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On \mathcal{L} -induced \mathcal{T}_n -modules

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A large, faint watermark of the Uppsala University seal is visible in the bottom right corner of the page. It features a sunburst in the center, surrounded by the Latin text 'SIGILLUM UNIVERSITATIS UPSALIENSIS' and 'VERITAS' at the bottom.

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Abstract

In this thesis we study the structure of \mathcal{T}_n -modules \mathcal{L} -induced from trivial modules over maximal subgroups and prove their irreducibility in a combinatorial way.

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1 Introduction

In this paper we settle a problem stated in [GM09, 11.6.2], it reduces determining the simplicity of a certain module to determining if a certain matrix has full rank. Since the simplicity of this module already is known, we give an alternative proof. This paper is mostly self-contained, and requires some basic abstract and linear algebra. In the first section the reader is introduced to elementary semigroup theory, ideals and representations. The \mathcal{L} -induced modules are also

introduced. The second section focuses on the structure and results specific to the full transformation monoid \mathcal{T}_n . The third section concerns the statement and proof of the problem.

2 A primer on semigroup theory

2.1 Semigroups

A semigroup is a set S with a binary operation $S \times S \rightarrow S$, written $(r, s) \rightarrow rs$, which is associative, that is

$$r(st) = (rs)t$$

for all r, s, t in S . Examples of semigroups are:

- Any group.
- The semigroup of all $n \times n$ matrices over some field under multiplication.
- The set of positive integers, reals or rational numbers under addition.
- Transformations of a set, with composition as multiplication.

An element e is a left identity if $es = s$, a right identity if $se = s$ and an identity if $se = es = s$ for all s in S . A semigroup can have several left or right identities, but an element which is both is unique, since if e and f are two different identities we have $e = ef = f$. If S has an identity, it is called a monoid.

In the rest of the paper, sometimes we will need a monoid to make our definitions less painful. We define S^1 to be the monoid constructed from a semigroup in the following canonical way:

$$S^1 = \begin{cases} S \cup \{1\} & \text{if } S \text{ has no identity element} \\ S & \text{otherwise} \end{cases}$$

where 1 is an element disjoint from S , acting as the identity on S .

An idempotent is an element e which satisfies $e^2 = e$. An identity is of course an idempotent, but the converse need not be true. Given a semigroup S , we define $\mathcal{E}(S)$ to be the set of idempotents in S .

2.2 Ideals

We define products of semigroup subsets A and B by

$$AB = \{st \mid s \in A, t \in B\}.$$

A subset $I \subset S$ is a left ideal if $SI \subset I$, a right ideal if $IS \subset I$ or an ideal if it is both a left and a right ideal. A principal ideal is the one generated by a single element called a generator. To ensure that the ideal contains the element generating it we shall work over the monoid S^1 .

Definition 2.1. We introduce the standard semigroup relations introduced by Green in [Gre51]. Two elements a, b are called

1. \mathcal{L} -equivalent, or $a\mathcal{L}b$ if $S^1a = S^1b$.
2. \mathcal{R} -equivalent, or $a\mathcal{R}b$ if $aS^1 = bS^1$.
3. \mathcal{J} -equivalent, or $a\mathcal{J}b$ if $S^1aS^1 = S^1bS^1$.

That is, a, b generate the same left, right and two-sided ideals, respectively.

The fact that these define equivalence relations is clear. We also note that $a\mathcal{L}b$ implies $ax\mathcal{L}bx$ and $a\mathcal{R}b$ implies $xa\mathcal{R}xb$ for all x in S . We call \mathcal{L} and \mathcal{R} right and left compatible, also called right and left congruences. We shall also consider the relation \mathcal{H} , where $a\mathcal{H}b$ if both $a\mathcal{L}b$ and $a\mathcal{R}b$ are true. Since the intersection of equivalence relations also is an equivalence relation so is \mathcal{H} .

Next we consider the join, or product of the relations \mathcal{L} and \mathcal{R} , denoted $\mathcal{L} \circ \mathcal{R}$; the relation defined by

$$a(\mathcal{L} \circ \mathcal{R})b \iff \text{there exists } x \in S \text{ such that } a\mathcal{L}x \text{ and } x\mathcal{R}b.$$

Lemma 2.2. The relations \mathcal{L} and \mathcal{R} commute, that is, $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

Proof. Let $a(\mathcal{L} \circ \mathcal{R})b$, then there exists $x \in S$ such that $a\mathcal{L}x$ and $x\mathcal{R}b$. Now by definition of \mathcal{L} and \mathcal{R} there are $u, v \in S^1$ such that $1a = ux$ and $xv = b1$. Let $y = av = uxv = ub$. Since \mathcal{L} is right compatible, $a\mathcal{L}x$ implies $av\mathcal{L}xv$, hence $y\mathcal{L}b$. Also $x\mathcal{R}b$ implies $ux\mathcal{R}ub$, hence $a\mathcal{R}y$. Therefore we have y such that $a\mathcal{R}y$ and $y\mathcal{L}b$, this implies $a(\mathcal{R} \circ \mathcal{L})b$. A dual argument shows that $a(\mathcal{R} \circ \mathcal{L})b$ implies $a(\mathcal{L} \circ \mathcal{R})b$, hence the relations commute. \square

Proposition 2.3. The relation $\mathcal{L} \circ \mathcal{R}$ is transitive.

Proof. Consider $a(\mathcal{L} \circ \mathcal{R})b$ and $b(\mathcal{L} \circ \mathcal{R})c$, this means that there exist $x, y \in S$ such that $a\mathcal{L}x$, $x\mathcal{R}b$, $b\mathcal{L}y$ and $y\mathcal{R}c$. The middle two relations imply that $x(\mathcal{R} \circ \mathcal{L})y$, and by Lemma 2.2 it follows that $x(\mathcal{L} \circ \mathcal{R})y$. Hence there exists z such that $x\mathcal{L}z$ and $z\mathcal{R}y$. Then $a\mathcal{L}x\mathcal{L}z$ and $z\mathcal{R}y\mathcal{R}c$; and by transitivity of \mathcal{L} and \mathcal{R} it follows that $a(\mathcal{L} \circ \mathcal{R})c$. \square

Note that $\mathcal{L} \circ \mathcal{R}$ is obviously symmetric and reflexive. This implies that $\mathcal{L} \circ \mathcal{R}$ is a well-defined equivalence relation. This relation will be denoted by \mathcal{D} . We have $\mathcal{D} \subseteq \mathcal{J}$ since \mathcal{D} is the smallest equivalence relation containing \mathcal{L} and \mathcal{R} , but so does \mathcal{J} . For finite semigroups, we have the following.

Proposition 2.4. In a finite semigroup $\mathcal{D} = \mathcal{J}$.

Proof. We already know that $\mathcal{D} \subseteq \mathcal{J}$, and only need to show that $\mathcal{J} \subseteq \mathcal{D}$. First we note the fact that every element of a finite semigroup S must have a power that is an idempotent. Assuming $a\mathcal{J}b$ by the definition there must exist elements $x, y, z, u \in S$ such that $xay = b$ and $zbu = a$. It follows that $(xz)^nb(uy)^n = b$ for every positive integer n . By the above there is a positive integer m such that $(uy)^m$ is an idempotent. Multiplying from the right we get $b(uy)^m = b$. Then for every $s \in S^1$ we have

$$bs = b(uy)^ms = buy(uy)^{m-1}s$$

This means that for every s , there exists $t = y(uy)^{m-1}s$ such that $bs = but$, therefore $b\mathcal{R}bu$. Dually it can be shown that $zb\mathcal{L}b$. It follows that $a\mathcal{L}bu$. Hence $a\mathcal{D}b$, and $\mathcal{J} \subseteq \mathcal{D}$. \square

Since all relations we have defined so far are equivalence relations, we will denote by $\mathcal{K}(a)$ the equivalence class containing the element a under the relation $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}, \mathcal{H}\}$.

Proposition 2.5. $a\mathcal{D}b$ is equivalent to $\mathcal{L}(a) \cap \mathcal{R}(b)$ being nonempty

Proof. By the definition of \mathcal{D} , if there exists a c such that $a\mathcal{L}c$ and $c\mathcal{R}b$, we must have $c \in \mathcal{L}(a) \cap \mathcal{R}(b)$ \square

Lemma 2.6 (Green's Lemma). Let $a\mathcal{R}b$, then there exists u, v such that $au = b$ and $bv = a$. Furthermore $\lambda_u : x \mapsto xu$ and $\lambda_v : y \mapsto yv$, that map $\mathcal{L}(a)$ to $\mathcal{L}(b)$ and $\mathcal{L}(b)$ to $\mathcal{L}(a)$ are mutually inverse bijections. Also \mathcal{R} -classes are preserved under both λ_u and λ_v . That is $x\mathcal{R}xu$ and $y\mathcal{R}yv$ for $x \in \mathcal{L}(a)$ and $y \in \mathcal{L}(b)$.

Proof. If $aS^1 = bS^1$, there must be elements u, v in S^1 such that $au = b1$ and $a1 = bv$. Then $a\mathcal{L}x$ implies $au\mathcal{L}xu = b\mathcal{L}xu$ and $b\mathcal{L}y$ implies $bv\mathcal{L}yv = a\mathcal{L}yv$. For being bijective, let $x \in \mathcal{L}(a)$, then we have $x = pa$ for some p , and

$$\lambda_v(\lambda_u(pa)) = pa uv = pbv = pa = x$$

This means $\lambda_v \circ \lambda_u$ is the identity on $\mathcal{L}(a)$ and $\lambda_u \circ \lambda_v$ is the identity on $\mathcal{L}(b)$, and therefore both λ_u and λ_v are bijective. For the last statement $a\mathcal{R}b$ implies $a\mathcal{R}au$. Since $a\mathcal{L}x$ there exists $x' \in S$ such that $a = x'x$. Since \mathcal{R} is a left congruence it follows that $x'\mathcal{R}x'xu$ which implies $x\mathcal{R}xu$. \square

We also have the following dual result, the proof is analogous.

Lemma 2.7 (Green's Lemma (dual)). Let $a\mathcal{L}b$, then there exists u, v such that $ua = b$ and $vb = a$. Furthermore $\mu_u : x \mapsto ux$ and $\mu_v : y \mapsto vy$, that map $\mathcal{R}(a)$ to $\mathcal{R}(b)$ and $\mathcal{R}(b)$ to $\mathcal{R}(a)$ are mutually inverse bijections. Also \mathcal{L} -classes are preserved under both μ_u and μ_v . That is $x\mathcal{L}ux$ and $y\mathcal{L}vy$ for $x \in \mathcal{R}(a)$ and $y \in \mathcal{R}(b)$.

Combining these results it is clear that any two \mathcal{H} -classes in the same \mathcal{D} -class have bijective mappings between them and therefore the same number of elements.

Theorem 2.8 (Green's Theorem). An \mathcal{H} -class H in a semigroup contains an idempotent if and only if it is a group.

Proof. If H contains an idempotent e , we have e^2 in H . This means in particular that $e\mathcal{L}e^2$ and $e\mathcal{R}e^2$. Now by Green's lemmas, we get that λ_e and μ_e are bijective on H . This means that $eh \in H$ and $he \in H$. The same argument applied in the same way gives that λ_h and μ_h are bijective on H . Thus the action of H on itself is closed and bijective, and a bijective action on a set is just a group. Formally since all elements h act bijectively on H , there must exist e, z in H such that $ex = xe = x$ and $xz = zx = e$ for all $x \in H$. \square

The structure theory for Green's relation gives us an intuitive notion called *egg-box diagram* of a semigroup. Clifford and Preston write the following in [CP61]:

It is a great help to visualize a \mathcal{D} -class D of a semigroup S in the following way, which we call the *egg-box picture*. Imagine the elements of D arranged in a rectangular pattern, like an egg-box, the columns corresponding to \mathcal{R} -classes and the rows to the \mathcal{L} -classes contained in D^* . Each cell of the egg-box corresponds to a \mathcal{H} -class contained in D , and the foregoing remark[†] shows that no cell is empty.

This is the picture we (and they) have in mind.

	$\mathcal{D}(\alpha):$		$\mathcal{R}(\alpha)$
$\mathcal{L}(\alpha)$		$\mathcal{H}(\alpha)$	

2.3 Modules and representations

A *representation* of S is a semigroup homomorphism $\phi : S \rightarrow \text{End}_{\mathbb{C}}(V)$, to the semigroup of linear operators over V , additionally it should map an eventual identity of S to the identity transformation on V .

We can also define an S -module by a map $\cdot : S \times V \rightarrow V$ such that

1. $s \cdot (v + w) = s \cdot v + s \cdot w$
2. $s \cdot (tv) = (st) \cdot v$
3. $s \cdot (\lambda v) = \lambda(s \cdot v)$
4. $1 \cdot v = v$ if S has an identity 1

with s, t in S , v, w in V , and λ in \mathbb{C} .

These two definitions are equivalent. Let ϕ be such a homomorphism, then $s \cdot v = \phi(s)v$ fulfills all the axioms for a S -module. Conversely, the action $s \cdot$ is a linear operator by 1 and 3. Statement 2 gives the homomorphism property. A submodule W of a module V is a subspace $W \subset V$ such that $S \cdot W \subseteq W$. If the only submodules are zero or the whole space, we call it simple. The corresponding representation is called irreducible. If a module can be decomposed into a direct sum of simple modules, it is called semisimple. Simple modules generally correspond to the building blocks of modules, this intuition makes it natural to try classifying all simple modules of some kind. A linear mapping $f : V \rightarrow W$ between S -modules is called an S -homomorphism if f commutes with the action of S , that is $f(s \cdot v) = s \cdot f(v)$, for all s in S and v in v . If this is an isomorphism then V and W are isomorphic modules.

2.4 \mathcal{L} -induced Modules

Consider the representation theory of finite groups where a natural problem is, given a subgroup $H \leq G$ and a representation of H , to try to extend the representation to the whole group G . A naive extension could be to define $g \cdot v = 0$ for all $g \notin H$, but this would not work since group elements need

^{*}We have switched rows and columns in the quote, as their book [CP61] defines composition in \mathcal{T}_n in the opposite order.

[†]Clifford and Preston are referring to their version of Proposition 2.5 above.

to be invertible. There is hope however. We know that subgroups partition a group into cosets $\{gH \mid g \in G\}$. Fix an index set I with representatives $A = \{a_1, \dots, a_{|G|/|H|}\}$ with one from each coset; let M be a H -module and consider the space

$$V_A(M) = \bigoplus_{i \in I} M^{(i)}.$$

Let $\eta_i : M \rightarrow M^{(i)}$ be the canonical injection on the i :th component. Then we can define a G -module on this space by

$$g \cdot \eta_i(v) = \eta_j(h \cdot v) \quad \text{where } ga_i = a_j h \text{ for some } j \in I \text{ and } h \in H$$

since ga_i is in some coset, we can always find such a_j and h .

For our purposes this construction is used as an motivational example; proofs and properties of this construction are omitted, for details see e.g. [Sag01].

Analogously, we can define a similar thing for a semigroup S . Let V be a $\mathcal{H}(e)$ -module for some idempotent e in S . Then similarly to the group construction, the \mathcal{L} -class $\mathcal{L}(e)$ is partitioned into different \mathcal{H} -classes, each corresponding to an \mathcal{R} -class. Let $A = \{a_i \mid i \in I\}$ be a set of representatives from each \mathcal{H} -class. Let y be an element in $\mathcal{L}(e)$. Since $a_i \mathcal{L}e$ and $a_i e = a_i$, we have that λ_{a_i} is a bijection $\mathcal{R}(e) \rightarrow \mathcal{R}(a_i)$ by Green's lemma. Since it preserves \mathcal{L} -classes we have that any $y \in \mathcal{L}(e)$ can be written in the form $a_i h$ for some $a_i \in A$ and $h \in \mathcal{H}(e)$. This means we can extend the action of $\mathcal{H}(e)$ in a canonical way to the whole of S . But since sa_i will not always be in $\mathcal{L}(e)$, we are forced to define our action to be zero if this happens. Consider the vector space with $|I|$ copies of M

$$V_A(M) = \bigoplus_{i \in I} M^i$$

where $\eta_i : M \rightarrow V(M)$ is the canonical injection into the i th component. Define an S -module over $V_A(M)$ by

$$s \cdot \eta_i(v) = \begin{cases} \eta_j(h \cdot v) & \text{if } sa_i = a_j h \text{ for some } j \in I \text{ and } h \in \mathcal{H}(e). \\ 0 & \text{if } sa_i \notin \mathcal{L}(e). \end{cases}$$

Proposition 2.9. The definition above turns $V_A(M)$ into an S -module.

Proof. The linearity follows from fact that M is an $\mathcal{H}(e)$ -module. To show associativity we note that $S^1 sta_i \subset S^1 ta_i \subset S^1 a_i = S^1 e$. This means that $sta_i \in \mathcal{L}(e)$ implies $ta_i \in \mathcal{L}(e)$. So we assume that $sta_i, ta_i \in \mathcal{L}(e)$. This means that $ta_i = a_j g$, $sa_i = a_k h$ and $sta_i = a_l z$ for some $i, j, k, l \in I$ and $g, h, z \in H$. We have

$$a_l z = sta_i = sa_j g = a_k h g$$

this means that $sta_i = a_k h g$ and

$$st \cdot \eta_i(v) = \eta_k(h \cdot (g \cdot v)) = s \cdot \eta_j(g \cdot v) = s \cdot (t \cdot \eta_i(v)).$$

□

Proposition 2.10. If we induce a module using two different set of representatives $\{a_i\}$ and $\{a'_i\}$, the results are isomorphic.

Proof. Every element a'_i satisfies $a'_i = a_i g_i$ for some $g_i \in \mathcal{H}(e)$. Define a linear mapping

$$f_i(\eta_i(v)) = \eta_i(g_i^{-1} \cdot v)$$

and extend it linearly to $f = \oplus_{i \in I} f_i$ so $f : V_A(M) \rightarrow V_{A'}(M)$. Since g_i is a group element, f is an isomorphism. Next we have to show that f commutes with the action of S . If $sa_i \notin \mathcal{L}(e)$, then $s \cdot \eta_i(g_i^{-1} \cdot v) = s \cdot \eta_i(v) = 0$. Assuming that $sa_i = a_j x$ for some $j \in I$ and $x \in \mathcal{H}(e)$, we get

$$sa'_i = sa_i g_i = a_j x g_i = a'_j g_j^{-1} x g_i.$$

This means that

$$\begin{aligned} s \cdot f_i(\eta_i(v)) &= \eta_j(g_j^{-1} x g_i(g_i^{-1} \cdot v)) \\ &= \eta_j(g_j^{-1}(x \cdot v)) = f_j(\eta_j(x \cdot v)) \\ &= f_j(s \cdot \eta_i(v)) \end{aligned}$$

and it follows that f is an S -module isomorphism. \square

Proposition 2.11. If e and e' are idempotents such that $e\mathcal{D}e'$, then for any $\mathcal{H}(e)$ -module M there exists a $\mathcal{H}(e')$ -module M' such that $V(M) \cong V(M')$

Proof. Let A be a set of representatives as earlier. By the previous theorem we can assume that $e \in A$. Since A is a transversal we note that there is an $l \in I$ such that $a_l \mathcal{R} e'$. It follows that there exists $u, v \in S$ such that $a_l v = e'$ and $e' u = a_l$. Note that $u = a_l$. Define a new set of representatives $A' = \{a'_i = a_i v \mid i \in I\}$. We note that λ_v maps $\mathcal{L}(e)$ onto $\mathcal{L}(e')$, a_i upon a'_i and especially a_l upon e' . Let $M' = M^{(l)}$ and see that this is an $\mathcal{H}(e')$ -module. Extend this module to the S -module $V_{A'}(M')$ as before, and denote its canonical injection by η'_i . Then define a mapping $f : V_A(M) \rightarrow V_{A'}(M')$ by

$$f(\eta_i(v)) = \eta'_i(\eta_l(v)).$$

We need to prove that it commutes with the action of S , so note that

$$sa'_i = sa_i v = a_j x v = a_j v u x v = a'_j(u x v).$$

for $x \in \mathcal{H}(e')$. Since $u x v u = u x = a_l x$, it follows that

$$\begin{aligned} s \cdot f(\eta_i(v)) &= \eta'_j(u x v \cdot \eta_l(v)) \\ &= \eta'_j(\eta_l(x \cdot v)) \\ &= f(\eta_j(x \cdot v)) \\ &= f(s \cdot \eta_i(v)) \end{aligned}$$

which proves the statement. \square

3 The full transformation monoid \mathcal{T}_n

In the following section, if nothing else is said, S is assumed to be the transformation monoid \mathcal{T}_n for some fixed integer n . Also let \mathbf{N} be the set $\{1, \dots, n\}$.

3.1 Structure

The full transformation monoid \mathcal{T}_n consists of functions from \mathbf{N} to \mathbf{N} . We write an element α of \mathcal{T}_n as

$$\alpha = (y_1, \dots, y_n)$$

which is the transformation defined by $\alpha(i) = y_i$ for all $i \in \mathbf{N}$. There is a complete characterization of finite semigroups, much like Cayley's theorem for groups, which gives us some motivation for the importance of \mathcal{T}_n .

Theorem 3.1 (Cayley). Every finite semigroup S with n elements is isomorphic to a subsemigroup of \mathcal{T}_n (or \mathcal{T}_{n+1} if S doesn't have an identity).

Proof. Let S be any semigroup, from which we will construct a set X_S and associate a function $f_s : X_S \rightarrow X_S$ to each s , by letting X_S be the set of elements in S . Now by noticing how every element s acts, we can define $f_s(x_i) = sx_i$, this is the left regular representation. Now if S didn't have an identity element, we can adjoin it in the canonical way described earlier, and then do the same embedding. \square

Let α be an element of \mathcal{T}_n then we denote the *image* of α by $\text{im}(\alpha)$. Note that $\text{im}(\alpha) = \alpha(\mathbf{N})$. The *rank* of an element α is the cardinality of the image. That is $\text{rank}(\alpha) = |\text{im}(\alpha)|$. We note that an element α gives rise to a partition of \mathbf{N} with classes of elements sent to the same element in the image. Call this equivalence relation π_α . It is defined as

$$a \pi_\alpha b \iff \alpha(a) = \alpha(b)$$

for $a, b \in \mathbf{N}$. By a *partition* of \mathbf{N} we mean a set of sets $p_j \subset \mathbf{N}$ such that $p_j \cap p_k = \emptyset$ and $\bigcup_j p_j = \mathbf{N}$, we will write a partition as $p = p_1 | \dots | p_k$ often omitting the brackets when writing out the sets. The partition of \mathbf{N} corresponding to an element α in \mathcal{T}_n will be denoted ρ_α . Given two partitions we say that $\pi_\alpha \subset \pi_\beta$ if $a \pi_\alpha b$ implies $a \pi_\beta b$ for all $a, b \in \mathbf{N}$. This happens when ρ_β is *coarser* than ρ_α .

3.2 Green's relations on \mathcal{T}_n

We begin our classification of Green's relations on \mathcal{T}_n by considering its principal ideals. In the following section, let S be a subsemigroup of \mathcal{T}_n .

Theorem 3.2. For each α the right principal ideals in \mathcal{T}_n are of the form

$$\alpha S = \{\beta \in S : \text{im}(\beta) \subset \text{im}(\alpha)\}.$$

Proof. Let X be the set on the right-hand side. For all $s \in S$ we have $\text{im}(\alpha s) \subset \text{im}(\alpha)$, and thus $\alpha S \subset X$. To show $X \subset \alpha S$ take some β in X . Since $\text{im}(\beta) \subset \text{im}(\alpha)$ we can for each b in $\text{im}(\beta)$ choose some $a_b \in \mathbf{N}$ such that $\alpha(a_b) = b$. Now let $\gamma \in S$ be the transformation $\gamma(x) = a_b$ when $\beta(x) = b$. Then $\alpha\gamma = \beta$. \square

Theorem 3.3. For each α the left principal ideals in S are of the form

$$S\alpha = \{\beta \in S : \pi_\alpha \subset \pi_\beta\}.$$

Proof. Let X be the set on the right-hand side. Let $\beta \in S\alpha$ with $\beta = s\alpha$ for some $s \in S$. Assuming that $x \pi_\alpha y$ it follows that

$$\beta(x) = s(\alpha(x)) = s(\alpha(y)) = \beta(y),$$

therefore $x \pi_\beta y$ implies $x \pi_\alpha y$ so $S\alpha \subset X$. Now, assuming that $\beta \in X$, define $t \in \mathcal{T}_n$ as

$$t(\alpha(x)) = \beta(x).$$

on elements in $\text{im}(\alpha)$. For elements not in $\text{im}(\alpha)$ extend it arbitrarily. This is well defined since if $\alpha(x) = \alpha(y)$, then $\beta(x) = \alpha(y)$ by assumption. It follows that $\beta = t\alpha \in S\alpha$. \square

Theorem 3.4. For each α the two-sided principal ideals in \mathcal{T}_n are of the form

$$S\alpha S = \{\beta \in S : \text{rank}(\beta) \leq \text{rank}(\alpha)\}. \quad (3.1)$$

Proof. The fact that $\text{rank}(S\alpha S) \leq \text{rank}(\alpha)$ is clear. Now for β on the right-hand side of (3.1), we want to show $\beta \in S\alpha S$. Let $\text{im}(\alpha) = \{a_1, \dots, a_k\}$ and $\text{im}(\beta) = \{b_1, \dots, b_m\}$ where $m \leq k$. Choose an element c_i in the preimage of every a_i , and define $t(y) = c_j$ for all y in the preimage of b_j . Next define $s(a_i) = b_i$. It follows that $s(\alpha(t(x))) = \beta(x)$ and therefore $X \subseteq S\alpha S$. \square

Now we can classify Green's relations on \mathcal{T}_n .

Theorem 3.5. Green's relations on \mathcal{T}_n are completely determined by the following, where $a, b \in \mathcal{T}_n$.

- (i) $a\mathcal{L}b$ if and only if $\rho_a = \rho_b$.
- (ii) $a\mathcal{R}b$ if and only if $\text{im}(a) = \text{im}(b)$.
- (iii) $a\mathcal{H}b$ if and only if $\text{im}(a) = \text{im}(b)$ and $\rho_a = \rho_b$.
- (iv) $a\mathcal{D}b$ if and only if $\text{rank}(a) = \text{rank}(b)$.
- (v) $a\mathcal{J}b$ if and only if $\text{rank}(a) = \text{rank}(b)$.

Proof. Follows from Theorems 3.2, 3.3, 3.4 and Proposition 2.4. \square

Corollary 3.6. The \mathcal{D} -classes of \mathcal{T}_n can be indexed by \mathbf{N} , we denote them by \mathcal{D}_k for some $k \in \mathbf{N}$.

Corollary 3.7. \mathcal{D}_k has $\binom{n}{k}$ different \mathcal{R} -classes, in correspondence with k -subsets of \mathbf{N} .

By $S(n, k)$ we mean the *Stirling numbers of the second kind*, the number of ordered k -partitions on a set of size n .

Corollary 3.8. \mathcal{D}_k has $S(n, k)$ different \mathcal{L} -classes, in correspondance with k -partitions of \mathbf{N} .

For our analysis of induced modules, we need the structure of the \mathcal{H} -classes. Remarkably, \mathcal{H} -classes with an idempotent are isomorphic to \mathcal{S}_k and, by Greens lemma, isomorphic to each other.

Theorem 3.9. For \mathcal{T}_n , \mathcal{H} -classes in \mathcal{D}_k containing an idempotent are groups, and all isomorphic to \mathcal{S}_k .

Proof. Let H be an \mathcal{H} -class with an idempotent. By Theorem 3.5(iii) elements have the same image and partition of the image. Thus every element in H can be considered as an assignment of elements in the image to blocks in the partition. Since H contains an idempotent every block must contain exactly one element from the image. Thus we can consider H as group of permutations on $\text{im}(h)$ so we have a bijection between elements $h \in H$ and $s \in \mathcal{S}_{\{a_1, \dots, a_k\}} \cong \mathcal{S}_k$. \square

For our study, we also note the following combinatorial principle that determine the structure of our egg-box and the location of the idempotents. We characterize the idempotent elements in \mathcal{T}_n and how they relate to Green's relations.

Proposition 3.10. The idempotents of \mathcal{T}_n are the elements α which are the identity when restricted to $\text{im}(\alpha)$

Proof. If $e^2 = e$ for some idempotent e of \mathcal{T}_n , then $e(x) = e(e(x))$, for $x \in \text{im}(e)$. If $\alpha|_{\text{im}(\alpha)}$ is the identity for some $\alpha \in \mathcal{T}_n$, then $\alpha(\alpha(x)) = \alpha(x)$ for all $x \in \mathbf{N}$. \square

The following corollary is used very often in the next section.

Lemma 3.11. Fix a \mathcal{L} and a \mathcal{R} class in $\mathcal{D}_k \subset \mathcal{T}_n$ corresponding to $\rho_\alpha = \{p_1, \dots, p_k\}$ and $\text{im}(\alpha) = \{a_1, \dots, a_k\}$. Then the \mathcal{H} -class in the intersection of these contains an idempotent if and only if every p_i contains exactly one unique a_j .

Proof. Since every part of the partition contains a unique part of the image, the \mathcal{H} -class contains the element which assigns sends p_i to a_i , and thus is an idempotent by Proposition 3.10. \square

Here are some other properties of the idempotents, we use the same notation as the lemma above.

Proposition 3.12. Every \mathcal{R} -class in \mathcal{D}_k has k^{n-k} idempotents.

Proof. Idempotents must fix the restricted image by Proposition 3.10, for the $n - k$ elements left we can choose any of the k elements in the image. \square

Proposition 3.13. Every \mathcal{L} -class in \mathcal{D}_k corresponding to the partition $p_1 | \dots | p_k$ has $|p_1| \cdot \dots \cdot |p_k|$ idempotents.

Proof. To contain an idempotent, every p_i should contain exactly one a_i . So the number of idempotents is the number of ways to pick one element from each p_i \square

Example 3.14. This is the egg-box diagram for \mathcal{D}_2 in \mathcal{T}_4 . Idempotent elements are denoted by $*$.

\mathcal{D}_2	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
1 234	$\begin{smallmatrix} * (1222) \\ (2111) \end{smallmatrix}$	$\begin{smallmatrix} * (1333) \\ (3111) \end{smallmatrix}$	$\begin{smallmatrix} * (1444) \\ (4111) \end{smallmatrix}$	$\begin{smallmatrix} (2333) \\ (3222) \end{smallmatrix}$	$\begin{smallmatrix} (2444) \\ (4222) \end{smallmatrix}$	$\begin{smallmatrix} (3444) \\ (4333) \end{smallmatrix}$
2 234	$\begin{smallmatrix} (2122) \\ * (1211) \end{smallmatrix}$	$\begin{smallmatrix} (3133) \\ (1311) \end{smallmatrix}$	$\begin{smallmatrix} (4144) \\ (1411) \end{smallmatrix}$	$\begin{smallmatrix} * (3233) \\ (2322) \end{smallmatrix}$	$\begin{smallmatrix} * (4244) \\ (2422) \end{smallmatrix}$	$\begin{smallmatrix} (4344) \\ (3433) \end{smallmatrix}$
3 124	$\begin{smallmatrix} (2212) \\ (1121) \end{smallmatrix}$	$\begin{smallmatrix} (3313) \\ * (1131) \end{smallmatrix}$	$\begin{smallmatrix} (4414) \\ (1141) \end{smallmatrix}$	$\begin{smallmatrix} (3323) \\ * (2232) \end{smallmatrix}$	$\begin{smallmatrix} (4424) \\ (2242) \end{smallmatrix}$	$\begin{smallmatrix} * (4434) \\ (3343) \end{smallmatrix}$
4 124	$\begin{smallmatrix} (2221) \\ (1112) \end{smallmatrix}$	$\begin{smallmatrix} (3331) \\ (1113) \end{smallmatrix}$	$\begin{smallmatrix} (4441) \\ * (1114) \end{smallmatrix}$	$\begin{smallmatrix} (3332) \\ (2223) \end{smallmatrix}$	$\begin{smallmatrix} (4442) \\ * (2224) \end{smallmatrix}$	$\begin{smallmatrix} (4443) \\ * (3334) \end{smallmatrix}$
12 34	$\begin{smallmatrix} (1122) \\ (2211) \end{smallmatrix}$	$\begin{smallmatrix} * (1133) \\ (3311) \end{smallmatrix}$	$\begin{smallmatrix} * (1144) \\ (4411) \end{smallmatrix}$	$\begin{smallmatrix} * (2233) \\ (3322) \end{smallmatrix}$	$\begin{smallmatrix} * (2244) \\ (4422) \end{smallmatrix}$	$\begin{smallmatrix} (3344) \\ (4433) \end{smallmatrix}$
13 24	$\begin{smallmatrix} * (1212) \\ (2121) \end{smallmatrix}$	$\begin{smallmatrix} (1313) \\ (3131) \end{smallmatrix}$	$\begin{smallmatrix} * (1414) \\ (4141) \end{smallmatrix}$	$\begin{smallmatrix} (2323) \\ * (3232) \end{smallmatrix}$	$\begin{smallmatrix} (2424) \\ (4242) \end{smallmatrix}$	$\begin{smallmatrix} * (3434) \\ (4343) \end{smallmatrix}$
14 23	$\begin{smallmatrix} * (1221) \\ (2112) \end{smallmatrix}$	$\begin{smallmatrix} * (1331) \\ (3113) \end{smallmatrix}$	$\begin{smallmatrix} (1441) \\ (4114) \end{smallmatrix}$	$\begin{smallmatrix} (2332) \\ (3223) \end{smallmatrix}$	$\begin{smallmatrix} (2442) \\ * (4224) \end{smallmatrix}$	$\begin{smallmatrix} (3443) \\ * (4334) \end{smallmatrix}$

4 \mathcal{T}_n modules \mathcal{L} -induced from the trivial \mathcal{S}_k module

4.1 Introduction

The following theorem shows why \mathcal{L} -induction is useful, we can defer the classification of simple \mathcal{T}_n -modules to the classification of simple \mathcal{S}_k -modules, by the following theorem.

Theorem 4.1. The \mathcal{T}_n -module $V(M)$ is simple if and only if $k = n$ or M is not isomorphic to the sign module.

Proof. The theorem is proved in [HZ57] and is stated as above in [GM09, 11.6.1i]. \square

Investigating a statement in [GM09], we shall consider a special case of Theorem 4.1, modules induced from the trivial module. For the rest of the section let M be the trivial \mathcal{S}_k -module over \mathbb{C} for some k . For $V(M)$ to be simple is equivalent to the fact that there exists some idempotent e such that

$$e \cdot v \neq 0$$

for every vector non-zero $v \in V(M)$. This is because e will project $V(M)$ onto $M^{\text{im}(e)}$, and then we can span the entirety of $V(M)$ by multiplying with all the a_i .

To satisfy this condition we notice the following facts about $V(\mathbb{C}_{\text{triv}})$.

$$\begin{array}{ll}
e \cdot \eta_i(v) = \eta_i(v) & \text{if } e \text{ is an idempotent and } e\mathcal{R}a_i \\
s \cdot \eta_i(v) = \eta_i(v) & \text{if } s\mathcal{L}e \text{ and } e\mathcal{R}a_i \\
s \cdot \eta_i(v) = 0 & \text{otherwise}
\end{array}$$

The first one follows from the fact that $e\mathcal{R}a_i$ implies $e\mathcal{R}ea_i$. This means that $x \rightarrow xa_i$ maps $\mathcal{L}(e) \rightarrow \mathcal{L}(a_i)$ and therefore $ea_i \in \mathcal{L}(a_i)$. The second statement follows since if $s\mathcal{L}e$, then $sa_i\mathcal{L}ea_i$, so $sa_i\mathcal{L}a_i$. The last statement follows since

if s is in a \mathcal{H} -class without an idempotent; and $s\mathcal{R}a_i$ then $s^2\mathcal{R}sa_i$, but $s^2 \notin \mathcal{D}_k$ since it must have collapsed some elements in its image.

By $M_{n,k}$ we mean the $S(n,k) \times \binom{n}{k}$ matrix with a 1 in the ij :th position if the \mathcal{H} -class corresponding to the intersection of the i :th \mathcal{L} -class and j :th \mathcal{R} -class contains an idempotent.

Proposition 4.2. The module $V(\mathbb{C}_{\text{triv}})$ is simple if and only if $M_{n,k}$ has rank $\binom{n}{k}$ and $k > 1$.

Proof. For simplicity, we will use k -subsets of \mathbf{N} as index sets I , since they index \mathcal{R} -classes. Then an element $s \cdot v$ will look like the following

$$s \cdot \sum_{S \in I} c_S v_S = \sum_{e \in \mathcal{E}(\mathcal{L}(s))} c_{\text{im}(e)} s \cdot v_{\text{im}(e)} = \left(\sum_{e \in \mathcal{E}(\mathcal{L}(s))} c_{\text{im}(e)} \right) \cdot v_{\text{im}(s)},$$

where the first equality is because if $\mathcal{L}(s) \cap \mathcal{R}(a_S)$ does not contain an idempotent v_S will be annihilated according to above, and the second equality is by linearity. This means that finding idempotents such that $e \cdot v \neq 0$ is equivalent to finding an \mathcal{L} -class such that the vector of idempotents has nonzero scalar product with v . For this to always be possible, we require that the matrix $M_{n,k}$ has maximal rank. \square

The following is an easy corollary of the theory in Section 3, but is restated in matrix language.

Corollary 4.3. The structure of $M_{n,k}$ is as follows

- (a) $M_{n,k}$ has $\binom{n}{k}$ rows, each corresponding to a k -subset s_i of \mathbf{N} .
- (b) $M_{n,k}$ has $S(n,k)$ columns, each corresponding to a k -partition ρ_i of \mathbf{N} .
- (c) There is a 1 in the a_{ij} :th position if the s_j intersects every part $p \in \rho_i$ in exactly one element, and otherwise a zero.

Proof. Follows from Corollary 3.7, Corollary 3.8, Lemma 3.11 and the fact that \mathcal{H} -classes separate idempotents. \square

By $M_{n,k}(\dots s_i \dots)$ we mean the square sub-matrix of $M_{n,k}$ constructed from the \mathcal{L} and \mathcal{R} classes with partitions p_i and subsets s_i respectively. The aim of the rest of this section is to prove the following theorem

Theorem 4.4. The matrix $M_{n,k}$ has rank $\binom{n}{k}$ for $k > 1$.

Proof. To prove that every $M_{n,k}$ has full rank, we will proceed by constructing a sequence of square sub-matrices $X_{n,k} \subset M_{n,k}$ of size $\binom{n}{k} \times \binom{n}{k}$ by picking the appropriate partitions. To construct this sequence we proceed by induction with that hypothesis and $X_{n-1,k-1}$ and $X_{n-1,k}$ are invertible. This is well defined since $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. By analogy with Pascal's triangle, every matrix will be defined in a unique way from the two elements above it in the triangle. The base cases are $X_{n,2}$ and $X_{n,n}$. \square

4.1.1 Lemmas

First we need the following two lemmas.

Lemma 4.5. Let s_i be a collections of k -element subsets, and ρ_i a collection of k -part partitions. Then

$$M_{n,k} \begin{pmatrix} s_1 & \cdots & s_{\binom{n}{k}} \\ \rho_1 & \cdots & \rho_{\binom{n}{k}} \end{pmatrix} = M_{n+1,k} \begin{pmatrix} s_1 & \cdots & s_{\binom{n}{k}} \\ \phi(\rho_1) & \cdots & \phi(\rho_{\binom{n}{k}}) \end{pmatrix}$$

Where every $\phi(\rho_i)$ is the partition ρ_i , with a new element $(n+1)$ adjoined to some arbitrary part of each partition.

Proof. The set s_i doesn't contain $n+1$, so we can adjoin it to the partitions however we like and still get the same matrix. In other words for all $p \in \rho_i$ we have $|s_j \cap p| = |s_j \cap (p \cup \{n+1\})|$ since $s_i \cap \{n+1\} = \emptyset$. \square

Lemma 4.6. Let s_i be a collections of k -element subsets, and ρ_i a collection of k -part partitions. Then

$$M_{n,k} \begin{pmatrix} s_1 & \cdots & s_{\binom{n}{k}} \\ \rho_1 & \cdots & \rho_{\binom{n}{k}} \end{pmatrix} = M_{n+1,k+1} \begin{pmatrix} s_1 \cup (n+1) & \cdots & s_{\binom{n}{k}} \cup (n+1) \\ [\rho_1|(n+1)] & \cdots & [\rho_{\binom{n}{k}}|(n+1)] \end{pmatrix}$$

Proof. For all parts $p \in \rho_i$, we have $|(s_i \cup \{n+1\}) \cap p| = |s_i \cap p|$, since $p \cap \{n+1\} = \emptyset$. When $p = \{n+1\}$ we have $|(s_i \cup \{n+1\}) \cap \{n+1\}| = 1$. Hence the logical value of the intersection property (Corollary 4.3(c)) is preserved. \square

4.1.2 Base cases

Now we can construct the base cases $X_{n,2}$ and $X_{n,n}$. The base case $X_{n,n}$ is trivial with

$$X_{n,n} = M_{n,n} \begin{pmatrix} \{1, \dots, n\} \\ 1 | \dots | n \end{pmatrix} = (1)$$

For $k=2$ the matrices has a very complicated structure. By picking partitions in correspondence with the subsets by

$$\{a, b\} \mapsto [1, \dots, a, (b+1), \dots, n | (a+1), \dots, b]$$

we can show invertibility for $k=2$ by another induction. It is conjectured in [GM09] that this matrix will always have a power of two as determinant, we shall soon see that is the case. Before proving this, we investigate some examples.

$X_{3,2}$ is invertible and $\det(X_{3,2}) = 2$.

$$X_{3,2} = M_{3,2} = \begin{bmatrix} & 2|31 & 23|1 & 3|12 \\ \{1,2\} & 1 & 1 & 0 \\ \{1,3\} & 0 & 1 & 1 \\ \{2,3\} & 1 & 0 & 0 \end{bmatrix}$$

$X_{4,2}$ is invertible with $\det(X_{4,2}) = 8$.

$$X_{4,2} = \left[\begin{array}{c|cccccc} & 2|341 & 23|41 & 234|1 & 3|412 & 34|12 & 4|123 \\ \hline \{1,2\} & 1 & 1 & 1 & 0 & 0 & 0 \\ \{1,3\} & 0 & 1 & 1 & 1 & 1 & 0 \\ \{1,4\} & 0 & 0 & 1 & 0 & 1 & 1 \\ \{2,3\} & 1 & 0 & 0 & 1 & 1 & 0 \\ \{2,4\} & 1 & 1 & 0 & 0 & 1 & 1 \\ \{3,4\} & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right]$$

Now we begin to see a pattern, in particular $X_{n,2}$ is structured as follows

$$X_{n,2} = \left[\begin{array}{c|cccc|cccc} & 2|3\dots n1, \dots, 234\dots n|1 & 3|1..n, \dots, n|1..(n-1) \\ \hline \{1,2\} & \mathbf{1} & \dots & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \{1,3\} & & & & & & \\ \vdots & & & & & & \\ \{1,n\} & & U & & & B & \\ \hline \{2,3\} & & & & & & \\ \vdots & & L & & & B' & \\ \{2,n\} & & & & & & \\ \{3,4\} & & & & & & \\ \vdots & & C & & & D & \\ \{n-1,n\} & & & & & & \end{array} \right]$$

Where thick lines delimit square submatrices on the diagonal. The subsets are split in two parts, one containing the element 1 and one not. The columns are the partition constructed earlier corresponding to subsets. Dotted lines are useful delimiters in the rest of the proof. We shall prove that this matrix is invertible, by showing the following facts

- The submatrix U has ones above the diagonal and zeroes on and below the diagonal.
- The submatrix L has ones on and below diagonal and zeroes above the diagonal.
- The submatrices B and B' are identical.
- The matrix $\begin{pmatrix} B' \\ D \end{pmatrix} = X_{n-1,2}$.

It will follow that we can reduce the matrix to a block-diagonal form, which is invertible if and only if $X_{n-1,2}$ is.

In the matrix U , we notice that 1 is fixed, hence an entry in the ij :th position is one if $|\{i+2\} \cap \{2, \dots, j+1\}| = 1$, that is $j > i+1$. The opposite happens in L when we fix 2.

In the right column, by construction we always have 1 and 2 in the same partition, this gives $B = B'$.

Next we consider the submatrix $\begin{bmatrix} B' \\ D \end{bmatrix}$. Let q be the cyclic permutation $(n \dots 1)$ that sends $p \rightarrow p-1$ and $1 \rightarrow n$ in both partitions and subsets, then

$$\begin{aligned} & M_{n,2} \begin{pmatrix} \{2,3\} & \cdots & \{p,q\} & \cdots & \{n-1,n\} \\ 3|1..n & \cdots & (1..p(q+1)..n|(p+1)..q) & \cdots & 1..(n-1)|n \end{pmatrix} \\ & \xrightarrow{q} M_{n,2} \begin{pmatrix} \{1,2\} & \cdots & \{p-1,q-1\} & \cdots & \{n-2,n-1\} \\ 2|1..n & \cdots & (1..(p-1)q..n|p..(q-1)) & \cdots & 1..n|(n-1) \end{pmatrix} \end{aligned}$$

This is permitted since we are just changing symbols. Then by Lemma 4.5 this is equivalent to

$$\begin{aligned} M_{n-1,2} \begin{pmatrix} \{1,2\} & \cdots & \{p-1,q-1\} \\ 2|1..(n-1) & \cdots & 1..(p-1)q..(n-1)|p..(q-1) \\ \cdots & 1..(n-2)|(n-1) \\ \cdots & \{n-2,n-1\} \end{pmatrix} &= X_{n-1,2}. \end{aligned}$$

This means that the matrix $\begin{bmatrix} B' \\ D \end{bmatrix} = X_{n-1,2}$. Finally we show the invertibility of the matrix, by subtracting the rows containing B' from the ones containing B . This is the somewhat simplified result, where solid lines delimit the square submatrices on the diagonal.

$$\left[\begin{array}{cc|c} 1 & 1 & \\ & \ddots & 0 \\ -1 & 1 & \\ \hline & L & \\ & C & X_{n-1,2} \end{array} \right]$$

This means that invertibility of $X_{n-1,2}$ implies invertibility of $X_{n,2}$. Let U be the submatrix in the upper left corner, then adding the furthestmost right column to the rest we get twos on the diagonal except for the last column. Since U is of size $n-1$ this gives us $\det(U) = 2^{n-2}$. It follows that

$$\det(X_{n,2}) = \det(U) \det(X_{n-1,2}) = 2^{n-2} \det(X_{n-1,2}) = 2^{(n-2)(n-1)/2}$$

for $n > 2$. We give as example $X'_{4,2}$ which is $X_{4,2}$ after this reduction.

$$X'_{4,2} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

4.1.3 Induction step

Now there is only the induction step left. We construct $X_{n,k}$ as a kind of extension of the matrices $X_{n-1,k-1}$ and $X_{n-1,k}$. We divide the subsets into two

groups, one containing n and the other not containing n . We notice that the sizes of those subsets are $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$, and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Now by our assumption we have sub-matrices $X_{n-1,k-1}$ and $X_{n-1,k}$ that are invertible. Now let α_i and β_i be the partitions in $X_{n-1,k-1}$ and $X_{n-1,k}$ respectively, and ϕ as in Lemma 4.5. Then

$$X_{n,k} = \left[\begin{array}{c|c|c} & \alpha_1|n, \dots, \alpha_{\binom{n-1}{k-1}}|n & \phi(\beta_1), \dots, \phi(\beta_{\binom{n-1}{k}}) \\ \hline \{1, \dots, k-1, n\} & & \\ \vdots & & \\ \{n-k, \dots, n\} & A & B \\ \hline \{1, \dots, k-1, n-1\} & & \\ \vdots & & \\ \{n-k-1, \dots, n-1\} & C & D \end{array} \right]$$

Now, by straightforward application of Lemma 4.5 and Lemma 4.6 we have $A = X_{n-1,k-1}$ and $D = X_{n-1,k}$. We also have $C = 0$ because n always is in its own partition, but not in the subsets. Then, putting k elements in $k-1$ parts, we must have two elements in the same partition by the pigeonhole principle. This reasoning is only possible when $k > 2$, because otherwise we only have one partition $\alpha_{\binom{n-1}{1}}$, which is why the case $k = 2$ must be proven by another method.

It follows that $\det(X_{n,k}) = \det(X_{n-1,k-1}) \det(X_{n-1,k})$ for $k > 2$. This implies that if we construct our sequence of matrices with base cases $X_{n,2}$ and $X_{n,n}$ as above, the determinant will always be a power of two.

Here is an example of the induction step for constructing $X_{4,3}$ from $X_{3,2}$ and $X_{3,3}$

$$\begin{aligned} & \left[\begin{array}{c|c|c|c} X_{3,2} : & 13|2 & 23|1 & 12|3 \\ \hline 12 & 1 & 1 & 0 \\ 13 & 0 & 1 & 1 \\ 23 & 1 & 0 & 1 \end{array} \right], \left[\begin{array}{c|c} X_{3,3} : & 1|2|3 \\ \hline 123 & 1 \end{array} \right] \\ & \quad \Downarrow \\ & \left[\begin{array}{c|c|c|c|c} X_{4,3} : & 13|2|4 & 23|1|4 & 12|3|4 & 1|2|3|4 \\ \hline 124 & 1 & 1 & 0 & 1 \\ 134 & 0 & 1 & 1 & 0 \\ 234 & 1 & 0 & 1 & 0 \\ 123 & 0 & 0 & 0 & 1 \end{array} \right] \end{aligned}$$

4.2 Alternative proof in the case $k = 2$

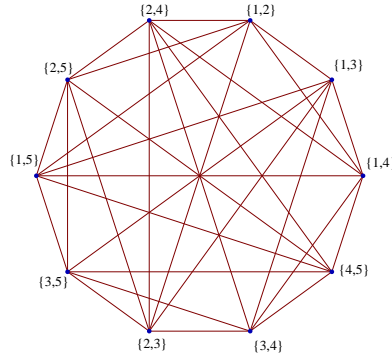
For $n > 4$ there is another interpretation. We construct a matrix by selecting partitions $pq|(\mathbf{N} \setminus \{p, q\})$ corresponding to $\{p, q\}$. We get the matrix

$$Y_{n,2} = M_{n,2} \left(\begin{array}{ccc|c} \{1, 2\} & \{1, 3\} & \cdots & \{n-1, n\} \\ \hline 12|3 \dots n & 13|24 \dots n & \cdots & (n-1)n|1 \dots (n-2) \end{array} \right)$$

Example 4.7. As an example, $Y_{5,2}$ looks like this:

$\{1,2\}$	0	1	1	1	1	0	1	1	0	0
$\{1,3\}$	1	0	1	1	0	1	1	0	1	0
$\{1,4\}$	1	1	0	0	1	1	1	0	0	1
$\{1,5\}$	1	1	0	0	1	1	0	1	1	0
$\{2,3\}$	1	0	1	1	0	1	0	1	0	1
$\{2,4\}$	0	1	1	1	1	0	0	0	1	1
$\{2,5\}$	1	1	1	0	0	0	0	1	1	1
$\{3,4\}$	1	0	0	1	1	0	1	0	1	1
$\{3,5\}$	0	1	0	1	0	1	1	1	0	1
$\{4,5\}$	0	0	1	0	1	1	1	1	1	0

This matrix has a very rich structure, but showing that this matrix is invertible is not as straightforward as the earlier construction. This matrix is square, symmetric and has zeroes on the diagonal, so we can regard it as the adjacency matrix of a non-directed graph without loops. This graph can be regarded as a graph with vertices $V = \{S \subset \mathbf{N} : |S| = 2\}$ with edges between them if they intersect in exactly one element. This graph is called the triangular graph, or the $(n, 2)$ -Johnson graph. For details see [GR01]. For example, the graph associated with $Y_{5,2}$ looks like the following



A well known way of constructing this graph, is taking the line graph of the complete graph K_n . The complete graph is just a set of vertices, with an edge between every pair of vertices. The line graph is constructed by assigning one vertex to each edge, and putting edges between them if the edges share an endpoint. That $Y_{n,2}$ is the line graph of K_n is quite clear, since every edge is just a pair of vertices, and they have an edge between them in the line graph if they have a vertex in common, that is intersect in exactly one element.

We notice that our graph satisfies the following

1. Every vertex has degree $k = 2(n - 2)$
2. Every two adjacent vertices have $\lambda = n - 2$ common neighbours
3. Every two non-adjacent vertices have $\mu = 4$ common neighbours

The graph-theoretic viewpoint shows its usefulness in the following theorem.

Theorem 4.8. The adjacency matrix of $Y_{n,2}$ has eigenvalues -2 , $n - 4$ and $2n - 4$.

Proof. In $Y_{n,2}^2$ the uv -entry is the number of walks of length two from node u to v . Let I be the identity matrix of appropriate size, and let J be a matrix consisting of all ones. Investigating two particular nodes in a graph, they are either the same, adjacent, or non-adjacent. If they are the same, any path goes to a neighbour and back, thus there are k paths. If they are adjacent, there are λ path through its common neighbours. If they are non-adjacent, there are μ paths through common neighbours. This gives us the following equation:

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

which can be rewritten as

$$A^2 - (\mu - \lambda)A - (n - \lambda)I = cJ.$$

Since the graph is regular, we know that A has an eigenvalue n with eigenvector consisting of all ones. Since the matrix is symmetric any other eigenvector must be orthogonal to this one, call this eigenvector z . Then $cJz = 0$, and any other eigenvalue θ must satisfy

$$\theta^2 - (\mu - \lambda)\theta - (n - \lambda) = 0$$

Inserting our parameters and solving the equation gives the other eigenvalues. \square

This gives us an alternative proof of the base cases in the induction for theorem Theorem 4.4. The determinant does not however, need to be a power of 2. Since all eigenvalues of $Y_{n,2}$ are different from zero for $n > 4$, we have the following.

Corollary 4.9. For $n > 4$, the matrix $Y_{n,2}$ is invertible.

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6 Notation

S	Any semigroup
S^1	A semigroup with an identity element adjoined
$\mathcal{E}(S)$	The set of idempotents in S
\mathbf{N}	The set $\{1, \dots, n\}$
$a\mathcal{L}b$	If $S^1a = S^1b$
$a\mathcal{R}b$	If $aS^1 = bS^1$
$a\mathcal{H}b$	If $a\mathcal{L}b$ and $a\mathcal{R}b$
$a(\mathcal{L} \circ \mathcal{R})b$	If there exists $z \in S$ such that $a\mathcal{L}z$ and $z\mathcal{R}b$
$a\mathcal{D}b$	If $a(\mathcal{L} \circ \mathcal{R})b$
$a\mathcal{J}b$	If $S^1aS^1 = S^1bS^1$
$\mathcal{K}(a)$	The equivalence class of a in $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$
$\text{im}(a)$	The image of the element $a \in \mathcal{T}_n$
π_a	The equivalence relation $x \pi_a y$ if $a(x) = a(y)$
ρ_a	The partition corresponding to π_a
$\text{rank}(a)$	The size of the image $ \text{im}(a) $
\mathcal{T}_n	The full transformation semigroup on $\{1, \dots, n\}$
$M_{n,k}$	The egg-box matrix for \mathcal{D}_k in \mathcal{T}_n
$M_{n,k}(\dots)$	The square submatrix of $M_{n,k}$ indexed by subsets and partitions

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