2016

Mathematical Proofs

CHAPTER 3 – SETS (EXERCISE SOLUTIONS)

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Section 1: Describing a Set

Exercises

1) Which of the following are sets?
   a) 1, 2, 3 Not a set
   b) {1, 2}, 3 Not a set
   c) {{1}, 2}, 3 Not a set
   d) {1, {2,}, 3} Set
   e) {1, 2, a, b} Set

2) Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S: p(x)\}$, where p(x) is some condition on x.
   a) $A = \{1, 2, 3\} = \{x \in S: x > 0\}$
   b) $B = \{0, 1, 2, 3\} = \{x \in S: x \geq 0\}$
   c) $C = \{-2, -1\} = \{x \in S: x < 0\}$
   d) $D = \{-2, 2, 3\} = \{x \in S: |x| \geq 2\}$

3) Determine the cardinality of each of the following sets:
   a) $A = \{1, 2, 3, 4, 5\}$ $|A| = 5$
   b) $B = \{0, 2, 4, \dots, 20\}$ $|B| = 11$
   c) $C = \{25, 26, 27, \dots, 75\}$ $|C| = 51$
   d) $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$ $|D| = 2$
   e) $E = \{\emptyset\}$ $|E| = 1$
   f) $F = \{2, \{2, 3, 4\}\}$ $|F| = 2$

4) Write each of the following sets by listing its elements within braces.
   a) $A = \{n \in \mathbb{Z}: -4 < n \leq 4\} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
   b) $B = \{n \in \mathbb{Z}: n^2 < 5\} = \{-2, -1, 0, 1, 2\}$
   c) $C = \{n \in \mathbb{N}: n^3 < 100\} = \{1, 2, 3, 4\}$
   d) $D = \{x \in \mathbb{R}: x^2 - x = 0\} = \{0, 1\}$
   e) $E = \{x \in \mathbb{R}: x^2 + x = 0\} = \{-1, 0\}$

5) Write each of the following sets in the form $\{x \in \mathbb{Z}: p(x)\}$, where p(x) is a property concerning x.
   a) $A = \{-1, -2, -3, \dots\} = \{x \in \mathbb{Z}: x < 0\}$
   b) $B = \{-3, -2, \dots, 3\} = \{x \in \mathbb{Z}: |x| \leq 3\}$
   c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbb{Z}: 0 < |x| \leq 2\}$

6) The set $E = \{2x : x \in \mathbb{Z}\}$ can be described by listing its elements, namely $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$. List the elements of the following sets in a similar manner.
   a) $A = \{2x + 1 : x \in \mathbb{Z}\} = \{\dots, -3, -1, 1, 3, 5, \dots\}$
   b) $B = \{4n : n \in \mathbb{Z}\} = \{\dots, -8, -4, 0, 4, 8, \dots\}$
   c) $C = \{3q + 1 : q \in \mathbb{Z}\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$

7) The set $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbb{Z}\} = \{2x : x \in \mathbb{Z}\}$. Describe the following sets in a similar manner.
   a) $A = \{\dots, -4, -1, 2, 5, 8, \dots\} = \{3x - 1 : x \in \mathbb{Z}\}$
   b) $B = \{\dots, -10, -5, 0, 5, 10, \dots\} = \{5x : x \in \mathbb{Z}\}$
   c) $C = \{1, 8, 27, 64, 125, \dots\} = (x^3 : x \in \mathbb{N}\}$

8) Let $A = \{n \in \mathbb{Z} : 2 \le |n| < 4\}, B = \{x \in \mathbb{Q} : 2 < x \le 4\}, C = \{x \in \mathbb{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$ $and\ D = \{x \in \mathbb{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0.$
   a) Describe the set A by listing its elements.
      i) $A = \{-2, -3, 2, 3\}$
   b) Give an example of three elements that belong to B but do not belong to A.
      i) $\frac{5}{2}, \frac{7}{3}, \frac{9}{4}$
   c) Describe the set C by listing its elements.
      i) $C = \{\sqrt{2}, 2\}$
   d) Describe the set D in another manner.
      i) $D = \{2\}$
   e) Determine the cardinality of the sets A, C and D.
      i) $|A| = 4; |C| = 2; |D| = 1$

9) For $A = \{2, 3, 5, 7, 8, 10, 13\}$, let $B = \{x \in A : x = y + z, where\ y, z \in A\}$ and $C = \{r \in B : r + s \in B\ for\ some\ s \in B\}$. Determine C.
   a) $B = \{5, 7, 8, 10, 13\}$
   b) $C = \{5, 8\}$ (because $5 + 8 = 13\ and\ 8 + 5 = 13$)

Section 2: Subsets

Exercises

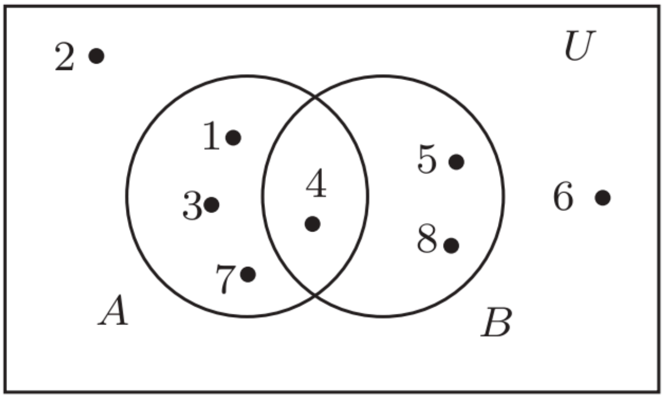
10) Give examples of three sets A, B and C such that
   a) $A \subseteq B \subset C$
      i) $A = \{1\}; B = \{1\}; C = \{1, 2\}$
   b) $A \in B, B \in C\ and\ A \notin C$
      i) $A = \{1\};\ B = \{\{1\}\}; C = \{\{\{1\}\}\}$
   c) $A \in B\ and\ A \subset C$
      i) $A = \{1\}; B = \{\{1\}\}; C = \{1, 2\}$

11) Let (a, b) be an open interval of real numbers and let $c \in (a, b)$. Describe an open interval I centered at c such that $I \subseteq (a, b)$.
   a) Let $r = \min(c - a, b - c)$, then $I = (c - r, c + r)$

12) Which of the following sets are equal?
   a) $A = \{n \in \mathbb{Z} : |n| < 2\} = \{-1, 0, 1\}$
   b) $B = \{n \in \mathbb{Z} :\ n^3 = n\} = \{-1, 0, 1\}$
   c) $C = \{n \in \mathbb{Z} :\ n^2 \le n\} = \{0, 1\}$

d) $D = \{n \in \mathbb{Z} : n^2 \leq 1\} = \{-1, 0, 1\}$
e) $E = \{-1, 0, 1\}$
f) Conclusion: The elements in $\{A, B, D, E\}$ are equal and C is on its own.

13) For a universal set $U = \{1, 2, \dots, 8\}$ and two sets $A = \{1, 3, 4, 7\}$ and $B = \{4, 5, 8\}$, draw a Venn diagram that represents these sets



14) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for
a) $A = \{1, 2\}$;
 i) $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$;
 ii) $|\mathcal{P}(a)| = 2^{|A|} = 2^2 = 4$
b) $A = \{\varnothing, 1, \{a\}\}$;
 i) $\mathcal{P}(A) = \{\varnothing, \{\varnothing\}, \{1\}, \{\{a\}\}, \{\varnothing, 1\}, \{\varnothing, \{a\}\}, \{1, \{a\}\}, \{\varnothing, 1, \{a\}\}\}$
 ii) $|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8$

15) Find $\mathcal{P}(A)$ for $A = \{0, \{0\}\}$.
a) $\mathcal{P}(A) = \{\varnothing, \{0\}, \{\{0\}\}, \{0, \{0\}\}\} = \{\varnothing, \{0\}, \{\{0\}\}, A\}$

16) Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.
a) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$
b) $\mathcal{P}(\mathcal{P}(\{1\})) = \{\varnothing, \{\varnothing\}, \{\{1\}\}, \{\varnothing, \{1\}\}\}$
c) $|\mathcal{P}(\mathcal{P}(\{1\}))| = 2^{|P(\{1\})|} = 2^2 = 4$

17) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{0, \varnothing, \{\varnothing\}\}$.
a) $\mathcal{P}(A) = \{\varnothing, \{0\}, \{\varnothing\}, \{\{\varnothing\}\}, \{0, \varnothing\}, \{0, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}, A\}$
b) $|\mathcal{P}(a)| = 2^{|A|} = 2^3 = 8$

18) For $A = \{x : x = 0 \ or \ x \in \mathcal{P}(\{0\})\}$, determine $\mathcal{P}(A)$.
a) $\mathcal{P}(\{0\}) = \{\varnothing, \{0\}\}$
b) $A = \{\varnothing, 0, \{0\}\}$
c) $\mathcal{P}(A) = \{\varnothing, \{\varnothing\}, \{0\}, \{\{0\}\}, \{\varnothing, 0\}, \{\varnothing, \{0\}\}, \{0, \{0\}\}, A\}$

19) Give an example of a set S such that
   a) $S \subseteq \mathcal{P}(\mathbb{N})$
      i) $S = \emptyset$
   b) $S \in \mathcal{P}(\mathbb{N})$
      i) $S = \{1\}$
   c) $S \subseteq \mathcal{P}(\mathbb{N})\ and\ |S| = 5$
      i) $S = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}$
   d) $S \in \mathcal{P}(\mathbb{N})\ and\ |S| = 5$
      i) $S = \{1,2,3,4,5\}$

20) Determine whether the following statements are true or false.
   a) If $\{1\} \in \mathcal{P}(A), then\ 1 \in A\ but\ \{1\} \notin A$
      i) **False**, e.g. $A = \{1, \{1\}\}$
   b) If A, B and C are sets such that $A \subset \mathcal{P}(B) \subset C\ and\ |A| = 2$, then $|C|$ can be $5\ but\ |C|\ cannot\ be\ 4$.
      i) **True**. If $|A| = 2$, then the cardinality of $\mathcal{P}(B) = 2^2 = 4$. Since $\mathcal{P}(B)$ is a proper subset of C, C must at least have a cardinality of 5.
   c) If a set B has one more element than a set A, then $\mathcal{P}(B)$ has at least two more elements than $\mathcal{P}(A)$.
      i) **False**, if $A = \emptyset$ then $|\mathcal{P}(A)| = 2^{|\emptyset|} = 2^0 = 1$ and $|\mathcal{P}(B)| = 2^{|\emptyset|+1} = 2^1 = 2$. (It is true if $A \neq \emptyset$)
   d) If four sets A, B, C and D are subsets of {1, 2, 3} such that $|A| = |B| = |C| = |D| = 2$, then at least two of these sets are equal.
      i) **True**. Different combinations of {1, 2, 3} with cardinality 2: $\frac{3!}{(3-2)!*2!} = \frac{3!}{2!} = \frac{3*2}{2} = 3$. Namely {1, 2}, {1, 3} and {2, 3}.

21) Three subsets A, B and C of $\{1,2,3,4,5\}$ have the same cardinality. Furthermore,
   a) 1 belongs to A and B but not to C.
   b) 2 belongs to A and C but not to B.
   c) 3 belongs to A and exactly one of B and C.
   d) 4 belongs to an even number of A, B and C.
   e) 5 belongs to an odd number of A, B and C.
   f) The sums of the elements in two of the sets A, B and C differ by 1.
   g) <u>What is B?</u>
      i) $A = \{1,2,3\}$
      ii) $B = \{1,4,5\}$
      iii) $C = \{2,3,4\}$

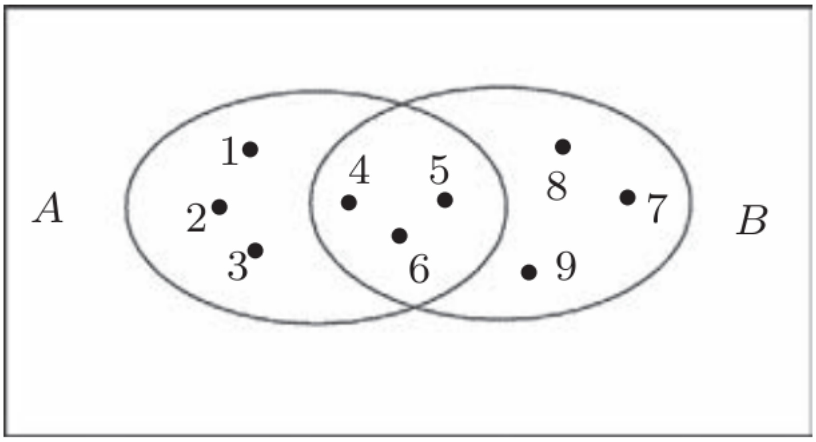
Section 3: Set Operations

Exercises

22) Let $U = \{1, 3, \dots, 15\}$ be the universal set, $A = \{1,5,9,13\}$, and $B = \{3,9,15\}$. Determine the following.
   a) $A \cup B = \{1,3,5,9,13,15\}$
   b) $A \cap B = \{9\}$

c) $A - B = \{1, 5, 13\}$
d) $B - A = \{3, 15\}$
e) $\bar{A} = U - A = \{3, 7, 11, 15\}$
f) $A \cap \bar{B} = A - B = \{1, 5, 13\}$

23) Give examples of two sets A and B such that $|A - B| = |A \cap B| = |B - A| = 3$. Draw the accompanying Venn diagram.
a) $A = \{1, 2, 3, 4, 5, 6\}; B = \{4, 5, 6, 7, 8, 9\}$

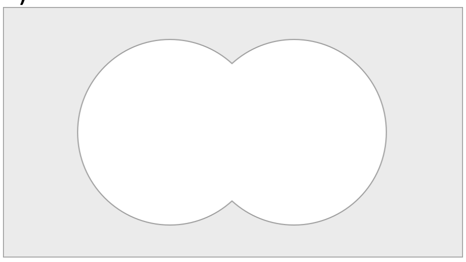


24) Give examples of three sets A, B and C such that $B \neq C$ but $B - A = C - A$
a) $A = \{1, 2\}$
b) $B = \{1, 2, 3\}$
c) $C = \{1, 3\}$

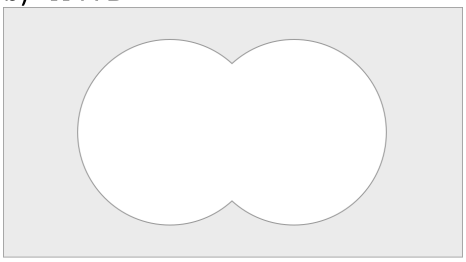
25) Give examples of three sets A, B and C such that
a) $A \in B, A \subseteq C \; and \; B \nsubseteq C$
 i) $A = \{1\}$
 ii) $B = \{\{1\}\}$
 iii) $C = \{1\}$
b) $B \in A, B \subset C \; and \; A \cap C \neq \emptyset$
 i) $A = \{\{1\}\}$
 ii) $B = \{1\}$
 iii) $C = \{1, \{1\}\}$
c) $A \in B, B \subseteq C \; and \; A \nsubseteq C$
 i) $A = \{1\}$
 ii) $B = \{\{1\}\}$
 iii) $C = \{\{1\}\}$

26) Let U be a universal set and let A and B be two subsets of U. Draw a Venn diagram for each of the following sets.

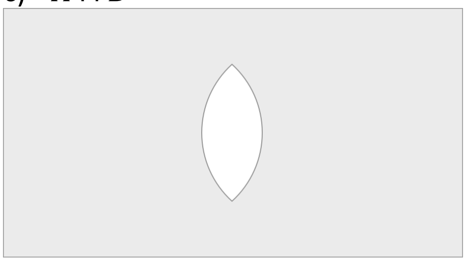
a) $\overline{A \cup B}$



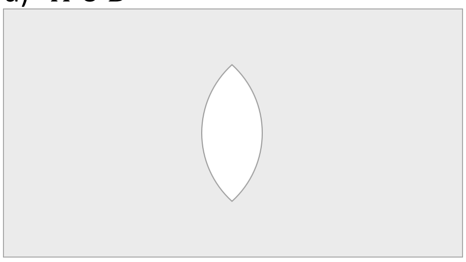
b) $\bar{A} \cap \bar{B}$



c) $\overline{A \cap B}$



d) $\bar{A} \cup \bar{B}$



27) Give an example of a universal set U, two sets A and B and accompanying Venn diagram such that $|A \cap B| = |A - B| = |B - A| = |\overline{A \cup B}| = 2$

a) $U = \{1, 2, \ldots, 8\};\ A = \{1, 2, 3, 4\};\ B = \{3, 4, 5, 6\}$