Mathematical Proofs

Chapter 1 – Exercise solutions and notes

Section 1:

Exercises

1) Which of the following sentences are statements? Indicate their truth value.
   a) False
   b) True
   c) Not a statement
   d) Not a statement
   e) Not a statement
   f) Not a statement
   g) Not a statement

2) Consider the sets A, B, C and D…
   a) True, since an integer *n* can be found for $1 + 3n = 25$
   b) False, since $33 \notin D$
   c) False, since $22 \in A$
   d) True, since all prime numbers except 2 are uneven
   e) True, since Ø has no elements
   f) False since 53 is a prime thus $53 \in C$

3) Which of the following statements are true?
   a) False, since Ø has no elements
   b) True since Ø is contained in {Ø}
   c) True since sets are unordered
   d) False since * is not equal to the set {Ø}
   e) True since Ø has no elements
   f) False since 1 is not a set

4) $x(x-1) = 6 \rightarrow x^2 - x = 6 : x \in \mathbb{R}$
   a) $T: \{3, -2\}$
   b) $F: \{x \in \mathbb{R} : x \neq 3, -2\}$

5) $3x - 2 > 4 : x \in \mathbb{Z}$
   a) T: $\{x \in \mathbb{Z} : x > 2\}$
   b) F: $\{x \in \mathbb{Z} : x \leq 2\}$

6) $P(A)$: $A \subseteq \{1, 2, 3\}$ over $S = P(\{1, 2, 4\})$
   a) $A \subseteq \{1, 2\}$
   b) $A \not\supseteq \{4\}$
   c) $A = \{4\}$

7) $P(n): n\ and\ n+2\ are\ primes$ "twin primes"
   a) $n = \{3, 5, 11, 17, 29, 41, \dots\}$

8) $P(n): \frac{n^2+5n+6}{2}\ is\ even$
   a) $S_1 = \{1, 2, 5\}$
   b) $S_2 = \{3, 4, 7\}$

9) $P(n): n < 6$

10) $P(n) and\ Q(n) : n \in S = \{2, 4, 6, 8\}$
   a) $P(n): \frac{n}{2}\ is\ uneven$
   b) $Q(n): n \in \{2, 8\}$

Section 2: The Negation of a Statement

Exercises

11) State the negation of each statement.
   a) $\sqrt{2}$ is not a rational number
   b) 0 is a negative integer
   c) 111 is not a prime number

12) Complete the truth table.

P	Q	~P	~Q
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

13) State the negation of each of the following statements
   a) The real number r is larger than $\sqrt{2}$
   b) The absolute value of the real number a is at least 3
   c) At most one of the triangles angles is 45°
   d) The area of the circle is less than 9π
   e) The sides of the triangle are of different lengths
   f) The point P in the plane lies inside the circle C

14) State the negation of each of the following statements
   a) At most one of my library books is overdue
   b) None (or both) of my friends misplaced his homework assignment
   c) Some expected that to happen
   d) It's often that my instructor teaches that course
   e) It's not surprising that two students received the same exam score

Section 3: The Disjunction and Conjunction of Statements

Notes

$$P\ or\ Q \rightarrow P \vee Q$$
$$P\ and\ Q \rightarrow P \wedge Q$$

Exercises

15) Complete the truth table.

P	Q	~Q	$P \wedge (\sim Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

16) For the sets A and B, consider the statements…

a) False
b) True
c) False
d) False
e) True

17) Let P: 15 is odd and Q: 21 is prime

a) True
b) False
c) False
d) True

18) $P(A): A \cap \{2,4,6\} = \emptyset \quad and \quad Q(A): A \neq \emptyset$

a) $P(A) \wedge Q(A) = True\ when\ A \in \{1,3,5\}$
b) $P(A) \vee \sim Q(A)) = True\ when\ A \in \{1,3,5\}\ or\ A = \emptyset$
c) $\sim P(A) \wedge \sim Q(A) = True\ when\ A = \{\}$

Section 4: The Implication

Notes

Table 1 - Implication truth table

P	Q	P => Q
True	True	True
True	False	False
False	True	True
False	False	True

Exercises

19) Consider the statements P: 17 is even and Q: 19 is prime. Write each statement in words and indicate whether it is true or false.
   a) $\sim P$: 17 is odd (True)
   b) $P \vee Q$: 17 is even or 19 is prime (True – 19 is prime)
   c) $P \wedge Q$: 17 is even and 19 is prime (False – 17 is odd)
   d) $P \Rightarrow Q$: If 17 is even, then 19 is prime (True – 19 is prime)

20) For statements P and Q, construct a truth table for (P => Q) => (~P)

P => Q	~P	(P => Q) => (~P)
True	False	False
False	False	True
True	True	True
True	True	True

21) Consider the statements P: $\sqrt{2}$ is rational and Q: $\frac{22}{7}$ is rational. Write each of the following statements in words and indicate whether it is true or false.
   a) $P \Rightarrow Q$: If $\sqrt{2}$ is rational, then $\frac{22}{7}$ is rational (True)
   b) $Q \Rightarrow P$: If $\frac{22}{7}$ is rational, then $\sqrt{2}$ is rational (False – $\sqrt{2}$ is not rational)
   c) $(\sim P) \Rightarrow (\sim Q)$: If $\sqrt{2}$ is irrational, then $\frac{22}{7}$ is irrational (False – $\frac{22}{7}$ is not irrational)
   d) $(\sim Q) \Rightarrow (\sim P)$: If $\frac{22}{7}$ is irrational, then $\sqrt{2}$ is irrational (True – $\sqrt{2}$ is irrational)

22) Consider the statements:

$$P{:}\sqrt{2}\ is\ rational.\quad Q{:}\frac{2}{3}\ is\ rational.\quad R{:}\sqrt{3}\ is\ rational.$$

   a) $(P \wedge Q) \Rightarrow R$: If $\sqrt{2}$ and $\frac{2}{3}$ are rational, then $\sqrt{3}$ is rational (True - $\sqrt{2}$ is not rational)
   b) $(P \wedge Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ and $\frac{2}{3}$ are rational, then $\sqrt{3}$ is irrational (True - $\sqrt{2}$ is not rational)
   c) $((\sim P) \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is irrational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational (False - $\sqrt{3}$ is not rational)
   d) $(P \vee Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ or $\frac{2}{3}$ is rational, then $\sqrt{3}$ is irrational (True - $\sqrt{3}$ is irrational)

23) Suppose that $\{S_1, S_2\}$ is a partition of a set S and $x \in S$. Which of the following are true?
   a) If we know that $x \notin S_1$ then x must belong to S_2. (True)
   b) It's possible that $x \notin S_1$ and $x \notin S_2$. (False)
   c) Either $x \notin S_1\ or\ x \notin S_2$. (True)
   d) Either $x \in S_1\ or\ x \in S_2$. (True)
   e) It's possible that $x \in S_1\ and\ x \in S_2$. (False)

24) Two sets A and B are nonempty disjoint subsets of a set S. If $x \in S$, then which of the following are true?
   a) It's possible that $x \in A \cap B$. (False – A and B are disjoint)
   b) If x is an element of A, then x can't be an element of B. (True – A and B are disjoint)

c) If x is not an element of A, then x must be an element of B. (False – It is possible that $A \cup B \neq S$)
d) It's possible that $x \notin A$ and $x \notin B$. (True – It's possible that $A \cup B \neq S$)
e) For each nonempty set C, either $x \in A \cap C$ or $x \in B \cap C$. (False – It is possible that $A \cup B \neq S$)
f) For some nonempty set C, both $x \in A \cup C$ and $x \in B \cup C$. (True if C contains x, False otherwise)

25) A college student makes the following statement: If I receive an A in both Calculus I and Discrete Mathematics this semester, then I'll take either Calculus II or Computer Programming this summer.

$$P: A\ in\ Calculus\ I\ and\ Discrete\ Mathematics$$
$$Q: Takes\ Calculus\ II\ or\ Computer\ Programming$$

a) P is false and Q is true. (True)
b) P is true and Q is false. (False)
c) P is false and Q is true. (True)
d) P is true and Q is true. (True)
e) P is false and Q is false. (True)

26) A college student makes the following statement: If I don't see my advisor today, then I'll see her tomorrow.

$$P: Don't\ see\ advisor\ today$$
$$Q: See\ advisor\ tomorrow$$

a) P is true and Q is false. (False)
b) P is false and Q is true. (True)
c) P is true and Q is true AND P is false and Q is false. (True)
d) P is true and Q is false. (False)

27) The instructor of a computer science class announces…
a) Alice => Ben
b) Ben => Cindy
c) Cindy => Don
d) The two students who attend are Cindy and Don

28) Consider the statement (implication): If Bill takes Sam to the concert, then Sam will take Bill to dinner.

$$P: Bill\ takes\ Sam\ to\ concert$$
$$Q: Sam\ takes\ bill\ to\ dinner$$

a) Q only if P. (False – P can be false and Q true and the implication still holds)
b) Either ~P or Q. (False – The $\sim P \wedge Q$ scenario is also true)
c) P is true. (False – Q doesn't happen)
d) P is true and Q is true. (True)
e) P is true and Q is false. (False)
f) P is false. (True)
g) P is false. (True)

29) Let P and Q be statements. Which of the following implies that $P \vee Q$ is false?
   a) $(\sim P) \vee (\sim Q)$ is false. (False – P or Q can be true)
   b) $(\sim P) \vee Q$ is true. (False – Q can be true)
   c) $(\sim P) \wedge (\sim Q)$ is true. (True – both P and Q must be false)
   d) $Q \Rightarrow P$ is true. (False – P or Q can be true)
   e) $P \wedge Q$ is false. (False – one of them can be true)

Section 5: More on Implications

Notes

$$\mathbb{R}: Real\ numbers\ (all\ real\ numbers)$$
$$\mathbb{Q}: Rational\ numbers$$
$$\mathbb{N}: Natural\ numbers\ (positive\ integers, starting\ from\ 1)$$
$$\mathbb{Z}: Integers\ (positive\ and\ negative\ including\ 0)$$

Exercises

30) Consider the open sentences $P(n): 5n + 3\ is\ prime$. And $Q(n): 7n + 1\ is\ prime$. Both over the domain $\mathbb{N}$. State in words.
   a) $P(n) \Rightarrow Q(n)$: If $5n + 3$ is prime, then $7n + 1$ is prime.
   b) $P(2) \Rightarrow Q(2)$: If 13 is prime, then 15 is prime. (False – 15 is not prime)
   c) $P(6) \Rightarrow Q(6)$: if 33 is prime, then 43 is prime. (True – 33 is not prime)

31) In each of the following, two open sentences P(x) and Q(x) over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.
   a) $P(x): |x| = 4; Q(x): x = 4; S = \{-4, -3, 1, 4, 5\}$
      i) $S_{true} = \{-3, 1, 4, 5\}$
   b) $P(x): x^2 = 16; Q(x): |x| = 4; S = \{-6, -4, 0, 3, 4, 8\}$
      i) $S_{true} = \{-6, -4, 0, 3, 4, 8\}$ aka $true\ for\ all\ x \in S$
   c) $P(x): x > 3; Q(x): 4x - 1 > 12; S = \{0, 2, 3, 4, 6\}$
      i) $S_{true} = \{0, 2, 3, 4, 6\}$ aka $true\ for\ all\ x \in S$

32) In each of the following, two open sentences P(x) and Q(x) over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.
   a) $P(x): x - 3 = 4; Q(x): x \geq 8; S = \mathbb{R}$
      i) True for $x \neq 7$
   b) $P(x): x^2 \geq 1; Q(x): x \geq 1; S = \mathbb{R}$
      i) True for $x > -1$
   c) $P(x): x^2 \geq 1; Q(x): x \geq 1; S = \mathbb{N}$
      i) True for all $x \in S$
   d) $P(x): x \in [-1, 2]; Q(x): x^2 \leq 2; S = [-1, 1]$
      i) True for all $x \in S$ since Q(x) is true for all $x \in S$

33) In each of the following, two open sentences P(x, y) and Q(x, y) are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x, y) \Rightarrow Q(x, y)$ for the given values of x and y.
   a) $P(x, y): x^2 - y^2 = 0;\ Q(x, y): x = y; (x, y) \in \{(1, -1), (3,4), (5,5)\}$

i) True for $(x, y) \in \{(3,4), (5,5)\}$

b) $P(x, y): |x| = |y|; Q(x, y): x = y; (x, y) \in \{(1,2), (2, -2), (6,6)\}$
   i) True for $(x, y) \in \{(1,2), (6,6)\}$
c) $P(x, y): x^2 + y^2 = 1; Q(x, y): x + y = 1; (x, y) \in \{(1, -1), (-3,4), (0, -1), (1,0)\}$
   i) True for $(x, y) \in \{(1, -1), (-3,4), (1,0)\}$

34) Each of the following describes an implication. Write the implication in the form "if, then."
a) If a point on the straight line is given by $2y + x - 3 = 0$ and x is an integer, then y an integer.
b) If n is odd then n^2 is odd.
c) If $3n + 7$ is even and $n \in \mathbb{Z}$, then n is odd.
d) If $f(x) = cosx$, then $f'(x) = -sinx$
e) If the circumference of C is 4π, then the area of C is 4π
f) If n^3 is even, then n is even.

Section 6: The Biconditional

Notes

$$Biconditional: (P \Rightarrow Q) \wedge (Q \Rightarrow P) = P \Leftrightarrow Q$$

P	Q	P => Q	Q => P	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$$The\ biconditional\ P \Leftrightarrow Q\ is\ often\ stated\ as: P = Q$$

Exercises

35) Let P: 18 is odd and Q: 25 is even. State $P \Leftrightarrow Q$ in words. Is $P \Leftrightarrow Q$ true or false?
a) 18 is odd if and only if 25 is even.
b) True (both are false)

36) Let P(x): x is odd and $Q(x): x^2 is\ odd$. Be open sentences over the domain $\mathbb{Z}$. State $P(x) \Leftrightarrow Q(x)$ in two ways: (1) using "if and only if" and (2) using "necessary and sufficient".
a) x is odd if and only if x^2 is odd
b) x being odd is a necessary and sufficient condition for x^2 being odd

37) For the open sentences $P(x): |x - 3| < 1; Q(x): x \in \{2, 4\}$. Over the domain $\mathbb{R}$, state the biconditional $P(x) \Leftrightarrow Q(x)$ in two different ways.
a) $|x - 3| < 1$ if and only if $x \in \{2,4\}$
b) The condition $|x - 3| < 1$ is necessary and sufficient for $x \in \{2,4\}$

38) Consider the open sentences: $P(x): x = -2; Q(x): x^2 = 4$ over the domain $S = \{-2, 0, 2\}$. State each of the following in words and determine all values of $x \in S$ for which the resulting statement is true.
   a) $\sim P(x)$
      i) $x \neq -2$
      ii) True for all $x \in \{0, 2\}$
   b) $P(x) \vee Q(x)$
      i) $x = -2 \text{ or } x^2 = 4$
      ii) True for all $x \in \{-2, 2\}$
   c) $P(x) \wedge Q(x)$
      i) $x = -2 \text{ and } x^2 = 4$
      ii) True for x = -2
   d) $P(x) \Rightarrow Q(x)$
      i) If $x = -2$ then $x^2 = 4$
      ii) True for all $x \in S$
   e) $Q(x) \Rightarrow P(x)$
      i) If $x^2 = 4$ then $x = -2$
      ii) True for $x \in \{-2, 0\}$
   f) $P(x) \Leftrightarrow Q(x)$
      i) $x = -2$ if and only if $x^2 = 4$
      ii) True for all $x \in \{-2, 0\}$

39) For the following open sentences $P(x)$ and Q(x) over domain S, determine all values of $x \in S$ for which the biconditional $P(x) \Leftrightarrow Q(x)$ is true.
   a) $P(x): |x| = 4; Q(x): x = 4; S = \{-4, -3,1,4,5\}$
      i) True for all $x \in \{-3, 1, 4, 5\}$
      ii) Alt. notation: True for all $x \in S - \{-4\}$
   b) $P(x): x \geq 3; Q(x): 4x - 1 > 12; S = \{0,2,3,4,6\}$
      i) True for all $x \in \{0, 2, 4, 6\}$
      ii) Alt. notation: True for all $x \in S - \{3\}$
   c) $P(x): x^2 = 16; Q(x): x^2 - 4x = 0; S = \{-6, -4,0,3,4,8\}$
      i) True for all $x \in \{-6, 3, 4, 8\}$
      ii) Alt. notation: True for all $x \in S - \{-4, 0\}$

40) In each of the following, two open sentences P(x,y) and Q(x,y) are given, where the domain of both x and y is $\mathbb{Z}$. Determine the truth value of $P(x, y) \Leftrightarrow Q(x, y)$ for the given values of x and y.
   a) $P(x, y): x^2 - y^2 = 0; Q(x, y): x = y; (x, y) \in \{(1, -1), (3,4), (5,5)\}$
      i) True for all $(x, y) \in \{(3,4), (5,5)\}$
   b) $P(x, y): |x| = |y|; Q(x, y): x = y; (x, y) \in \{(1,2), (2, -2), (6,6)\}$
      i) True for all $(x, y) \in \{(1,2), (6,6)\}$
   c) $P(x, y): x^2 + y^2 = 1; Q(x, y): x + y = 1; (x, y) \in \{(1, -1), (-3,4), (0, -1), (1,0)\}$
      i) True for all $(x, y) \in \{(1, -1), (1,0)\}$

41) Determine all values of n in the domain $S = \{1,2,3\}$ for which the following is a true statement: A necessary and sufficient condition for $\frac{n^3+n}{2}$ to be even is that $\frac{n^2+n}{2}$ is odd.
   a) $n = 1$
      i) $\frac{2}{2}$ is even and $\frac{2}{2}$ is odd. (False)
   b) $n = 2$
      i) $\frac{10}{2}$ is even and $\frac{6}{2}$ is odd. (False)
   c) $n = 3$
      i) $\frac{90}{2}$ is even and $\frac{12}{2}$ is odd (True – both are false)

42) Determine all values of n in the domain $S = \{2,3,4\}$ for which the following is a true statement: The integer $\frac{n(n-1)}{2}$ is odd if and only if $\frac{n(n+1)}{2}$ is even.
   a) $n = 2$
      i) $\frac{2}{2}$ is odd if and only if $\frac{6}{2}$ is even. (False)
   b) $n = 3$
      i) $\frac{6}{2}$ is odd if and only if $\frac{12}{2}$ is even. (True)
   c) $n = 4$
      i) $\frac{12}{2}$ is odd if and only if $\frac{20}{2}$ is even. (False)

43) Let $S = \{1,2,3\}$. Consider the following open sentences over the domain S. Determine three distinct elements a, b, c in S such that…

$$P(n): \frac{(n+4)(n+5)}{2} \ is\ odd$$

$$Q(n): 2^{n-2} + 3^{n-2} + 6^{n-2} > (2.5)^{n-1}$$

   a) $P(a) \Rightarrow Q(a)$ is false
      i) $P(1)$ is true and $Q(1)$ is false
      ii) $a = 1$
   b) $Q(b) \Rightarrow P(b)$ is false
      i) $P(3)$ is false and $Q(3)$ is true
      ii) $b = 3$
   c) $P(a) \Leftrightarrow Q(a)$ is true
      i) $P(2)$ is true and $Q(2)$ is true
      ii) $c = 2$

44) Let $S = \{1,2,3,4\}$. Consider the following open sentences over the domain S. Determine four distinct elements a, b, c, d in S such that…

$$P(n): \frac{n(n-1)}{2} \ is\ even$$
$$Q(n): 2^{n-2} - (-2)^{n-2} \ is\ even$$
$$R(n): 5^{n-1} + 2^{n} \ is\ prime$$

Table 2 - Results of P(n), Q(n) and R(n) given n in {1, 2, 3, 4}

n	P(n)	Q(n)	R(n)
1	0 True	1 False	3 True
2	1 False	0 True	9 False
3	3 False	4 True	33 False
4	6 True	0 True	141 False

a) $P(a) \Rightarrow Q(a)$ is false
   i) $a \in \{1\} \rightarrow a = 1$
b) $Q(b) \Rightarrow P(b)$ is true
   i) $b \in \{1, 4\} \rightarrow b = 4$
c) $P(c) \Leftrightarrow R(c)$ is true
   i) $c \in \{1, 2, 3\} \rightarrow c = 2$
d) $Q(d) \Leftrightarrow R(d)$ is false
   i) $d \in \{1, 2, 3, 4\} \rightarrow d = 3$

45) Let $P(n)$: $2^n - 1$ is a prime; $Q(n)$: $n\ is\ a\ prime$. Be open sentences over the domain $S = \{2,3,4,5,6,11\}$. Determine all values of $n \in S$ for which $P(n) \Leftrightarrow Q(n)$ is a true statement.
a) $2: 2^2 - 1\ is\ a\ prime\ if\ and\ only\ if\ 2\ is\ a\ prime$
   i) True (both statements are true)
b) $3: 2^3 - 1\ is\ a\ prime\ if\ and\ only\ if\ 3\ is\ a\ prime$
   i) True (both statements are true)
c) $4: 2^4 - 1\ is\ a\ prime\ if\ and\ only\ if\ 4\ is\ a\ prime$
   i) True (both statements are false)
d) $5: 2^5 - 1\ is\ a\ prime\ if\ and\ only\ if\ 5\ is\ a\ prime$
   i) True (both statements are true)
e) $6: 2^6 - 1\ is\ a\ prime\ if\ and\ only\ if\ 6\ is\ a\ prime$
   i) True (both statements are false)
f) $11: 2^{11} - 1\ is\ a\ prime\ if\ and\ only\ if\ 11\ is\ a\ prime$
   i) False ($2^{11} - 1$ is not a prime but 11 is)
g) SUMMARY: True for all $n \in S - \{11\}$

Section 7: Tautologies and Contradictions

Notes

$$Tautology: A\ compound\ statement\ which\ is\ always\ true, e.g. P \vee \sim P$$
$$Contradiction: A\ compound\ statement\ which\ is\ always\ false, e.g. P \wedge \sim P$$

Exercises

46) For statements P and Q, show that $P \Rightarrow (P \vee Q)$ is a tautology

P	Q	$(P \vee Q)$	$P \Rightarrow (P \vee Q)$
T	T	T	**T**
T	F	T	**T**
F	T	T	**T**
F	F	F	**T**

47) For statements P and Q, show that $\big(P \wedge (\sim Q)\big) \wedge (P \wedge Q)$ is a contradiction

P	Q	$P \wedge Q$	$P \wedge (\sim Q)$	$\big(P \wedge (\sim Q)\big) \wedge (P \wedge Q)$
T	T	T	F	**F**
T	F	F	T	**F**
F	T	F	F	**F**
F	F	F	F	**F**

48) For statements P and Q, show that $\big(P \wedge (P \Rightarrow Q)\big) \Rightarrow Q$ is a tautology. Then state the compound statement in words. (This is an important logical argument form, called **modus ponens.**)

a) If P is true and P implies Q, then Q is true.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$\big(P \wedge (P \Rightarrow Q)\big) \Rightarrow Q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

49) For statements P, Q and R, show that $\big((P \Rightarrow Q) \wedge (Q \Rightarrow R)\big) \Rightarrow (P \Rightarrow R)$ is a tautology. Then state this compound statement in words. (This is another important logical argument form, called **syllogism**.)

a) If P implies Q and Q implies R, then P implies R

P	Q	R	$(P \Rightarrow Q)$	$(Q \Rightarrow R)$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$(P \Rightarrow R)$	$\big((P \Rightarrow Q) \wedge (Q \Rightarrow R)\big) \Rightarrow \cdots$
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	F	T	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	F	T	**T**
F	F	T	T	T	T	T	**T**
F	F	F	T	T	T	T	**T**

50) Let R and S be compound statements involving the same compound statements. If R is a tautology and S is a contradiction, then what can be said of the following?

a) $R \vee S$ is true, since R is always true
b) $R \wedge S$ is false, since S is always false
c) $R \Rightarrow S$ is false, since $'true \Rightarrow false'$ is false
d) $S \Rightarrow R$ is true, since $'false \Rightarrow true'$ is true

Section 8: Logical Equivalence

Notes

$$Logical\ equivalence: P \Rightarrow Q \equiv (\sim P) \vee Q$$

Exercises

51) For statements P and Q, the implication $(\sim P) \Rightarrow (\sim Q)$ is called the inverse of the implication $P \Rightarrow Q$.
a) Use a truth table to show that these statements are not logically equivalent

P	Q	$P \Rightarrow Q$	$(\sim P) \Rightarrow (\sim Q)$
T	T	T	T
T	F	**F**	**T**
F	T	**T**	**F**
F	F	T	T

b) Find another implication that is logically equivalent to $(\sim P) \Rightarrow (\sim Q)$ and verify your answer
i) $P \vee (\sim Q) \equiv Q \Rightarrow P \equiv (\sim P) \Rightarrow (\sim Q)$

P	Q	$(\sim P) \Rightarrow (\sim Q)$	$P \vee (\sim Q)$	$Q \Rightarrow P$
T	T	**T**	**T**	**T**
T	F	**T**	**T**	**T**
F	T	**F**	**F**	**F**
F	F	**T**	**T**	**T**

52) Let P and Q be statements.
a) Is $\sim(P \vee Q)$ logically equivalent to $(\sim P) \vee (\sim Q)$? Explain.
i) They are logically equivalent since each statement is only true when both Q and P are false, and true otherwise.
b) What can you say about the biconditional $\sim(P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$?
i) The biconditional is a tautology since $\sim(P \vee Q) \equiv ((\sim P) \vee (\sim Q))$

53) For statements P, Q and R, use a truth table to show that each of the following pairs of statements is logically equivalent.
a) $(P \wedge Q) \Leftrightarrow P\ and\ P \Rightarrow Q$

P	Q	$P \wedge Q$	$(P \wedge Q) \Leftrightarrow P$	$P \Rightarrow Q$
T	T	T	**T**	**T**
T	F	F	**F**	**F**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

$P \Rightarrow (Q \vee R)$ and $(\sim Q) \Rightarrow ((\sim P) \vee R)$

P	Q	R	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$(\sim P) \vee R$	$(\sim Q) \Rightarrow ((\sim P) \vee R)$
T	T	T	T	**T**	T	**T**
T	T	F	T	**T**	F	**T**
T	F	T	T	**T**	T	**T**
T	F	F	F	**F**	F	**F**
F	T	T	T	**T**	T	**T**
F	T	F	T	**T**	T	**T**
F	F	T	T	**T**	T	**T**
F	F	F	F	**T**	T	**T**

54) For statements P and Q, show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are logically equivalent

P	Q	$P \wedge (\sim P)$	$(\sim Q) \Rightarrow (P \wedge (\sim P))$
T	**T**	F	**T**
T	**F**	F	**F**
F	**T**	F	**T**
F	**F**	F	**F**

55) For statements P, Q and R, show that $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	**T**	T	T	**T**
T	T	F	T	**F**	F	F	**F**
T	F	T	T	**T**	T	T	**T**
T	F	F	T	**F**	F	T	**F**
F	T	T	T	**T**	T	T	**T**
F	T	F	T	**F**	T	F	**F**
F	F	T	F	**T**	T	T	**T**
F	F	F	F	**T**	T	T	**T**

56) Two compound statements S and T are composed of the same component statements P, Q and R. If S and T are not logically equivalent, then what can we conclude from this?

a) $S \Leftrightarrow T$ is not a tautology

57) Five compound statements S_1, S_2, S_3, S_4 and S_5 are all composed of the same component statements P and Q whose truth tables have identical first and fourth rows. Show that at least two of these five statements are logically equivalent.

S1	S2	S3	S4	S5
T	T	T	T	T
T	T	F	F	X
T	F	T	F	X
F	F	F	F	F

Section 9: Some Fundamental Properties of Logical Equivalence

Notes

Theorem 18

(1) Commutative Laws
   (a) $P \vee Q \equiv Q \vee P$
   (b) $P \wedge Q \equiv Q \wedge P$
(2) Associative Laws
   (a) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
   (b) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
(3) Distributive Laws
   (a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   (b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(4) De Morgan's Laws
   (a) $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
   (b) $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

Theorem 21

(1) For statements P and Q,
   (a) $\sim(P \Rightarrow Q) \equiv P \wedge (\sim Q)$
   (b) $\sim(P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$

Exercises

58) Verify the following laws stated in Theorem 18:

a) Let P, Q and R be statements. Then $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   i) The first statement is true if P or Q and R, or all three are true. Equivalently statement two is true only if both of the parenthesized statements are true. This requires either P to be true (since a P is in both statements), or Q and R to be true (since there is one of each in the statements). Thus the second statement is also true if P or Q and R, or all three are true.
b) Let P and Q be statements. Then $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
   i) The first statement is true only if P and Q are false. We can easily see that this is the case for statement two as well.

59) Write negations of the following open sentences.

a) Either x=0 or y=o
   i) Using De Morgan's Law (a): Both $x \neq 0 \; and \; y \neq 0$
b) The integers a and b are both even
   i) Using De Morgan's Law (b): Either the integer a is odd or the integer b is odd.

60) Consider the implication: If x and y are even, then xy is even.

a) State the implication using "only if": x and y are even only if xy is even
b) State the converse of the implication: xy is even only if x and y are even
c) State the implication as a disjunction: x and y are odd or xy is even

$$Theorem\ 17: P \Rightarrow Q \equiv (\sim P) \vee Q$$

d) State the negation of the implication as a conjunction: x and y are even and xy is odd

61) For a real number x, let $P(x): x^2 = 2\ and\ Q(x): x = \sqrt{2}$. State the negation of the biconditional $P \Leftrightarrow Q$ in words.

a) Biconditional: $x^2 = 2$ if and only if $x = \sqrt{2}$
b) Negation: $\sim(P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$, using De Morgan's Law (b)
c) Result: Either both $x^2 = 2$ and $x \neq \sqrt{2}$, or both $x = \sqrt{2}$ and $x^2 \neq 2$

62) Let P and Q be statements. Show that $[(P \vee Q) \wedge \sim(P \wedge Q)] \equiv \sim(P \Leftrightarrow Q)$

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$	$\sim(P \Leftrightarrow Q)$
T	T	T	T	**F**	**F**
T	F	T	F	**T**	**T**
F	T	T	F	**T**	**T**
F	F	F	F	**F**	**F**

63) Let $n \in \mathbb{Z}$. For which implication is its negation the following? The integer $3n + 4$ is odd and $5n - 6$ is even

a) The negated statement has the form $P \wedge Q$
   i) $P: 3n + 4\ is\ odd; Q: 5n - 6$ is even
b) Using Theorem 21 (a): $\sim(P \Rightarrow Q) \equiv P \wedge (\sim Q)$
c) Thus the original implication is: If $3n + 4$ is odd, then $5n - 6$ is odd.

64) For which biconditional is its negation the following? $n^3 and\ 7n + 2\ are\ odd\ or\ n^3 and\ 7n + 2\ are\ even$

a) The negated statement has the form: $(P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$
   i) $P: n^3\ is\ odd; Q: 7n + 2\ is\ even$
b) Using Theorem 21 (b): $\sim(P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$
c) Thus the original biconditional is: n^3 is odd if and only if $7n + 2$ is even