The effect of different pressure distributions on the deformations of an elastic-walled tube

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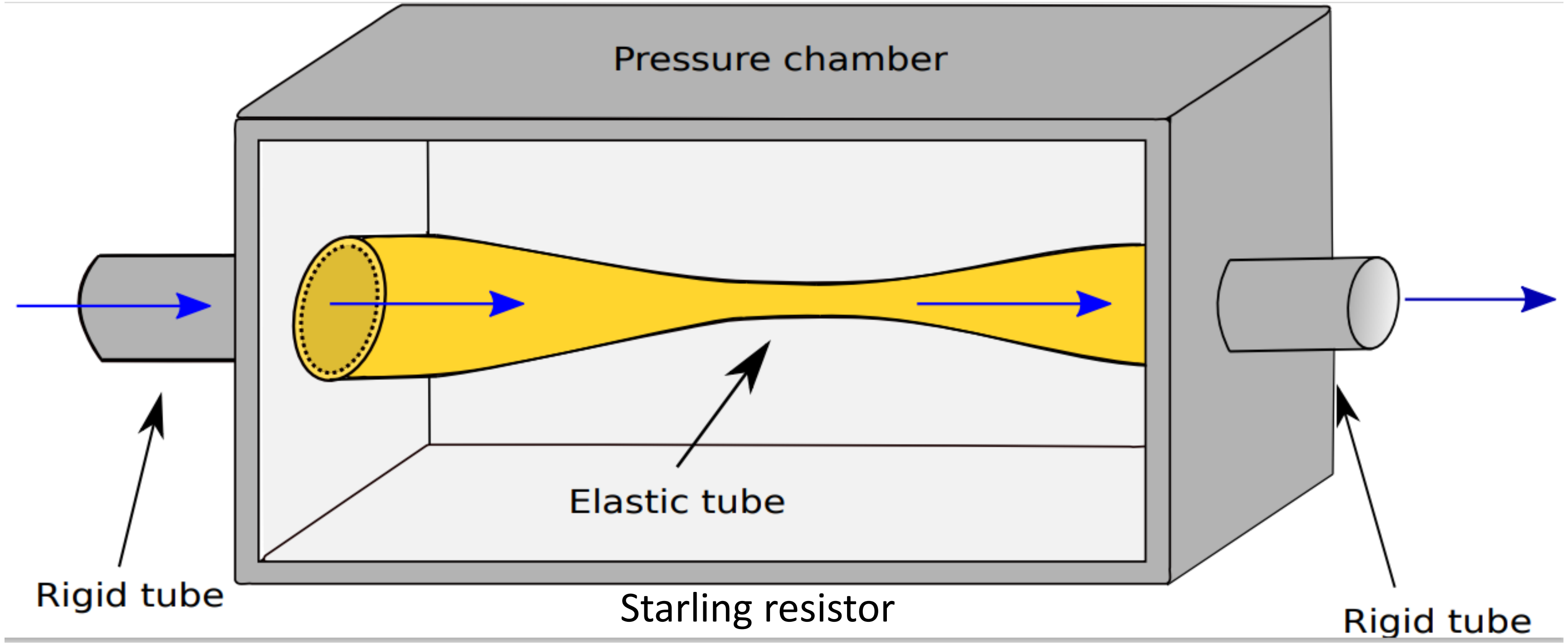
Model

Model used is based on a starling resistor set-up



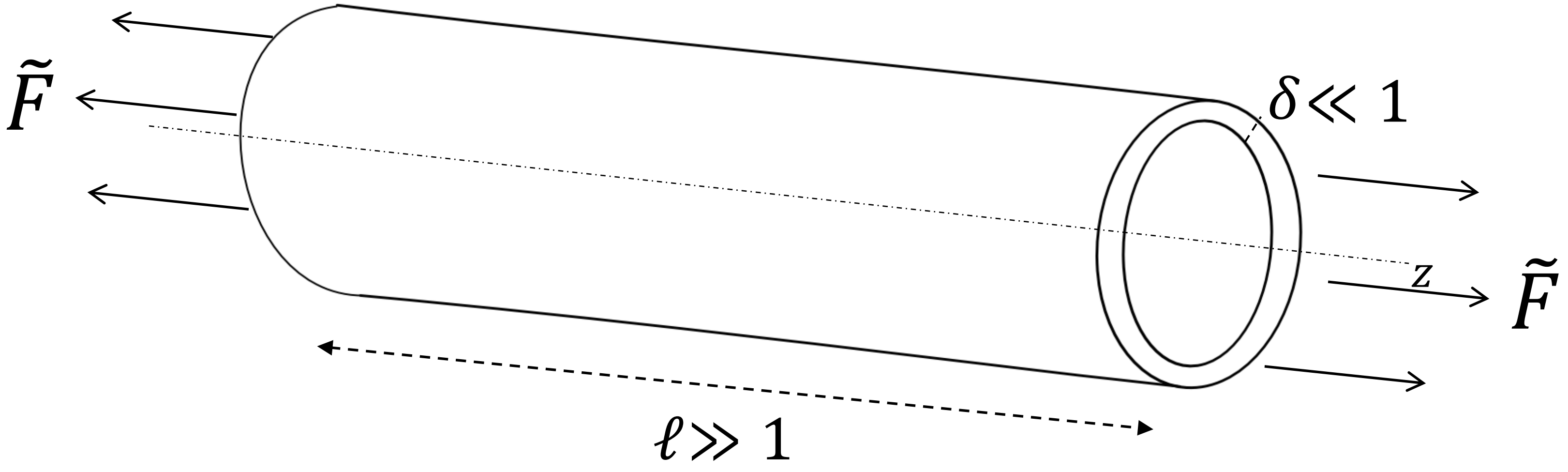
Starling resistor

<u>Applications of this model</u>

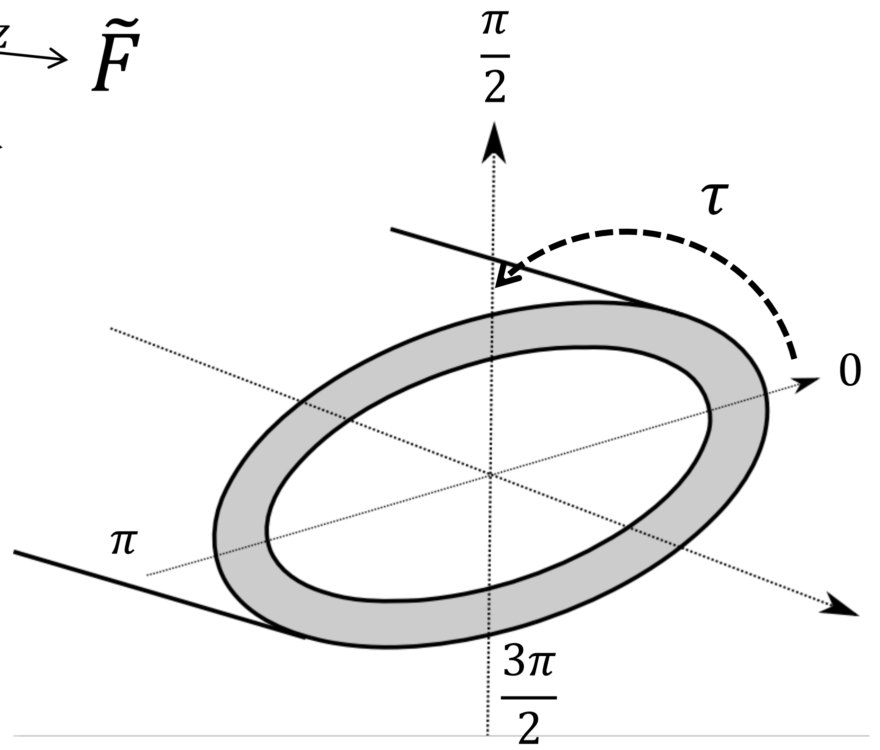
- Modelling blood flow through veins/arteries
- Modelling airflow in the lungs

Physical set-up and asymptotic regime

$\tilde{P}(\tau, z)$ (Pressure difference between inside and outside of tube)



Adopt set-up of Netherwood and Whittaker (2023)



τ is measured around the midplane of the tube

Expression for the change in cross-sectional area

Cross-sectional area change:

$$A - \bar{A} = \sum_{n=1}^{\infty} A_n$$

$$\tilde{F}\frac{d^2 A_n}{dz^2} - \lambda_n A_n = -Q_n \bar{A} t_n$$

Subject to $A_n = 0$ on $z = 0,1$

ODE for A_n

$$Q_n = -\tanh^2 2\sigma_0 \int_0^{\frac{\pi}{2}} \frac{1}{h}\frac{\partial}{\partial \tau}\left(\frac{\tilde{P}(\tau, z)}{\bar{B}(\tau)}\right) Y_n(\tau)\, d\tau$$

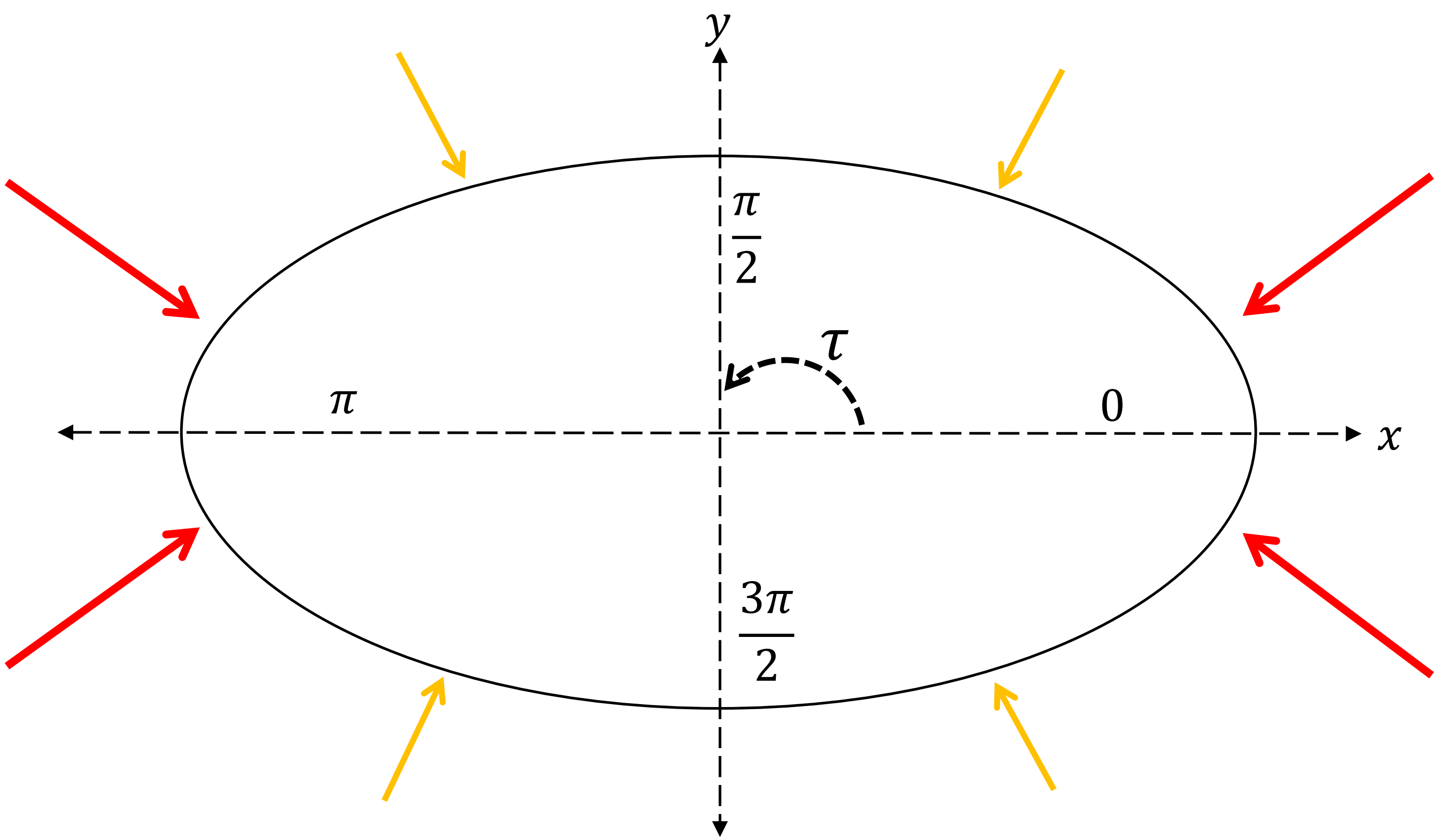
σ_0: Ellipticity parameter

σ_0 small : very elliptical cross-section

σ_0 large : very circular cross-section

Assumed symmetry of the transmural pressure $\widetilde{P}$

Pressure must be even and π-periodic

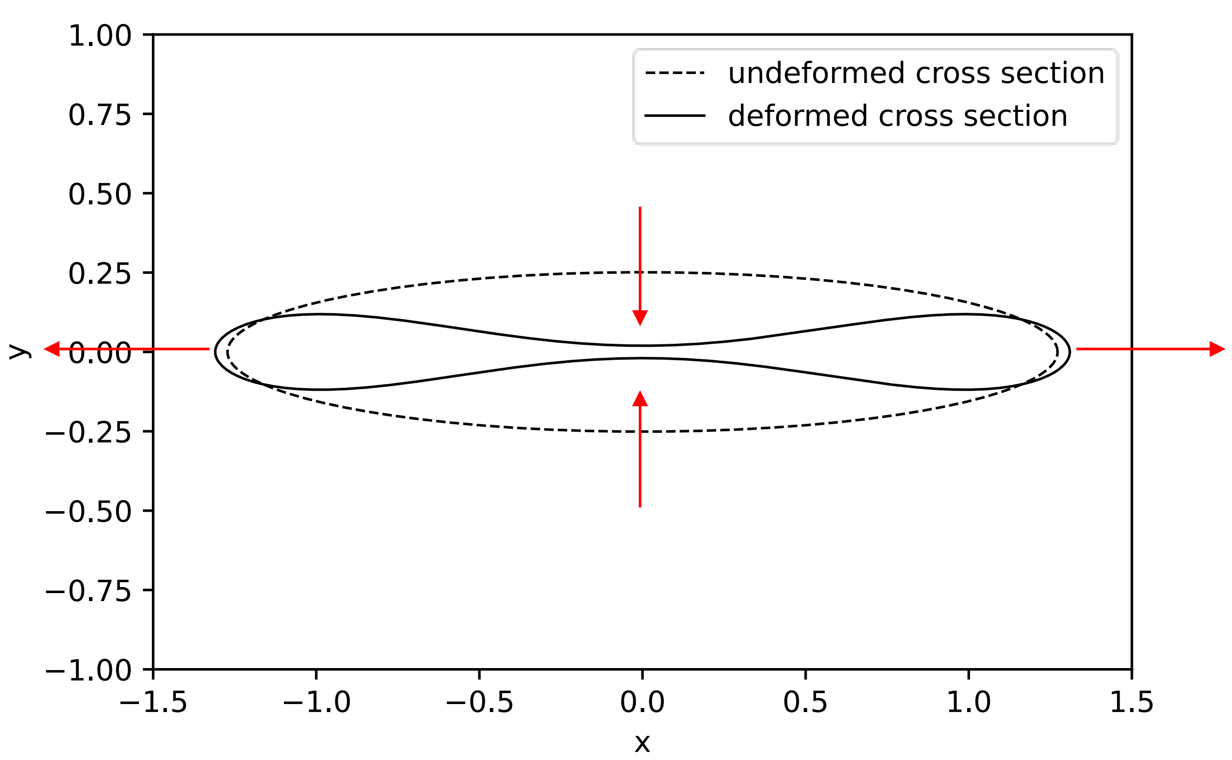


Symmetry $\Longrightarrow$ Restrict azimuthal domain to $\tau \in \left(0, \frac{\pi}{2}\right)$

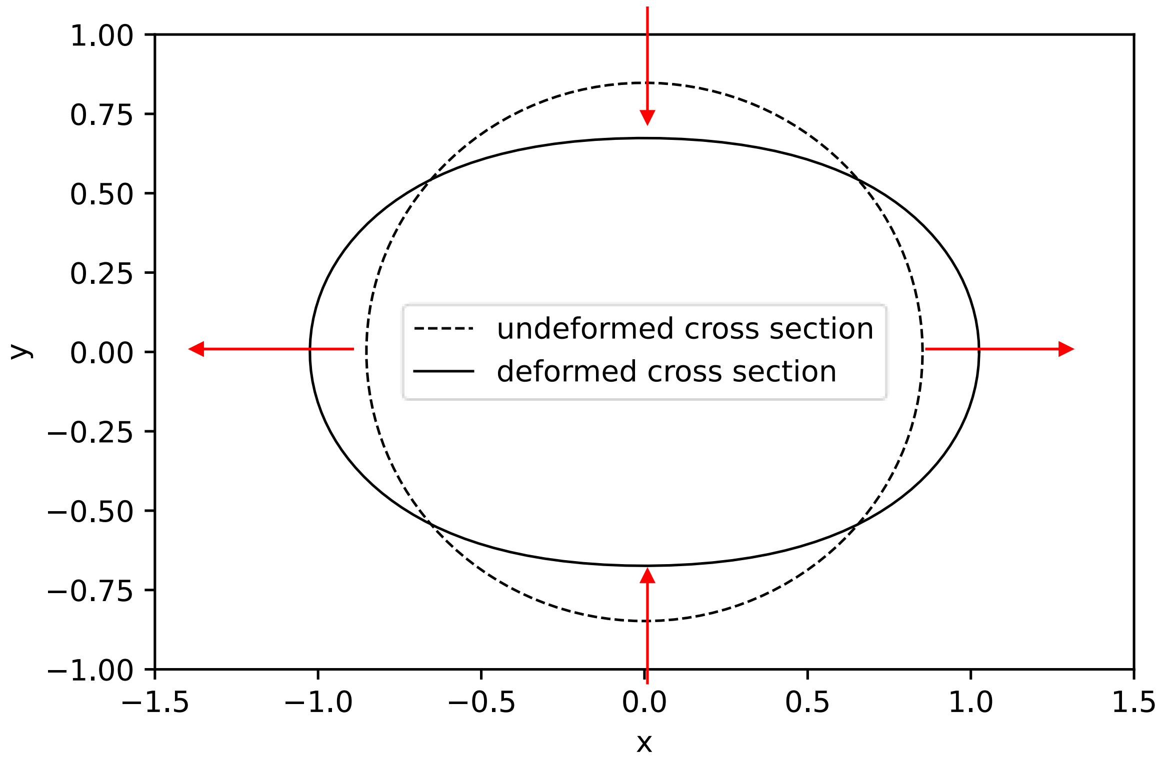
Solution for azimuthally varying transmural pressure

Example: $\tilde{P} = 6\cos 2\tau$

$\sigma_0 = 0.2$



$\sigma_0 \to \infty$ (Initially circular cross-section)



A unique family of transmural pressures

Can we impose a special pressure that excites only a single azimuthal eigenmode?

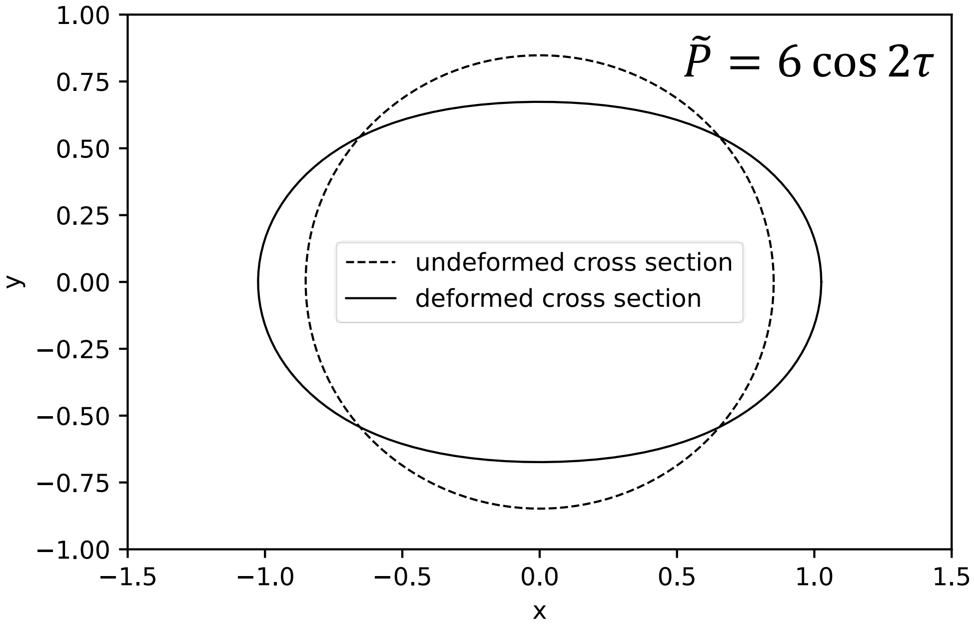
Find a pressure which gives a non-zero contribution from the mode n with $Q_m = 0$ for all $m \neq n$

Solve:

$$\frac{\partial}{\partial \tau}(\tilde{p}h^3) = \gamma \mathscr{F}(Y_m)$$

For a circular cross-section:

$$\tilde{p} = \gamma \cos 2n\tau + C_I$$



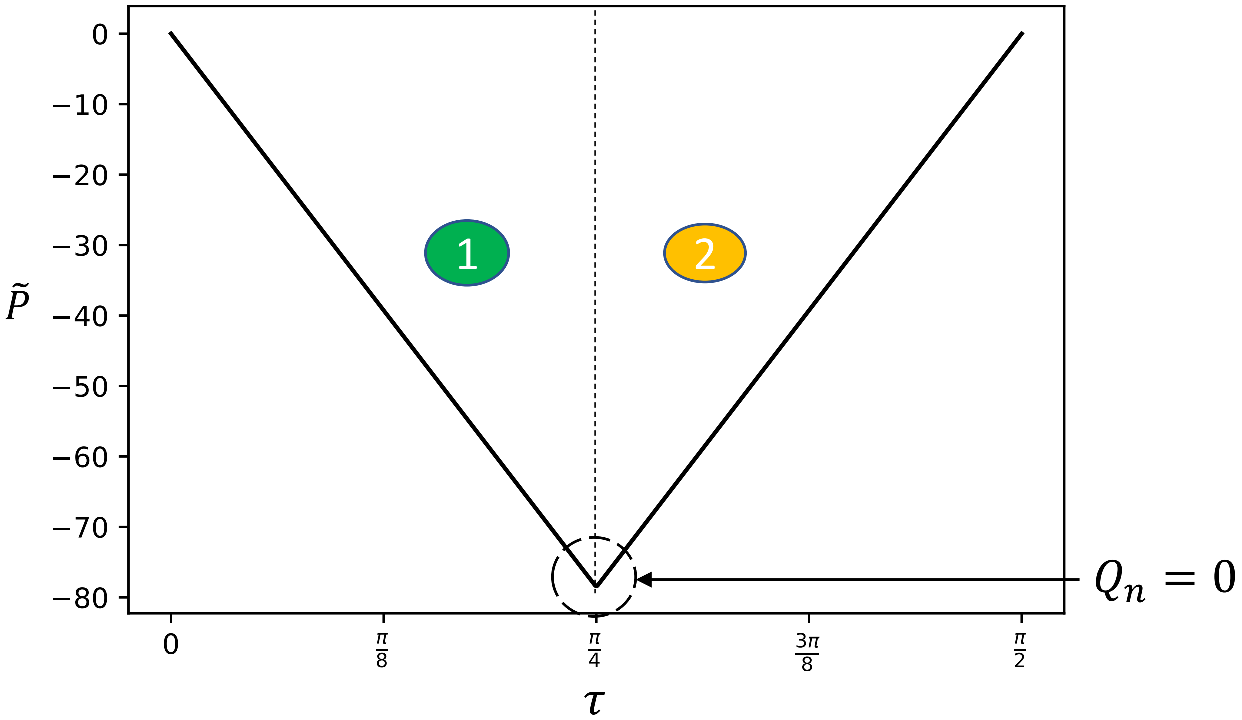
Analytic solution of the deformation of an initially cylindrical tube:

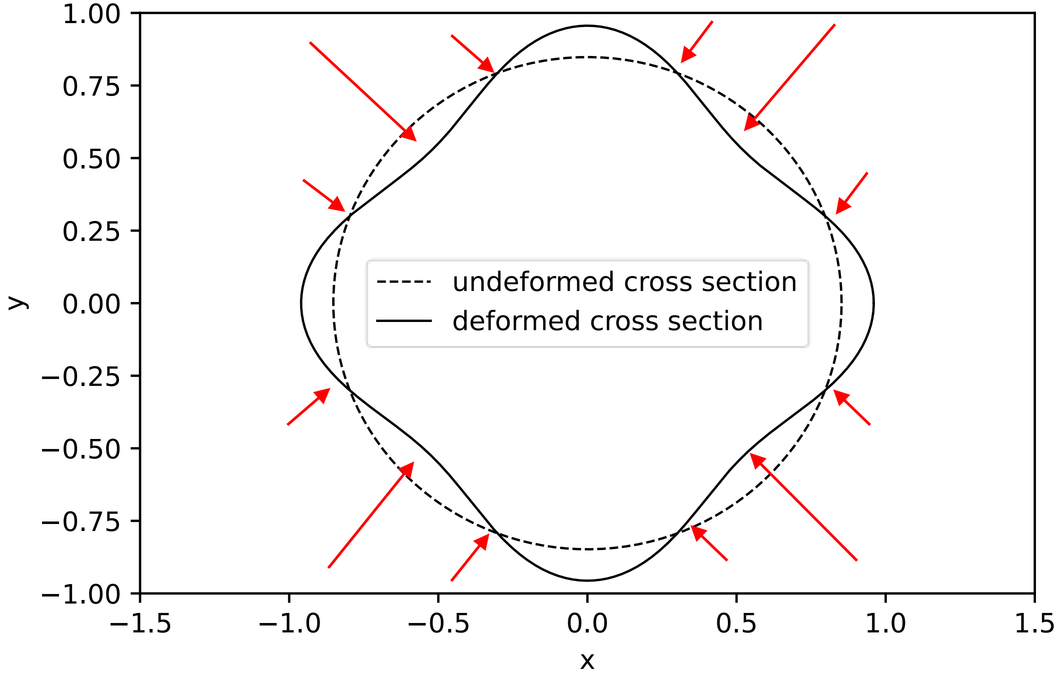
$$\underline{\boldsymbol{r}} = a\begin{pmatrix}\cos\tau\\ \sin\tau\\ \ell z\end{pmatrix} + \frac{2n\epsilon a\gamma}{\lambda_n(1+4n^2)}\left(2n\cos(2n\tau)\begin{pmatrix}\cos\tau\\ \sin\tau\\ 0\end{pmatrix} - \sin(2n\tau)\begin{pmatrix}-\sin\tau\\ \cos\tau\\ 0\end{pmatrix}\right)\left(1 - \frac{\cosh\left(\sqrt{\frac{\lambda_n}{\tilde{F}}}\left(z-\frac{1}{2}\right)\right)}{\cosh\left(\sqrt{\frac{\lambda_n}{4\tilde{F}}}\right)}\right)$$

Form of the pressure was also found for any elliptical cross-section, however this needed to be determined numerically.

Sharp changes in transmural pressure

We consider a transmural pressure in which the derivative has a discontinuity at $\tau = \frac{\pi}{4}$





$$\frac{Q_n}{\tanh 2\sigma_0} = \int_0^{\frac{\pi}{4}-\epsilon} \left(\tilde{p}' - \frac{1}{\cosh 2\sigma_0} \left[\tilde{p}' \cos 2\tau - 3\tilde{p} \sin 2\tau \right] \right) Y_n d\tau \quad 1$$

$$+ \int_{\frac{\pi}{4}+\epsilon}^{\frac{\pi}{2}-\epsilon} \left(\tilde{p}' - \frac{1}{\cosh 2\sigma_0} \left[\tilde{p}' \cos 2\tau - 3\tilde{p} \sin 2\tau \right] \right) Y_n d\tau \quad 2$$

Conclusion

Summary

- Solutions to azimuthally varying pressures for different elliptical cross-sections
- Found a family of pressures which gives solutions on just one azimuthal mode
- Found the analytic solution to the deformation of an initially cylindrical tube under a specific pressure
- Looked at a form of pressure with a discontinuity in the derivative

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References

Netherwood and Whittaker (2023) – A New Solution for the Deformations of an Initially Elliptical Elastic-walled Tube. Q. J. Mech. Appl. 10.1093

R. J. Whittaker, M. Heil, O. E. Jensen and S. L. Waters, A rational derivation of a tube law from shell theory, Q. J. Mech. Appl. Math. 63 (2010) 465–496.