

《现代磁性材料原理与应用》课程考查论文

**A Brief Investigation of the Principles of  
Ferromagnetism**

**铁磁性原理初探**

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# **A Brief Investigation of the Principles of Ferromagnetism**

**Abstract:** In part I of the paper, the method of molecular field theory will be introduced to formulate the origin of ferromagnetism which is the most conspicuous of all the magnetic phenomena. The fact that this method was successful in construing the Curie law was later eclipsed by its failure to predict the other ferromagnetic phenomena and its obvious deviation from the quantitative analysis of ferromagnetism using quantum mechanics which is indispensable for an accurate theoretical explanation of magnetism, for, according to Bohr-van Leeuwen theorem, magnetism could not have existed in the classical regime. In part II, rigorous solution to the Ising model, which is the simplest model for describing ferromagnetism, will be given in one-dimensional case to illustrate the intrinsic inadequacies of classical theory. Phase transition which is impossible in one-dimensional Ising model will emerge in two-dimensional case whose exact solution will be derived in part III. And the conclusion is in part IV.

## **Introduction**

In this short paper, the principles of magnetism will be investigated, first using the method of mean field theory<sup>[1]</sup> which was initially invented by Van der Waals to study the physical properties of real gases in the nineteenth century and reinvented by Weiss in 1907 to elucidate the mechanism for the Curie law which depicts the behavior of ferromagnets, i.e., above the Curie temperature, the magnetization of the ferromagnets will be effectively vanishing, the magnetization is nonzero only when the temperature falls below the Curie temperature. According to this theory, spontaneous magnetization is an inevitable consequence of the “internal molecular field” (which is actually the interaction of electron spins in the ferromagnets, considering the notion of electron spin was nonexistent until 1925), as long as the temperature of the ferromagnets is low enough. However, the ferromagnetic materials such as iron, nickel, and cobalt usually do not exhibit spontaneous magnetization even at low temperatures. To get out of this dilemma, Weiss introduced the postulate of magnetic domains, according to which, the actual nonexistence of spontaneous magnetization arises from the mutual cancellation of magnetization in adjacent magnetic domains. A magnetic domain is a localized region where the magnetization is saturated, though the orientations of the magnetization may point in different directions. A justification of this hypothesis will be given from the perspective of thermodynamics. By invoking the thermodynamic potentials, we show that the existence of magnetic domains is more favorable than homogeneous magnetization for sake of the thermal stability of the magnetic systems. Rigorous theoretical derivation and experimental verification of the magnetic domains may be found in reference[2].

The classical description of ferromagnetism, which is essentially a quantum mechanical phenomenon, is deemed to be intrinsically inadequate for a quantitative analysis even though it

may provide us an expedient way to study it. The simplest model that captures the quantum mechanical quintessence yet can be solved exactly in one and two dimensions is the Ising model, which was first suggested by Wilhelm Lenz who later gave it as a problem to his graduate student Ernst Ising after whom the model was finally named. The exact solution of this model in one dimension was obtained by Ising himself in 1925. However, much to his chagrin, he found that there is no phase transition in this model in one dimension, i.e., there can be no ferromagnetism. He further ratiocinated, incorrectly, that there can be no phase transition in three dimensions either. Feeling disappointed, he concluded that this model is not suitable for describing ferromagnetism and abandoned it, never returning to this subject in the rest of his life. But in 1936, Peierls reasoned that spontaneous magnetization is inevitable in two- and three-dimensional Ising model as long as the temperature is low enough, and later Kramers and Wannier<sup>[3]</sup> introduced the matrix formalism that was to have profound influences on the final solution of the 2-dimensional Ising model to attack this enigma. Wielding ponderous matrices, they exactly located the transition temperature on the basis of the assumption that the transition point exists and is unique. Although their heuristic argument is successful in many aspects, this method fails to determine the nature of the singularity which is of great interest for statistical physicists. The calculation of the exact solution to two-dimensional Ising model is quite a challenge and has been the focus of physicists and mathematicians alike for many years. The puzzle was finally resolved by Lars Onsager whose recondite calculation baffled all the coeval physicists. In the classical paper published in 1944<sup>[4]</sup>, Onsager gave the partition function of the system in a closed form, demonstrating explicitly that phase transition (from paramagnetism to ferromagnetism) occurs at the critical temperature located previously by Kramers and Wannier and determining the nature of the singularity which is actually logarithmic in contrary to the results of all the preceding approximations. Given that before Onsager there were prevalent controversies among physicists about whether statistical physics in which all the functions are analytic is adequate for describing phase transitions where singularities are ineluctable, the significance of Onsager's solution can never be overestimated. The reason for the emergence of phase transition in thermodynamic systems was later formulated by T. D. Lee, and C. N. Yang, in two papers published in 1952<sup>[5],[6]</sup>.

In spite of the spectacular success of Onsager's solution, the lengthy calculation performed by him is much too esoteric for the general physicists to digest. Fortunately, Onsager's calculation was later greatly simplified by his student Bruria Kaufman<sup>[11]</sup> who gave an ingenious and illuminating interpretation of the abstruse method used by him. In part III we will focus on the method introduced by Kaufman to give the exact solution to two-dimensional Ising model with the meticulous and possibly unnecessary mathematical details omitted. Itinerant ferromagnetism, which is critical for the description of magnetism in metals and is possibly more important than the ferromagnetism in Ising model, will only be given a brief discussion in part IV for want of length and the lack of a mature theory for it.

## **I. Mean Field Theory for Ferromagnetism<sup>[1]</sup>**

We assume that electrons on different lattice sites interact via their spins, and then the interaction

energy of two electrons with spins  $\vec{S}_1, \vec{S}_2$  has the form:

$$H = -J \vec{S}_1 \cdot \vec{S}_2, \quad (1.1)$$

where  $J$  is a positive constant that depends on the distance between the two electrons. Then the total energy of the system is:

$$\hat{H} = -\frac{1}{2} \sum_{l,l'} J_{ll'} \vec{S}_l \cdot \vec{S}_{l'}, \quad (1.2)$$

with the symbol  $l$  enumerating the lattice sites. Due to the symmetric property of the interaction,  $J_{ll'} = J_{l'l}$ . We set  $J_{ll} = 0$  to avoid the interaction of the electron with itself. This is the Heisenberg model for ferromagnetism, simple as it may seem, it has defied all possible exact solutions. A simplification of this intricate model is the Ising model, whose Hamiltonian is:

$$\hat{H} = -\frac{1}{2} \sum_{l,l'} J_{ll'} S_l^z S_{l'}^z. \quad (1.3)$$

This model is substantially simpler than the Heisenberg model, for it only takes into account the interaction of electrons in one single direction. When a magnetic field is applied, the Hamiltonian is:

$$\hat{H} = -\frac{1}{2} \sum_{l,l'} J(l-l') \sigma_l \sigma_{l'} - h \sum_l \sigma_l. \quad (1.4)$$

Here we have used  $\sigma_l = \frac{2}{\hbar} S_l^z$  to denote electron spin,  $\sigma_l = \pm 1$  for every electron. The

translational invariance of the lattices dictates that  $J(l-l') = J(l'-l) = \frac{\hbar^2}{4} J_{ll'}$ . The physical significance of equation (1.4) can be interpreted as the existence of an internal field (the name that Weiss gave it) which acts on a chosen spin  $\sigma_l$ , and the internal field has the form:

$$h_l = h + \sum_{l'} J(l-l') \sigma_{l'}. \quad (1.5)$$

The average value of the internal field is

$$\langle h_l \rangle = h + \sum_{l'} J(l-l') \langle \sigma_{l'} \rangle = h + \tilde{J} m \quad (1.6)$$

In the equation (1.6),  $m = \langle \sigma_l \rangle$  is the average value of spin which is position independent due to the translational invariance and  $\tilde{J} = \sum_{l'} J(l-l')$ .

If we introduce the approximation  $\sigma_l \sigma_{l'} \sim -\langle \sigma_l \rangle \langle \sigma_{l'} \rangle + \langle \sigma_l \rangle \sigma_{l'} + \sigma_l \langle \sigma_{l'} \rangle$ , then equation (1.4) can be recast as

$$\hat{H} = \frac{1}{2} m^2 N \tilde{J} - \sum_i \sigma_i (h + \tilde{J} m). \quad (I.7)$$

Then the partition function of the system is

$$\begin{aligned} Z &= \text{Tr} \left( e^{-\beta \hat{H}} \right) = e^{-\frac{1}{2} \beta m^2 N \tilde{J}} \left( \sum_{\sigma_i = \pm 1} e^{\beta \sigma_i (h + \tilde{J} m)} \right)^N \\ &= \left( e^{-\frac{1}{2} \beta m^2 \tilde{J}} 2 \cosh \left( \beta (h + \tilde{J} m) \right) \right)^N, \end{aligned} \quad (I.8)$$

where  $\beta = 1/kT$ ,  $k$  is Boltzmann constant and  $T$  is the absolute temperature.

From  $m = \frac{1}{N} kT \frac{\partial}{\partial h} \log Z$ , we finally get an implicit equation for the average value of spin  $m$ :

$$m = \tanh \left( \beta (\tilde{J} m + h) \right) \quad (I.9)$$

Let us now introduce the denotation:

$$T_c = \frac{\tilde{J}}{k}, \tau = \frac{T - T_c}{T_c} \quad (I.10)$$

We will solve the equation (I.9) in two cases:

**A).** The applied field  $h = 0$ .

In this case, the implicit equation for  $m$  is

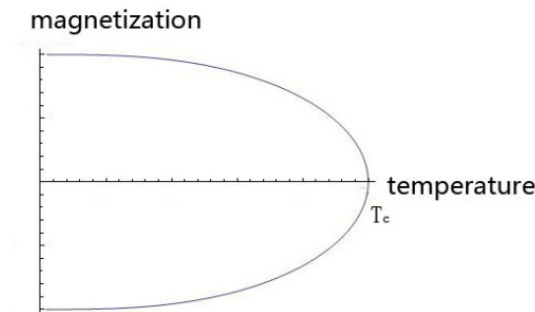
$$m = \tanh \left( \beta \tilde{J} m \right) = \tanh \left( \frac{T_c}{T} m \right) \quad (I.11)$$

We can see that  $m$  is a function of temperature  $T$ , the explicit expression for which is

$$T = \frac{T_c m}{\text{arc tanh } m} = \frac{2T_c m}{\log \left( \frac{1+m}{1-m} \right)} \quad (I.12)$$

The magnetization is proportional to the average value of electron spins with the coefficients being the Bohr magneton, so from now on we will simply refer to  $m$  as the magnetization unless otherwise specified.

The figure of magnetization as a function of temperature is plotted as below:



Curie law is demonstrated vividly by the above picture: Magnetization is vanishing above a critical temperature called the Curie temperature; only when the temperature is lowered below the critical temperature does the magnetization become finite. In the vicinity of the critical point, the relation between the magnetization and temperature can be approximated as a power law:

$$m \sim (-3\tau)^{1/2} = \sqrt{\frac{3}{T_c}} (T_c - T)^{1/2} \quad (\text{I.13})$$

The free energy is:

$$\begin{aligned} F &= -kT \log Z = -NkT \left( -\frac{1}{2} \beta m^2 \tilde{J} + \log(2 \cosh \beta m \tilde{J}) \right) \\ &= Nk \left( \frac{1}{2} m^2 T_c - T \log \left( 2 \cosh \left( \frac{T_c}{T} m \right) \right) \right) \\ &= Nk \left( \frac{1}{2} m^2 T_c - T \log 2 + \frac{1}{2} T \log(1 - m^2) \right) \end{aligned} \quad (\text{I.14})$$

Then the specific heat is:

$$\begin{aligned} c &= -T \frac{\partial^2 F}{\partial T^2} = -Nk T_c m \frac{\partial m}{\partial T} \\ &= -\frac{1}{2} Nk m \frac{\log \left( \frac{1+m}{1-m} \right)}{1 - \frac{1}{1-m^2} \frac{2m}{\log \left( \frac{1+m}{1-m} \right)}} \end{aligned} \quad (\text{I.15})$$

When the  $T > T_c$ , the magnetization vanishes identically, therefore the specific heat which is proportional to  $m$  is also zero; when the temperature is slightly below  $T_c$ , that is, when

$\tau = \frac{T - T_c}{T_c} \rightarrow 0^-$ ,  $m$  is slightly above zero, the specific heat at this critical point is:

$$c|_{\tau \rightarrow 0^-} = -\frac{1}{2} Nk \lim_{m \rightarrow 0} m \frac{\log \left( \frac{1+m}{1-m} \right)}{1 - \frac{1}{1-m^2} \frac{2m}{\log \left( \frac{1+m}{1-m} \right)}} = \frac{3}{2} Nk \quad (\text{I.16})$$

Then in the vicinity of the critical point, the specific heat of the system exhibits a jump,

$$\Delta c = c_{\tau \rightarrow 0^-} - c_{\tau \rightarrow 0^+} = \frac{3}{2} Nk \quad (\text{I.17})$$

It shall be seen that although the prediction of the existence of a discontinuity in the specific heat is correct, the nature of the discontinuity is incorrect according to the exact theory.

Having expatiated upon the case with the absence of an applied field, now we will focus our attention on the case when an external field is present.

**B).** The applied field  $h \neq 0$ .

In this case, the implicit equation for  $m$  is:

$$m = \tanh\left(\frac{T_c}{T}m + \frac{h}{kT}\right) \quad (I.18)$$

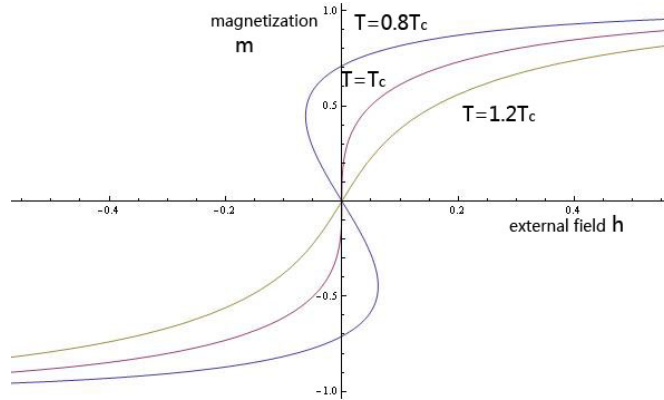
Then the explicit relation between  $m$  and  $T$  is:

$$T = \frac{T_c m + h/k}{\operatorname{arc} \tanh m} = \frac{2(T_c m + h/k)}{\log\left(\frac{1+m}{1-m}\right)} \quad (I.19)$$

An obvious paradox arises from (I.19), for, as is clear from this equation, when  $m \rightarrow 0^-$ , the absolute temperature will diverge to negative infinity, which is impossible. This is the so called negative-temperature problem which deserves our attention, but now we shall omit a meticulous discussion of it and just concentrate our attention on what is most intriguing to us at the moment: how does the external field affect the magnetization of the ferromagnetic specimen. To answer this question, we need to solve the equation (I.18) as a function of the external field  $h$  and the magnetization  $m$ . The explicit relation between  $h$  and  $m$  is:

$$h = kT_c \left( \frac{T}{T_c} \operatorname{arc} \tanh m - m \right) \quad (I.20)$$

We now plot the magnetization  $m$  as a function of external field as below:



As is clear from the figure above, when  $T > T_c$ , the magnetization is a monotonic function of the external field; while below the Curie temperature, the magnetization may be a multi-valued function of the external field when the intensity of the applied field is below a critical value  $h_c$ , which may be calculated to be:

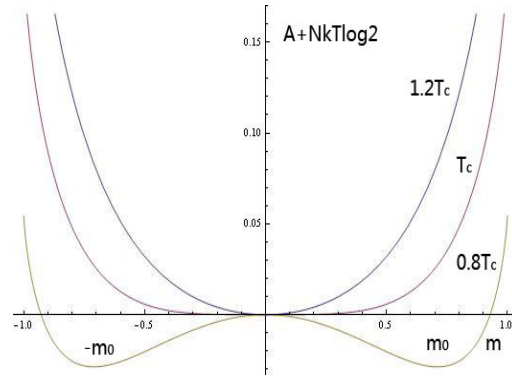
$$h_c = kT_c \left( \frac{T}{T_c} \operatorname{arc} \tanh \left( \sqrt{1 - \frac{T}{T_c}} \right) - \sqrt{1 - \frac{T}{T_c}} \right). \quad (I.21)$$

When  $T < T_c$  and  $0 < h < h_c$ , a given value of  $h$  correspond to three values of magnetization

$m$ , one positive and the other two negative. Theoretically, any of these three values are acceptable, however, as we shall see, only the positive value is realizable if the system is to be thermodynamically stable. To prove this, we are obliged to consider the Helmholtz free energy:

$$\begin{aligned}
 A &= F + Nm h = -kT \log Z + Nm h \\
 &= NkT_c \left( -\frac{m^2}{2} - \frac{T}{T_c} \log \frac{2}{\sqrt{1-m^2}} + \frac{1}{2} \frac{T}{T_c} m \log \left( \frac{1+m}{1-m} \right) \right), \\
 A + NkT \log 2 &= NkT_c \left( -\frac{m^2}{2} + \frac{1}{2} \frac{T}{T_c} \log(1-m^2) + \frac{1}{2} \frac{T}{T_c} m \log \left( \frac{1+m}{1-m} \right) \right)
 \end{aligned} \tag{1.22}$$

The figure of the function is plotted as below with differing temperatures:



By invoking the relation

$$h = \frac{1}{N} \left( \frac{\partial A}{\partial m} \right)_T, \tag{1.23}$$

we can see that when  $T \geq T_c$ , corresponding every  $h$ , there exists one unique  $m$ ; when  $T < T_c$ , for an applied field that is not strong enough (without loss of generality, we now assume that  $h > 0$ ), there are three eligible  $m$ , one of them of is positive and the other two are negative. From the above figure, we can easily see that the free energy corresponding to the positive  $m$  is lower than that corresponding to the negative  $m$ , therefore we can conclude that for a positive (negative) applied field only the configuration with a positive (negative)  $m$  is stable. When the applied field  $h = 0$ ,  $m = \pm m_0$  are equally acceptable as they correspond to an equal free energy.

So in the absence of an external field a fraction of the bulk has the magnetization  $m_0$  and the rest may have the opposite magnetization  $-m_0$ , from hence springs the notion of magnetic domains which are in thermal equilibrium due to the equal free energy they share.



## II. Ising Model for Ferromagnetism (One-Dimensional Case)

In the mean field theory of ferromagnetism, we can see that the emergence of spontaneous magnetization depends only on the temperature and nothing else. In this part we shall show that actually in one-dimensional Ising model with the absence of an applied field, no phase transition will occur no matter how low the temperature may be. In the Ising model, the interactions of electrons are restricted to their nearest neighbors. The Hamiltonian of one dimensional Ising model is:

$$\hat{H} = -\sum_{i=1}^{N-1} J \sigma_i \sigma_{i+1} - \sum_{i=1}^N h \sigma_i \quad (\text{II.1})$$

where  $N$  is the number of lattice sites, the constant  $J > 0$  is the interaction energy (which we assume to be equal for any pair of adjacent electrons) of nearest electrons, and  $h$  represents the applied field. The partition function of one dimensional Ising model with  $N + 1$  lattice sites is:

$$\begin{aligned} Z_{N+1} &= \text{Tr} \left( e^{-\beta \hat{H}} \right) = \text{Tr} \left[ \exp \left( \sum_{i=1}^N \beta J \sigma_i \sigma_{i+1} + \sum_{i=1}^{N+1} \beta h \sigma_i \right) \right] \\ &= \sum_{\{\sigma_i = \pm 1\}} \exp \left( \sum_{i=1}^N \beta J \sigma_i \sigma_{i+1} + \sum_{i=1}^{N+1} \beta h \sigma_i \right) \end{aligned} \quad (\text{II.2})$$

where the summation is over all possible configurations, the total number of which is  $2^{N+1}$  due to the two possible orientations that each electron spin can adopt independently of others. Now our major task is to calculate the analytic form of the partition. To do this, we will adopt the transfer matrix method [7], [8]. First, we make the following definitions:

$$\begin{aligned} Z_{N+1}^+ &= \sum_{\{\sigma_i = \pm 1\}, \sigma_{N+1}=1} \exp \left( \sum_{i=1}^N \beta J \sigma_i \sigma_{i+1} + \sum_{i=1}^{N+1} \beta h \sigma_i \right) \\ Z_{N+1}^- &= \sum_{\{\sigma_i = \pm 1\}, \sigma_{N+1}=-1} \exp \left( \sum_{i=1}^N \beta J \sigma_i \sigma_{i+1} + \sum_{i=1}^{N+1} \beta h \sigma_i \right) \end{aligned} \quad (\text{II.3})$$

Then, it is clear from this definition that  $Z_{N+1} = Z_{N+1}^+ + Z_{N+1}^-$ .

Direct calculation leads to the following iteration relations:

$$\begin{aligned} Z_{N+1}^+ &= e^{\beta J + \beta h} Z_N^+ + e^{-\beta J + \beta h} Z_N^-; \\ Z_{N+1}^- &= e^{-\beta J - \beta h} Z_N^+ + e^{\beta J - \beta h} Z_N^- \end{aligned} \quad (\text{II.4})$$

When written in the matrix form, equation (II.4) can become more illuminating:

$$\begin{pmatrix} Z_{N+1}^+ \\ Z_{N+1}^- \end{pmatrix} = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J + \beta h} \\ e^{-\beta J - \beta h} & e^{\beta J - \beta h} \end{pmatrix} \begin{pmatrix} Z_N^+ \\ Z_N^- \end{pmatrix} \quad (\text{II.5})$$

Now define

$$\chi_N = \begin{pmatrix} Z_N^+ \\ Z_N^- \end{pmatrix}, \quad \text{and} \quad M = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J + \beta h} \\ e^{-\beta J - \beta h} & e^{\beta J - \beta h} \end{pmatrix}, \quad (\text{II.6})$$

then equation (II.5) may be written more compactly as:

$$\chi_{N+1} = M \chi_N \quad (\text{II.7})$$

Through iterations of equation (II.7), we can finally get

$$\chi_{N+1} = M \chi_N = \cdots = M^N \chi_1 \quad (\text{II.8})$$

And the first term may be easily calculated to be

$$\chi_1 = \begin{pmatrix} Z_1^+ \\ Z_1^- \end{pmatrix} = \begin{pmatrix} e^{\beta h} \\ e^{-\beta h} \end{pmatrix} \quad (\text{II.9})$$

Having done this, our mission is reduced to calculating the powers of the matrix  $M$ . In order to make things appear clearer, we shall make use of a little trick to deal with the ubiquitous exponential functions. It is obvious that the identity  $e^x = \cosh x + \sinh x = \cosh x (1 + \tanh x)$

holds. With the invocation of this identity, the matrix  $M$  may be written as

$$M = \cosh(\beta J) \cosh(\beta h) \begin{pmatrix} (1+A)(1+B) & (1-A)(1+B) \\ (1-A)(1-B) & (1+A)(1-B) \end{pmatrix}; \quad (\text{II.10})$$

$$A = \tanh \beta J, \quad B = \tanh \beta h$$

Then

$$M^N = [\cosh(\beta J) \cosh(\beta h)]^N \begin{pmatrix} (1+A)(1+B) & (1-A)(1+B) \\ (1-A)(1-B) & (1+A)(1-B) \end{pmatrix}^N \quad (\text{II.11})$$

$$= [\cosh(\beta J) \cosh(\beta h)]^N U \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} U^{-1}$$

where the eigenvalues are

$$\lambda_{1,2} = 1 + A \pm \sqrt{(1-A)^2 + 4AB^2}, \lambda_1 > \lambda_2 \quad (\text{II.12})$$

and the transformation matrix is

$$U = \begin{pmatrix} \frac{B + AB + \sqrt{(1-A)^2 + 4AB^2}}{(1-A)(1-B)} & \frac{B + AB - \sqrt{(1-A)^2 + 4AB^2}}{(1-A)(1-B)} \\ 1 & 1 \end{pmatrix}, \quad (\text{II.13})$$

$$U^{-1} = \begin{pmatrix} \frac{(1-A)(1-B)}{2\sqrt{(1-A)^2 + 4AB^2}} & -\frac{B + AB - \sqrt{(1-A)^2 + 4AB^2}}{2\sqrt{(1-A)^2 + 4AB^2}} \\ -\frac{(1-A)(1-B)}{2\sqrt{(1-A)^2 + 4AB^2}} & \frac{B + AB + \sqrt{(1-A)^2 + 4AB^2}}{2\sqrt{(1-A)^2 + 4AB^2}} \end{pmatrix}$$

Substitute equations (II.9) ~ (II.13) into equation (II.8) and note that  $Z_{N+1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} Z_{N+1}^+ \\ Z_{N+1}^- \end{pmatrix}$ , then,

we finally get:

$$Z_{N+1} = [\cosh(\beta J)]^N [\cosh(\beta h)]^{N+1} \lambda_1^N \times \frac{1}{\sqrt{(1-A)^2 + 4AB^2}} \left[ \left( 1 - A + 2AB^2 + \sqrt{(1-A)^2 + 4AB^2} \right) + \left( \frac{\lambda_2}{\lambda_1} \right)^N \left( -1 + A - 2AB^2 + \sqrt{(1-A)^2 + 4AB^2} \right) \right] \quad (\text{II.14})$$

We now define the free energy per lattice site in the thermodynamic limit as

$$F = -kT \lim_{N \rightarrow \infty} \frac{1}{N+1} \log Z_{N+1} \quad (\text{II.15})$$

After substituting equation (II.14) into (II.15), we have

$$F = -kT \left( \log \cos \beta J + \log \cos \beta h + \log \left( 1 + A + \sqrt{(1-A)^2 + 4AB^2} \right) \right) \quad (\text{II.16})$$

Substitute the explicit expressions for  $A$  and  $B$  into equation (II.16), then we obtain the free energy per lattice site in a closed form:

$$F = -J - kT \log \left( \cosh \beta h + \sqrt{(\cosh \beta h)^2 - 2e^{-2\beta J} \sinh 2\beta J} \right) \quad (\text{II.17})$$

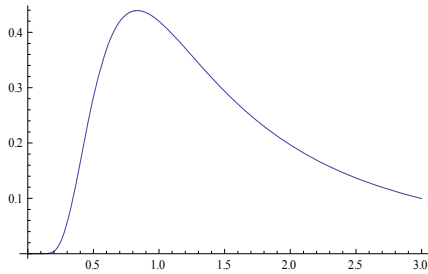
When there is no applied field, the free energy per lattice site is reduced to

$$F = -kT \log(2 \cosh \beta J) \quad (\text{II.18})$$

According to the nomenclature for the order of phase transitions that was proposed by Ehrenfest<sup>[1]</sup>, we can determine the order of phase transitions by differentiating the free energy and see when a singularity arises. The specific heat of the one-dimensional Ising model with the absence of an applied field is:

$$c = -T \frac{\partial^2 F}{\partial T^2} = \frac{J^2}{kT^2} \frac{1}{\cosh^2 \left( \frac{J}{kT} \right)} \quad (\text{II.19})$$

The specific heat versus temperature is plotted as below:

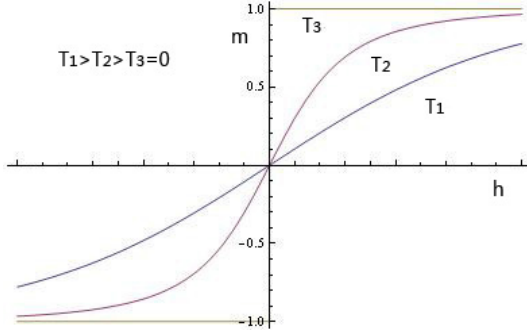


It can be seen clearly from the above figure that the specific heat is smooth for all finite temperatures, hence no phase transition occurs and ultimately no spontaneous magnetization is expected.

The expectation value of the spin in the thermodynamic limit, i.e., the magnetization of the one-dimensional Ising model, is:

$$\begin{aligned}
\langle \sigma \rangle = m(h; T) &= \lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr} \left( \hat{\rho} \sum_{i=1}^N \sigma_i \right) = -\frac{\partial F}{\partial h} \\
&= \frac{\sinh \beta h}{\sqrt{(\cosh \beta h)^2 - 2e^{-2\beta J} \sinh(2\beta J)}}
\end{aligned} \tag{II.20}$$

The magnetization as a function of the external field with the temperature as a parameter is plotted as follows:



This figure further corroborates the assertion that one-dimensional Ising model cannot accommodate spontaneous magnetization: the magnetization is vanishing for all values of temperature except when the absolute temperature is identically zero, in which case the magnetization equation (II.20) degenerates into a step function.

The exact solution of two-dimensional Ising model in the absence of applied field which was obtained in 1944 by Lars Onsager<sup>[4]</sup> is much more difficult to calculate. With the omission of the lengthy calculation which is postponed to part III, the final solution will be directly presented here. The free energy per lattice site is<sup>[7],[9]</sup>:

$$-\frac{F}{kT} = \log 2 + \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} dq_1 dq_2 \log \left[ (\cosh 2\beta J)^2 - \sinh 2\beta J (\cos q_1 + \cos q_2) \right] \tag{II.21}$$

A singularity arises when the argument of the logarithmic function vanishes, i.e., when

$$q_1 = q_2 = 2\pi, \quad \sinh 2\beta J = 1 \tag{II.22}$$

From equation (II.22), the transition temperature can be determined by

$$\frac{J}{kT_c} = \frac{1}{2} \log(1 + \sqrt{2}) = 0.44069 \tag{II.23}$$

It can be shown that as  $T \rightarrow T_c$ , the free energy and its first order derivative are continuous while the specific heat which is proportional to the second order derivative of the free energy diverges logarithmically, i.e., a second order phase transition occurs. Thus, Onsager has definitely demonstrated that statistical mechanics is powerful enough to describe phase transition and critical phenomena, resolving a decades-long debate among the physicists and inaugurating a new era for statistical physics.

### III. Ising Model for Ferromagnetism (Two-Dimensional Case)<sup>[11]</sup>

The two-dimensional Ising model we shall study here is a rectangular crystal lattice with  $m$  rows and  $n$  columns. Then the Hamiltonian for this model is (in this part the external field will always be absent):

$$\hat{H} = - \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j \quad (\text{III.1})$$

Just as has already been stated, the interactions of lattice sites are restricted only to nearest neighbors. The configuration of the crystal can be specified by stating all the values of the spin on the lattice sites. A more compact way to specify the configuration of the crystal is to first specify the configuration of each row and then integrate all the rows together. Since there are  $n$  lattice sites in each row, there are  $2^n$  possible configurations,  $1 \leq \nu_j \leq 2^n, 1 \leq j \leq m$ . And then the configuration of the crystal is given by the set  $\{\nu_1, \nu_2, \dots, \nu_m\}$ .

The energy due to interactions within a row may be denoted by  $E(\nu_i)$ ; the energy due to interaction between two adjacent rows is denoted by  $E(\nu_i, \nu_{i+1})$ .

The explicit expression for the interaction energy in the  $l$ th row is

$$E(\nu_l) = -J \sum_{i=1}^n \sigma_i^l \sigma_{i+1}^l, \quad \sigma_{n+1}^l = \sigma_1^l \quad (\text{III.2})$$

The explicit expression for the interaction energy between the  $l$ th row and  $(l+1)$ th row is

$$E(\nu_l, \nu_{l+1}) = -J \sum_{i=1}^n \sigma_i^{l+1} \sigma_i^l \quad (\text{III.3})$$

Then the energy of a configuration of the crystal is

$$E_c = \sum_{i=1}^m E(\nu_i) + \sum_{i=1}^m E(\nu_i, \nu_{i+1}) \quad (\text{III.4})$$

In the above formula, periodic condition has been introduced, that is,  $\nu_{m+1} = \nu_1$ . Then the partition function of the system is

$$Z = \sum_{\{\sigma=\pm 1\}} e^{-E_c/kT} = \sum_{\{\sigma=\pm 1\}} \exp \left[ - \left( \sum_{i=1}^m E(\nu_i) + \sum_{i=1}^m E(\nu_i, \nu_{i+1}) \right) / kT \right] \quad (\text{III.5})$$

If we introduce the abbreviations:

$$\begin{aligned} (V_l)_{\nu_l \nu_{l+1}} &= e^{-E(\nu_l, \nu_{l+1})/kT} \\ (V_l')_{\nu_l \nu_l} &= e^{-E(\nu_l)/kT} \end{aligned} \quad (\text{III.6})$$

Then the partition function can be expressed as:

$$Z = \sum_{\{V_i\}} (V_2)_{V_1 V_1} (V_1')_{V_1 V_2} \cdots (V_2)_{V_m V_m} (V_1')_{V_m V_1} = \text{Tr}(V_2 V_1')^m \quad (\text{III.7})$$

It has already been shown by Onsager that

$$\begin{aligned} V_2 &= \exp\left(H' \sum_{r=1}^n s_r s_{r+1}\right), & H' &= \frac{J'}{kT} \\ V_1' &= (2 \sinh 2H)^{n/2} \exp\left(H^* \sum_{r=1}^n C_r\right), & H &= \frac{J}{kT}, \quad e^{-2H} = \tanh H^* \end{aligned} \quad (\text{III.8})$$

Here  $S_r$  and  $C_r$  are  $2^n$ -dimensional matrices,

$$\begin{aligned} s_r &= \overbrace{\text{I} \times \text{I} \times \cdots \times \text{I}}^{r-1} \times s \times \overbrace{\text{I} \times \text{I} \times \cdots \times \text{I}}^{n-r} \\ C_r &= \overbrace{\text{I} \times \text{I} \times \cdots \times \text{I}}^{r-1} \times C \times \overbrace{\text{I} \times \text{I} \times \cdots \times \text{I}}^{n-r} \quad 1 \leq r \leq n \end{aligned} \quad (\text{III.9})$$

The matrices  $\text{I}$ ,  $s$  and  $C$  are

$$\text{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In equation (III.9), the Kronecker direct product is used.

We now define

$$V_1 = \exp\left(H^* \sum_{r=1}^n C_r\right) \quad (\text{III.10})$$

Then the partition function is

$$Z = (2 \sinh 2H)^{mn/2} \text{Tr}(V_2 V_1)^m = (2 \sinh 2H)^{mn/2} \sum_{i=1}^{2^n} \lambda_i^m \quad (\text{III.11})$$

where  $\lambda_i$  are the eigenvalues of the matrix  $V = V_2 V_1$ . To calculate the exact partition function,

we have only to calculate the eigenvalues  $\lambda_i$ . To do this, we define the matrices:

$$\begin{aligned} \Gamma_{2r-1} &= \overbrace{C \times C \times \cdots \times C}^{r-1} \times s \times \overbrace{\text{I} \times \text{I} \times \cdots \times \text{I}}^{n-r} = P_r \\ \Gamma_{2r} &= \overbrace{C \times C \times \cdots \times C}^{r-1} \times i s C \times \overbrace{\text{I} \times \text{I} \times \cdots \times \text{I}}^{n-r} = Q_r \quad 1 \leq r \leq n \end{aligned} \quad (\text{III.12})$$

It can be easily verified that these matrices satisfy the anticommutation relations:

$$\Gamma_l \Gamma_k + \Gamma_k \Gamma_l = 2\delta_{kl} \quad 1 \leq k, l \leq 2n \quad (\text{III.13})$$

The significance of these matrices lies in that any  $2^n$ -dimensional matrix can be expressed by the products of these matrices. Specifically, the matrices  $V_1, V_2$  can be expressed by them. It can be shown that

$$-iP_r Q_r = C_r = \overbrace{\mathbf{I} \times \mathbf{I} \times \cdots \times \mathbf{I}}^{r-1} \times C \times \overbrace{\mathbf{I} \times \mathbf{I} \times \cdots \times \mathbf{I}}^{n-r} \quad (\text{III.14})$$

Therefore

$$V_1 = \prod_{r=1}^n \exp(-iH^* P_r Q_r) \quad (\text{III.15})$$

And from

$$\begin{aligned} P_{r+1} Q_r &= -is_r s_{r+1} = -is_{r+1} s_r \quad 1 \leq r \leq n-1 \\ P_1 Q_n C_1 C_2 \cdots C_n &\equiv P_1 Q_n U = is_1 s_n = is_n s_1 \end{aligned} \quad (\text{III.16})$$

we can get

$$V_2 = \prod_{r=1}^{n-1} \exp(iH' P_{r+1} Q_r) \exp(-iH' P_1 Q_n U)$$

Using these matrices, we can rewrite the matrix  $V = V_2 V_1$  as

$$V \equiv \frac{1}{2}(\mathbf{I} + U)V^+ + \frac{1}{2}(\mathbf{I} - U)V^- \quad (\text{III.17})$$

Here

$$\begin{aligned} V^+ &= \prod_{r=1}^n \exp(-iH^* P_r Q_r) \cdot \prod_{r=1}^{n-1} \exp(iH' P_{r+1} Q_r) \cdot \exp(-iH' P_1 Q_n U) \\ V^- &= \prod_{r=1}^n \exp(-iH^* P_r Q_r) \cdot \prod_{r=1}^{n-1} \exp(iH' P_{r+1} Q_r) \cdot \exp(iH' P_1 Q_n U) \end{aligned} \quad (\text{III.18.a})$$

and here

$$U = \overbrace{C \times C \times \cdots \times C}^n = \begin{pmatrix} & & C \\ & \ddots & \\ C & & \end{pmatrix}_{2^n \times 2^n} \quad (\text{III.18.b})$$

Now we will concentrate our attention instead on the calculation of the eigenvalues of the matrices

$V^+, V^-$ , as we shall see that the eigenvalues of  $V$  can be obtained from the eigenvalues of

$V^+, V^-$ . To calculate the eigenvalues of  $V^+, V^-$ , the method of spinor analysis shall be employed.

To understand spinor analysis, we shall observe that equation (III.13) still holds when the matrices

$\Gamma_k$  undergoes a similarity transformation, that is, when we set  $\Gamma_k^* = S \Gamma_k S^{-1}$ ,  $1 \leq k \leq n$ ,

$$\Gamma_l^* \Gamma_k^* + \Gamma_k^* \Gamma_l^* = 2\delta_{kl} \quad 1 \leq k, l \leq 2n \quad (\text{III.19})$$

It can be proved that the converse of the above theorem is also true, i.e.,

**Theorem:** If two sets of matrices  $\Gamma_k, \Gamma_k^*$  both obey the anticommutation relations (III.13), then

a orthogonal transformation  $S$  can be found such that  $\Gamma_k^* = S \Gamma_k S^{-1}$ .

Consider the following orthogonal transformation:

$$\begin{aligned}
K: \Gamma_k &\rightarrow \cos \Theta \Gamma_k - \sin \Theta \Gamma_l \equiv \Gamma_k^* & \Gamma_i &\rightarrow \Gamma_i \\
&\Gamma_l \rightarrow \sin \Theta \Gamma_k + \cos \Theta \Gamma_l \equiv \Gamma_l^* & i &\neq k, l
\end{aligned} \tag{III.20}$$

We now define  $S(K) = \exp\left(\frac{\theta}{2} \Gamma_k \Gamma_l\right)$ , and note that

$$(\Gamma_k \Gamma_l)^2 = \Gamma_k \Gamma_l \Gamma_k \Gamma_l = -\Gamma_k \Gamma_k \Gamma_l \Gamma_l = -I$$

then

$$\begin{aligned}
S(K) &= I + \frac{\theta}{2} \Gamma_k \Gamma_l - \frac{1}{2!} \left(\frac{\theta}{2}\right)^2 I - \frac{1}{3!} \left(\frac{\theta}{2}\right)^3 \Gamma_k \Gamma_l + \dots \\
&= \cos\left(\frac{\theta}{2}\right) I + \sin\left(\frac{\theta}{2}\right) \Gamma_k \Gamma_l
\end{aligned} \tag{III.21}$$

Therefore

$$\begin{aligned}
&S(K) \cdot \Gamma_k \cdot S(K)^{-1} \\
&= \left[ \cos\left(\frac{\theta}{2}\right) I + \sin\left(\frac{\theta}{2}\right) \Gamma_k \Gamma_l \right] \cdot \Gamma_k \cdot \left[ \cos\left(\frac{\theta}{2}\right) I - \sin\left(\frac{\theta}{2}\right) \Gamma_k \Gamma_l \right] \\
&= \cos \Theta \Gamma_k - \sin \Theta \Gamma_l = \Gamma_k^*
\end{aligned} \tag{III.22}$$

Similarly,  $S(K) \Gamma_l S(K)^{-1} = \sin \Theta \Gamma_k + \cos \Theta \Gamma_l$ . Then the transformation  $K$  is a rotation in  $2^n$ -dimensional space and the matrix  $S(K)$  is called the spin representation of the rotation  $K$ .

The matrix  $S(K)$  has a particularly simple form when we let  $\Gamma_k, \Gamma_l$  take the following values:

$$\begin{aligned}
\Gamma_k &= P_r^* = g P_r g = \overbrace{s \times s \cdots \times s}^{r-1} \times C \times \overbrace{I \times \cdots \times I}^{n-r} \\
\Gamma_l &= Q_r^* = g Q_r g = \overbrace{s \times s \cdots \times s}^{r-1} \times i C s \times \overbrace{I \times \cdots \times I}^{n-r} \\
\text{here } g &= 2^{-n/2} (C + s) \times (C + s) \times \cdots \times (C + s) = g^{-1}
\end{aligned}$$

In this case,

$$S(K) = \exp\left(\frac{\theta}{2} P_r^* Q_r^*\right) = \overbrace{I \times I \times \cdots \times I}^{r-1} \times \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \times \overbrace{I \times \cdots \times I}^{n-r} \tag{III.23}$$

Consider now a product of  $n$  commuting plane rotations. The  $2n$   $\Gamma$  matrices are grouped into pairs, and each pair  $(\Gamma_{r_1}, \Gamma_{r_2})$  is rotated by an angle  $\theta_r$ , as in (III.20). We have

$$K = \prod_{r=1}^n K_r \tag{III.24}$$

The eigenvalues of  $K_r$  are  $e^{\pm i\theta_r}$ .

It can be proved that  $S(O') S(O'') = S(O'O'')$ , therefore



$$S(K) = \prod_{r=1}^n S(K_r) = \prod_{r=1}^n \exp\left(\theta_r/2 \Gamma_{r_1} \Gamma_{r_2}\right) \quad (\text{III.25})$$

In the special case  $\Gamma_{r_1} = P_r^*, \Gamma_{r_2} = Q_r^*$ , we have

$$\begin{aligned} S(K) &= \prod_{r=1}^n \exp\left(\theta_r/2 \Gamma_{r_1} \Gamma_{r_2}\right) \\ &= \begin{pmatrix} e^{-i\theta_1/2} & 0 \\ 0 & e^{i\theta_1/2} \end{pmatrix} \times \begin{pmatrix} e^{-i\theta_2/2} & 0 \\ 0 & e^{i\theta_2/2} \end{pmatrix} \times \cdots \times \begin{pmatrix} e^{-i\theta_n/2} & 0 \\ 0 & e^{i\theta_n/2} \end{pmatrix} \end{aligned} \quad (\text{III.26})$$

Then obviously, the eigenvalues of  $S(K)$  are

$$\lambda = \exp\left[i/2(\pm\theta_1 \pm \theta_2 \pm \cdots \pm \theta_n)\right] \quad (\text{III.27})$$

All sign combinations are to be taken, giving the  $2^n$  eigenvalues of the matrix  $S(K)$ . Even for a more general choice of  $\Gamma_{r_1}, \Gamma_{r_2}$  the eigenvalues of  $S(K)$  are still the same, because an arbitrary

set of matrices  $\Gamma_{r_1}, \Gamma_{r_2}$  can be obtained from the set of  $P_r^*, Q_r^*$  via a similarity transformation which leaves the eigenvalues invariant. Therefore once we know the eigenvalues of the rotation matrices which constitute the matrix  $S(K)$ , we can obtain the eigenvalues of  $S(K)$  immediately through the formula (III.27).

Having familiarized ourselves with some elements of spinor analysis, we can now proceed to calculate the eigenvalues of  $V^+$  and  $V^-$ . To do this, we shall rewrite  $V^+$  and  $V^-$  in the form that is most suitable for spinor analysis:

$$V^- = V_1 V_2^-, \quad V^+ = V_1 V_2^+ \quad (\text{III.28})$$

Here,

$$\begin{aligned} V_2^- &= \prod_{r=1}^n \exp\left(iH' P_{r+1} Q_r\right), \\ V_2^+ &= \prod_{r=1}^{n-1} \exp\left(iH' P_{r+1} Q_r\right) \cdot \exp\left(-iH' P_1 Q_n U\right) \end{aligned} \quad (\text{III.29})$$

Definition:

$$V_0^- = V_1^{1/2} \cdot V_2^- \cdot V_1^{1/2} = V_1^{-1/2} \cdot V^- \cdot V_1^{1/2} \quad (\text{III.30})$$

Then it is clear that the matrix  $V_0^-$  has the same eigenvalues as matrix  $V^-$  since they differ from each other only by a similarity transformation.

$$V_0^- = \prod_{r=1}^n \exp\left(-\frac{iH^*}{2} P_r Q_r\right) \cdot \prod_{r=1}^n \exp\left(iH' P_{r+1} Q_r\right) \cdot \prod_{r=1}^n \exp\left(-\frac{iH^*}{2} P_r Q_r\right) \quad (\text{III.31})$$

The first (and also last) term in equation (III.31) is the spin representation of the rotation matrix  $R_1$ :

$$R_1 = \begin{pmatrix} A & \dots & 0 \\ & A & \\ \vdots & & \ddots & \vdots \\ & & & A \\ 0 & \dots & & A \end{pmatrix}, \text{ here } A = \begin{pmatrix} \cosh H^* & -i \sinh H^* \\ i \sinh H^* & \cosh H^* \end{pmatrix}$$

The middle term is the spin representation of the rotation matrix  $R_2$  :

$$R_2 = \begin{pmatrix} \cosh 2H' & 0 & i \sinh 2H' \\ & M & \\ 0 & M & 0 \\ & & \ddots & \\ -i \sinh 2H' & 0 & \cosh 2H' \end{pmatrix}, \text{ here } M = \begin{pmatrix} \cosh 2H' & -i \sinh 2H' \\ i \sinh 2H' & \cosh 2H' \end{pmatrix}$$

Then we have

$$V_0^- = S(R_0^-) = S(R_1)S(R_2)S(R_1) = S(R_1 R_2 R_1) \quad (\text{III.32})$$

After some matrix multiplications, we finally have:

$$R_0^- = R_1 R_2 R_1 = \begin{pmatrix} a & b & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & b^\dagger \\ b^\dagger & a & b & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & b^\dagger & a & b & 0 & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ b & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & b^\dagger & a \end{pmatrix} \quad (\text{III.32})$$

where the matrices  $a, b, b^\dagger$  are defined as:

$$a = \begin{pmatrix} \cosh 2H' \cosh 2H^* & -i \cosh 2H' \sinh 2H^* \\ i \cosh 2H' \sinh 2H^* & \cosh 2H' \cosh 2H^* \end{pmatrix}$$

$$b = \begin{pmatrix} -\frac{1}{2} \sinh 2H' \sinh 2H^* & i \sinh 2H' \sinh^2 H^* \\ -i \sinh 2H' \cosh^2 H^* & -\frac{1}{2} \sinh 2H' \sinh 2H^* \end{pmatrix}$$

$$b^\dagger = \begin{pmatrix} -\frac{1}{2} \sinh 2H' \sinh 2H^* & i \sinh 2H' \cosh^2 H^* \\ -i \sinh 2H' \sinh^2 H^* & -\frac{1}{2} \sinh 2H' \sinh 2H^* \end{pmatrix}$$

The matrix  $R_0^-$  is a Toeplitz matrix or more specifically a cyclic matrix whose eigenvalues have already been calculated out by mathematicians. According to known theorems about cyclic matrix, the  $2n$  eigenvalues of  $R_0^-$  are the eigenvalues of  $n$  two-dimensional matrices

$$\alpha_{2r} = \varepsilon^{2(n-1)r} b^\dagger + \varepsilon^{2r} b + a, \quad 1 \leq r \leq n, \quad \varepsilon = e^{i\pi/n} \quad (\text{III.33})$$

Since  $\det(\alpha_{2r}) = 1$  for all  $1 \leq r \leq n$ , the eigenvalues of the matrix  $\alpha_{2r}$  can be written as  $e^{\pm\gamma_{2r}}$ , and the trace of the matrix is

$$\frac{1}{2} \text{Tr}(\alpha_{2r}) = \cosh \gamma_{2r} = \cosh 2H^* \cosh 2H' - \sinh 2H^* \sinh 2H' \cos(2r\pi/n) \quad (\text{III.34})$$

Therefore, the eigenvalues of  $\alpha_{2r}$  can be uniquely identified via equation (III.34). Having determined the eigenvalues of the rotation matrix  $R_0^-$ , we can directly write the eigenvalues of matrix  $V_0^-$  which is the spin representation of  $R_0^-$  by invoking the equation (III.27). The eigenvalues of  $V_0^-$  (also of  $V^-$ ) are

$$\lambda^- = e^{(\pm\gamma_2 \pm \gamma_4 \pm \dots \pm \gamma_{2n})/2} \quad (\text{III.35})$$

Similarly, we define

$$\begin{aligned} V_0^+ &= V_1^{1/2} \cdot V_2^+ \cdot V_1^{1/2} \\ &= \prod_{r=1}^n \exp\left(-\frac{iH^*}{2} P_r Q_r\right) \cdot \prod_{r=1}^{n-1} \exp(iH' P_{r+1} Q_r) \cdot \exp(-iH' P_1 Q_n) \cdot \prod_{r=1}^n \exp\left(-\frac{iH^*}{2} P_r Q_r\right) \end{aligned} \quad (\text{III.36})$$

Then obviously,

$$V_0^+ = S(R_0^+) \quad (\text{III.37})$$

And the rotation matrix is

$$R_0^+ = \begin{pmatrix} a & b & 0 & 0 & 0 & . & . & . & 0 & -b^\dagger \\ b^\dagger & a & b & 0 & 0 & . & . & . & 0 & 0 \\ 0 & b^\dagger & a & b & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -b & 0 & 0 & 0 & 0 & . & . & . & b^\dagger & a \end{pmatrix}$$

The eigenvalues of  $R_0^+$  are comprised of the eigenvalues of the matrices

$$\alpha_{2r-1} = -\varepsilon^{(2r-1)(n-1)} b^\dagger + \varepsilon^{2r-1} b + a, \quad 1 \leq r \leq n, \quad \varepsilon = e^{i\pi/n} \quad (\text{III.38})$$

Because  $\det(\alpha_{2r-1}) = 1$ , the eigenvalues of  $\alpha_{2r-1}$  can be written as  $e^{\pm\gamma_{2r-1}}$ , and we have the equations of eigenvalues that resemble equation (III.34):

$$\begin{aligned} \frac{1}{2} \text{Tr}(\alpha_{2r-1}) &= \cosh \gamma_{2r-1} \\ &= \cosh 2H^* \cosh 2H' - \sinh 2H^* \sinh 2H' \cos((2r-1)\pi/n) \end{aligned} \quad (\text{III.39})$$

Then the eigenvalues of  $V_0^+$  (also of  $V^+$ ) are

$$\lambda^+ = e^{(\pm\gamma_1 \pm \gamma_3 \pm \dots \pm \gamma_{2n-1})/2} \quad (\text{III.40})$$

Hitherto, we have obtained the eigenvalues of  $V^+$  and  $V^-$ , now we shall begin to determine the

eigenvalues of  $V$  which is related to  $V^+$  and  $V^-$  via equation (III.17). It is easy to verify that

$$[U, V^+] = 0, [U, V^-] = 0, [(I+U) \cdot V^+, (I-U) \cdot V^-] = 0$$

Thus  $\frac{1}{2}(I+U)$  and  $V^+$  can be diagonalized simultaneously, and  $\frac{1}{2}(I-U)$  and  $V^-$  can be diagonalized simultaneously. Then

$$\begin{aligned} \frac{1}{2}(I+U) \cdot V^+ &= T_1^{-1} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \Lambda^+ T_1 \\ \frac{1}{2}(I-U) \cdot V^- &= T_2^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \Lambda^- T_2 \end{aligned}$$

Therefore,  $\frac{1}{2}(I+U)$  picks half of the eigenvalues of  $V^+$  and  $\frac{1}{2}(I-U)$  picks half of the eigenvalues of  $V^-$ . From this argument, we can conclude that  $V^+$  contributes  $2^{n-1}$  eigenvalues to  $V$  and  $V^-$  contributes the other  $2^{n-1}$  eigenvalues to  $V$ . It has already been determined by Kaufman<sup>[11]</sup> that half of the eigenvalues of  $V$  are of the form  $e^{(\pm\gamma_1 \pm \gamma_3 \pm \dots \pm \gamma_{2n-1})/2}$  and the other half of the eigenvalues of  $V$  are of the form  $e^{(\pm\gamma_2 \pm \gamma_4 \pm \dots \pm \gamma_{2n})/2}$ ; in each eigenvalue, only an even number of minus signs appears in the exponent.

At last, we get the exact partition function for the two-dimensional Ising model

$$\begin{aligned} Z &= (2 \sinh 2H)^{mn/2} \sum_{i=1}^{2^n} \lambda_i^m \\ &= (2 \sinh 2H)^{mn/2} \left( \sum e^{m(\pm\gamma_1 \pm \gamma_3 \pm \dots \pm \gamma_{2n-1})/2} + \sum e^{m(\pm\gamma_2 \pm \gamma_4 \pm \dots \pm \gamma_{2n})/2} \right) \end{aligned} \quad (\text{III.41})$$

Equation (III.41) can be written more compactly as

$$Z = \frac{1}{2} (2 \sinh 2H)^{mn/2} \left\{ \prod_{r=1}^n \left( 2 \cosh \frac{m}{2} \gamma_{2r} \right) + \prod_{r=1}^n \left( 2 \sinh \frac{m}{2} \gamma_{2r} \right) \right. \\ \left. + \prod_{r=1}^n \left( 2 \cosh \frac{m}{2} \gamma_{2r-1} \right) + \prod_{r=1}^n \left( 2 \sinh \frac{m}{2} \gamma_{2r-1} \right) \right\} \quad (\text{III.42})$$

The free energy per lattice site in the thermodynamic limit is defined as

$$F = -kT \lim_{m, n \rightarrow \infty} \frac{1}{mn} \log Z \quad (\text{III.43})$$

Then

$$-\frac{F}{kT} = \lim_{m, n \rightarrow \infty} \frac{1}{mn} \log Z = \frac{1}{2} \log 2 \sinh 2H + \lim_{n \rightarrow \infty} \frac{1}{n} \log \lambda_{\max} \quad (\text{III.44})$$

Now, we have only to determine the maximum value of the eigenvalues. Without loss of generality,

we assume here that all  $\gamma_k > 0$ . It is then obvious that

$$\lambda_{\max}^+ = e^{(\gamma_1 + \gamma_3 + \dots + \gamma_{2n-1})/2}$$

$$\lambda_{\max}^- = e^{(\gamma_2 + \gamma_4 + \dots + \gamma_{2n})/2}$$

As shown by Kaufman,  $\lambda_{\max}^-/\lambda_{\max}^+ \rightarrow 1$  for low temperature and  $\lambda_{\max}^-/\lambda_{\max}^+ < 1$  for high temperature. Thus,

$$-\frac{F}{kT} = \frac{1}{2} \log 2 \sinh 2H + \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{r=1}^n \gamma_{2r-1} \quad (\text{III.45})$$

In the special case when  $H = J/kT = H' = J'/kT$ , that is, the interaction is isotropic, we have

$$\cosh \gamma_{2r-1} = \coth 2H \cosh 2H - \cos \left[ (2r-1)\pi/n \right]$$

And

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{r=1}^n \gamma_{2r-1} \\ &= \frac{1}{4\pi} \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{2\pi}{n} \log \left[ \cosh \gamma_{2r-1} + \sqrt{\cosh^2 \gamma_{2r-1} - 1} \right] \\ &= \frac{1}{4\pi} \int_0^{2\pi} \log \left( \coth 2H \cosh 2H - \cos \omega + \sqrt{(\coth 2H \cosh 2H - \cos \omega)^2 - 1} \right) d\omega \end{aligned}$$

So, we get

$$-\frac{F}{kT} = \frac{1}{2} \log 2 \sinh 2H + \frac{1}{4\pi} \int_0^{2\pi} \gamma(\omega) d\omega \quad (\text{III.46})$$

Where

$$\cosh \gamma(\omega) = \coth 2H \cosh 2H - \cos \omega$$

Since  $\gamma(2\pi - \omega) = \gamma(\omega)$ , then equation (III.46) can be rewritten as

$$-\frac{F}{kT} = \frac{1}{2} \log 2 \sinh 2H + \frac{1}{2\pi} \int_0^\pi \gamma(\omega) d\omega \quad (\text{III.47})$$

To reconcile equation (III.47) with (II.21), we shall resort to the arguments by Kerson Huang<sup>[12]</sup>, according to whom, we have the identity

$$|\gamma| = \frac{1}{\pi} \int_0^\pi \log(2 \cosh \gamma - 2 \cos t) dt$$

Then, equation (III.47) can be simplified into

$$\begin{aligned} -\frac{F}{kT} &= \frac{1}{2} \log(2 \sinh 2H) + \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \log[2 \cosh(\gamma(\omega)) - 2 \cos t] dt d\omega \\ &= \log 2 + \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \log[\cosh^2 2H - \sinh 2H (\cos \omega + \cos t)] dt d\omega \end{aligned} \quad (\text{III.48})$$

When  $\omega = t = 2\pi$  and  $\sinh 2H = 1$ , the argument of the logarithmic function vanishes, and singularity is expected at this point. It will be shown that this indeed corresponds to phase transition, and the Curie temperature is determined by the equation

$$\sinh(2J/kT_c) = 1.$$

Equation (III.48) can be further simplified into

$$-\frac{F}{kT} = \log(2 \cosh(2H)) + \frac{1}{2\pi} \int_0^\pi \log \frac{1}{2} \left( 1 + \sqrt{1 - \kappa^2 \sin^2 \varphi} \right) d\varphi \quad (\text{III.49})$$

Here

$$\kappa = \frac{2}{\coth 2H \cosh 2H} = \frac{2}{\coth \frac{2J}{kT} \cosh \frac{2J}{kT}}$$

Now we again write  $\beta = 1/kT$ , and then the internal energy is

$$\begin{aligned} E &= \frac{\partial}{\partial \beta} (\beta F) \\ &= -J \coth 2\beta J \left[ 1 + \frac{2}{\pi} (2 \tanh^2 2\beta J - 1) \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \kappa^2 \sin^2 \varphi}} \right] \\ &\equiv -J \coth 2\beta J \left[ 1 + \frac{2}{\pi} \kappa' K_1(\kappa) \right] \end{aligned} \quad (\text{III.50})$$

Here, we have defined  $\kappa' = 2 \tanh^2 2\beta J - 1$ , and the first kind complete elliptical integral

$K_1(\kappa)$  is employed<sup>[13]</sup>,

$$K_1(\kappa) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \kappa^2 \sin^2 \varphi}}$$

The specific heat is

$$c = \frac{\partial E}{\partial T} = \frac{2k}{\pi} (\beta J \coth 2\beta J)^2 \left\{ 2K_1(\kappa) - 2E_1(\kappa) - (1 - \kappa') \left[ \frac{\pi}{2} + \kappa' K_1(\kappa) \right] \right\} \quad (\text{III.51})$$

Here,  $E_1(\kappa)$  is the complete elliptic integral of the second kind<sup>[13]</sup>:

$$E_1(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 \varphi} d\varphi$$

In the vicinity of the Curie point,  $\kappa = 1$  and the specific heat diverges logarithmically:

$$\frac{1}{k} c \approx \frac{2}{\pi} \left( \frac{2J}{kT_c} \right)^2 \left[ -\log \left| 1 - \frac{T}{T_c} \right| + \log \left( \frac{kT_c}{2J} \right) - \left( 1 + \frac{\pi}{4} \right) \right] \quad (\text{III.52})$$

From (III.50) and (III.52), we can see that near the Curie point the internal energy is continuous while the specific heat is logarithmically divergent, which demonstrates that the two-dimensional Ising model has undergone a second order phase transition in which no latent heat is released or absorbed.

Although the two-dimensional Ising model without an external field is solvable, the exact solution of this model with the presence of an applied magnetic field has yet to emerge. The spontaneous magnetization of two-dimensional Ising model which is usually defined as

$$M_s = \lim_{H \rightarrow 0^+} M(H)$$

has to be modified since we do not know the functional form the magnetization. The result for the spontaneous magnetization, which was first announced in 1948 by Onsager who never published the detailed calculation of it and was later calculated out by C. N. Yang in 1952<sup>[10]</sup> to whom this result is generally attributed, is:

$$M_s(T) = \begin{cases} 0 & \text{for } T \geq T_c \\ (1 - x^{-4})^{1/8} & \text{for } T < T_c \end{cases}$$

Here

$$x = \sinh(2\beta J).$$

In the immediate vicinity of the critical point, the spontaneous magnetization can be approximated as the power law:

$$M_s(T) = \left(1 - \sinh^{-4}\left(\frac{2J}{kT}\right)\right)^{1/8} \approx \left(-\frac{8\sqrt{2}J}{kT_c}\tau\right)^{1/8} = \left(-4\sqrt{2}\log(1 + \sqrt{2})\tau\right)^{1/8} \quad (\text{III.53})$$

We can see clearly that this differs substantially from the spontaneous magnetization equation (I.13) which is obtained via the crude molecular field theory.

The spontaneous magnetization in Ising model originates from the Pauli Exclusion Principle, which states that no two identical fermions can occupy the same state. Therefore in the crystal lattices, if two electrons were to occupy the same lattice site, they would have to possess opposite spins. But as two electrons approach each other, the Coulomb repulsion between them will increase indefinitely. So, in order to minimize the total energy of the system, the electrons would tend to repel each other, and Pauli Exclusion Principle dictates that when all the electrons have the same spin, they would automatically repel each other, from whence springs the sought-after ferromagnetism.

## IV. Conclusions

In this short paper, the origin of ferromagnetism has been explored first using the mean field approximation and then using the Ising model. The Ising model for ferromagnetism has been solved exactly both for one- and two-dimensional cases. From the calculation, we can see that the mean field approximation is adequate for a qualitative explanation for ferromagnetism, and a

quantitative analysis demands the solution of the more accurate Ising model. However, as we can see, the Ising model is effective only for the insulating or dielectric systems in which the electrons are localized around a lattice site. When the electrons are allowed to move among the crystal lattices, as in the case of metals, Ising model has to be replaced by the Hubbard model<sup>[14]</sup> which takes into account the kinetic energy and the Coulomb repulsions of the electrons. It was not until 1995<sup>[15]</sup> that a rigorous proof was given that, in certain lattices, spin-independent Coulomb interaction can give rise to ferromagnetism in itinerant electron systems. The recent observation of itinerant ferromagnetism in a Fermi gas of ultracold atoms<sup>[16]</sup> where the lattice structure is absent may have a profound influence on our understanding of ferromagnetism whose abysmal mystery and enigmatic beauty will continue to intrigue the scientists all over the world.

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