

# Mathematical Proofs

CHAPTER 3 – SETS (EXERCISE SOLUTIONS)

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## Section 1: Describing a Set

Exercises

1) Which of the following are sets?

- a)  $1, 2, 3$                 Not a set
- b)  $\{1, 2\}, 3$             Not a set
- c)  $\{\{1\}, 2\}, 3$         Not a set
- d)  $\{1, \{2\}, 3\}$         Set
- e)  $\{1, 2, a, b\}$         Set

2) Let  $S = \{-2, -1, 0, 1, 2, 3\}$ . Describe each of the following sets as  $\{x \in S : p(x)\}$ , where  $p(x)$  is some condition on  $x$ .

- a)  $A = \{1, 2, 3\} = \{x \in S : x > 0\}$
- b)  $B = \{0, 1, 2, 3\} = \{x \in S : x \geq 0\}$
- c)  $C = \{-2, -1\} = \{x \in S : x < 0\}$
- d)  $D = \{-2, 2, 3\} = \{x \in S : |x| \geq 2\}$

3) Determine the cardinality of each of the following sets:

- a)  $A = \{1, 2, 3, 4, 5\}$                  $|A| = 5$
- b)  $B = \{0, 2, 4, \dots, 20\}$             $|B| = 11$
- c)  $C = \{25, 26, 27, \dots, 75\}$         $|C| = 51$
- d)  $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$         $|D| = 2$
- e)  $E = \{\emptyset\}$                          $|E| = 1$
- f)  $F = \{2, \{2, 3, 4\}\}$                 $|F| = 2$

4) Write each of the following sets by listing its elements within braces.

- a)  $A = \{n \in \mathbb{Z} : -4 < n \leq 4\} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
- b)  $B = \{n \in \mathbb{Z} : n^2 < 5\} = \{-2, -1, 0, 1, 2\}$
- c)  $C = \{n \in \mathbb{N} : n^3 < 100\} = \{1, 2, 3, 4\}$
- d)  $D = \{x \in \mathbb{R} : x^2 - x = 0\} = \{0, 1\}$
- e)  $E = \{x \in \mathbb{R} : x^2 + x = 0\} = \{-1, 0\}$

5) Write each of the following sets in the form  $\{x \in \mathbb{Z} : p(x)\}$ , where  $p(x)$  is a property concerning  $x$ .

- a)  $A = \{-1, -2, -3, \dots\} = \{x \in \mathbb{Z} : x < 0\}$
- b)  $B = \{-3, -2, \dots, 3\} = \{x \in \mathbb{Z} : |x| \leq 3\}$
- c)  $C = \{-2, -1, 1, 2\} = \{x \in \mathbb{Z} : 0 < |x| \leq 2\}$

6) The set  $E = \{2x : x \in \mathbb{Z}\}$  can be described by listing its elements, namely  $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ . List the elements of the following sets in a similar manner.

- a)  $A = \{2x + 1 : x \in \mathbb{Z}\} = \{\dots, -3, -1, 1, 3, 5, \dots\}$
- b)  $B = \{4n : n \in \mathbb{Z}\} = \{\dots, -8, -4, 0, 4, 8, \dots\}$
- c)  $C = \{3q + 1 : q \in \mathbb{Z}\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$

- 7) The set  $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$  of even integers can be described by means of a defining condition by  $E = \{y = 2x : x \in \mathbb{Z}\} = \{2x : x \in \mathbb{Z}\}$ . Describe the following sets in a similar manner.
- $A = \{\dots, -4, -1, 2, 5, 8, \dots\} = \{3x - 1 : x \in \mathbb{Z}\}$
  - $B = \{\dots, -10, -5, 0, 5, 10, \dots\} = \{5x : x \in \mathbb{Z}\}$
  - $C = \{1, 8, 27, 64, 125, \dots\} = \{x^3 : x \in \mathbb{N}\}$
- 8) Let  $A = \{n \in \mathbb{Z} : 2 \leq |n| < 4\}$ ,  $B = \{x \in \mathbb{Q} : 2 < x \leq 4\}$ ,  $C = \{x \in \mathbb{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$  and  $D = \{x \in \mathbb{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\}$ .
- Describe the set A by listing its elements.
    - $A = \{-2, -3, 2, 3\}$
  - Give an example of three elements that belong to B but do not belong to A.
    - $\frac{5}{2}, \frac{7}{3}, \frac{9}{4}$
  - Describe the set C by listing its elements.
    - $C = \{\sqrt{2}, 2\}$
  - Describe the set D in another manner.
    - $D = \{2\}$
  - Determine the cardinality of the sets A, C and D.
    - $|A| = 4; |C| = 2; |D| = 1$
- 9) For  $A = \{2, 3, 5, 7, 8, 10, 13\}$ , let  $B = \{x \in A : x = y + z, \text{ where } y, z \in A\}$  and  $C = \{r \in B : r + s \in B \text{ for some } s \in B\}$ . Determine C.
- $B = \{5, 7, 8, 10, 13\}$
  - $C = \{5, 8\}$  (because  $5 + 8 = 13$  and  $8 + 5 = 13$ )

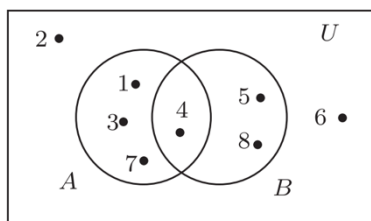
## Section 2: Subsets

### Exercises

- 10) Give examples of three sets A, B and C such that
- $A \subseteq B \subset C$ 
    - $A = \{1\}; B = \{1\}; C = \{1, 2\}$
  - $A \in B, B \in C$  and  $A \notin C$ 
    - $A = \{1\}; B = \{\{1\}\}; C = \{\{\{1\}\}\}$
  - $A \in B$  and  $A \subset C$ 
    - $A = \{1\}; B = \{\{1\}\}; C = \{1, 2\}$
- 11) Let  $(a, b)$  be an open interval of real numbers and let  $c \in (a, b)$ . Describe an open interval I centered at c such that  $I \subseteq (a, b)$ .
- Let  $r = \min(c - a, b - c)$ , then  $I = (c - r, c + r)$
- 12) Which of the following sets are equal?
- $A = \{n \in \mathbb{Z} : |n| < 2\} = \{-1, 0, 1\}$
  - $B = \{n \in \mathbb{Z} : n^3 = n\} = \{-1, 0, 1\}$
  - $C = \{n \in \mathbb{Z} : n^2 \leq n\} = \{0, 1\}$

- d)  $D = \{n \in \mathbb{Z} : n^2 \leq 1\} = \{-1, 0, 1\}$   
 e)  $E = \{-1, 0, 1\}$   
 f) Conclusion: The elements in  $\{A, B, D, E\}$  are equal and C is on its own.

13) For a universal set  $U = \{1, 2, \dots, 8\}$  and two sets  $A = \{1, 3, 4, 7\}$  and  $B = \{4, 5, 8\}$ , draw a Venn diagram that represents these sets



14) Find  $\mathcal{P}(A)$  and  $|\mathcal{P}(A)|$  for

- a)  $A = \{1, 2\}$ ;  
 i)  $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ ;  
 ii)  $|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$   
 b)  $A = \{\emptyset, 1, \{a\}\}$ ;  
 i)  $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{1\}, \{\{a\}\}, \{\emptyset, 1\}, \{\emptyset, \{a\}\}, \{1, \{a\}\}, \{\emptyset, 1, \{a\}\}\}$   
 ii)  $|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8$

15) Find  $\mathcal{P}(A)$  for  $A = \{0, \{0\}\}$ .

- a)  $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, \{0, \{0\}\}\} = \{\emptyset, \{0\}, \{\{0\}\}, A\}$

16) Find  $\mathcal{P}(\mathcal{P}(\{1\}))$  and its cardinality.

- a)  $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$   
 b)  $\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$   
 c)  $|\mathcal{P}(\mathcal{P}(\{1\}))| = 2^{|\mathcal{P}(\{1\})|} = 2^2 = 4$

17) Find  $\mathcal{P}(A)$  and  $|\mathcal{P}(A)|$  for  $A = \{0, \emptyset, \{\emptyset\}\}$ .

- a)  $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$   
 b)  $|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8$

18) For  $A = \{x : x = 0 \text{ or } x \in \mathcal{P}(\{0\})\}$ , determine  $\mathcal{P}(A)$ .

- a)  $\mathcal{P}(\{0\}) = \{\emptyset, \{0\}\}$   
 b)  $A = \{\emptyset, 0, \{0\}\}$   
 c)  $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{0\}, \{\{0\}\}, \{\emptyset, 0\}, \{\emptyset, \{0\}\}, \{0, \{0\}\}, A\}$

19) Give an example of a set S such that

- a)  $S \subseteq \mathcal{P}(\mathbb{N})$   
 i)  $S = \emptyset$   
 b)  $S \in \mathcal{P}(\mathbb{N})$   
 i)  $S = \{1\}$

- c)  $S \subseteq \mathcal{P}(\mathbb{N})$  and  $|S| = 5$   
 i)  $S = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$   
 d)  $S \in \mathcal{P}(\mathbb{N})$  and  $|S| = 5$   
 i)  $S = \{1, 2, 3, 4, 5\}$

20) Determine whether the following statements are true or false.

- a) If  $\{1\} \in \mathcal{P}(A)$ , then  $1 \in A$  but  $\{1\} \notin A$   
 i) **False**, e.g.  $A = \{1, \{1\}\}$   
 b) If  $A, B$  and  $C$  are sets such that  $A \subset \mathcal{P}(B) \subset C$  and  $|A| = 2$ , then  $|C|$  can be 5 but  $|C|$  cannot be 4.  
 i) **True**. If  $|A| = 2$ , then the cardinality of  $\mathcal{P}(B) = 2^2 = 4$ . Since  $\mathcal{P}(B)$  is a proper subset of  $C$ ,  $C$  must at least have a cardinality of 5.  
 c) If a set  $B$  has one more element than a set  $A$ , then  $\mathcal{P}(B)$  has at least two more elements than  $\mathcal{P}(A)$ .  
 i) **False**, if  $A = \emptyset$  then  $|\mathcal{P}(A)| = 2^{|\emptyset|} = 2^0 = 1$  and  $|\mathcal{P}(B)| = 2^{|\emptyset|+1} = 2^1 = 2$ . (It is true if  $A \neq \emptyset$ )  
 d) If four sets  $A, B, C$  and  $D$  are subsets of  $\{1, 2, 3\}$  such that  $|A| = |B| = |C| = |D| = 2$ , then at least two of these sets are equal.  
 i) **True**. Different combinations of  $\{1, 2, 3\}$  with cardinality 2:  $\frac{3!}{(3-2)! \cdot 2!} = \frac{3!}{2!} = \frac{3 \cdot 2}{2} = 3$ .  
 Namely  $\{1, 2\}, \{1, 3\}$  and  $\{2, 3\}$ .

21) Three subsets  $A, B$  and  $C$  of  $\{1, 2, 3, 4, 5\}$  have the same cardinality. Furthermore,

- a) 1 belongs to  $A$  and  $B$  but not to  $C$ .  
 b) 2 belongs to  $A$  and  $C$  but not to  $B$ .  
 c) 3 belongs to  $A$  and exactly one of  $B$  and  $C$ .  
 d) 4 belongs to an even number of  $A, B$  and  $C$ .  
 e) 5 belongs to an odd number of  $A, B$  and  $C$ .  
 f) The sums of the elements in two of the sets  $A, B$  and  $C$  differ by 1.  
 g) What is  $B$ ?  
 i)  $A = \{1, 2, 3\}$   
 ii)  $B = \{1, 4, 5\}$   
 iii)  $C = \{2, 3, 4\}$

### Section 3: Set Operations

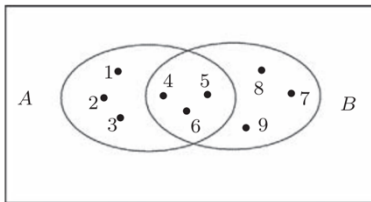
#### Exercises

22) Let  $U = \{1, 3, \dots, 15\}$  be the universal set,  $A = \{1, 5, 9, 13\}$ , and  $B = \{3, 9, 15\}$ . Determine the following.

- a)  $A \cup B = \{1, 3, 5, 9, 13, 15\}$   
 b)  $A \cap B = \{9\}$   
 c)  $A - B = \{1, 5, 13\}$   
 d)  $B - A = \{3, 15\}$   
 e)  $\bar{A} = U - A = \{3, 7, 11, 15\}$   
 f)  $A \cap \bar{B} = A - B = \{1, 5, 13\}$

23) Give examples of two sets  $A$  and  $B$  such that  $|A - B| = |A \cap B| = |B - A| = 3$ . Draw the accompanying Venn diagram.

a)  $A = \{1, 2, 3, 4, 5, 6\}; B = \{4, 5, 6, 7, 8, 9\}$



24) Give examples of three sets  $A$ ,  $B$  and  $C$  such that  $B \neq C$  but  $B - A = C - A$

a)  $A = \{1, 2\}$

b)  $B = \{1, 2, 3\}$

c)  $C = \{1, 3\}$

25) Give examples of three sets  $A$ ,  $B$  and  $C$  such that

a)  $A \in B, A \subseteq C$  and  $B \not\subseteq C$

i)  $A = \{1\}$

ii)  $B = \{\{1\}\}$

iii)  $C = \{1\}$

b)  $B \in A, B \subset C$  and  $A \cap C \neq \emptyset$

i)  $A = \{\{1\}\}$

ii)  $B = \{1\}$

iii)  $C = \{1, \{1\}\}$

c)  $A \in B, B \subseteq C$  and  $A \not\subseteq C$

i)  $A = \{1\}$

ii)  $B = \{\{1\}\}$

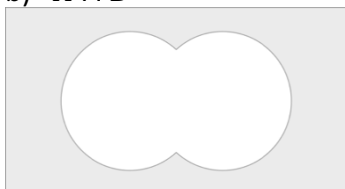
iii)  $C = \{\{1\}\}$

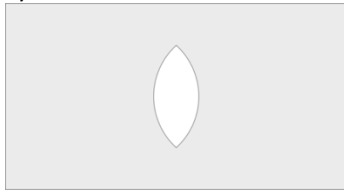
26) Let  $U$  be a universal set and let  $A$  and  $B$  be two subsets of  $U$ . Draw a Venn diagram for each of the following sets.

a)  $\overline{A \cup B}$



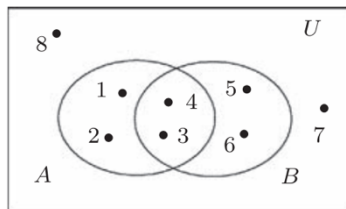
b)  $\bar{A} \cap \bar{B}$



c)  $\overline{A \cap B}$ d)  $\overline{A} \cup \overline{B}$ 

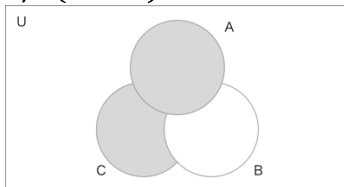
27) Give an example of a universal set  $U$ , two sets  $A$  and  $B$  and accompanying Venn diagram such that  $|A \cap B| = |A - B| = |B - A| = |\overline{A \cup B}| = 2$

a)  $U = \{1, 2, \dots, 8\}$ ;  $A = \{1, 2, 3, 4\}$ ;  $B = \{3, 4, 5, 6\}$

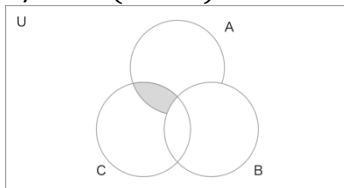


28) Let  $A$ ,  $B$  and  $C$  be nonempty subsets of a universal set  $U$ . Draw a Venn diagram for each of the following set operations.

a)  $(C - B) \cup A$



b)  $C \cap (A - B)$



29) Let  $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ .

a) Determine which of the following are elements of  $A$ :  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$

i)  $\emptyset$  and  $\{\emptyset\}$  are elements of  $A$

b) Determine  $|A| = 3$

c) Determine which of the following are subsets of  $A$ :  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$

i)  $\emptyset, \{\emptyset\}$  and  $\{\emptyset, \{\emptyset\}\}$  are subsets of  $A$

For (d)-(i), determine the indicated sets.



- d)  $\emptyset \cap A = \emptyset$
- e)  $\{\emptyset\} \cap A = \{\emptyset\}$
- f)  $\{\emptyset, \{\emptyset\}\} \cap A = \{\emptyset, \{\emptyset\}\}$
- g)  $\emptyset \cup A = A$
- h)  $\{\emptyset\} \cup A = A$
- i)  $\{\emptyset, \{\emptyset\}\} \cup A = A$

30) Let  $A = \{x \in \mathbb{R} : |x - 1| \leq 2\}$ ,  $B = \{x \in \mathbb{R} : |x| \geq 1\}$  and  $C = \{x \in \mathbb{R} : |x + 2| \leq 3\}$ .

- a) Express A, B and C using interval notation.
  - i)  $A = \{x \in \mathbb{R} : -1 \leq x \leq 3\} = [-1, 3]$
  - ii)  $B = (-\infty, -1] \cup [1, \infty)$
  - iii)  $C = \{x \in \mathbb{R} : -5 \leq x \leq 1\} = [-5, 1]$

31) Give an example of four different sets A, B, C and D such that (1)  $A \cup B = \{1, 2\}$  and  $C \cap D = \{2, 3\}$  and (2) if B and C are interchanged and  $\cup$  and  $\cap$  are interchanged, then we get the same result ( $A \cap C = \{1, 2\}$  and  $B \cup D = \{2, 3\}$ ).

- a)  $A = \{1, 2\}$
- b)  $B = \{2\}$
- c)  $C = \{1, 2, 3\}$
- d)  $D = \{2, 3\}$

32) Give an example of four different subsets A, B, C and D of  $\{1, 2, 3, 4\}$  such that all intersections of two subsets are different.

- a)  $A = \{1, 2, 3\}$
- b)  $B = \{2, 4\}$
- c)  $C = \{2, 3, 4\}$
- d)  $D = \{1, 3, 4\}$ 
  - i)  $A \cap B = B \cap A = \{2\}$
  - ii)  $A \cap C = C \cap A = \{2, 3\}$
  - iii)  $A \cap D = D \cap A = \{1, 3\}$
  - iv)  $B \cap C = C \cap B = \{2, 4\}$
  - v)  $B \cap D = D \cap B = \{4\}$
  - vi)  $C \cap D = D \cap C = \{3, 4\}$

33) Give an example of two nonempty sets A and B such that  $\{A \cup B, A \cap B, A - B, B - A\}$  is the power set of some set.

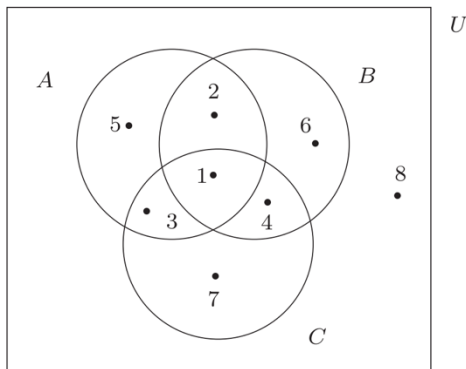
- a)  $A = \{1\}$
- b)  $B = \{2\}$
- c)  $\mathcal{P}(\{\{1, 2\}\}) = \{\{1, 2\}, \emptyset, \{1\}, \{2\}\}$

34) Give examples of two subsets A and B of  $\{1, 2, 3\}$  such that all of the following sets are different:  $A \cup B, A \cup \bar{B}, \bar{A} \cup B, \bar{A} \cup \bar{B}, A \cap B, A \cap \bar{B}, \bar{A} \cap B, \bar{A} \cap \bar{B}$ .

- a)  $A = \{1, 2\}$
- b)  $B = \{2, 3\}$
- c) Then the different results are:  $\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{2\}, \{1\}, \{3\}, \emptyset$

35) Give examples of a universal set  $U$  and sets  $A$ ,  $B$  and  $C$  such that each of the following sets contains exactly one element:  $A \cap B \cap C$ ,  $(A \cap B) - C$ ,  $(A \cap C) - B$ ,  $(B \cap C) - A$ ,  $A - (B \cup C)$ ,  $B - (A \cup C)$ ,  $C - (A \cup B)$ ,  $\overline{A \cup B \cup C}$ . Draw the accompanying Venn diagram.

- $U = \{1, 2, \dots, 8\}$
- $A = \{1, 2, 3, 5\}$
- $B = \{1, 2, 4, 6\}$
- $C = \{1, 3, 4, 7\}$
- Then the different results are:  $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}$ .



## Section 4: Indexed Collections of Sets

### Exercises

36) For a real number  $r$ , define  $S_r$  to be the interval  $[r - 1, r + 2]$ . Let  $A = \{1, 3, 4\}$ . Determine  $\bigcup_{\alpha \in A} S_\alpha$  and  $\bigcap_{\alpha \in A} S_\alpha$ .

- $\bigcup_{\alpha \in A} S_\alpha = [0, 3] \cup [2, 5] \cup [3, 6] = [0, 6]$
- $\bigcap_{\alpha \in A} S_\alpha = [0, 3] \cap [2, 5] \cap [3, 6] = \{3\}$

37) Let  $A = \{1, 2, 5\}$ ,  $B = \{0, 2, 4\}$ ,  $C = \{2, 3, 4\}$  and  $S = \{A, B, C\}$ . Determine  $\bigcup_{X \in S} X$  and  $\bigcap_{X \in S} X$ .

- $\bigcup_{X \in S} X = \{1, 2, 5\} \cup \{0, 2, 4\} \cup \{2, 3, 4\} = \{0, 1, 2, \dots, 5\}$
- $\bigcap_{X \in S} X = \{1, 2, 5\} \cap \{0, 2, 4\} \cap \{2, 3, 4\} = \{2\}$

38) For a real number  $r$ , define  $A_r = \{r^2\}$ ,  $B_r$  as the closed interval  $[r - 1, r + 1]$  and  $C_r$  as the interval  $(r, \infty)$ . For  $S = \{1, 2, 4\}$ , determine

- $\bigcup_{\alpha \in S} A_\alpha$  and  $\bigcap_{\alpha \in S} A_\alpha$ 
  - $\bigcup_{\alpha \in S} A_\alpha = \{1^2\} \cup \{2^2\} \cup \{4^2\} = \{1, 4, 16\}$
  - $\bigcap_{\alpha \in S} A_\alpha = \emptyset$
- $\bigcup_{\alpha \in S} B_\alpha$  and  $\bigcap_{\alpha \in S} B_\alpha$ 
  - $\bigcup_{\alpha \in S} B_\alpha = [0, 2] \cup [1, 3] \cup [3, 5] = [0, 5]$
  - $\bigcap_{\alpha \in S} B_\alpha = \emptyset$
- $\bigcup_{\alpha \in S} C_\alpha$  and  $\bigcap_{\alpha \in S} C_\alpha$ 
  - $\bigcup_{\alpha \in S} C_\alpha = (1, \infty) \cup (2, \infty) \cup (4, \infty) = (1, \infty)$
  - $\bigcap_{\alpha \in S} C_\alpha = (4, \infty)$

39) Let  $A = \{a, b, \dots, z\}$  be the set consisting of the letters of the alphabet. For  $\alpha \in A$ , let  $A_\alpha$  consist of  $\alpha$  and the two letters that follow it, where  $A_y = \{y, z, a\}$  and  $A_z = \{z, a, b\}$ . Find a

set  $S \subseteq A$  of smallest cardinality such that  $\bigcup_{\alpha \in S} A_\alpha = A$ . Explain why your set  $S$  has the required properties.

a)  $S = \{a, d, g, j, m, p, s, v, y\}; |S| = 9$

b) 26 letters in the alphabet divided by 3 (cardinality of any  $A_\alpha$ )  $= \frac{26}{3} = 8,6 \approx 9$

40) For  $i \in \mathbb{Z}$ , let  $A_i = \{i - 1, i + 1\}$ . Determine the following:

a)  $\bigcup_{i=1}^5 A_{2i} = \{1, 3\} \cup \{3, 5\} \cup \dots \cup \{9, 11\} = \{1, 3, \dots, 11\}$

b)  $\bigcup_{i=1}^5 (A_i \cap A_{i+1}) = (\{0, 2\} \cap \{1, 3\}) \cup (\{1, 3\} \cap \{2, 4\}) \cup \dots \cup (\{4, 6\} \cap \{5, 7\}) = \emptyset$

c)  $\bigcup_{i=1}^5 (A_{2i-1} \cap A_{2i+1}) = (\{0, 2\} \cap \{2, 4\}) \cup (\{2, 4\} \cap \{4, 6\}) \cup \dots \cup (\{8, 10\} \cap \{10, 12\}) = \{2, 4, \dots, 10\}$

41) For each of the following, find an indexed collection  $\{A_n\}_{n \in \mathbb{N}}$  of distinct sets (that is, no two sets are equal) satisfying the given conditions.

a)  $\bigcap_{n=1}^{\infty} A_n = \{0\}$  and  $\bigcup_{n=1}^{\infty} A_n = [0, 1]$

i)  $\{A_n\}_{n \in \mathbb{N}}$ , where  $A_n = \left\{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{n}\right\} = \left[0, \frac{1}{n}\right]$

b)  $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$  and  $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$

i)  $\{A_n\}_{n \in \mathbb{N}}$ , where  $A_n = \{a \in \mathbb{Z} : |a| \leq n\}$

42) For each of the following collections of sets, define a set  $A_n$  for each  $n \in \mathbb{N}$  such that the indexed collection  $\{A_n\}_{n \in \mathbb{N}}$  is precisely the given collection of sets. Then find both the union and intersection of the indexed collections of sets.

a)  $\left\{[1, 2 + 1), \left[1, 2 + \frac{1}{2}\right), \left[1, 2 + \frac{1}{3}\right), \dots\right\}$

i)  $\{A_n\}_{n \in \mathbb{N}}$  where  $A_n = \left\{x \in \mathbb{R} : 1 \leq x < 2 + \frac{1}{n}\right\} = \left[1, 2 + \frac{1}{n}\right)$

b)  $\left\{(-1, 2), \left(-\frac{3}{2}, 4\right), \left(-\frac{5}{3}, 6\right), \left(-\frac{7}{4}, 8\right), \dots\right\}$

i)  $\{A_n\}_{n \in \mathbb{N}}$  where  $A_n = \left\{x \in \mathbb{R} : -\frac{2n-1}{n} < x < 2n\right\} = \left(-\frac{2n-1}{n}, 2n\right)$

43) For  $r \in \mathbb{R}^+$ , let  $A_r = \{x \in \mathbb{R} : |x| < r\}$ . Determine  $\bigcup_{r \in \mathbb{R}^+} A_r$  and  $\bigcap_{r \in \mathbb{R}^+} A_r$ .

a)  $\bigcup_{r \in \mathbb{R}^+} A_r = \mathbb{R}$

b)  $\bigcap_{r \in \mathbb{R}^+} A_r = \{0\}$

44) Each of the following sets is a subset of  $A = \{1, 2, \dots, 10\}$ :  $A_1 = \{1, 5, 7, 9, 10\}$ ,  $A_2 = \{1, 2, 3, 8, 9\}$ ,  $A_3 = \{2, 4, 6, 8, 9\}$ ,  $A_4 = \{2, 4, 8\}$ ,  $A_5 = \{3, 6, 7\}$ ,  $A_6 = \{3, 8, 10\}$ ,  $A_7 = \{4, 5, 7, 9\}$ ,  $A_8 = \{4, 5, 10\}$ ,  $A_9 = \{4, 6, 8\}$ ,  $A_{10} = \{5, 6, 10\}$ ,  $A_{11} = \{5, 8, 9\}$ ,  $A_{12} = \{6, 7, 10\}$ ,  $A_{13} = \{6, 8, 9\}$ .

a) Find a set  $I \subseteq \{1, 2, \dots, 13\}$  such that for every two distinct elements  $j, k \in I$ ,  $A_j \cap A_k = \emptyset$  and  $|\bigcup_{i \in I} A_i|$  is maximum.

i)  $I = \{1, 4\}$

45) For  $n \in \mathbb{N}$ , let  $A_n = \left(-\frac{1}{n}, 2 - \frac{1}{n}\right)$ . Determine  $\bigcup_{n \in \mathbb{N}} A_n$  and  $\bigcap_{n \in \mathbb{N}} A_n$ .

a)  $\bigcup_{n \in \mathbb{N}} A_n = (-1, 2)$

b)  $\bigcap_{n \in \mathbb{N}} A_n = [0, 1]$

## Section 5: Partitions of Sets

Exercises

- 46) Which of the following are partitions of  $A = \{a, b, c, d, e, f, g\}$ ? For each collection of subsets that is not a partition of A, explain your answer.
- a)  $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$ ; **This is a partition of A**
  - b)  $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$ ; This is not a partition, because  $\forall s \in S_2, g \notin s$
  - c)  $S_3 = \{A\}$ ; **This is a partition of A**
  - d)  $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$ ; This is not a partition because  $\emptyset \in S_4$
  - e)  $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$ ; This is not a partition because b is in  $\{b, f\}$  and in  $\{b, g\}$ .
- 47) Which of the following sets are partitions of  $A = \{1, 2, 3, 4, 5\}$ ?
- a)  $S_1 = \{\{1, 3\}, \{2, 5\}\}$ ; This is not a partition since the element 4 belongs to no element of  $S_1$ .
  - b)  $S_2 = \{\{1, 2\}, \{3, 4, 5\}\}$ ; **This is a partition**
  - c)  $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ ; This is not a partition, since 2, 3 and 4 appear in multiple elements of  $S_3$ .
  - d)  $S_4 = A$ ; This is not a partition, since it isn't a collection of subsets of A, but the set A itself.
- 48) Let  $A = \{1, 2, 3, 4, 5, 6\}$ . Give an example of a partition S of A such that  $|S| = 3$
- a)  $S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$
- 49) Give an example of a set A with  $|A| = 4$  and two disjoint partitions  $S_1$  and  $S_2$  of A with  $|S_1| = |S_2| = 3$
- a)  $A = \{1, 2, 3, 4\}$
  - b)  $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$
  - c)  $S_2 = \{\{1, 2\}, \{3\}, \{4\}\}$
- 50) Give an example of a partition of  $\mathbb{N}$  into three subsets.
- a)  $S = \{\{1\}, \{x \in \mathbb{N} : x > 1 \text{ and } x \text{ is even}\}, \{x \in \mathbb{N} : x > 1 \text{ and } x \text{ is odd}\}\}$
- 51) Give an example of a partition of  $\mathbb{Q}$  into three subsets.
- a)  $S = \{\{x \in \mathbb{Q} : x < 0\}, \{0\}, \{x \in \mathbb{Q} : x > 0\}\}$
- 52) Give an example of three sets  $A, S_1, S_2$  such that  $S_1$  is a partition of A,  $S_2$  is a partition of  $S_1$  and  $|S_2| < |S_1| < |A|$ .
- a)  $A = \{1, 2, 3, 4\}$
  - b)  $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$
  - c)  $S_2 = \{\{\{1\}, \{2\}\}, \{\{3, 4\}\}\}$
  - d) Then  $|A| = 4; |S_1| = 3; |S_2| = 2$ ; and  $|S_2| < |S_1| < |A|$ .
- 53) Give an example of a partition of  $\mathbb{Z}$  into four subsets.
- a)  $S = \{A_1, A_2, A_3, A_4\}$ 
    - i)  $A_1 = \{x \in \mathbb{Z} : x < 0 \text{ and } x \text{ is odd}\}$
    - ii)  $A_2 = \{x \in \mathbb{Z} : x < 0 \text{ and } x \text{ is even}\}$

- iii)  $A_3 = \{x \in \mathbb{Z} : x \geq 0 \text{ and } x \text{ is odd}\}$
- iv)  $A_4 = \{x \in \mathbb{Z} : x \geq 0 \text{ and } x \text{ is even}\}$

- 54) Let  $A = \{1, 2, \dots, 12\}$ . Give an example of a partition  $S$  of  $A$  satisfying the following requirements: (i)  $|S| = 5$ , (ii) there is a subset  $T$  of  $S$  such that  $|T| = 4$  and  $|\bigcup_{X \in T} X| = 10$  and (iii) there is no element  $B \in S$  such that  $|B| = 3$ .
- a)  $S = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$
  - b)  $T = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\}$
  - c)  $\bigcup_{X \in T} X = \{1, 2, 3, \dots, 10\}$
- 55) A set  $S$  is partitioned into two subsets  $S_1$  and  $S_2$ . This produces a partition  $\mathcal{P}_1$  of  $S$  where  $\mathcal{P}_1 = \{S_1, S_2\}$  and so  $|\mathcal{P}_1| = 2$ . One of the sets in  $\mathcal{P}_1$  is then partitioned into two subsets, producing a partition  $\mathcal{P}_2$  of  $S$  with  $|\mathcal{P}_2| = 3$ . A total of  $|\mathcal{P}_1|$  sets in  $\mathcal{P}_2$  are partitioned into two new subsets each, producing a partition  $\mathcal{P}_3$  of  $S$ . Next, a total of  $|\mathcal{P}_2|$  sets in  $\mathcal{P}_3$  are partitioned into two new subsets, each producing a partition  $\mathcal{P}_4$  of  $S$ . This is continued until partition  $\mathcal{P}_6$  of  $S$ . What is  $|\mathcal{P}_6|$ ?
- a)  $|\mathcal{P}_1| = 2$
  - b)  $|\mathcal{P}_2| = 3$
  - c)  $|\mathcal{P}_3| = 5$  (since  $|\mathcal{P}_1|$  subsets are partitioned into two new subsets each)
  - d)  $|\mathcal{P}_4| = 8$  (since  $|\mathcal{P}_2|$  subsets are partitioned into two new subsets each)
  - e)  $|\mathcal{P}_5| = 13$  (since  $|\mathcal{P}_3|$  subsets are partitioned into two new subsets each)
  - f)  $|\mathcal{P}_6| = 21$  (since  $|\mathcal{P}_4|$  subsets are partitioned into two new subsets each)
- 56) We mentioned that there are three ways that a collection  $S$  of subsets of a nonempty set  $A$  is defined to be a partition of  $A$ . **Definition 1:** The collection  $S$  consists of pairwise disjoint nonempty subsets of  $A$  and every element of  $A$  belongs to a subset in  $S$ . **Definition 2:** The collection  $S$  consists of nonempty subsets of  $A$  and every element of  $A$  belongs to exactly one subset in  $S$ . **Definition 3:** The collection  $S$  consists of subsets of  $A$  satisfying the three properties (1) every subset in  $S$  is nonempty, (2) every two subsets of  $A$  are equal or disjoint and (3) the union of all subsets in  $S$  is  $A$ .
- a) Show that any collection  $S$  of subsets of  $A$  satisfying Definition 1 satisfies Definition 2.
    - i) In definition 1, the subsets of  $A$  in  $S$  are pairwise disjoint and every element of  $A$  belongs to a subset in  $S$ .
    - ii) Since the subsets are pairwise disjoint, each element of  $A$  is contained in only one subset of  $S$ .
    - iii) This is the same as saying, each element of  $A$  belongs to exactly one subset in  $S$ , which is the premise of definition 2.
  - b) Show that any collection  $S$  of subsets of  $A$  satisfying Definition 2 satisfies Definition 3.
    - i) In definition 2, the union of all subsets in  $S$  is  $A$ . Thus any element  $x \in A$  can be found in a subset of  $S$ .
    - ii) Definition 3 also states that the subsets of  $S$  must be pairwise disjoint, which together with previous statement implies that and every element of  $A$  belongs to exactly one subset in  $S$ .
  - c) Show that any collection  $S$  of subsets of  $A$  satisfying Definition 3 satisfies Definition 1.

- i) In definition 3, the union of all subsets in  $S$  is  $A$ . Thus any element  $x \in A$  can be found in a subset of  $S$ . This is the same as stated in definition 1: every element of  $A$  belongs to a subset in  $S$ .
- ii) Definition 1 also states: the collection  $S$  consists of pairwise disjoint nonempty subsets of  $A$ . This is the same as in definition 3: every subset in  $S$  is nonempty and every two subsets of  $A$  are equal or disjoint.

## Section 6: Cartesian Products of Sets

### Exercises

57) Let  $A = \{x, y, z\}$  and  $B = \{x, y\}$ . Determine  $A \times B$ .

a)  $A \times B = \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$

58) Let  $A = \{1, \{1\}, \{\{1\}\}\}$ . Determine  $A \times A$ .

a)  $A \times A = \{(1, 1), (1, \{1\}), (1, \{\{1\}\}), (\{1\}, 1), (\{1\}, \{1\}), (\{1\}, \{\{1\}\}), (\{\{1\}\}, 1), (\{\{1\}\}, \{1\}), (\{\{1\}\}, \{\{1\}\})\}$

59) For  $A = \{a, b\}$ , determine  $A \times \mathcal{P}(A)$ .

a)  $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

b)  $A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, \{a, b\}), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, \{a, b\})\}$

60) For  $A = \{\emptyset, \{\emptyset\}\}$ , determine  $A \times \mathcal{P}(A)$ .

a)  $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

b)  $A \times \mathcal{P}(A) = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\emptyset, \{\{\emptyset\}\}), (\emptyset, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\emptyset\}, \{\{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\})\}$

61) For  $A = \{1, 2\}$  and  $B = \{\emptyset\}$ , determine  $A \times B$  and  $\mathcal{P}(A) \times \mathcal{P}(B)$ .

a)  $A \times B = \{(1, \emptyset), (2, \emptyset)\}$

b)  $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

c)  $\mathcal{P}(B) = \{\emptyset, \{\emptyset\}\}$

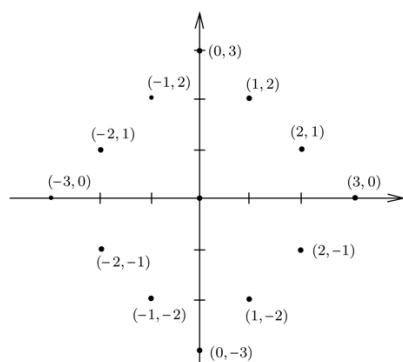
d)  $\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\{1\}, \emptyset), (\{1\}, \{\emptyset\}), (\{2\}, \emptyset), (\{2\}, \{\emptyset\}), (\{1, 2\}, \emptyset), (\{1, 2\}, \{\emptyset\})\}$

62) Describe the graph of the circle whose equation is  $x^2 + y^2 = 4$  as a subset of  $\mathbb{R} \times \mathbb{R}$ .

a)  $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 4\}$

63) For the elements of the set  $S = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x| + |y| = 3\}$ . Plot the corresponding points in the Euclidean  $xy$ -plane.

a)  $S = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x| + |y| = 3\} = \{(-3, 0), (-2, -1), (-2, 1), (-1, -2), (-1, 2) \dots\}$



64) For  $A = \{1, 2\}$  and  $B = \{1\}$ , determine  $\mathcal{P}(A \times B)$ .

- a)  $A \times B = \{(1, 1), (2, 1)\}$   
 b)  $\mathcal{P}(A \times B) = \{\emptyset, \{(1, 1)\}, \{(2, 1)\}, \{(1, 1), (2, 1)\}\}$

65) For  $A = \{x \in \mathbb{R} : |x - 1| \leq 2\}$  and  $B = \{y \in \mathbb{R} : |y - 4| \leq 2\}$ , give a geometric description of the points in the xy-plane belonging to  $A \times B$ .

- a)  $A \times B = [-1, 3] \times [2, 6]$  which is the set of all points within the square bounded by  $x = -1, x = 3, y = 2$  and  $y = 6$ .

66) For  $A = \{a \in \mathbb{R} : |a| \leq 1\}$  and  $B = \{b \in \mathbb{R} : |b| = 1\}$ , give a geometric description of the points in the xy-plane belonging to  $(A \times B) \cup (B \times A)$ .

- a)  $(A \times B) \cup (B \times A) = ([-1, 1] \times \{-1, 1\}) \cup (\{-1, 1\} \times [-1, 1])$ , which is the set of all points outlining the sides of the square bounded by  $x = -1, x = 1, y = -1$  and  $y = 1$ .

### Additional Exercises

67) The set  $T = \{2k + 1 : k \in \mathbb{Z}\}$  can be described as  $T = \{\dots, -3, -1, 1, 3, \dots\}$ . Describe the following sets in a similar manner.

- a)  $A = \{4k + 3 : k \in \mathbb{Z}\} = \{\dots, -5, -1, 3, 7, \dots\}$   
 b)  $B = \{5k - 1 : k \in \mathbb{Z}\} = \{\dots, -6, -1, 4, 9, \dots\}$

68) Let  $S = \{-10, -9, \dots, 9, 10\}$ . Describe each of the following sets as  $\{x \in S : p(x)\}$  where  $p(x)$  is some condition on  $x$ .

- a)  $A = \{-10, -9, \dots, -1, 1, \dots, 9, 10\} = \{x \in S : x \neq 0\}$   
 b)  $B = \{-10, -9, \dots, -1, 0\} = \{x \in S : x \leq 0\}$   
 c)  $C = \{-5, -4, \dots, 0, 1, \dots, 7\} = \{x \in S : -5 \leq x \leq 7\}$   
 d)  $D = \{-10, -9, \dots, 4, 6, 7, \dots, 10\} = \{x \in S : x \neq 5\}$

69) Describe each of the following sets by listing its elements within braces.

- a)  $\{x \in \mathbb{Z} : x^3 - 4x = 0\} = \{-2, 0, 2\}$   
 b)  $\{x \in \mathbb{R} : |x| = -1\} = \emptyset$   
 c)  $\{m \in \mathbb{N} : 2 < m \leq 5\} = \{3, 4, 5\}$   
 d)  $\{n \in \mathbb{N} : 0 \leq n \leq 3\} = \{1, 2, 3\}$   
 e)  $\{k \in \mathbb{Q} : k^2 - 4 = 0\} = \{-2, 2\}$   
 f)  $\{k \in \mathbb{Z} : 9k^2 - 3 = 0\} = \emptyset$

g)  $\{k \in \mathbb{Z} : 1 \leq k^2 \leq 10\} = \{-3, -2, -1, 1, 2, 3\}$

70) Determine the cardinality of each of the following sets.

- |   |            |
|---|------------|
| a) $A = \{1, 2, 3, \{1, 2, 3\}, 4, \{4\}\}$   | $ A  = 6$  |
| b) $B = \{x \in \mathbb{R} :  x  = -1\} = \emptyset$  | $ B  = 0$  |
| c) $C = \{m \in \mathbb{N} : 2 < m \leq 5\} = \{3, 4, 5\}$                                    | $ C  = 3$  |
| d) $D = \{n \in \mathbb{N} : n < 0\} = \emptyset$   | $ D  = 0$  |
| e) $E = \{k \in \mathbb{N} : 1 \leq k^2 \leq 100\} = \{1, 2, \dots, 10\}$                     | $ E  = 10$ |
| f) $F = \{k \in \mathbb{Z} : 1 \leq k^2 \leq 100\} = \{-10, -9, \dots, -1, 1, \dots, 9, 10\}$ | $ F  = 20$ |

71) For  $A = \{-1, 0, 1\}$  and  $B = \{x, y\}$ , determine  $A \times B$ .

a)  $A \times B = \{(-1, x), (-1, y), (0, x), (0, y), (1, x), (1, y)\}$

72) Let  $U = \{1, 2, 3\}$  be the universal set and let  $A = \{1, 2\}$ ,  $B = \{2, 3\}$  and  $C = \{1, 3\}$ . Determine the following.

- a)  $(A \cup B) - (B \cap C) = U - \{3\} = A$   
 b)  $\bar{A} = U - A = \{3\}$   
 c)  $\overline{B \cup C} = U - (B \cup C) = U - U = \emptyset$   
 d)  $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$

73) Let  $A = \{1, 2, \dots, 10\}$ . Give an example of two sets  $S$  and  $B$  such that  $S \subseteq \mathcal{P}(A)$ ,  $|S| = 4$ ,  $B \in S$  and  $|B| = 2$ .

- a)  $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \dots\}$   
 b)  $S = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$   $|S| = 4$  and  $S \subseteq \mathcal{P}(A)$   
 c)  $B = \{1, 2\}$   $|B| = 2$  and  $B \in S$

74) For  $A = \{1\}$  and  $C = \{1, 2\}$ , give an example of a set  $B$  such that  $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$ .

- a)  $\mathcal{P}(A) = \{\emptyset, \{1\}\}$   
 b)  $\mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$   
 c)  $B = \{\emptyset, \{1\}, \{2\}\}$

75) Give examples of two sets  $A$  and  $B$  such that  $A \cap \mathcal{P}(A) \in B$  and  $\mathcal{P}(A) \subseteq A \cup B$ .

- a)  $A = \{1\}$   $\mathcal{P}(A) = \{\emptyset, \{1\}\}$   $A \cap \mathcal{P}(A) = \emptyset$   
 b)  $B = \{\emptyset, \{1\}\}$   $A \cup B = \{\emptyset, 1, \{1\}\}$

76) Which of the following sets are equal?

- a)  $A = \{n \in \mathbb{Z} : -4 \leq n \leq 4\} = \{-4, -3, -2, \dots, 4\}$   
 b)  $B = \{x \in \mathbb{N} : 2x + 2 = 0\} = \emptyset$   
 c)  $C = \{x \in \mathbb{Z} : 3x - 2 = 0\} = \emptyset$   
 d)  $D = \{x \in \mathbb{Z} : x^3 = 4x\} = \{-2, 0, 2\}$   
 e)  $E = \{-2, 0, 2\}$   
 f) Thus  $B$  and  $C$  are equal,  $D$  and  $E$  are equal and  $A$  is not equal to any of the other sets.

77) Let  $A$  and  $B$  be subsets of some unknown universal set  $U$ . Suppose that  $\bar{A} = \{3, 8, 9\}$ ,  $A - B = \{1, 2\}$ ,  $B - A = \{8\}$  and  $A \cap B = \{5, 7\}$ . Determine  $U$ ,  $A$  and  $B$ .



- a) From  $\bar{A} = \{3, 8, 9\}$ , we know:  $U = \{3, 8, 9\}, A = \{\}, B = \{\}$   
 b) From  $A - B = \{1, 2\}$ , we add:  $U = \{1, 2, 3, 8, 9\}, A = \{1, 2\}, B = \{\}$   
 c) From  $B - A = \{8\}$ , we add:  $U = \{1, 2, 3, 8, 9\}, A = \{1, 2\}, B = \{8\}$   
 d) From  $A \cap B = \{5, 7\}$  we add:  $U = \{1, 2, 3, 5, 7, 8, 9\}, A = \{1, 2, 5, 7\}, B = \{5, 7, 8\}$

78) Let  $I$  denote the interval  $[0, \infty)$ . For each  $r \in I$ , define

$$A_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = r^2\}$$

$$B_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq r^2\}$$

$$C_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 > r^2\}$$

**I am very uncertain about this exercise; the answers are most likely wrong.**

- a) Determine  $\bigcup_{r \in I} A_r$  and  $\bigcap_{r \in I} A_r$ .  
 i)  $\bigcup_{r \in I} A_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (|x| = |r| \text{ and } y = 0) \text{ or } (x = 0 \text{ and } |y| = |r|)\}$   
 (1) e.g.  $A_0 = \{(0, 0)\}, A_1 = \{(-1, 0), (1, 0), (0, -1), (0, 1)\}, A_2 = \{(-2, 0), (2, 0), (0, -2), (0, 2)\} \dots$   
 ii)  $\bigcap_{r \in I} A_r = \emptyset$   
 b) Determine  $\bigcup_{r \in I} B_r$  and  $\bigcap_{r \in I} B_r$   
 i)  $\bigcup_{r \in I} B_r = \{(x, y) \in \mathbb{R} \times \mathbb{R}\}$   
 ii)  $\bigcap_{r \in I} B_r = \{(0, 0)\}$   
 c) Determine  $\bigcup_{r \in I} C_r$  and  $\bigcap_{r \in I} C_r$   
 i)  $\bigcup_{r \in I} C_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x, y) \neq (0, 0)\}$   
 ii)  $\bigcap_{r \in I} C_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = \infty\}$

79) Give an example of four sets  $A_1, A_2, A_3, A_4$  such that  $|A_i \cap A_j| = |i - j|$  for every two integers  $i$  and  $j$  with  $1 \leq i < j \leq 4$ .

- a)  $A_1 = \{1, 2\}$   
 b)  $A_2 = \{1, 3\}$   
 c)  $A_3 = \{1, 4\}$   
 d)  $A_4 = \{1, 5\}$   
 e) Then  $\forall i, j \in 1 \leq i < j \leq 4, |A_i \cap A_j| = |i - j| = 1$

80) Give an example of two problems suggested by exercise 79.

- a) Solve one of the problems.

81) Let  $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}$  and  $C = \{1, 2, 3, 4, 5\}$ . For the sets  $S$  and  $T$  described below, explain whether  $|S| < |T|$ ,  $|S| > |T|$  or  $|S| = |T|$ .

- a) Let  $B$  be the universal set and let  $S$  be the set of all subsets  $X$  of  $B$  for which  $|X| \neq |\bar{X}|$ . Let  $T$  be the set of 2 element subsets of  $C$ .  
 i)  $|S| = 1 + \frac{4!}{(4-1)!*1!} + \frac{4!}{(4-3)!*3!} = 4 + 4 = 9$   
 ii)  $|T| = \frac{5!}{(5-2)!*2!} = 10$   
 iii)  $|S| < |T|$   
 b) Let  $S$  be the set of all partitions of the set  $A$  and let  $T$  be the set of 4-element subsets of  $C$ .  
 i)  $|T| = \frac{5!}{(5-4)!*4!} = 5$

- ii)  $S = \{\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1, 2, 3\}\}\}$
- iii)  $|S| = |T|$
- c) Let  $S = \{(b, a) : b \in B, a \in A, a + b \text{ is odd}\}$  and let  $T$  be the set of all nonempty proper subsets of  $A$ .
- i)  $|T| = \frac{3!}{(3-2)!*2!} + \frac{3!}{(3-1)!*1!} = 3 + 3 = 6$
- ii)  $|S| = |\{(1,2), (2,1), (2,3), (3,2), (4,1), (4,3)\}|$
- iii)  $|S| = |T|$