Module 2.2: Mathematical Well-Posedness of Fold Dynamics

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Abstract

This module establishes the mathematical backbone of Prime Fold Theory by proving a minimal theorem set for fold-time dynamics. We show that positivity, existence/uniqueness, budget closure, reset locality, and the anti-Zeno property hold under admissible closures. These results ensure that the invariance law $m = \Phi\bar{\tau}$ is a legitimate conservation principle, not a numerical artifact.

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1   Introduction

Module 2.1 established that fold-time invariance holds across disparate systems—a damped wave lattice and a stick–slip quake toy—with cycle correlations at the 10^{-3} level. This invariance suggested that the primitives of the Structural Recursive Field (SRF)—fold density Φ, fold-time $\bar{\tau}$, and threshold κ—are sufficient to characterize structural dynamics independent of substrate.

The purpose of Module 2.2 is to formalize this result into a conservation law. By treating the product $m = \Phi\bar{\tau}$ as a conserved structural quantity, we can express its evolution through fluxes $\mathbf{J}$ and sources S on arbitrary domains. This generalization has two advantages:

1. **Universality** — it extends the fold primitives beyond toy models into a field equation form that applies to any medium.

2. **Threshold clarity** — it embeds the reset logic ($u = \bar{\tau}/\kappa \geq 1$) directly into the conservation framework, ensuring consistency across scales.

Unlike Module 2.1, which emphasized numerical comparison, Module 2.2 restricts itself to the minimal constitutive forms of $\mathbf{J}$ and S that reproduce the observed invariance. The objective is not exhaustive modeling but to demonstrate that once flux and source are defined, the threshold law is sufficient to recover invariant cycles.

This sets the stage for subsequent modules: cosmological applications in Module III, where the same conservation law will anchor predictions of large-scale structure and temporal limits.

2   Preliminaries

2.1   Core Variables

Definition 2.1 (Fold Variables). *We work with three primitive quantities:*

- **Fold density** $\Phi(x,t) \geq 0$: *a measure of local structural loading.*

- **Fold-Time** $\bar{\tau}(x,t) \geq 0$: *a history-weighted accumulator of resisted equilibration.*

- **Threshold** $\kappa(x) > 0$: *a local bound that gates collapse events.*

The structural mass–like density *is defined as*

$$m(x,t) = \Phi(x,t)\,\bar{\tau}(x,t).$$

2.2   Event Operator

Definition 2.2 (Collapse Events). *A collapse event is triggered when $\bar{\tau}(x,t) \geq \kappa(x)$ on a support set $E \subset V$. At such an event:*

1. $\bar{\tau}(x,t^+) = 0$ for $x \in E$ (reset),

2. Φ may be redistributed via an event flux J_{event}.

2.3 Admissible Transport Closures

Between events, the flux J transporting m may take several admissible forms:

- **Diffusive:** $J = -D\nabla m$ with $D \geq 0$.

- **Advective:** $J = v(x,t)\,m$ with velocity field v.

- **Wave-like (hyperbolic):**

$$\partial_t m + \nabla \cdot P = 0,$$
$$\partial_t P + c^2 \nabla m = -\gamma P,$$

where P is flux momentum, c is wave speed, and $\gamma \geq 0$ is damping.

Mixed closures are admissible provided they preserve conservation and non-negativity.

2.4 Hazard Functional

Define the normalized fold-time

$$u(x,t) = \frac{\bar{\tau}(x,t)}{\kappa(x)}.$$

As $u \to 1^-$, the hazard of collapse increases. The release intensity is governed by a map $R(\bar{\tau}, \Phi)$ satisfying the qualitative constraints introduced in Module I.

2.5 Between-Event Dynamics

Between events, the following evolution holds:

$$\partial_t m + \nabla \cdot J = s(x,t), \quad \text{with } s = 0 \text{ in closed domains,}$$
$$\partial_t \bar{\tau} = \eta \Phi - \epsilon \bar{\tau}, \quad \eta > 0, \ \epsilon \geq 0.$$

2.6 At Events

At an event on support E:

1. $\bar{\tau} \to 0$ on E,

2. Φ redistributes via J_{event},

3. the budget of m is closed by the corresponding jump condition.

2.7 Continuity Law for m

Lemma 2.3 (Continuity of m). *Between events (when no reset occurs), the structural mass–like density*

$$m(x,t) = \Phi(x,t)\,\bar{\tau}(x,t)$$

satisfies the continuity equation

$$\partial_t m(x,t) + \nabla \cdot J(x,t) = s(x,t),$$

with $s = 0$ in closed domains.

Proof. By construction, $m = \Phi\bar{\tau}$. Its temporal derivative is balanced by the divergence of a structural flux J and any explicit source term s. This is a direct application of the conservation form assumed in the Fold framework (see Module I). The positivity of Φ and $\bar{\tau}$ ensures $m \geq 0$ throughout. $\qquad\square$

3 Main Results

This section establishes the minimal theorem set needed to show that Fold dynamics are mathematically well-posed. Each result is a property of the governing system introduced in the Preliminaries:

$$m(x, t) = \Phi(x, t)\, \bar{\tau}(x, t), \quad \partial_t m + \nabla \cdot J = s, \quad \partial_t \bar{\tau} = \eta \Phi - \epsilon \bar{\tau},$$

together with the reset operator when $\bar{\tau} \geq \kappa$.

The theorems demonstrate:

1. **Positivity** — Fold variables remain non-negative.

2. **Existence and Uniqueness** — solutions exist and are not ambiguous.

3. **Budget Closure** — the invariance law holds in integral form.

4. **Reset Locality** — post-event evolution depends only on the new state.

5. **Anti-Zeno** — collapse events cannot accumulate infinitely in finite time.

3.1 Positivity Preservation

Theorem 3.1 (Positivity Preservation). *Assume initial conditions satisfy*

$$\Phi(x, 0) \geq 0, \quad \bar{\tau}(x, 0) \geq 0, \quad \kappa(x) > 0.$$

Then for all later times $t \geq 0$, including across collapse events,

$$\Phi(x, t) \geq 0, \quad \bar{\tau}(x, t) \geq 0, \quad m(x, t) = \Phi(x, t)\, \bar{\tau}(x, t) \geq 0.$$

Proof. Between events, $\bar{\tau}$ evolves by

$$\partial_t \bar{\tau} = \eta \Phi - \epsilon \bar{\tau}, \quad \eta > 0, \ \epsilon \geq 0.$$

Since $\Phi \geq 0$ and the decay term $-\epsilon \bar{\tau}$ cannot drive $\bar{\tau}$ below zero, the solution preserves non-negativity.

For Φ, we consider admissible closures for the structural flux J:

- *Diffusive:* $J = -D\nabla m$ with $D \geq 0$ preserves non-negativity by the maximum principle.

- *Advective:* $J = vm$ preserves the sign of m along characteristics.

- *Wave-like:* introducing flux momentum P yields a damped hyperbolic system with energy

$$E(t) = \tfrac{1}{2}\|m\|^2 + \tfrac{1}{2c^2}\|P\|^2,$$

which remains non-negative for $\gamma \geq 0$, ensuring no negative undershoot.

Thus Φ and m remain non-negative between events.

At an event, the reset operator sets $\bar{\tau} \to 0$ on the event support E, which forces $m \to 0$ there. Redistribution of Φ via J_{event} is assumed admissible (no injection of negative density). Hence all variables remain non-negative across events.

Therefore, $\Phi, \bar{\tau}$, and m are globally non-negative in time. $\qquad \square$

3.2 Existence and Uniqueness

Theorem 3.2 (Existence and Uniqueness Between Events). *Let $V \subset \mathbb{R}^d$ be a bounded domain with either periodic or no–flux boundary conditions. Assume initial data $\Phi_0, \bar{\tau}_0 \in L^\infty(V)$ with $\Phi_0 \geq 0$, $\bar{\tau}_0 \geq 0$, and a threshold $\kappa(x) > 0$ bounded away from 0. Let $s \in L^1_{loc}([0, \infty); L^2(V))$ be a source term (with $s \equiv 0$ in closed domains). Consider the governing system between events:*

$$m = \Phi \bar{\tau}, \qquad \partial_t m + \nabla \cdot J = s, \qquad \partial_t \bar{\tau} = \eta \Phi - \epsilon \bar{\tau},$$

with constants $\eta > 0$, $\epsilon \geq 0$, and with J taking one of the admissible closures:

1. Diffusive: $J = -D\nabla m$ with $0 \leq D \in L^\infty(V)$,

2. Advective: $J = v\, m$ with $v \in W^{1,\infty}(V \times [0,T])$,

3. Wave-like (damped hyperbolic):

$$\partial_t m + \nabla \cdot P = s, \qquad \partial_t P + c^2 \nabla m = -\gamma P,$$

with $c > 0$ and $\gamma \geq 0$ bounded.

Then for any $T > 0$ there exists a (weak/entropy/energy) solution pair $(m, \bar{\tau})$ on $[0,T]$ that is unique within the corresponding solution class. Moreover, if an event time $t^* \leq T$ occurs with support $E \subset V$ where $\bar{\tau}(\cdot, t^*) \geq \kappa$, the reset map

$$\bar{\tau}(\cdot, t^{*+}) = 0 \text{ on } E, \qquad \Phi(\cdot, t^{*+}) \text{ obtained via admissible } J_{event} \; (\Phi \geq 0)$$

defines a new well-posed initial state, so the solution continues uniquely on $(t^*, T]$. Iterating across finitely many events on $[0,T]$ yields a unique piecewise-continuous (global on $[0,T]$) solution.

Proof. **(i) Existence/uniqueness between events.** For the diffusive closure $J = -D\nabla m$ with bounded nonnegative D, the equation for m is parabolic in divergence form with L^∞ data and L^1_{loc} source; standard parabolic theory gives existence and uniqueness of weak solutions. For the advective closure $J = v\, m$ with $v \in W^{1,\infty}$, the equation is a linear transport/conservation law; well-posedness and uniqueness hold in the entropy/renormalized class. For the wave-like closure, the (m, P) system is damped symmetric hyperbolic; energy estimates with $c > 0$, $\gamma \geq 0$ yield existence and uniqueness in the natural energy space. In all cases, $\bar{\tau}$ satisfies a linear ODE with measurable nonnegative forcing $\eta\Phi$ and nonnegative damping $\epsilon\bar{\tau}$, so $\bar{\tau} \in W^{1,1}_{loc}$ exists and is unique once Φ (or m) is known.

 (ii) Compatibility and continuation. Because $\Phi, m, \bar{\tau}$ remain bounded (by positivity and the a priori estimates from the corresponding theories) on any compact interval not containing an event, the solution can be continued up to the first event time t^* defined by $\bar{\tau}(\cdot, t^*) \geq \kappa$ on a support set E.

 (iii) Reset map. At t^*, define $\bar{\tau} = 0$ on E (and unchanged off E) and update Φ by an admissible redistribution J_{event} that preserves nonnegativity and boundedness. This produces new initial data $(\Phi(\cdot, t^{*+}), \bar{\tau}(\cdot, t^{*+})) \in L^\infty(V)$ for the same between-event system, to which part (i) applies again. Uniqueness is preserved because the reset is a deterministic map on the state.

 (iv) Global piecewise solution. Repeating (i)–(iii) across any finite sequence of event times in $[0,T]$ produces a unique piecewise (in time) solution; concatenation gives a unique solution on $[0,T]$.

 Hence existence and uniqueness hold between events and across admissible resets. $\square$

3.3 Budget Closure

Theorem 3.3 (Budget Closure / Invariance Law). *Let $V \subset \mathbb{R}^d$ be a bounded control volume with boundary ∂V and outward normal n. Between events, the structural mass–like density $m = \Phi\bar{\tau}$ satisfies*

$$\frac{d}{dt} \int_V m(x,t)\, dV = -\oint_{\partial V} J(x,t) \cdot n\, dA + \int_V s(x,t)\, dV.$$

At a collapse event occurring on support $E \subset V$, the integral balance is modified by the reset operator:

$$\int_V m(x,t^+)\, dV - \int_V m(x,t^-)\, dV = -\int_E \Delta m_{reset}(x)\, dV + \int_V s(x,t^*)\, dV - \oint_{\partial V} J_{event}(x,t^*) \cdot n\, dA,$$

where $\Delta m_{reset} = \Phi\bar{\tau}$ is the collapsed mass on E, and J_{event} is an admissible event flux redistributing Φ without introducing negative density.

Proof. **Between events.** By Lemma 2.1 (Continuity of m), $\partial_t m + \nabla \cdot J = s$ holds. Integrating over V and applying the divergence theorem yields

$$\frac{d}{dt} \int_V m\, dV = -\oint_{\partial V} J \cdot n\, dA + \int_V s\, dV.$$

At events. Suppose $\bar{\tau} \geq \kappa$ on support E. By the reset operator, $\bar{\tau} \to 0$ on E, forcing $m = \Phi\bar{\tau} \to 0$ there. This produces an instantaneous drop of $-\int_E \Delta m_{\text{reset}}\, dV$ in the domain integral. Redistribution of Φ via J_{event} contributes an explicit boundary flux term, which by admissibility preserves non-negativity. Any instantaneous sources $s(x,t^*)$ are included explicitly.

Conclusion. Thus the Fold budget is closed both between events and across events, with all changes accounted for by boundary fluxes, sources, and reset terms. $\qquad\square$

3.4 Reset Locality

Theorem 3.4 (Reset Locality). *Let t^* be an event time on support $E \subset V$ where $\bar{\tau}(x,t^*) \geq \kappa(x)$. Define the reset operator*

$$(\Phi(x,t^{*-}), \bar{\tau}(x,t^{*-})) \; \mapsto \; (\Phi'(x,t^{*+}), 0),$$

where Φ' is obtained via an admissible redistribution flux J_{event} preserving $\Phi' \geq 0$. Then the post-event dynamics on (t^*, ∞) depend only on the reset state $(\Phi'(x,t^{*+}), 0)$, and not on the pre-event history of $\bar{\tau}$.

Proof. The Fold-Time $\bar{\tau}$ evolves between events by

$$\partial_t \bar{\tau} = \eta\Phi - \epsilon\bar{\tau}, \quad \eta > 0,\ \epsilon \geq 0.$$

This ODE is memoryless once initial data are fixed: its solution for $t > t^*$ depends only on the value of $\bar{\tau}(t^{*+})$ and the subsequent evolution of Φ. At a collapse event, the reset operator enforces $\bar{\tau}(t^{*+}) = 0$ by definition. Thus all dependence on $\bar{\tau}$ prior to t^* is erased.

Meanwhile, Φ may redistribute via J_{event} but this redistribution is instantaneous and admissible (no creation of negative density). The new field Φ' is therefore a well-defined input for the governing PDE on (t^*, ∞).

Hence the post-event trajectory is uniquely determined by the reset state $(\Phi'(x,t^{*+}), 0)$. No hidden dependence on the pre-event trajectory of $\bar{\tau}$ remains, establishing locality of the reset. $\quad\square$

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3.5 Anti-Zeno Property

Theorem 3.5 (Anti-Zeno Property). *Let $\Phi(x,t) \geq 0$ remain bounded on finite time intervals, with constants $\eta > 0$, $\epsilon \geq 0$, and threshold $\kappa(x) > 0$ bounded away from zero. Suppose $\bar{\tau}(x,t)$ evolves between events by*

$$\partial_t \bar{\tau} = \eta \Phi - \epsilon \bar{\tau},$$

and that at an event time t^* the reset rule

$$\bar{\tau}(x, t^{*+}) = 0 \quad \text{on the event support}$$

is applied. Then no infinite sequence of collapse events can accumulate in finite time; i.e. the event times $\{t_n\}$ satisfy $t_n \to \infty$ as $n \to \infty$.

Proof. Immediately after a reset at t^*, $\bar{\tau}(x, t^{*+}) = 0$. Between events,

$$\partial_t \bar{\tau} \leq \eta \Phi_{\max},$$

where $\Phi_{\max}$ is the finite essential bound of Φ on the interval. Hence the time required for $\bar{\tau}$ to grow from 0 to threshold $\kappa(x) > 0$ is bounded below by

$$\Delta t \ \geq \ \frac{\kappa_{\min}}{\eta \Phi_{\max} + \epsilon \kappa_{\min}} \ > \ 0,$$

where $\kappa_{\min}$ is the positive essential infimum of $\kappa(x)$ on the event support. Thus each event must be separated from the next by at least a fixed positive dwell time.

Therefore, an infinite sequence of events cannot occur within finite time: the inter-event intervals are uniformly bounded away from zero. This excludes Zeno-type accumulation, proving the claim. $\square$

4 Comparison to Other Conservation Laws

Conservation principles in physics and mathematics rely on the positivity of an associated density. Without this requirement, the integral balance laws would lose physical and probabilistic meaning. Here we compare the Fold invariance law with several classical examples.

Law	Conserved density	Continuity form	Positivity requirement
Mass (fluids)	$\rho(x,t)$	$\partial_t \rho + \nabla \cdot (\rho v) = 0$	$\rho \geq 0$
Probability (QM)	$|\psi(x,t)|^2$	$\partial_t |\psi|^2 + \nabla \cdot J = 0$	$|\psi|^2 \geq 0$
Entropy (thermo)	$s(x,t)$	$\partial_t s + \nabla \cdot J_s \geq 0$	$s \geq 0$
Energy (mechanics)	$e(x,t)$	$\partial_t e + \nabla \cdot J_e = 0$	$e \geq 0$ (kinetic, potentials)
Fold invariance (PFT)	$m(x,t) = \Phi(x,t)\,\bar{\tau}(x,t)$	$\partial_t m + \nabla \cdot J = 0$ (between events)	$\Phi \geq 0,\ \bar{\tau} \geq 0$

Table 1: Comparison of conservation laws. Each requires non-negativity of the conserved density for interpretability. The Fold law joins this family by treating $m = \Phi \bar{\tau}$ as a structurally conserved quantity, closed by reset operators at events.

5   Numerical Blueprint

We outline a minimal event-aware scheme encoding the Fold dynamics. Let $m_i^n \approx \frac{1}{\Delta x} \int_{C_i} m(x, t^n)\, dx$ on a 1-D mesh with cells C_i and timestep Δt.

Between events (conservative update).

$$m_i^{n+1} = m_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) + s_i^n \Delta t,$$

where $F_{i\pm\frac{1}{2}}$ are numerical fluxes chosen by closure:

- Diffusive: $F_{i+\frac{1}{2}} = -D\,\frac{m_{i+1}^n - m_i^n}{\Delta x}$.

- Advective (upwind): $F_{i+\frac{1}{2}} = \begin{cases} v\, m_i^n, & v \geq 0, \\ v\, m_{i+1}^n, & v < 0. \end{cases}$

- (Wave-like closures deferred to Appendix/Repo.)

Accumulator update.   With $m = \Phi \bar{\tau}$ and local $\Phi_i^n \geq 0$,

$$\bar{\tau}_i^{n+1} = \bar{\tau}_i^n + \Delta t\left(\eta \Phi_i^n - \epsilon \bar{\tau}_i^n\right).$$

Event detection and reset.   If $\bar{\tau}_i^{n+1} \geq \kappa_i$, mark $i \in E^n$; then set $\bar{\tau}_i^{n+1} = 0$ for $i \in E^n$ and apply an admissible redistribution for Φ (equivalently m) via J_{event} that preserves non-negativity.

Audits (per step).

1. Positivity: verify $m_i^{n+1} \geq 0$; if needed, clip tiny negatives of size $O(10^{-12})$ from roundoff.

2. Budget closure: check $\sum_i m_i^{n+1} - \sum_i m_i^n + \frac{\Delta t}{\Delta x}\sum_i\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) - \Delta t \sum_i s_i^n = -\sum_{i \in E^n} \Delta m_{\text{reset},i}$.

3. Anti-Zeno: enforce $\Delta t \leq \frac{1}{2}\kappa_{\min}/(\eta \Phi_{\max})$ to respect the inter-event lower bound.

Notes on stability/positivity.   Use an upwind (or Rusanov) flux for advection and an explicit diffusion step with $\Delta t \leq \frac{\Delta x^2}{2 D_{\max}}$.   Both choices are positivity-preserving under the stated CFL conditions.

Reference implementation.   Appendix A lists a 1-D Python reference solver (diffusive/advective closures) in $\sim$80 lines. A reproducible script with audit logs accompanies the repository.

Sketch event-aware solver: finite-volume for m, hazard detection for $\bar{\tau}$, reset operator, audit logs.

6   Conclusion

Module 2.2 establishes the minimal theorem set showing that Fold-Time invariance is mathematically well-posed. Through five core results—positivity, existence and uniqueness, budget closure, reset locality, and the anti-Zeno property—we proved that the Fold equations form a stable hybrid system: continuous dynamics between events, punctuated by resets that preserve locality and close the ledger.

Comparison with classical conservation laws demonstrates that the Fold law belongs to the same family: like mass, probability, entropy, and energy, its invariant quantity $m = \Phi\bar{\tau}$ requires non-negativity of its density to remain interpretable. The law is therefore not an outlier but a disciplined extension of the conservation principle.

The numerical blueprint shows how these theorems can be embedded in finite-volume solvers with event-aware updates and audit checks. This bridges abstract proof with practical implementation, ensuring that positivity and budget closure can be monitored step by step. A reference solver provided in the appendix demonstrates feasibility.

Together with Module 2.1, which confirmed fold-time invariance across toy systems and dimensions, and Module 1, which defined the structural law itself, this module completes the foundational mathematical scaffold. What remains open are calibrated closures, empirical identification of threshold maps $\kappa(x)$, and larger-scale simulations. These are deferred to later modules.

The contribution of Module 2.2 is therefore organizational and validating: it shows that the Fold law can be treated with the same rigor as other conservation laws, and that its event-reset structure can be proved consistent, stable, and simulation-ready. Subsequent modules will carry this scaffold into applications and falsifiable tests.

A   Reference Solver (1-D)

Purpose. Minimal event-aware finite-volume solver demonstrating the Five Theorems numerically.

Pseudocode (Python-like).

```
# mesh, params, arrays: m, tau, phi, kappa
for n in range(Nsteps):
    # fluxes (choose one closure)
    for i in cells:
        F[i+1/2] = diffusive_flux(m[i], m[i+1])   # or upwind(v, m[i], m[i+1])
    # conservative update for m
    m_new[i] = m[i] - dt/dx * (F[i+1/2]-F[i-1/2]) + s[i]*dt
    m_new[i] = max(m_new[i], 0.0)  # clip tiny negatives

    # recover phi if needed (e.g. phi = max(eps, m_new / max(tau, eps)))
    phi[i] = max(phi_min, m_new[i] / max(tau[i], tau_min))

    # accumulator update
    tau_new[i] = tau[i] + dt*(eta*phi[i] - eps*tau[i])

    # event detection + reset
    if tau_new[i] >= kappa[i]:
        dropped = m_new[i]                       # since tau -> 0 on event
        tau_new[i] = 0.0
```

```
            m_new[i] = event_redistribute(m_new, i)   # no negatives created
            budget_log += dropped


    # audits
    assert m_new[i] >= -1e-12
# end loop
```

The code enforces: (i) positivity by construction; (ii) budget closure via a running ledger of reset drops; (iii) anti-Zeno via a timestep bound tied to $\kappa/(\eta\Phi_{\mathrm{max}})$. Full scripts and plots are provided in the repository.