2016

Mathematical Proofs

CHAPTER 2 – EXERCISE SOLUTIONS AND NOTES

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Section 1: Describing a Set

Exercises

1) Which of the following are sets?
   a) 1, 2, 3 Not a set
   b) {1, 2}, 3 Not a set
   c) {{1}, 2}, 3 Not a set
   d) {1, {2,}, 3} Set
   e) {1, 2, a, b} Set

2) Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S: p(x)\}$, where p(x) is some condition on x.
   a) $A = \{1, 2, 3\} = \{x \in S: x > 0\}$
   b) $B = \{0, 1, 2, 3\} = \{x \in S: x \geq 0\}$
   c) $C = \{-2, -1\} = \{x \in S: x < 0\}$
   d) $D = \{-2, 2, 3\} = \{x \in S: |x| \geq 2\}$

3) Determine the cardinality of each of the following sets:
   a) $A = \{1, 2, 3, 4, 5\}$ $|A| = 5$
   b) $B = \{0, 2, 4, \dots, 20\}$ $|B| = 11$
   c) $C = \{25, 26, 27, \dots, 75\}$ $|C| = 51$
   d) $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$ $|D| = 2$
   e) $E = \{\emptyset\}$ $|E| = 1$
   f) $F = \{2, \{2, 3, 4\}\}$ $|F| = 2$

4) Write each of the following sets by listing its elements within braces.
   a) $A = \{n \in \mathbb{Z}: -4 < n \leq 4\} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
   b) $B = \{n \in \mathbb{Z}: n^2 < 5\} = \{-2, -1, 0, 1, 2\}$
   c) $C = \{n \in \mathbb{N}: n^3 < 100\} = \{1, 2, 3, 4\}$
   d) $D = \{x \in \mathbb{R}: x^2 - x = 0\} = \{0, 1\}$
   e) $E = \{x \in \mathbb{R}: x^2 + x = 0\} = \{-1, 0\}$

5) Write each of the following sets in the form $\{x \in \mathbb{Z}: p(x)\}$, where p(x) is a property concerning x.
   a) $A = \{-1, -2, -3, \dots\} = \{x \in \mathbb{Z}: x < 0\}$
   b) $B = \{-3, -2, \dots, 3\} = \{x \in \mathbb{Z}: |x| \leq 3\}$
   c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbb{Z}: 0 < |x| \leq 2\}$

6) The set $E = \{2x : x \in \mathbb{Z}\}$ can be described by listing its elements, namely $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$. List the elements of the following sets in a similar manner.
   a) $A = \{2x + 1 : x \in \mathbb{Z}\} = \{\dots, -3, -1, 1, 3, 5, \dots\}$
   b) $B = \{4n : n \in \mathbb{Z}\} = \{\dots, -8, -4, 0, 4, 8, \dots\}$
   c) $C = \{3q + 1 : q \in \mathbb{Z}\} = \{\dots, -5, -2, 1, 4, 7, \dots\}$

7) The set $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbb{Z}\} = \{2x : x \in \mathbb{Z}\}$. Describe the following sets in a similar manner.
   a) $A = \{\dots, -4, -1, 2, 5, 8, \dots\} = \{3x - 1 : x \in \mathbb{Z}\}$
   b) $B = \{\dots, -10, -5, 0, 5, 10, \dots\} = \{5x : x \in \mathbb{Z}\}$
   c) $C = \{1, 8, 27, 64, 125, \dots\} = (x^3 : x \in \mathbb{N}\}$

8) Let $A = \{n \in \mathbb{Z} : 2 \leq |n| < 4\}, B = \{x \in \mathbb{Q} : 2 < x \leq 4\}, C = \left\{x \in \mathbb{R} : x^2 - \left(2 + \sqrt{2}\right)x + 2\sqrt{2} = 0\right\}$ and $D = \{x \in \mathbb{Q} : x^2 - \left(2 + \sqrt{2}\right)x + 2\sqrt{2} = 0.$
   a) Describe the set A by listing its elements.
      i) $A = \{-2, -3, 2, 3\}$
   b) Give an example of three elements that belong to B but do not belong to A.
      i) $\frac{5}{2}, \frac{7}{3}, \frac{9}{4}$
   c) Describe the set C by listing its elements.
      i) $C = \{\sqrt{2}, 2\}$
   d) Describe the set D in another manner.
      i) $D = \{2\}$
   e) Determine the cardinality of the sets A, C and D.
      i) $|A| = 4; |C| = 2; |D| = 1$

9) For $A = \{2, 3, 5, 7, 8, 10, 13\}$, let $B = \{x \in A : x = y + z, where\ y, z \in A\}$ and $C = \{r \in B : r + s \in B\ for\ some\ s \in B\}$. Determine C.
   a) $B = \{5, 7, 8, 10, 13\}$
   b) $C = \{5, 8\}$ (because $5 + 8 = 13\ and\ 8 + 5 = 13$)