2**: The collection S consists of nonempty subsets of A and every element of A belongs to exactly one subset in S. **Definition 3**: The collection S consists of subsets of A satisfying the three properties (1) every subset in S is nonempty, (2) every two subsets of A are equal or disjoint and (3) the union of all subsets in S is A.
a) Show that any collection S of subsets of A satisfying Definition 1 satisfies Definition 2.
   i) In definition 1, the subsets of A in S are pairwise disjoint and every element of A belongs to a subset in S.
   ii) Since the subsets are pairwise disjoint, each element of A is contained in only one subset of S.
   iii) This is the same as saying, each element of A belongs to exactly one subset in S, which is the premise of definition 2.
b) Show that any collection S of subsets of A satisfying Definition 2 satisfies Definition 3.
   i) In definition 3, the union of all subsets in S is A. Thus any element x$\in A$ can be found in a subset of S.
   ii) Definition 3 also states that the subsets of S must be pairwise disjoint, which together with previous statement implies that and every element of A belongs to exactly one subset in S.
c) Show that any collection S of subsets of A satisfying Definition 3 satisfies Definition 1.

i) In definition 3, the union of all subsets in S is A. Thus any element x$\in A$ can be found in a subset of S. This is the same as stated in definition 1: every element of A belongs to a subset in S.
ii) Definition 1 also states: the collection S consists of pairwise disjoint nonempty subsets of A. This is the same as in definition 3: every subset in S is nonempty and every two subsets of A are equal or disjoint.

## Section 6: Cartesian Products of Sets

### Exercises

57) Let $A = \{x, y, z\}$ and $B = \{x, y\}$. Determine $A \times B$.
   a) $A \times B = \{(x,x),(x,y),(y,x),(y,y),(z,x),(z,y)\}$

58) Let $A = \{1, \{1\}, \{\{1\}\}\}$. Determine $A \times A$.
   a) $A \times A =$
   $\{(1,1),(1,\{1\}),(1,\{\{1\}\}),(\{1\},1),(\{1\},\{1\}),(\{1\},\{\{1\}\}),(\{\{1\}\},1),(\{\{1\}\},\{1\}),(\{\{1\}\},\{\{1\}\})\}$

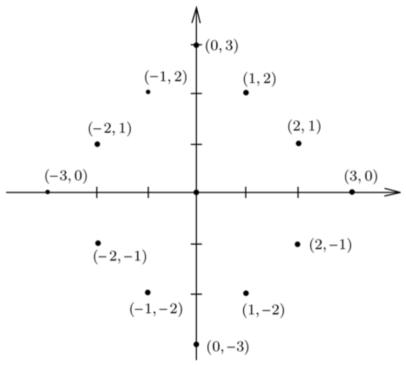
59) For $A = \{a, b\}$, determine $A \times \mathcal{P}(A)$.
   a) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$
   b) $A \times \mathcal{P}(A) = \{(a,\emptyset),(a,\{a\}),(a,\{b\}),(a,\{a,b\}),(b,\emptyset),(b,\{a\}),(b,\{b\}),(b,\{a,b\})\}$

60) For $A = \{\emptyset, \{\emptyset\}\}$, determine $A \times \mathcal{P}(A)$.
   a) $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
   b) $A \times \mathcal{P}(A) =$
   $\{(\emptyset,\emptyset),(\emptyset,\{\emptyset\}),(\emptyset,\{\{\emptyset\}\}),(\emptyset,\{\emptyset,\{\emptyset\}\}),(\{\emptyset\},\emptyset),(\{\emptyset\},\{\emptyset\}),(\{\emptyset\},\{\{\emptyset\}\}),(\{\emptyset\},\{\emptyset,\{\emptyset\}\})\}$

61) For $A = \{1,2\}$ and $B = \{\emptyset\}$, determine $A \times B$ and $\mathcal{P}(A) \times \mathcal{P}(B)$.
   a) $A \times B = \{(1,\emptyset),(2,\emptyset)\}$
   b) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
   c) $\mathcal{P}(B) = \{\emptyset, \{\emptyset\}\}$
   d) $\mathcal{P}(A) \times \mathcal{P}(B) =$
   $\{(\emptyset,\emptyset),(\emptyset,\{\emptyset\}),(\{1\},\emptyset),(\{1\},\{\emptyset\}),(\{2\},\emptyset),(\{2\},\{\emptyset\}),(\{1,2\},\emptyset),(\{1,2\},\{\emptyset\})\}$

62) Describe the graph of the circle whose equation is $x^2 + y^2 = 4$ as a subset of $\mathbb{R} \times \mathbb{R}$.
   a) $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 4\}$

63) For the elements of the set $S = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : |x| + |y| = 3\}$. Plot the corresponding points in the Euclidean xy-plane.
   a) $S = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : |x| + |y| = 3\} = \{(-3,0),(-2,-1),(-2,1),(-1,-2),(-1,2) \ldots\}$




64) For $A = \{1, 2\}$ and $B = \{1\}$, determine $\mathcal{P}(A\times B)$.
   a) $A\times B = \{(1,1), (2,1)\}$
   b) $\mathcal{P}(A\times B) = \{\emptyset, \{(1,1)\}, \{(2,1)\}, \{(1,1), (2,1)\}\}$

65) For $A = \{x \in \mathbb{R} : |x - 1| \le 2\}$ and $B = \{y \in \mathbb{R} : |y - 4| \le 2\}$, give a geometric description of the points in the xy-plane belonging to $A\times B$.
   a) $A\times B = [-1, 3]\times[2, 6]$ which is the set of all points within the square bounded by $x = -1, x = 3, y = 2$ and $y = 6$.

66) For $A = \{a \in \mathbb{R} : |a| \le 1\}$ and $B = \{b \in \mathbb{R} : |b| = 1\}$, give a geometric description of the points in the xy-plane belonging to $(A\times B) \cup (B\times A)$.
   a) $(A\times B) \cup (B\times A) = ([-1, 1]\times\{-1, 1\}) \cup (\{-1, 1\}\times[-1,1])$, which is the set of all points outlining the sides of the square bounded by $x = -1, x = 1, y = -1\ and\ y = 1$.

## Additional Exercises

67) The set $T = \{2k + 1 : k \in \mathbb{Z}\}$ can be described as $T = \{\dots, -3, -1, 1, 3, \dots\}$. Describe the following sets in a similar manner.
   a) $A = \{4k + 3 : k \in \mathbb{Z}\} = \{\dots, -5, -1, 3, 7, \dots\}$
   b) $B = \{5k - 1 : k \in \mathbb{Z}\} = \{\dots, -6, -1, 4, 9, \dots\}$

68) Let $S = \{-10, -9, \dots, 9, 10\}$. Describe each of the following sets as $\{x \in S : p(x)\}$ where p(x) is some condition on x.
   a) $A = \{-10, -9, \dots, -1, 1, \dots, 9, 10\} = \{x \in S : x \ne 0\}$
   b) $B = \{-10, -9, \dots, -1, 0\} = \{x \in S : x \le 0\}$
   c) $C = \{-5, -4, \dots, 0, 1, \dots, 7\} = \{x \in S : -5 \le x \le 7\}$
   d) $D = \{-10, -9, \dots, 4, 6, 7, \dots, 10\} = \{x \in S : x \ne 5\}$

69) Describe each of the following sets by listing its elements within braces.
   a) $\{x \in \mathbb{Z} : x^3 - 4x = 0\} = \{-2, 0, 2\}$
   b) $\{x \in \mathbb{R} : |x| = -1\} = \emptyset$
   c) $\{m \in \mathbb{N} : 2 < m \le 5\} = \{3, 4, 5\}$
   d) $\{n \in \mathbb{N} : 0 \le n \le 3\} = \{1, 2, 3\}$
   e) $\{k \in \mathbb{Q} : k^2 - 4 = 0\} = \{-2, 2\}$
   f) $\{k \in \mathbb{Z} : 9k^2 - 3 = 0\} = \emptyset$

g) $\{k \in \mathbb{Z} : 1 \leq k^2 \leq 10\} = \{-3, -2, -1, 1, 2, 3\}$

70) Determine the cardinality of each of the following sets.
a) $A = \{1, 2, 3, \{1, 2, 3\}, 4, \{4\}\}$ $|A| = 6$
b) $B = \{x \in \mathbb{R} : |x| = -1\} = \emptyset$ $|B| = 0$
c) $C = \{m \in \mathbb{N} : 2 < m \leq 5\} = \{3, 4, 5\}$ $|C| = 3$
d) $D = \{n \in \mathbb{N} : n < 0\} = \emptyset$ $|D| = 0$
e) $E = \{k \in \mathbb{N} : 1 \leq k^2 \leq 100\} = \{1, 2, \dots, 10\}$ $|E| = 10$
f) $F = \{k \in \mathbb{Z} : 1 \leq k^2 \leq 100\} = \{-10, -9, \dots, -1, 1, \dots, 9, 10\}$ $|F| = 20$

71) For $A = \{-1, 0, 1\}$ and $B = \{x, y\}$, determine $A \times B$.
a) $A \times B = \{(-1, x), (-1, y), (0, x), (0, y), (1, x), (1, y)\}$

72) Let $U = \{1, 2, 3\}$ be the universal set and let $A = \{1, 2\}, B = \{2, 3\}$ and $C = \{1, 3\}$. Determine the following.
a) $(A \cup B) - (B \cap C) = U - \{3\} = A$
b) $\bar{A} = U - A = \{3\}$
c) $\overline{B \cup C} = U - (B \cup C) = U - U = \emptyset$
d) $A \times B = \{(1, 2), (1, 3), (2,2), (2,3)\}$

73) Let $A = \{1, 2, \dots, 10\}$. Give an example of two sets S and B such that $S \subseteq \mathcal{P}(A), |S| = 4, B \in S \text{ and } |B| = 2$.
a) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \dots\}$
b) $S = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$ $|S| = 4 \text{ and } S \subseteq \mathcal{P}(A)$
c) $B = \{1, 2\}$ $|B| = 2 \text{ and } B \in S$

74) For $A = \{1\}$ and $C = \{1,2\}$, give an example of a set B such that $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$.
a) $\mathcal{P}(A) = \{\emptyset, \{1\}\}$
b) $\mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
c) $B = \{\emptyset, \{1\}, \{2\}\}$

75) Give examples of two sets A and B such that $A \cap \mathcal{P}(A) \in B$ and $\mathcal{P}(A) \subseteq A \cup B$.
a) $A = \{1\}$ $\mathcal{P}(A) = \{\emptyset, \{1\}\}$ $A \cap \mathcal{P}(A) = \emptyset$
b) $B = \{\emptyset, \{1\}\}$ $A \cup B = \{\emptyset, 1, \{1\}\}$