2016

Mathematical Proofs

CHAPTER 4 – DIRECT PROOF AND PROOF BY CONTRAPOSITIVE (EXERCISE SOLUTIONS)

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Section 1: Trivial and Vacuous Proofs

Exercises

1) Let $x \in \mathbb{R}$. Prove that if $0 < x < 1$, then $x^2 - 2x + 2 \neq 0$
   a) Since $x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1$, it follows that $x^2 - 2x + 2 \neq 0$ for all $x \in \mathbb{R}$. Hence the statement is true trivially.

2) Let $x \in \mathbb{N}$. Prove that if $|n-1| + |n+1| \leq 1$, then $|n^2 - 1| \leq 4$.
   a) Since $|n-1| \geq 0$ and $|n+1| \geq 2$, it follows that $|n-1| + |n+1| \geq 2$ and the statement $|n-1| + |n+1| \leq 1$ is false for all $n \in \mathbb{N}$. Hence the statement is true vacuously.

3) Let $r \in Q^+$. Prove that if $\frac{r^2+1}{r} \leq 1$, then $\frac{r^2+2}{r} \leq 2$.
   a) Note that $\frac{r^2+1}{r} = r + \frac{1}{r}$. If $r \geq 1$, then $r + \frac{1}{r} > 1$; while if $0 < r < 1$, then $\frac{1}{r} > 1$ and so $r + 1 > 1$. Thus $\frac{r^2+1}{r} \leq 1$ is false for all $r \in Q^+$ and so the statement is true vacuously.

4) Let $x \in \mathbb{R}$. Prove that if $x^3 - 5x - 1 \geq 0$, then $(x-1)(x-3) \geq -2$.
   a) Note that $(x-1)(x-3) = (x-2)^2 - 1$. Since $(x-2)^2 > 0$, it follows that $(x-2)^2 - 1 \geq -1 > -2$ and so the statement is true trivially.

5) Let $n \in \mathbb{N}$. Prove that if $n + \frac{1}{n} < 2$, then $n^2 + \frac{1}{n^2} < 4$.
   a) Since $n^2 + \frac{1}{n^2} = (n-1)^2 \geq 0$, it follows that $n^2 + 1 \geq 2n$ and so $n + \frac{1}{n} \geq 2$. Thus the statement is true vacuously.

6) Prove that if a, b and c are odd integers such that $a + b + c = 0$, then $abc < 0$. (You are permitted to use well-known properties of integers here.)
   a) Since the sum of any two odd integers is always even, and the sum of an even and an odd integer is always odd, the sum of $a + b + c$ will always be odd. Hence $a + b + c = 0$ is always false. Thus the statement is true vacuously.

7) Prove that if $x, y\ and\ z$ are three real numbers such that $x^2 + y^2 + z^2 < xy + xz + yz$, then $x + y + z > 0$
   a) Since $(x-y)^2 + (x-z)^2 + (y-z)^2 \geq 0$, it follows that $x^2 - 2xy + y^2 + x^2 - 2xz + z^2 + y^2 - 2yz + z^2 = 2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz \geq 0$ and so $x^2 + y^2 + z^2 \geq xy + xz + yz$. Thus the statement is true vacuously.

Section 2: Direct Proofs

Exercises

8) Prove that if x is an odd integer, then $9x + 5$ is even.
   a) Assume that x is an odd integer. Since x is odd, we can write $x = 2n + 1$ for some integer n. Now $9x + 5 = 9(2n+1) + 5 = 18n + 14 = 2(9n + 7)$. Since $9n + 7$ is an integer, $9x + 5$ is even.

9) Prove that if x is an even integer, then $5x - 3$ is an odd integer.
   a) Assume that x is an even integer. Since x is even, we can write $x = 2n$ for some integer n. Now $5x - 3 = 5(2n) - 3 = 10n - 3 = 2(5n - 2) + 1$. Since $2n - 2$ is an integer, $5x - 3$ is odd.

10) Prove that if a and c are odd integers, then $ab + bc$ is even for every integer b.
   a) Assume a and c are odd integers. Observe that if b is even, ab and bc will be even, and thus $ab + bc$ will be even. If b is odd, ab and bc will be odd, and thus $ab + bc$ will also be even.

11) Let $n \in \mathbb{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.
   a) Assume that $1 - n^2 > 0$. Then $n = 0$. Thus $3 \cdot 0 - 2 = -2$ is an even integer.

12) Let $x \in \mathbb{Z}$. Prove that if 2^{2x} is an odd integer, then 2^{-2x} is an odd integer.
   a) Assume 2^{2x} is odd. Then $x = 0$. Thus $2^{-2\cdot 0} = 1$ is an odd integer.

13) Let $S = \{0, 1, 2\}$ and let $n \in S$. Prove that if $\frac{(n+1)^2(n+2)^2}{4}$ is even, then $\frac{(n+2)^2(n+3)^2}{4}$ is even.
   a) Assume that $\frac{(n+1)^2(n+2)^2}{4}$ is even. Then $n = 2$ and $\frac{(n+2)^2(n+3)^2}{4} = \frac{16\cdot 25}{4} = 100$, which is even.

14) Let $S = \{1, 5, 9\}$. Prove that if $n \in S$ and $\frac{n^2+n-6}{2}$ is odd, then $\frac{2n^3+3n^2+n}{6}$ is even.
   a) Assume $n \in S$ and $\frac{n^2+n-6}{2}$ is odd. Then $n = 9$ and $\frac{2n^3+3n^2+n}{6} = \frac{2\cdot 9^3+3\cdot 9^2+9}{6} = \frac{1710}{6} = 285$. Thus the statement is false!

15) Let $A = \{n \in \mathbb{Z} : n > 2 \; and \; n \; is \; odd\}$ and $B = \{n \in \mathbb{Z} : n < 11\}$. Prove that if $n \in A \cap B, then \; n^2 - 2$ is prime.
   a) Assume that $n \in A \cap B = \{3, 5, 7, 9\}$. Then $\{n^2 - 2 : n \in A \cap B\} = \{7, 23, 47, 79\}$ are all primes.