The Effect of Different Pressure Distributions on the Deformations of an Elastic-Walled Tube

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Steady Dimensionless Set-up

We investigate the effect of different pressure distributions on the small-amplitude deformations of a long thin-walled elastic tube. We adopt the physical set up of Whittaker *et al* (1) involving an initially axially uniform elliptical tube undergoing an axial pre-stress $\tilde{F}$. The tube is deformed by a transmural pressure $\tilde{P}$ (the pressure difference between the inside and outside of the tube). Whittaker *et al* (1) showed that the dominant mechanism that balance the pressure are those of azimuthal bending and axial tension.

We follow Netherwood & Whittaker (2) to include azimuthal variation in the pressure, which is assumed to be even and π-periodic. This corresponds to mirror symmetry in the axis of the cross-section, so we restrict the azimuthal domain to $\tau \in \left(0, \frac{\pi}{2}\right)$. Applications of this model include predicting how veins, arteries and the airways collapse under pressure.

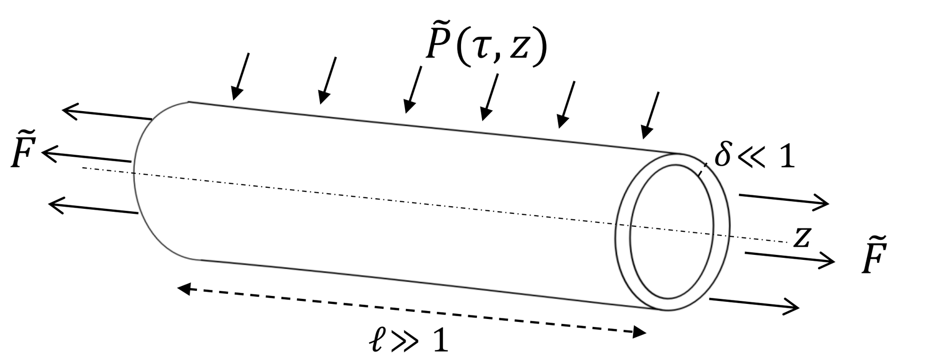

Fig 1: The initial elliptical configuration of a long thin-walled elastic tube under prestress. The tube will undergo deformations about this base-state.

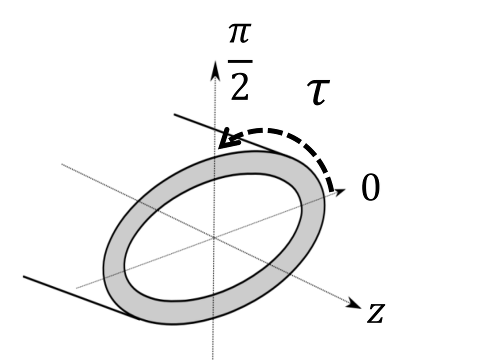

Fig 2: The (τ, z) co-ordinate system. Here τ is measured around the midplane of the tube, and z is aligned with the tubes central axis.

Solution for Elliptical Tube

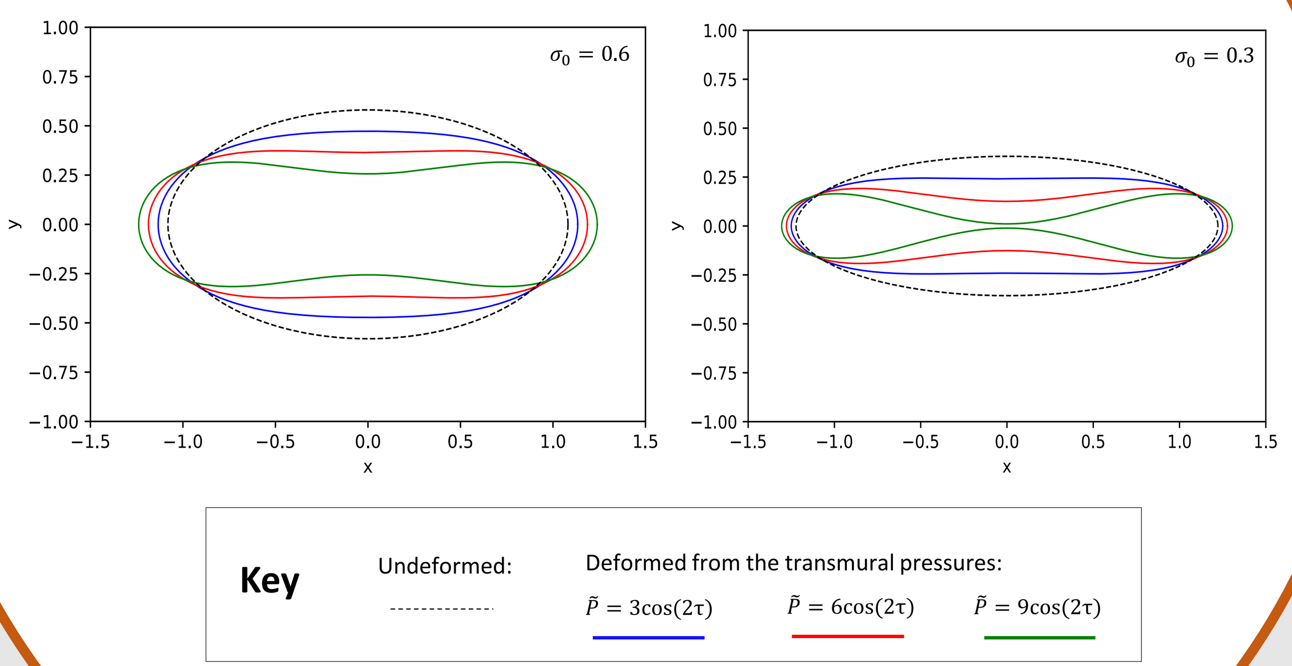

Fig 3: Cross-sectional deformations induced by azimuthally non-uniform transmural pressure with $\sigma_0 = 0.3, 0.6$ and $\tilde{F} = 1$.

Governing Equations for Area Change

Netherwood and Whittaker (2) showed that the change in cross-sectional area of the tube is given by

$$A - \bar{A} = \sum_{n=1}^{\infty} A_n \; ,$$

with each A_n satisfying:

$$\tilde{F}\frac{d^2 A_n}{dz^2} - \lambda_n A_n = \frac{-Q_n t_n}{\bar{A}} \quad \text{with} \quad A_n = 0 \;\text{ on }\; z = 0,1$$

and

$$Q_n = -\tanh^2 2\sigma_0 \int_0^{\frac{\pi}{2}} \frac{1}{h}\frac{\partial}{\partial \tau}\left(\frac{\tilde{P}(\tau, z)}{\bar{B}(\tau)}\right) Y_n(\tau)\, d\tau \; .$$

Here σ_0 is a measure of the ellipticity of the tube and the azimuthal functions $Y_n(\tau)$ and constants t_n are known and were obtained numerically by Netherwood and Whittaker (2) for each n. Once the A_n's have been determined, the deformations in the tube can be obtained by following the methodology of Netherwood and Whittaker (2).

The problem has been formulated in terms of A_n because for a fluid-conveying tube (e.g., blood vessels and the airways), changes in cross-sectional area displace fluid particles, resulting in fluid-structure interaction.

Analytic Solution for a Cylindrical Tube

For an initially cylindrical tube we showed that imposing a transmural pressure $\tilde{P} = \gamma\cos(2n\tau)$ results in $Q_m = 0$ for all $m \neq n$. This meant we could determine the deformation analytically.

$$\underline{r} = a\begin{pmatrix}\cos\tau\\ \sin\tau\\ \ell z\end{pmatrix} + \frac{2n\epsilon a\gamma}{\lambda_n(1+4n^2)}\left(2n\cos(2n\tau)\begin{pmatrix}\cos\tau\\ \sin\tau\\ 0\end{pmatrix} - \sin(2n\tau)\begin{pmatrix}-\sin\tau\\ \cos\tau\\ 0\end{pmatrix}\right)\left(1 - \frac{\cosh\left(\sqrt{\frac{\lambda_n}{\tilde{F}}}\left(z - \frac{1}{2}\right)\right)}{\cosh\left(\sqrt{\frac{\lambda_n}{4\tilde{F}}}\right)}\right)$$

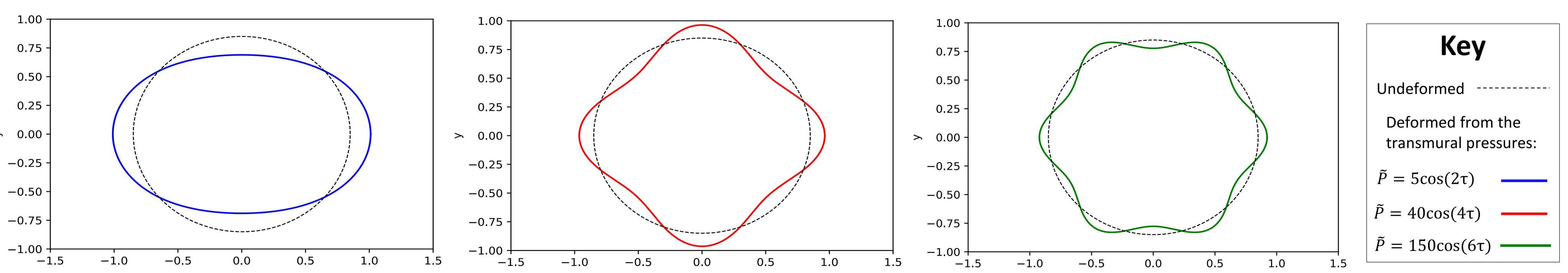

Fig 4: Cross-sectional deformations of an initially cylindrical tube using $\tilde{P} = \gamma\cos(2n\tau)$, with γ = 5, 40 and 150 respectively and $\tilde{F} = 1$. Values were chosen to exaggerate the deformation for better visualisation.

We also found forms of transmural pressure resulting in $Q_m = 0$ for all $m \neq n$ for an initially elliptical tube, however this needed to be determined numerically, and so were the resulting deformations.

Sharp Change in Transmural Pressure

We argued that Q_n is well behaved for pressure distributions with a discontinuous first derivative (i.e. for sharp changes in pressure). This was done by splitting up Q_n and summing over the domain.

As an example, a transmural pressure like in figure 5 squeezes the tube with maximum inward pressure at $\tau = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ creating a pinching like scenario.

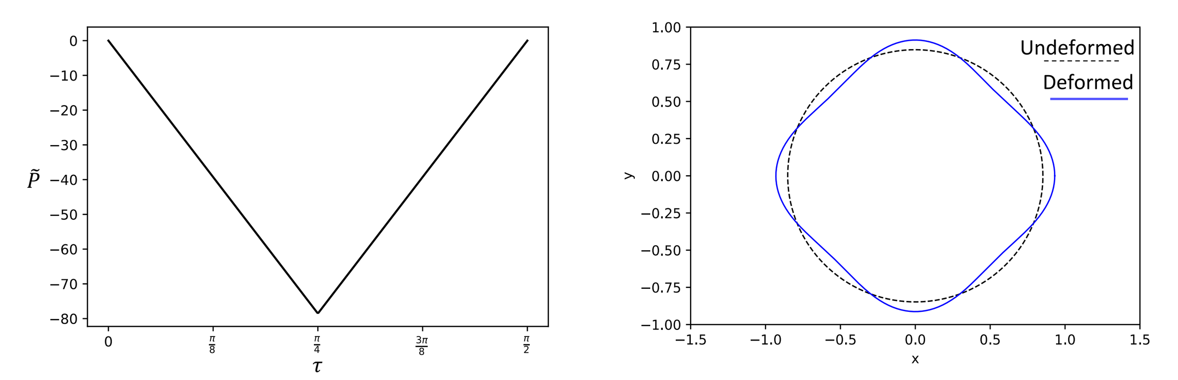

Fig 5: A pressure distribution containing a sharp change at $\tau = \frac{\pi}{4}$ (left) and the associated cross-sectional deformation (right).

References

(1) R. J. Whittaker, M. Heil, O. E. Jensen and S. L. Waters, A rational derivation of a tube law from shell theory, Q. J. Mech. Appl. Math. 63 (2010) 465–496.

(2) Netherwood and Whittaker (2023) – A New Solution for the Deformations of an Initially Elliptical Elastic-walled Tube. Q. J. Mech. Appl. 10.1093

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