Task: Fractions as Decimals

Setup: Long division easily gives you

$$801 \div 5 = 160 \ \text{Remainder} 1$$

Question: What if we continue past the units and break up according to tenths too?

Answer: 160.2

How about $801 \div 25$?

Answer: keep track of hundredths, 32.04

Challenge: If we're working with integers (whole numbers) will this always work out nice and neatly? Meaning, will this always stop exactly after a while? If so, why? If not, give a counter-example.

Answer: no, $1 \div 3 = 0.33333\ldots$ for instance.

Observe: This is not so dissimilar to $1 \div 5 = 0.20000\ldots$. In one case, 3's repeat, in the other 0's repeat.

Challenge: Is there a division that doesn't just have the same number go on forever? If so, give a counter-example. If not, why?

Answer: Encourage students to try some examples, using long division. Yes, $1 \div 7 = 0.14285714285714\ldots$ or $3 \div 11 = 0.2727272727\ldots$ for instance.

Observe: These instances do seem to repeat BLOCKS though:

$$1 \div 7 = 0.14285714285714\ldots \text{ or } 3 \div 11 = 0.2727272727\ldots \text{ for instance.}$$

Challenge: Will there always be a block that repeats? If so, why? If not, give a counter-example.

Hint: Consider what would happen if you keep applying long division ...

Answer: we would eventually have to have the same remainder that we've had at some earlier stage of the long division, and after that we will be forced to repeat. This must happen because there are only finitely many possible remainders. For $n \div d$ there are at most $d-1$ possibly different remainders, so the repeating section can be at most $d-1$ long, but we will always get a repeating section.

Challenge: So one integer divided by another always gives rise to a repeating section. What about the reverse? Given a repeating decimal expansion, does it necessarily come from a fraction?

Hint:

1. Consider $0.5252525\ldots$ and notice,

$$100 \times 0.52525252\ldots = 52.5252525\ldots$$

2. But,

$$52.52525\ldots - 0.52525\ldots = 25$$

3. So, that's

$$(100-1)x0.5252525\ldots = 25$$

4. So,

$$0.52525\ldots = \frac{25}{99}$$

And there was nothing special about $0.52525\ldots$ (try make similar construction with a number that repeats three numbers).

Post Hint Challenge: How could you use special expansions such as those in the hint that start with 0 and then immediately start repeating to say something about a more general expansion with a repeating piece after some point?

Answer: any random decimal expansion that, after some point, simply repeats a section again and again forever, could be thought of as a fraction plus one of these repeating chunks, and thus is a fraction plus a fraction. For example:

$$\begin{aligned} 23.8245121212\ldots &= 23.7033 + 0.121212\ldots \\ &= \frac{237033}{10000} + \frac{12}{99} \end{aligned}$$

Final Observation: So fractions have decimal expansions that repeat forever after some point and vice versa.

What about other decimal expansions? For instance:

$$0.10100100010000100001\ldots \text{ or } 0.12345678910111213\ldots?$$

Or any other random collection of digits that never repeats? $0.328975437\ldots$ or $3.14159\ldots$.

These are **IRRATIONAL NUMBERS**, and there are a lot of them! Way more than there are fractions, in fact. Hopefully this now seems reasonable, since not many random collections of digits repeat over and over forever after all!