2016

Mathematical Proofs

CHAPTER 1 – EXERCISE SOLUTIONS AND NOTES

LASSE HAMMER PRIEBE

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Section 1: Statements

Exercises

1) Which of the following sentences are statements? Indicate their truth value.
   a) The integer 123 is prime.
      i) False
   b) The integer 0 is even
      i) True
   c) Is $5 * 2 = 10$?
      i) Not a statement
   d) $x^2 - 4 = 0$
      i) Not a statement
   e) Multiply $5x + 2\ by\ 3$
      i) Not a statement
   f) $5x + 3$ is an odd integer
      i) Not a statement
   g) What an impossible question!
      i) Not a statement

2) Consider the sets A, B, C and D below. Which of the following statements are true? Give an explanation for each false statement.

$$A = \{1, 4, 7, 10, 13, 16, \dots\}; B = \{x \in \mathbb{Z}: x\ is\ odd\}$$
$$C = \{x \in \mathbb{Z}: x\ is\ prime\ and\ x \neq 2\}; D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\}$$

   a) $25 \in A$
      i) True, since an integer *n* can be found such that $1 + 3n = 25$
   b) $33 \in D$
      i) False, since $33 \notin D$
   c) $22 \notin A \cup D$
      i) False, since $22 \in A$
   d) $C \subseteq B$
      i) True, since all prime numbers except 2 are odd
   e) $\emptyset \in B \cap D$
      i) False, since the neither B or D contains the empty set Ø.
   f) $53 \notin C$
      i) False since 53 is a prime thus $53 \in C$

3) Which of the following statements are true? Give an explanation for each false statement.
   a) $\emptyset \in \emptyset$
      i) False, since the empty set Ø has no elements
   b) $\emptyset \in \{\emptyset\}$
      i) True.
   c) $\{1, 3\} = \{3,1\}$
      i) True (sets are unordered)
   d) $\emptyset = \{\emptyset\}$
      i) False since the empty set Ø is not equal to the set containing the empty set {Ø}

e) $\emptyset \subset \{\emptyset\}$
   i) True
f) $1 \subseteq \{1\}$
   i) False since 1 is not a set

4) Consider the open sentence $P(x)$: $x(x-1) = 6 \rightarrow x^2 - x = 6 : x \in \mathbb{R}$
   a) For what values of x is $P(x)$ a true statement?
      i) $\forall x \in \{3, -2\}$
   b) For what values of x is $P(x)$ a false statement?
      i) $\forall x \in \mathbb{R} - \{3, -2\}$

5) For the open sentence $P(x)$: $3x - 2 > 4 : x \in \mathbb{Z}$ determine:
   a) For what values of x is $P(x)$ a true statement?
      i) $\forall x \in \{\mathbb{Z} : x > 2\}$
   b) For what values of x is $P(x)$ a false statement?
      i) $\forall x \in \{\mathbb{Z} : x \leq 2\}$

6) For the open sentence $P(A)$: $A \subseteq \{1, 2, 3\}$ over $S = P(\{1, 2, 4\})$ determine.
   a) All $A \in S$ for which P(A) is true
      i) $P(A)$ is true for all $A \in P(\{1,2\})$
   b) All $A \in S$ for which P(A) is false
      i) $P(A)$ is false for all $A \in \{x \in P(\{1,2\}): x + \{4\}\}$
   c) All $A \in S$ for which $A \cap \{1,2,3\} = \emptyset$
      i) $A = \{4\}$

7) Let $P(n)$: $n\ and\ n + 2\ are\ primes$. Be an open statement over the domain $\mathbb{N}$. Find six positive integers n for which P(n) is true. If $n \in N$ such that P(n) is true, then the two integers n, n+2 are called twin primes. It has been conjectured that there are infinitely many twin primes.
   a) $n = \{3, 5, 11, 17, 29, 41, \dots\}$

8) Let $P(n)$: $\frac{n^2+5n+6}{2}\ is\ even$
   a) Find a set S_1 of three integers, such that P(n) is an open sentence over the domain S_1 and P(n) is true for each $n \in S_1$
      i) $S_1 = \{1, 2, 5\}$
   b) Find a set S_s of three integers, such that P(n) is an open sentence over the domain S_2 and P(n) is false for each $n \in S_2$
      i) $S_2 = \{3, 4, 7\}$

9) Find an open sentence P(n) over the domain $S = \{3, 5, 7, 9\}$ such that P(n) is true for half of the integers in S and false for the other half.
   a) $P(n)$: $n < 6$

10) Find two open sentences P(n) and Q(n), both over the domain $S = \{2, 4, 6, 8\}$, such that P(2) and Q(2) are both true, P(4) and Q(4) are both false, P(6) is true and Q(6) is false, while P(8) is false and Q(8) is true.

a) $P(n): \frac{n}{2}\ is\ uneven$
b) $Q(n): n \in \{2, 8\}$

Section 2: The Negation of a Statement

Exercises

11) State the negation of each statement.
   a) $\sqrt{2}$ is a rational number
      i) $\sqrt{2}$ is not a rational number
   b) 0 is not a negative integer
      i) 0 is a negative integer
   c) 111 is a prime number
      i) 111 is not a prime number

12) Complete the truth table.

P	Q	~P	~Q
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

13) State the negation of each of the following statements
   a) The real number r is at most $\sqrt{2}$
      i) The real number r is larger than $\sqrt{2}$
   b) The absolute value of the real number a less than 3
      i) The absolute value of the real number a is at least 3
   c) Two angles of the triangle are 45°
      i) At most one of the triangles angles is 45°
   d) The area of the circle is at least 9π
      i) The area of the circle is less than 9π
   e) Two sides of the triangle have the same lengths
      i) The sides of the triangle are of different lengths
   f) The point P in the plane lies outside of the circle C
      i) The point P in the plane lies inside the circle C

14) State the negation of each of the following statements
   a) At least two of my library books are overdue
      i) At most one of my library books is overdue
   b) One of my two friends misplaced his homework assignment
      i) One of two my friends did not misplace his homework assignment
   c) No one expected that to happen
      i) Some expected that to happen

d) It's not often that my instructor teaches that course
   i) It's often that my instructor teaches that course
e) It's surprising that two students received the same exam score
   i) It's not surprising that two students received the same exam score

Section 3: The Disjunction and Conjunction of Statements

Notes

$$P \text{ or } Q \rightarrow P \vee Q$$
$$P \text{ and } Q \rightarrow P \wedge Q$$

Exercises

15) Complete the truth table.

P	Q	~Q	$P \wedge (\sim Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

16) For the sets A and B, consider the statements…
   a) False
   b) True
   c) False
   d) False
   e) True

17) Let P: 15 is odd and Q: 21 is prime
   a) True
   b) False
   c) False
   d) True

18) $P(A): A \cap \{2,4,6\} = \emptyset \quad and \quad Q(A): A \neq \emptyset$
   a) $P(A) \wedge Q(A) = True\ when\ A \in \{1,3,5\}$
   b) $P(A) \vee \sim Q(A)) = True\ when\ A \in \{1,3,5\}\ or\ A = \emptyset$
   c) $\sim P(A) \wedge \sim Q(A) = True\ when\ A = \{\}$

Section 4: The Implication

Notes

Table 1 - Implication truth table

P	Q	P => Q
True	True	True
True	False	False

False	True	True
False	False	True

Exercises

19) Consider the statements P: 17 is even and Q: 19 is prime. Write each statement in words and indicate whether it is true or false.
   a) $\sim P$: 17 is odd (True)
   b) $P \vee Q$: 17 is even or 19 is prime (True – 19 is prime)
   c) $P \wedge Q$: 17 is even and 19 is prime (False – 17 is odd)
   d) $P \Rightarrow Q$: If 17 is even, then 19 is prime (True – 19 is prime)

20) For statements P and Q, construct a truth table for (P => Q) => (~P)

P => Q	~P	(P => Q) => (~P)
True	False	False
False	False	True
True	True	True
True	True	True

21) Consider the statements P: $\sqrt{2}$ is rational and Q: $\frac{22}{7}$ is rational. Write each of the following statements in words and indicate whether it is true or false.
   a) $P \Rightarrow Q$: If $\sqrt{2}$ is rational, then $\frac{22}{7}$ is rational (True)
   b) $Q \Rightarrow P$: If $\frac{22}{7}$ is rational, then $\sqrt{2}$ is rational (False – $\sqrt{2}$ is not rational)
   c) $(\sim P) \Rightarrow (\sim Q)$: If $\sqrt{2}$ is irrational, then $\frac{22}{7}$ is irrational (False – $\frac{22}{7}$ is not irrational)
   d) $(\sim Q) \Rightarrow (\sim P)$: If $\frac{22}{7}$ is irrational, then $\sqrt{2}$ is irrational (True – $\sqrt{2}$ is irrational)

22) Consider the statements:

$$P{:}\, \sqrt{2}\ is\ rational. \quad Q{:}\, \frac{2}{3}\ is\ rational. \quad R{:}\, \sqrt{3}\ is\ rational.$$

   a) $(P \wedge Q) \Rightarrow R$: If $\sqrt{2}$ and $\frac{2}{3}$ are rational, then $\sqrt{3}$ is rational (True - $\sqrt{2}$ is not rational)
   b) $(P \wedge Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ and $\frac{2}{3}$ are rational, then $\sqrt{3}$ is irrational (True - $\sqrt{2}$ is not rational)
   c) $((\sim P) \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is irrational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational (False - $\sqrt{3}$ is not rational)
   d) $(P \vee Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ or $\frac{2}{3}$ is rational, then $\sqrt{3}$ is irrational (True - $\sqrt{3}$ is irrational)

23) Suppose that $\{S_1, S_2\}$ is a partition of a set S and $x \in S$. Which of the following are true?
   a) If we know that $x \notin S_1$ then x must belong to S_2. (True)
   b) It's possible that $x \notin S_1$ and $x \notin S_2$. (False)
   c) Either $x \notin S_1\ or\ x \notin S_2$. (True)
   d) Either $x \in S_1\ or\ x \in S_2$. (True)

e) It's possible that $x \in S_1$ and $x \in S_2$. (False)

24) Two sets A and B are nonempty disjoint subsets of a set S. If $x \in S$, then which of the following are true?
   a) It's possible that $x \in A \cap B$. (False – A and B are disjoint)
   b) If x is an element of A, then x can't be an element of B. (True – A and B are disjoint)
   c) If x is not an element of A, then x must be an element of B. (False – It is possible that $A \cup B \neq S$)
   d) It's possible that $x \notin A$ and $x \notin B$. (True – It's possible that $A \cup B \neq S$)
   e) For each nonempty set C, either $x \in A \cap C$ or $x \in B \cap C$. (False – It is possible that $A \cup B \neq S$)
   f) For some nonempty set C, both $x \in A \cup C$ and $x \in B \cup C$. (True if C contains x, False otherwise)

25) A college student makes the following statement: If I receive an A in both Calculus I and Discrete Mathematics this semester, then I'll take either Calculus II or Computer Programming this summer.

$$P: A\ in\ Calculus\ I\ and\ Discrete\ Mathematics$$
$$Q: Takes\ Calculus\ II\ or\ Computer\ Programming$$

   a) P is false and Q is true. (True)
   b) P is true and Q is false. (False)
   c) P is false and Q is true. (True)
   d) P is true and Q is true. (True)
   e) P is false and Q is false. (True)

26) A college student makes the following statement: If I don't see my advisor today, then I'll see her tomorrow.

$$P: Don't\ see\ advisor\ today$$
$$Q: See\ advisor\ tomorrow$$

   a) P is true and Q is false. (False)
   b) P is false and Q is true. (True)
   c) P is true and Q is true AND P is false and Q is false. (True)
   d) P is true and Q is false. (False)

27) The instructor of a computer science class announces…
   a) Alice => Ben
   b) Ben => Cindy
   c) Cindy => Don
   d) The two students who attend are Cindy and Don

28) Consider the statement (implication): If Bill takes Sam to the concert, then Sam will take Bill to dinner.

$$P: Bill\ takes\ Sam\ to\ concert$$
$$Q: Sam\ takes\ bill\ to\ dinner$$

   a) Q only if P. (False – P can be false and Q true and the implication still holds)
   b) Either ~P or Q. (False – The $\sim P \wedge Q$ scenario is also true)

c) P is true. (False – Q doesn't happen)
d) P is true and Q is true. (True)
e) P is true and Q is false. (False)
f) P is false. (True)
g) P is false. (True)

29) Let P and Q be statements. Which of the following implies that $P \vee Q$ is false?
a) $(\sim P) \vee (\sim Q)$ is false. (False – P or Q can be true)
b) $(\sim P) \vee Q$ is true. (False – Q can be true)
c) $(\sim P) \wedge (\sim Q)$ is true. (True – both P and Q must be false)
d) $Q \Rightarrow P$ is true. (False – P or Q can be true)
e) $P \wedge Q$ is false. (False – one of them can be true)

Section 5: More on Implications

Notes

$$\mathbb{R}: Real\ numbers\ (all\ real\ numbers)$$
$$\mathbb{Q}: Rational\ numbers$$
$$\mathbb{N}: Natural\ numbers\ (positive\ integers, starting\ from\ 1)$$
$$\mathbb{Z}: Integers\ (positive\ and\ negative\ including\ 0)$$

Exercises

30) Consider the open sentences $P(n): 5n + 3\ is\ prime$. And $Q(n): 7n + 1\ is\ prime$. Both over the domain $\mathbb{N}$. State in words.
a) $P(n) \Rightarrow Q(n)$: If $5n + 3$ is prime, then $7n + 1$ is prime.
b) $P(2) \Rightarrow Q(2)$: If 13 is prime, then 15 is prime. (False – 15 is not prime)
c) $P(6) \Rightarrow Q(6)$: if 33 is prime, then 43 is prime. (True – 33 is not prime)

31) In each of the following, two open sentences P(x) and Q(x) over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.
a) $P(x): |x| = 4; Q(x): x = 4; S = \{-4, -3, 1, 4, 5\}$
   i) $S_{true} = \{-3, 1, 4, 5\}$
b) $P(x): x^2 = 16; Q(x): |x| = 4; S = \{-6, -4, 0, 3, 4, 8\}$
   i) $S_{true} = \{-6, -4, 0, 3, 4, 8\}$ aka $true\ for\ all\ x \in S$
c) $P(x): x > 3; Q(x): 4x - 1 > 12; S = \{0, 2, 3, 4, 6\}$
   i) $S_{true} = \{0, 2, 3, 4, 6\}$ aka $true\ for\ all\ x \in S$

32) In each of the following, two open sentences P(x) and Q(x) over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.
a) $P(x): x - 3 = 4; Q(x): x \geq 8; S = \mathbb{R}$
   i) True for $x \neq 7$
b) $P(x): x^2 \geq 1; Q(x): x \geq 1; S = \mathbb{R}$
   i) True for $x > -1$
c) $P(x): x^2 \geq 1; Q(x): x \geq 1; S = \mathbb{N}$
   i) True for all $x \in S$
d) $P(x): x \in [-1, 2]; Q(x): x^2 \leq 2; S = [-1, 1]$

i) True for all $x \in S$ since Q(x) is true for all $x \in S$

33) In each of the following, two open sentences P(x, y) and Q(x, y) are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x,y) \Rightarrow Q(x,y)$ for the given values of x and y.

a) $P(x,y): x^2 - y^2 = 0;\ Q(x,y): x = y; (x,y) \in \{(1,-1),(3,4),(5,5)\}$
   i) True for $(x,y) \in \{(3,4),(5,5)\}$
b) $P(x,y): |x| = |y|; Q(x,y): x = y; (x,y) \in \{(1,2),(2,-2),(6,6)\}$
   i) True for $(x,y) \in \{(1,2),(6,6)\}$
c) $P(x,y): x^2 + y^2 = 1; Q(x,y): x + y = 1; (x,y) \in \{(1,-1),(-3,4),(0,-1),(1,0)\}$
   i) True for $(x,y) \in \{(1,-1),(-3,4),(1,0)\}$

34) Each of the following describes an implication. Write the implication in the form "if, then."

a) If a point on the straight line is given by $2y + x - 3 = 0$ and x is an integer, then y an integer.
b) If n is odd then n^2 is odd.
c) If $3n + 7$ is even and $n \in \mathbb{Z}$, then n is odd.
d) If $f(x) = cosx$, then $f'(x) = -sinx$
e) If the circumference of C is 4π, then the area of C is 4π
f) If n^3 is even, then n is even.

Section 6: The Biconditional

Notes

$$Biconditional: (P \Rightarrow Q) \wedge (Q \Rightarrow P) = P \Leftrightarrow Q$$

P	Q	P => Q	Q => P	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$$The\ biconditional\ P \Leftrightarrow Q\ is\ often\ stated\ as: P = Q$$

Exercises

35) Let P: 18 is odd and Q: 25 is even. State $P \Leftrightarrow Q$ in words. Is $P \Leftrightarrow Q$ true or false?

a) 18 is odd if and only if 25 is even.
b) True (both are false)

36) Let P(x): x is odd and $Q(x): x^2 is\ odd$. Be open sentences over the domain $\mathbb{Z}$. State $P(x) \Leftrightarrow Q(x)$ in two ways: (1) using "if and only if" and (2) using "necessary and sufficient".

a) x is odd if and only if x^2 is odd
b) x being odd is a necessary and sufficient condition for x^2 being odd

37) For the open sentences $P(x): |x-3| < 1; Q(x): x \in \{2, 4\}$. Over the domain $\mathbb{R}$, state the biconditional $P(x) \Leftrightarrow Q(x)$ in two different ways.
   a) $|x-3| < 1$ if and only if $x \in \{2,4\}$
   b) The condition $|x-3| < 1$ is necessary and sufficient for $x \in \{2,4\}$

38) Consider the open sentences: $P(x): x = -2; Q(x): x^2 = 4$ over the domain $S = \{-2, 0, 2\}$. State each of the following in words and determine all values of $x \in S$ for which the resulting statement is true.
   a) $\sim P(x)$
      i) $x \neq -2$
      ii) True for all $x \in \{0, 2\}$
   b) $P(x) \vee Q(x)$
      i) $x = -2 \; or \; x^2 = 4$
      ii) True for all $x \in \{-2, 2\}$
   c) $P(x) \wedge Q(x)$
      i) $x = -2 \; and \; x^2 = 4$
      ii) True for x = -2
   d) $P(x) \Rightarrow Q(x)$
      i) If $x = -2$ then $x^2 = 4$
      ii) True for all $x \in S$
   e) $Q(x) \Rightarrow P(x)$
      i) If $x^2 = 4$ then $x = -2$
      ii) True for $x \in \{-2, 0\}$
   f) $P(x) \Leftrightarrow Q(x)$
      i) $x = -2$ if and only if $x^2 = 4$
      ii) True for all $x \in \{-2, 0\}$

39) For the following open sentences $P(x)$ and Q(x) over domain S, determine all values of $x \in S$ for which the biconditional $P(x) \Leftrightarrow Q(x)$ is true.
   a) $P(x): |x| = 4; Q(x): x = 4; S = \{-4, -3, 1, 4, 5\}$
      i) True for all $x \in \{-3, 1, 4, 5\}$
      ii) Alt. notation: True for all $x \in S - \{-4\}$
   b) $P(x): x \geq 3; Q(x): 4x - 1 > 12; S = \{0, 2, 3, 4, 6\}$
      i) True for all $x \in \{0, 2, 4, 6\}$
      ii) Alt. notation: True for all $x \in S - \{3\}$
   c) $P(x): x^2 = 16; Q(x): x^2 - 4x = 0; S = \{-6, -4, 0, 3, 4, 8\}$
      i) True for all $x \in \{-6, 3, 4, 8\}$
      ii) Alt. notation: True for all $x \in S - \{-4, 0\}$

40) In each of the following, two open sentences P(x,y) and Q(x,y) are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x, y) \Leftrightarrow Q(x, y)$ for the given values of x and y.
   a) $P(x, y): x^2 - y^2 = 0; Q(x, y): x = y; (x, y) \in \{(1, -1), (3,4), (5,5)\}$
      i) True for all $(x, y) \in \{(3,4), (5,5)\}$
   b) $P(x, y): |x| = |y|; Q(x, y): x = y; (x, y) \in \{(1,2), (2, -2), (6,6)\}$

   i) True for all $(x, y) \in \{(1,2), (6,6)\}$
c) $P(x, y): x^2 + y^2 = 1; Q(x, y): x + y = 1; (x, y) \in \{(1, -1), (-3,4), (0, -1), (1,0)\}$
   i) True for all $(x, y) \in \{(1, -1), (1,0)\}$

41) Determine all values of n in the domain $S = \{1,2,3\}$ for which the following is a true statement: A necessary and sufficient condition for $\frac{n^3+n}{2}$ to be even is that $\frac{n^2+n}{2}$ is odd.
a) $n = 1$
   i) $\frac{2}{2}$ is even and $\frac{2}{2}$ is odd. (False)
b) $n = 2$
   i) $\frac{10}{2}$ is even and $\frac{6}{2}$ is odd. (False)
c) $n = 3$
   i) $\frac{90}{2}$ is even and $\frac{12}{2}$ is odd (True – both are false)

42) Determine all values of n in the domain $S = \{2,3,4\}$ for which the following is a true statement: The integer $\frac{n(n-1)}{2}$ is odd if and only if $\frac{n(n+1)}{2}$ is even.
a) $n = 2$
   i) $\frac{2}{2}$ is odd if and only if $\frac{6}{2}$ is even. (False)
b) $n = 3$
   i) $\frac{6}{2}$ is odd if and only if $\frac{12}{2}$ is even. (True)
c) $n = 4$
   i) $\frac{12}{2}$ is odd if and only if $\frac{20}{2}$ is even. (False)

43) Let $S = \{1,2,3\}$. Consider the following open sentences over the domain S. Determine three distinct elements a, b, c in S such that…

$$P(n): \frac{(n+4)(n+5)}{2} \; is \; odd$$

$$Q(n): 2^{n-2} + 3^{n-2} + 6^{n-2} > (2.5)^{n-1}$$

a) $P(a) \Rightarrow Q(a)$ is false
   i) $P(1)$ is true and $Q(1)$ is false
   ii) $a = 1$
b) $Q(b) \Rightarrow P(b)$ is false
   i) $P(3)$ is false and $Q(3)$ is true
   ii) $b = 3$
c) $P(a) \Leftrightarrow Q(a)$ is true
   i) $P(2)$ is true and $Q(2)$ is true
   ii) $c = 2$

44) Let $S = \{1,2,3,4\}$. Consider the following open sentences over the domain S. Determine four distinct elements a, b, c, d in S such that…

$$P(n): \frac{n(n-1)}{2} \text{ is even}$$
$$Q(n): 2^{n-2} - (-2)^{n-2} \text{ is even}$$
$$R(n): 5^{n-1} + 2^n \text{ is prime}$$

Table 2 - Results of P(n), Q(n) and R(n) given n in {1, 2, 3, 4}

n	P(n)	Q(n)	R(n)
1	0 True	1 False	3 True
2	1 False	0 True	9 False
3	3 False	4 True	33 False
4	6 True	0 True	141 False

a) $P(a) \Rightarrow Q(a)$ is false
   i) $a \in \{1\} \rightarrow a = 1$
b) $Q(b) \Rightarrow P(b)$ is true
   i) $b \in \{1, 4\} \rightarrow b = 4$
c) $P(c) \Leftrightarrow R(c)$ is true
   i) $c \in \{1, 2, 3\} \rightarrow c = 2$
d) $Q(d) \Leftrightarrow R(d)$ is false
   i) $d \in \{1, 2, 3, 4\} \rightarrow d = 3$

45) Let $P(n)$: $2^n - 1$ is a prime; $Q(n)$: $n \text{ is a prime}$. Be open sentences over the domain $S = \{2,3,4,5,6,11\}$. Determine all values of $n \in S$ for which $P(n) \Leftrightarrow Q(n)$ is a true statement.
   a) $2: 2^2 - 1 \text{ is a prime if and only if } 2 \text{ is a prime}$
      i) True (both statements are true)
   b) $3: 2^3 - 1 \text{ is a prime if and only if } 3 \text{ is a prime}$
      i) True (both statements are true)
   c) $4: 2^4 - 1 \text{ is a prime if and only if } 4 \text{ is a prime}$
      i) True (both statements are false)
   d) $5: 2^5 - 1 \text{ is a prime if and only if } 5 \text{ is a prime}$
      i) True (both statements are true)
   e) $6: 2^6 - 1 \text{ is a prime if and only if } 6 \text{ is a prime}$
      i) True (both statements are false)
   f) $11: 2^{11} - 1 \text{ is a prime if and only if } 11 \text{ is a prime}$
      i) False ($2^{11} - 1$ is not a prime but 11 is)
   g) SUMMARY: True for all $n \in S - \{11\}$

Section 7: Tautologies and Contradictions

Notes

$$Tautology: A\ compound\ statement\ which\ is\ always\ true, e.g. P \vee \sim P$$
$$Contradiction: A\ compound\ statement\ which\ is\ always\ false, e.g. P \wedge \sim P$$

Exercises

46) For statements P and Q, show that $P \Rightarrow (P \vee Q)$ is a tautology

P	Q	$(P \vee Q)$	$P \Rightarrow (P \vee Q)$
T	T	T	**T**
T	F	T	**T**
F	T	T	**T**
F	F	F	**T**

47) For statements P and Q, show that $\big(P \wedge (\sim Q)\big) \wedge (P \wedge Q)$ is a contradiction

P	Q	$P \wedge Q$	$P \wedge (\sim Q)$	$\big(P \wedge (\sim Q)\big) \wedge (P \wedge Q)$
T	T	T	F	**F**
T	F	F	T	**F**
F	T	F	F	**F**
F	F	F	F	**F**

48) For statements P and Q, show that $\big(P \wedge (P \Rightarrow Q)\big) \Rightarrow Q$ is a tautology. Then state the compound statement in words. (This is an important logical argument form, called **modus ponens.**)

a) If P is true and P implies Q, then Q is true.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$\big(P \wedge (P \Rightarrow Q)\big) \Rightarrow Q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

49) For statements P, Q and R, show that $\big((P \Rightarrow Q) \wedge (Q \Rightarrow R)\big) \Rightarrow (P \Rightarrow R)$ is a tautology. Then state this compound statement in words. (This is another important logical argument form, called **syllogism**.)

a) If P implies Q and Q implies R, then P implies R

P	Q	R	$(P \Rightarrow Q)$	$(Q \Rightarrow R)$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$(P \Rightarrow R)$	$\big((P \Rightarrow Q) \wedge (Q \Rightarrow R)\big) \Rightarrow \cdots$
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	F	T	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	F	T	**T**
F	F	T	T	T	T	T	**T**

F	F	F	T	T	T	T	**T**

50) Let R and S be compound statements involving the same compound statements. If R is a tautology and S is a contradiction, then what can be said of the following?

a) $R \vee S$ is true, since R is always true
b) $R \wedge S$ is false, since S is always false
c) $R \Rightarrow S$ is false, since $'true \Rightarrow false'$ is false
d) $S \Rightarrow R$ is true, since $'false \Rightarrow true'$ is true

Section 8: Logical Equivalence

Notes

$$Logical\ equivalence: P \Rightarrow Q \equiv (\sim P) \vee Q$$

Exercises

51) For statements P and Q, the implication $(\sim P) \Rightarrow (\sim Q)$ is called the inverse of the implication $P \Rightarrow Q$.

a) Use a truth table to show that these statements are not logically equivalent

P	Q	$P \Rightarrow Q$	$(\sim P) \Rightarrow (\sim Q)$
T	T	T	T
T	F	**F**	**T**
F	T	**T**	**F**
F	F	T	T

b) Find another implication that is logically equivalent to $(\sim P) \Rightarrow (\sim Q)$ and verify your answer

i) $P \vee (\sim Q) \equiv Q \Rightarrow P \equiv (\sim P) \Rightarrow (\sim Q)$

P	Q	$(\sim P) \Rightarrow (\sim Q)$	$P \vee (\sim Q)$	$Q \Rightarrow P$
T	T	**T**	**T**	**T**
T	F	**T**	**T**	**T**
F	T	**F**	**F**	**F**
F	F	**T**	**T**	**T**

52) Let P and Q be statements.

a) Is $\sim(P \vee Q)$ logically equivalent to $(\sim P) \vee (\sim Q)$? Explain.

i) They are logically equivalent since each statement is only true when both Q and P are false, and true otherwise.

b) What can you say about the biconditional $\sim(P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$?

i) The biconditional is a tautology since $\sim(P \vee Q) \equiv ((\sim P) \vee (\sim Q))$

53) For statements P, Q and R, use a truth table to show that each of the following pairs of statements is logically equivalent.

a) $(P \wedge Q) \Leftrightarrow P\ and\ P \Rightarrow Q$

P	Q	$P \wedge Q$	$(P \wedge Q) \Leftrightarrow P$	$P \Rightarrow Q$

T	T	T	**T**	**T**
T	F	F	**F**	**F**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

$P \Rightarrow (Q \vee R)$ and $(\sim Q) \Rightarrow ((\sim P) \vee R)$

P	Q	R	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$(\sim P) \vee R$	$(\sim Q) \Rightarrow ((\sim P) \vee R)$
T	T	T	T	**T**	T	**T**
T	T	F	T	**T**	F	**T**
T	F	T	T	**T**	T	**T**
T	F	F	F	**F**	F	**F**
F	T	T	T	**T**	T	**T**
F	T	F	T	**T**	T	**T**
F	F	T	T	**T**	T	**T**
F	F	F	F	**T**	T	**T**

54) For statements P and Q, show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are logically equivalent

P	Q	$P \wedge (\sim P)$	$(\sim Q) \Rightarrow (P \wedge (\sim P))$
T	**T**	F	**T**
T	**F**	F	**F**
F	**T**	F	**T**
F	**F**	F	**F**

55) For statements P, Q and R, show that $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**F**	F	F	**F**
T	F	T	T	**T**	T	T	**T**
T	F	F	T	**F**	F	T	**F**
F	T	T	T	**T**	T	T	**T**
F	T	F	T	**F**	T	F	**F**
F	F	T	F	**T**	T	T	**T**
F	F	F	F	**T**	T	T	**T**

56) Two compound statements S and T are composed of the same component statements P, Q and R. If S and T are not logically equivalent, then what can we conclude from this?

a) $S \Leftrightarrow T$ is not a tautology

57) Five compound statements S_1, S_2, S_3, S_4 and S_5 are all composed of the same component statements P and Q whose truth tables have identical first and fourth rows. Show that at least two of these five statements are logically equivalent.

S1	S2	S3	S4	S5
T	T	T	T	T
T	T	F	F	X

T	F	T	F	X
F	F	F	F	F

Section 9: Some Fundamental Properties of Logical Equivalence

Notes

Theorem 18

(1) Commutative Laws
   (a) $P \vee Q \equiv Q \vee P$
   (b) $P \wedge Q \equiv Q \wedge P$
(2) Associative Laws
   (a) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
   (b) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
(3) Distributive Laws
   (a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   (b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(4) De Morgan's Laws
   (a) $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
   (b) $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

Theorem 21

(1) For statements P and Q,
   (a) $\sim(P \Rightarrow Q) \equiv P \wedge (\sim Q)$
   (b) $\sim(P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$

Exercises

58) Verify the following laws stated in Theorem 18:
   a) Let P, Q and R be statements. Then $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
      i) The first statement is true if P or Q and R, or all three are true. Equivalently statement two is true only if both of the parenthesized statements are true. This requires either P to be true (since a P is in both statements), or Q and R to be true (since there is one of each in the statements). Thus the second statement is also true if P or Q and R, or all three are true.
   b) Let P and Q be statements. Then $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
      i) The first statement is true only if P and Q are false. We can easily see that this is the case for statement two as well.

59) Write negations of the following open sentences.
   a) Either x=0 or y=o
      i) Using De Morgan's Law (a): Both $x \neq 0$ and $y \neq 0$
   b) The integers a and b are both even
      i) Using De Morgan's Law (b): Either the integer a is odd or the integer b is odd.

60) Consider the implication: If x and y are even, then xy is even.
   a) State the implication using "only if": x and y are even only if xy is even
   b) State the converse of the implication: xy is even only if x and y are even
   c) State the implication as a disjunction: x and y are odd or xy is even

$$Theorem\ 17: P \Rightarrow Q \equiv (\sim P) \vee Q$$

   d) State the negation of the implication as a conjunction: x and y are even and xy is odd

61) For a real number x, let $P(x): x^2 = 2\ and\ Q(x): x = \sqrt{2}$. State the negation of the biconditional $P \Leftrightarrow Q$ in words.
   a) Biconditional: $x^2 = 2$ if and only if $x = \sqrt{2}$
   b) Negation: $\sim(P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$, using De Morgan's Law (b)
   c) Result: Either both $x^2 = 2$ and $x \neq \sqrt{2}$, or both $x = \sqrt{2}$ and $x^2 \neq 2$

62) Let P and Q be statements. Show that $[(P \vee Q) \wedge \sim(P \wedge Q)] \equiv \sim(P \Leftrightarrow Q)$

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$	$\sim(P \Leftrightarrow Q)$
T	T	T	T	**F**	**F**
T	F	T	F	**T**	**T**
F	T	T	F	**T**	**T**
F	F	F	F	**F**	**F**

63) Let $n \in \mathbb{Z}$. For which implication is its negation the following? The integer $3n + 4$ is odd and $5n - 6$ is even
   a) The negated statement has the form $P \wedge Q$
      i) $P: 3n + 4\ is\ odd; Q: 5n - 6$ is even
   b) Using Theorem 21 (a): $\sim(P \Rightarrow Q) \equiv P \wedge (\sim Q)$
   c) Thus the original implication is: If $3n + 4$ is odd, then $5n - 6$ is odd.

64) For which biconditional is its negation the following? $n^3 and\ 7n + 2\ are\ odd\ or\ n^3 and\ 7n + 2\ are\ even$
   a) The negated statement has the form: $(P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$
      i) $P: n^3\ is\ odd; Q: 7n + 2\ is\ even$
   b) Using Theorem 21 (b): $\sim(P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$
   c) Thus the original biconditional is: n^3 is odd if and only if $7n + 2$ is even

Section 10: Quantified Statements

Exercises

65) Let S denote the set of odd integers and let $P(x): x^2 + 1\ is\ even; Q(x): x^2\ is\ even$ be open sentences over the domain S. State $\forall x \in S, P(x)$ and $\exists x \in S, Q(x)$ in words.
   a) For every odd integer x, the integer $x^2 + 1$ is even.
   b) An odd integer x exists, such that the integer x^2 is even.

66) Define an open sentence R(x) over some domain S and then state $\forall x \in S, R(x) and\ \exists x \in S, R(x)$
   a) Definitions: R(x): 2x + 1 is prime; S: The set of integers $\mathbb{Z}$
   b) $\forall x \in S, R(x)$: For every integer x, the integer $2x + 1$ is prime
   c) $\exists x \in S, R(x)$: For some integer x, the integer $2x + 1$ is prime

67) State the negations of the following quantified statements, where all sets are subsets of some universal set U.
   a) $\forall A \in U, A \cap \bar{A} = \emptyset$
      i) Negation: $\exists A \in U, A \cap \bar{A} \neq \emptyset$

b) $\exists A \in U, \bar{A} \subseteq A$
   i) Negation: $\forall A \in U, \bar{A} \nsubseteq A$

68) State the negations of the following quantified statements:
   a) For every rational number r, the number 1/r is rational.
      i) There exists a rational number r, such that the number 1/e is not rational.
   b) There exists a rational number r, such that $r^2 = 2$.
      i) For every rational number r, $r^2 \neq 2$

69) Let $P(n)\colon \frac{5n-6}{3}\ is\ an\ integer$. Be an open sentence over the domain $\mathbb{Z}$. Determine, with explanations, whether the following statements are true.
   a) $\forall n \in \mathbb{Z}, P(n)$
      i) False, since "$P(1)\colon -\frac{1}{3} is\ an\ integer$" is false.
   b) $\exists n \in \mathbb{Z}, P(n)$
      i) True, since "$P(3)\colon 3\ is\ an\ integer$" is true.

70) Determine the truth value of each of the following statements.
   a) $\exists x \in \mathbb{R}, x^2 - x = 0$; True (e.g. $1^2 - 1 = 0$)
   b) $\forall n \in \mathbb{N}, n + 1 \geq 2$; True ($\mathbb{N}$ is all positive integers ≥ 1 and $1 + 1 \geq 2$)
   c) $\forall x \in \mathbb{R}, \sqrt{x^2} = x$; False (False for all negative numbers)
   d) $\exists x \in \mathbb{Q}, 3x^2 - 27 = 0$; True (for -3 and 3)
   e) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y + 3 = 8$; True (e.g. $x = 2$ and $y = 3$)
   f) $\forall x, y \in \mathbb{R}, x + y + 3 = 8$; False (e.g. $1 + 1 + 3 \neq 8$)
   g) $\exists x, y \in \mathbb{R}, x^2 + y^2 = 9$; True (e.g. $1^2 + 3^2 = 9$)
   h) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 = 9$; False (e.g. $1^2 + 1^2 \neq 9$)

71) The statement: "For every integer m, either $m \leq 1\ or\ m^2 \geq 4$" can be expressed using a quantifier as: $\forall m \in \mathbb{Z}, m \leq 1\ or\ m^2 \geq 4$. Do this for the following two statements.
   a) There exists integers a and b such that both $ab < 0\ and\ a + b > 0$.
      i) $\exists a, b \in \mathbb{Z}, ab < 0\ and\ a + b > 0$
   b) For all real numbers x and y, $x \neq y\ implies\ that\ x^2 + y^2 > 0$.
      i) $\forall x, y \in \mathbb{R}, (x \neq y) \Rightarrow (x^2 + y^2 > 0)$
   c) Express in words the negations of the statements in (a) and (b).
      i) For all integers a and b, either $ab \geq 0\ or\ a + b \leq 0$. (De Morgan's Law – b)
      ii) Real numbers x and y exists, such that $x \neq y\ and\ x^2 + y^2 \leq 0$. (Theorem 21)
   d) Using quantifiers, express in symbols the negations of the statements in both (a) and (b).
      i) $\forall a, b \in \mathbb{Z}, ab \geq 0\ or\ a + b \leq 0$
      ii) $\exists x, y \in \mathbb{R}, x \neq y\ and\ x^2 + y^2 \leq 0$

72) Let P(x) and Q(x) be open sentences where the domain of the variable x is S. Which of the following implies that $(\sim P(x)) \Rightarrow Q(x)$ is false for some $x \in S$?
   a) $P(x) \wedge Q(x)$ is false for all $x \in S$.
      i) This does not, e.g. P can be true and Q false for all $x \in S$, thus the original statement would never be false.

b) $P(x)$ is true for all $x \in S$.
   i) This does not, since the original statement would always be true.
c) $Q(x)$ is true for all $x \in S$.
   i) This does not, since the original statement would always be true.
d) $P(x) \vee Q(x)$ is false for some $x \in S$.
   i) **This**, since this implies that P and Q will be false at the same time for some $x \in S$, which in turn implies that the original statement will be false for some ($since\ true \Rightarrow false\ is\ false$).
e) $P(x) \wedge \left(\sim Q(x)\right)$ is false for all $x \in S$.
   i) This does not, since it just implies that (P, Q) is never (true, false), which means that the original statement is never $false \Rightarrow false$ (which is true for implications).

73) Let P(x) and Q(x) be open sentences where the domain of the variable x is T. Which of the following implies that $P(x) \Rightarrow Q(x)$ is true for all $x \in T$?
a) $P(x) \wedge Q(x)$ is false for all $x \in S$.
   i) This does not, since it is possible that P is true and Q is false.
b) $Q(x)$ is true for all $x \in S$.
   i) **This**, since the implication will always be true.
c) $P(x)$ is false for all $x \in S$.
   i) **This**, since the implication will always be true.
d) $P(x) \wedge \left(\sim Q(x)\right)$ is true for some $x \in S$.
   i) This does not (It is the negation of the original implication).
e) $P(x)$ is true for all $x \in S$.
   i) This does not, since Q may be false and thus the original implication is false.
f) $\left(\sim P(x)\right) \wedge \left(\sim Q(x)\right)$ is false for all $x \in S$.
   i) This does not, since this statement is false even though both P and Q are false.

74) Consider the open sentence: $P(x, y, z)$: $(x-1)^2 + (y-2)^2 + (z-2)^2 > 0$. Where the domain of each of the variables x, y and z is $\mathbb{R}$.
a) Express the quantified statement $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, P(x, y, z)$ in words.
   i) For all real numbers x, y and z, $(x-1)^2 + (y-2)^2 + (z-2)^2 > 0$.
b) Is the quantified statement in (a) true or false? Explain.
   i) It is false since $P(1,2,2) = 0$
c) Express the negation of the quantified statement in (a) in symbols.
   i) $\exists x, y, z \in \mathbb{R}, \sim P(x, y, z)$
d) Express the negation of the quantified statement in (a) in word.
   i) Real numbers x, y and z exists, such that $(x-1)^2 + (y-2)^2 + (z-2)^2 \leq 0$
e) Is the negation of quantified statement in (a) true or false? Explain.
   i) It is true since the original statement was false. ($P(1,2,2) \leq 0$)

75) Consider the quantified statement: For every $s \in S$ and $t \in S$, $st - 2$ is prime. Where the domain $S = \{3,5,11\}$ and $P(s, t)$: $st - 2\ is\ prime$.
a) Express this quantified statement in symbols.
   i) $\forall s, t \in S, P(s, t)$
b) Is the quantified statement (a) true or false? Explain.

i) $P(3,3) = P(3,3) = 7\ is\ prime\ (true)$
ii) $P(3,5) = P(5,3) = 13\ is\ prime\ (true)$
iii) $P(3,11) = P(11,3) = 31\ is\ prime\ (true)$
iv) $P(5,11) = P(11,5) = 53\ is\ prime\ (true)$
v) $P(11,11) = P(11,11) = 119\ is\ prime\ (false)$
vi) In summary: the quantified statement is true for all combinations of s and t except (11,11), thus the statement is false.

c) Express the negation in symbols.
   i) $\exists s, t \in S, \sim(P(s,t))$
d) Express the negation in words.
   i) Numbers s and t in the domain S exists, such that $st - 2$ is not prime.
e) Is the negation true or false? Explain.
   i) It is true since the original is false ("P(11,11) is not prime" is true)

76) Let A be the set of circles in the plane with center (0, 0) and let B be the set of circles in the plane with center (1, 1). Furthermore, let $P(C_1, C_2)$: $C_1 and\ C_2$ have exactly two points in common. Be an open sentence where the domain of C_1 is A and the domain of C_2 is B.

a) Express the following quantified statement in words: $\forall C_1 \in A, \exists C_2 \in B, P(C_1, C_2)$.
   i) For every circle C_1 in the plane with center (0, 0) there exists some circle C_2 in the plane with center (1, 1), such that $C_1 and\ C_2$ have exactly two points in common.
b) Express the negation of the statement in symbols.
   i) $\exists C_1 \in A, \forall C_2 \in B, \left(\sim P(C_1, C_2)\right)$
c) Express the negation in words.
   i) A circle C_1 in the plane with center (0, 0) exists, such that every circle C_2 in the plane with center (1,1), $C_1 and\ C_2$ have exactly two points in common.

77) For a triangle T, let r(T) denote the ratio of the length of the longest side of T to the length of the smallest side of T. Let A denote the set of all triangles and let $P(T_1, T_2)$: $r(T_2) \geq r(T_1)$. Be an open sentence where the domain of both $T_1 and\ T_2$ is A.

a) Express the following quantified statement in words: $\exists T_1 \in A, \forall T_2 \in A, P(T_1, T_2)$
   i) There exists a triangle T_1 such that for every triangle T_2, $r(T_2) \geq r(T_1)$.
b) Express the negation in symbols.
   i) $\forall T_1 \in A, \exists T_2 \in A, \sim P(T_1, T_2)$
c) Express the negation in words.
   i) For every triangle T_1, there exists a triangle T_2 such that $r(T_2) < r(T_1)$.

78) Consider the open sentence $P(a, b)$: $\frac{a}{b} < 1$. Where the domain of $a\ is\ A = \{2, 3, 5\}$ and the domain of $b\ is\ B = \{2, 4, 6\}$.

a) State the quantified statement: $\forall a \in A, \exists b \in B, P(a, b)$. In words.
   i) For every integer a in A, there exists an integer b in B such that a/b < 1.
b) Show the statement is true.
   i) For a=2: $P(2, 4)$ is less than 1
   ii) For a=3: $P(3, 4)$ is less than 1
   iii) For a=5: $P(5, 6)$ is less than 1
   iv) Thus a number b exists for every a, such that the statement is true.

79) Consider the open sentence $Q(a, b)$: $a - b < 0$, where the domain of $a\ is\ A = \{3, 5, 8\}$ and the domain of b is $B = \{3, 6, 10\}$.
   a) State the quantified statement $\exists b \in B, \forall a \in A, Q(a, b)$ in words.
      i) There exists an integer b in B such that for every integer a in A, $a - b < 0$.
   b) Show the quantified statement (a) is true.
      i) When b is 10, the statement is true for all values of a. Thus b in B can be indeed be found to make the statement true for all a's.
      ii) $3 - 10 = -7; 5 - 10 = -5; 8 - 10 = -2$

Section 11: Characterizations

Exercises

80) Give a definition of each of the following and the state a characterization of each.
   a) Two lines in the plane are perpendicular
      i) Definition: Two lines in the plane are perpendicular if they intersect at a 90° angle.
      ii) Characterization: Two lines in the plane are perpendicular if and only if the slopes of the lines are opposite reciprocals.
   b) A rational number
      i) Definition: A rational number is a real number that can be expressed as a fraction of two integers.
      ii) Characterization: A real number r is rational if and only if it is not irrational.

81) Define an integer n to be odd if n is not even. State a characterization odd integers.
   a) An integer n is odd if and only if $\frac{n+1}{2}$ is an integer.
   b) An integer n is odd if and only in n^2 is odd.

82) Define a triangle to be isosceles if it has two equal sides. Which of the following statements are characterizations of isosceles triangles? If a statement is not a characterization of isosceles triangles, then explain why.
   a) If a triangle is equilateral, then it is isosceles.
      i) This is not a characterization, but an implication in the form $P \Rightarrow Q$, and a triangle being isosceles does not necessitate it being equilateral.
   b) A triangle T is isosceles if and only if T has two equal sides.
      i) This is not a characterization because it is equal to the definition.
   c) If a triangle has two equal sides, then it is isosceles.
      i) This is an implication.
   d) A triangle T is isosceles if and only if T is equilateral.
      i) This is a characterization.
   e) If a triangle has two equal angles, then it is isosceles.
      i) This is an implication.
   f) A triangle T is isosceles if and only if T has two equal angles.
      i) This is a characterization.

83) By definition, a right triangle is a triangle one of whose angles is a right angle. Also, two angles in a triangle are complementary if the sum of their degrees is 90°. Which of the following

statements are characterizations of a right triangle? If a statement is not a characterization of a right triangle, then explain why.

a) A triangle is a right triangle if and only if two of its sides are perpendicular.
   i) Characterization.
b) A triangle is a right triangle if and only if it has two complementary angles.
   i) Characterization (since it follows that the remaining angle is 90°).
c) A triangle is a right triangle if and only if its area is half of the product of the lengths of some pair of its sides.
   i) Characterization (the two sides perpendicular to each other, other triangles use height and base).
d) A triangle is a right triangle if and only if the square of the length of its longest side equals to the sum of the squares of the lengths of the two smallest sides.
   i) Characterization (Pythagoras).
e) A triangle is a right triangle if and only if twice of the area of the triangle equals the area of some triangle.
   i) False, this must be the case for other kinds of triangles as well.

84) Two distinct lines in the plane are defined to be parallel if and only if they don't intersect. Which of the following is a characterization of parallel lines?

a) Two distinct lines l_1 and l_2 are parallel if and only if any line l_3 that is perpendicular to l_1 is also perpendicular to l_2.
   i) Characterization.
b) Two distinct lines l_1 and l_2 are parallel if and only if any line distinct from l_1 and l_2 that doesn't intersect l_1 also doesn't intersect l_2.
   i) Characterization since this means that the third line is also parallel to the lines.
c) Two distinct lines l_1 and l_2 are parallel if and only if whenever a line l intersects l_1 in an acute angle α, then l also intersects l_2 at an acute angle α.
   i) Characterization.
d) Two distinct lines l_1 and l_2 are parallel if and only if whenever a point P is not on l_1, the point P is not on l_2.
   i) Not a characterization since the lines can be offset from each other.

Additional Exercises

85) Construct a truth table for $P \wedge \big(Q \Rightarrow (\sim P)\big)$

P	Q	$Q \Rightarrow (\sim P)$	$P \wedge \big(Q \Rightarrow (\sim P)\big)$
T	T	F	**F**
T	F	T	**T**
F	T	T	**F**
F	F	T	**F**

86) Given that the implication $(Q \vee R) \Rightarrow (\sim P)$ is false and Q is false, determine the truth values of R and P.

a) The only scenario in which an implication is false is: $true \Rightarrow false$.
b) Thus, **R and P must be true.**

87) Find a compound statement involving the component statements P and Q that has the truth table given below.

P	Q	~Q	
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

a) $P \vee (\sim Q)$

88) Determine the truth value of each of the following quantified statements:
a) $\exists x \in \mathbb{R}, x^3 + 2 = 0$
   i) True, e.g. $-\sqrt[3]{2}$
b) $\forall n \in \mathbb{N}, 2 \geq 3 - n$
   i) True.
c) $\forall x \in \mathbb{R}, |x| = x$
   i) False (negative numbers)
d) $\exists x \in \mathbb{Q}, x^4 - 4 = 0$
   i) False (solutions, e.g. $\sqrt{2}$, are irrational)
e) $\exists x, y \in \mathbb{R}, x + y = \pi$
   i) True, e.g. 0 + π.
f) $\forall x, y \in \mathbb{R}, x + y = \sqrt{x^2 + y^2}$
   i) False (negative numbers)

89) Rewrite each of the implications below using (1) only if and (2) sufficient.
a) If a function f is differentiable, then f is continuous
   i) The function f is differentiable only if f is continuous
   ii) The function f being differentiable is sufficient for f being continuous.
b) If $x = -5, then\ x^2 = 25$
   i) The number $x = -5$ only if $x^2 = 25$
   ii) $x = -5$ is sufficient for $x^2 = 25$

90) Let $P(n): n^2 - n + 5\ is\ a\ prime$. Be an open sentence over a domain S.
a) Determine the truth values of the quantified statements $\forall n \in S, P(n)$ and $\exists n \in S, \sim P(n)$ for $S = \{1, 2, 3, 4\}$
   i) $\forall n \in S, P(n)$ is true
      (1) $P(1) = 5\ is\ a\ prime$
      (2) $P(2) = 5\ is\ a\ prime$
      (3) $P(3) = 11\ is\ a\ prime$
      (4) $P(4) = 17\ is\ a\ prime$
   ii) $\exists n \in S, \sim P(n)$ is false because its negation is true
b) Determine the truth values of the quantified statements $\forall n \in S, P(n)$ and $\exists n \in S, \sim P(n)$ for $S = \{1, 2, 3, 4, 5\}$
   i) $\exists n \in S, \sim P(n)$ is true for n = 5
   ii) $\forall n \in S, P(n)$ is false because its negation is true

c) How are the statements in (a) and (b) related?
   i) They are negations of each other

91) For statements P, Q and R, show that:
   a) $\big((P \wedge Q) \Rightarrow R\big) \equiv \Big(\big(P \wedge (\sim R)\big) \Rightarrow (\sim Q)\Big)$
      i) $\big((P \wedge Q) \Rightarrow R\big)$
      ii) $(\sim(P \wedge Q) \vee R)$ Theorem 21-a
      iii) $\big((\sim P) \vee (\sim Q) \vee R\big)$ Theorem 18-4-b (De Morgan's Laws)
      iv) $\big((\sim P) \vee R \vee (\sim Q)\big)$ Theorem 18-1-a (and 18-2-a)
      v) $\Big(\sim\big(P \wedge (\sim R)\big) \vee (\sim Q)\Big)$ Theorem 18-4-b (De Morgan's Laws)
      vi) $\Big(\big(P \wedge (\sim R)\big) \Rightarrow (\sim Q)\Big)$ Theorem 21-a
   b) $\big((P \wedge Q) \Rightarrow R\big) \equiv \Big(\big(Q \wedge (\sim R)\big) \Rightarrow (\sim P)\Big)$
      i) $\big((P \wedge Q) \Rightarrow R\big)$
      ii) $(\sim(P \wedge Q) \vee R)$ Theorem 21-a
      iii) $\big((\sim P) \vee (\sim Q) \vee R\big)$ Theorem 18-4-b (De Morgan's Laws)
      iv) $\big((\sim Q) \vee R \vee (\sim P)\big)$ Theorem 18-1-a (and 18-2-a)
      v) $\Big(\sim\big(Q \wedge (\sim R)\big) \vee (\sim P)\Big)$ Theorem 18-4-b (De Morgan's Laws)
      vi) $\Big(\big(Q \wedge (\sim R)\big) \Rightarrow (\sim P)\Big)$ Theorem 21-a

92) For a fixed integer n, use Exercise 91 to restate the following implication in two different ways: If n is a prime and n > 2, then n is odd. (P: n is a prime; Q: n > 2; R: n is odd.)
   a) If n is a prime and n is even, then $n \leq 2$.
   b) If $n > 2$ and n is even, then n is not a prime.

93) For fixed integers n and m, use Exercise 91 to restate the following implication in two different ways: If m is even and n is odd, then m + n is odd. (P: m is even; Q: n is odd; R: m + n is odd.)
   a) If m is even and $m + n$ is even, then n is even.
   b) If n is odd and $m + n$ is even, then m is odd.

94) For a real-valued function f and a real number x, use Exercise 91 to restate the following implication in two different ways:

$$If\ f'(x) = 3x^2 - 2x\ and\ f(0) = 4, then\ f(x) = x^3 - x^2 + 4$$

   a) If $f'(x) = 3x^2 - 2x$ and $f(x) \neq x^3 - x^2 + 4$, then $f(0) \neq 4$.
   b) If $f(0) = 4$ and $f(x) \neq x^3 - x^2 + 4$, then $f'(x) \neq 3x^2 - 2x$.

95) For the set $S = \{1, 2, 3\}$, give an example of three open sentences P(n), Q(n) and R(n), each over the domain S, such that (1) each of P(n), Q(n) and R(n) is a true statement for exactly two elements of S, (2) all of the implications $P(1) \Rightarrow Q(1), Q(2) \Rightarrow R(2)\ and\ R(3) \Rightarrow P(3)$ are true, and (3) the converse of each implication in (2) is false.
   a) (3) the converse of the implications have to be false, thus $P(1), Q(2)$ and $R(3)$ must be false, while $Q(1), R(2)$ and $P(3)$ must be true.
      i) Thus: $P(n)$ must be a sentence that is false for $n = 1$, and true for all $n \in S - \{1\}$

ii) $Q(n)$ must be a sentence that is false for $n = 2$, and true for all $n \in S - \{2\}$
iii) $R(n)$ must be a sentence that is false for $n = 3$, and true for all $n \in S - \{3\}$

b) Final sentences for (1), (2) and (3):
   i) $P(n): n = 2\ or\ n = 3$
   ii) $Q(n): n = 1\ or\ n = 3$
   iii) $R(n): n = 1\ or\ n = 2$

96) Do there exist a set S of cardinality 2 and a set {P(n), Q(n), R(n)} of three open sentences over the domain S, such that (1) the implications $P(a) \Rightarrow Q(a), Q(b) \Rightarrow R(b), R(c) \Rightarrow P(c)$ are true, where $a, b, c \in S$ and (2) the converses of the implications in (1) are false? Necessarily, at least two of these elements a, b and c of S are equal.

a) (2) the converse of the implications have to be false, thus $P(a), Q(b)$ and $R(c)$ must be false, while $Q(a), R(b)$ and $P(c)$ must be true.
   i) Thus: $P(n)$ must be a sentence that is false for $n = a$, and true for $n = c$, thus $a \neq c$ and $a = b$ or $b = c$ since the cardinality of the set is 2.
   ii) $Q(n)$ must be a sentence that is false for $n = b$, and true for $n = a$, thus $a \neq b$. Combining this knowledge with previous (i): $b = c$
   iii) $R(n)$ must be true for $n = b$ and false for $n = c$, but this contradicts the previous claims, and thus a cardinality of 2 is too little for this logic to be possible.

b) Conclusion: No, there does not exists a set of cardinality 2 for (1) and (2).

97) Let $A = \{1, 2, \dots, 6\}$ and $B = \{1, 2, \dots, 7\}$. For $x \in A, let\ P(x): 7x + 4\ is\ odd$. For $y \in B, let\ Q(y): 5y + 9\ is\ odd$. Let $S = \{(P(x), Q(y)): x \in A, y \in B, P(x) \Rightarrow Q(y)\ is\ false\}$.

a) What is |S|?
   i) $P(x)$ is true for $x \in A - \{2, 4, 6\}$
   ii) $Q(y)$ is true for $y \in B - \{1, 3, 5, 7\}$
   iii) The combinations of $(P(x), Q(y))$ must be false for $P(x) \Rightarrow Q(y)$, thus $P(x)$ must be true and $Q(y)$ must be false.
   iv) Thus the possible combinations are $(P(1), Q(y)): y \in \{1, 3, 5, 7\}$ + $(P(3), Q(y)): y \in \{1, 3, 5, 7\}$ + $(P(5), Q(y)): y \in \{1, 3, 5, 7\}$
   v) This totals 12 different combinations, thus |S| = 12.

98) Let $P(x, y, z)$ be an open sentence, where the domain of x, y and z are A, B, and C, respectively.

a) State the quantified statement $\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z)$ in words.
   i) For every x in A and y in B, a z exists such that P(x, y, z) is true.

b) State the quantified statement $\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z)$ in words for $P(x, y, z): x = yz$
   i) For every x in A and y in B, a z exists such that $x = yz$.

c) Determine whether the quantified statement in (b) is true when $A = \{4, 8\}, B = \{2, 4\}\ and\ C = \{1, 2, 4\}$.
   i) $P(4, 2, 2)$ is true, $P(4, 4, 1)$ is true, $P(8, 2, 4)$ is true and $P(8, 4, 2)$ is true
   ii) Conclusion: the quantified statement in (b) is true.

99) Let $P(x, y, z)$ be an open sentence, where the domains of $x, y\ and\ z$ are $A, B\ and\ C$ respectively.

a) Express the negation of $\forall x \in A, \forall y \in B, \exists z \in C, P(x, y, z)$ in symbols.
   i) $\exists x \in A, \exists y \in B, \forall z \in C, \sim P(x, y, z)$
b) Express the negation in words.
   i) There exists some x in A and some y in B, such that for every z in C, $P(x, y, z)$ is true.
c) Determine whether $\exists x \in A, \exists y \in B, \forall z \in C, \sim P(x, y, z)$ is true when $P(x, y, z): x + z = y$. For $A = \{1,3\}, B = \{3,5,7\}\ and\ C = \{0,2,4,6\}$
   i) First of all $\sim P(x, y, z): x + z \neq y$
   ii) $\sim P(1, 3, 2)$ is false
   iii) $\sim P(1, 5, 4)$ is false
   iv) $\sim P(1, 7, 6)$ is false
   v) $\sim P(3, 3, 0)$ is false
   vi) $\sim P(3, 5, 2)$ is false
   vii) $\sim P(3, 7, 4)$ is false
   viii) Conclusion: an x in A and a y in B does not exists such that for all z in C the statement is true, thus the negated expression is false.

100) Write each of the following using “if, then”.

a) A sufficient condition for a triangle to be isosceles is that it has two equal angles.
   i) If a triangle is isosceles, then it has two equal angles.
b) Let C be a circle of diameter $\sqrt{\left(\frac{2}{\pi}\right)}$. Then the area of C is ½.
   i) If the diameter of a circle C is $\sqrt{\left(\frac{2}{\pi}\right)}$, then the area of C is ½.
c) The 4^{th} power of every odd integer is odd.
   i) If an integer n is odd, then the 4^{th} power of n is odd.
d) Suppose that the slope of a line l is 2. Then the equation of l is $y = 2x + b$ for some real number b.
   i) If the slope of a line l is 2, then the equation of l is $y = 2x + b$ for some real number b.
e) Whenever a and b are nonzero rational numbers, a/b is a nonzero rational number.
   i) If a and b are nonzero rational numbers, then a/b is a nonzero rational number.
f) For every three integers, there exist two of them whose sum is even.
   i) If x, y and z are integers, then there exists two of them whose sum is even.
g) A triangle is a right triangle if the sum of two of its angles is 90°.
   i) If the sum of two angles of a triangle T is 90, then T is a right triangle.
h) The number $\sqrt{3}$ is irrational.
   i) If a real number n = $\sqrt{3}$, then n is irrational.