# Quantum Hall Effect and its Topological Aspects

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(Dated: April 22, 2013)

## Abstract

Quantum Hall Effect(QHE) will be discussed in this paper. A brief description of the experimental background will be given. Theoretical proof of quantization of Hall conductivity was first given by Thouless *et al* through the introduction of the TKNN formula. In this paper, the TKNN formula will be derived using Kubo formula for direct current conductivity. Later, a topological argument will be employed to show that quantum Hall conductivity is a natural result of Chern's theorem in global differentail geometry. From the relation between current vector and external vector potential, an effective action will be given from which the response of the electron gas to the external driving field can be derived naturally using variational calculus. It will aslo be proved that the effective action that we write down is gauge invariant, just as it should be.

## I. INTRODUCTION

Integer quantum Hall effect(QHE) was discovered more than 30 years ago[1], and its discovery inaugurated a new era for condensed matter physics. Shortly after the observation of quantized Hall conductance, a famous formula, which is now known as TKNN formula[2], was proposed to explain this bizarre phenomenon. It was realized that topology played an important role in QHE, and the quantization of Hall conductance is actually a new kind of quantization, which is called topological quantization. The quantization of magnetic charge in the monopole model first studied by Dirac[3] was perhaps the first known example of topological quantization. With the actual observation of QHE, the insight of Dirac, which was further developed by C.N.Yang and T.S.WU[4], was firmly corroborated that topology should play a critical role in the study of physics.

Geometric phase in physics was first fleshed out by Berry in 1983[5], and immediately after its discovery association was made between QHE and Berry curvature[6]. It was shown that quantization of Hall conducatance is a corollary of a beautiful theorem in global differential geometry, which states that the integration of curvature on a closed two dimensional surface(or a manifold, to be more precise mathemtically)should be an integer multiple of $2\pi$. This theorem, known as Gauss-Bonnet theorem, was greatly generalized by S.S.Chern through his introduction of Chern characteristic classes and Chern numbers[7] for manifolds whose dimension could be an arbitrary integer number $n$. In this paper, it will be shown that the celebrated quantum Hall conductance is essentially $\frac{e^2}{h}$ times the first Chern number[8], which has already been proved by Chern to be an integer about forty years before the discovery of QHE.

As was noted repeatedly, the low energy effective action for QHE is Chern-Simons theory, from which all the physical properties of QHE can be readily derived[9]. In classical field theory, e.g., in electrodynamics and general relativity, we all work from the Lagrangian and then use it to derive all the results that of interest to us. But in condensed matter physics, we work in the reverse order. Here we have already known the so-called "Theory of Everything" in condensed matter systems, it is just the Schrodinger's equation. We then derive the low energy effective field theory from microscopic interactions between particles. That is, we derived the Lagrangian, rather than get it from guesswork(based on some basic principles of symmetry, of course), as was done in classical and quantum field theory.

The structure of this paper is as follows. In section II, a brief review of experimental results for QHE will be given. TKNN foromula for Hall conductivity will be derived in section III, and the proof that this Hall conductivity should be quantized is given in section IV. In section V, effective Lagrangian for QHE is derived, and its gauge invariance will be proved. Conclusion is in section VI.

## II. EXPERIMENTAL RESULTS FOR QHE

Quantum Hall effect was discovered by Klitzing in 1980 using metal-oxide-semiconductor field-effect transistor(MOSFET), at low temperature (the temperature of liquid Helium), and with application of strong magnetic field ($\simeq 15T$). A schematic diagram[10] for the experiment setup is shown in FIG. 1.

In the emergence of QHE, it is observed that longitudinal resistance $\rho_{xx}$ drops to zero, while transversal resistance $\rho_{xy}$ stays on a plateau, as shown in FIG. 2.

The importance of QHE lies in the fact that it provides an accurate determination of fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ whose physical significance can never be overestimated. Because fine structure constant aries in the context of relativistic quantum mechanics, traditionally the determination of it is always within the scope of relativistic physics, e.g., by measuring the anomalous magnetic moment of the electron, etc. QHE provides an alternative measurement of $\alpha$ with precision that can easily eclipse that obtained in quantum elctrodynamics by calculainting increasingly complicated Feynman diagrams. Because Hall conductivity is proportional to $\frac{e^2}{h}$, which will be proved in this paper, fine structure constant can readily be obtained once we know the speed of light, the value of which has already been exactly define

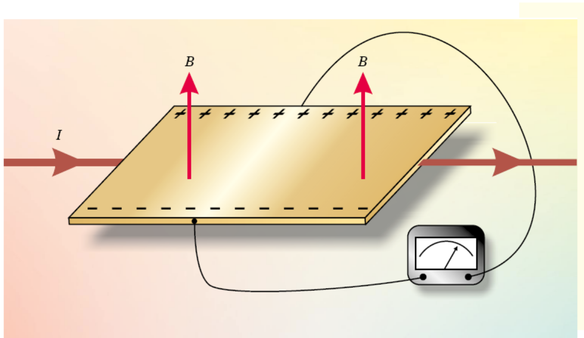


FIG. 1. A schematic diagram for the devices used for observation of QHE. Here 2D electron gas is restricted within a thin layer of MOSFET to which a perpendicular magnetic field is applied. Current flow in the $x$ direction is driven by an external electric field. We measure the current in the $y$ direction to get information about Hall conductivity.

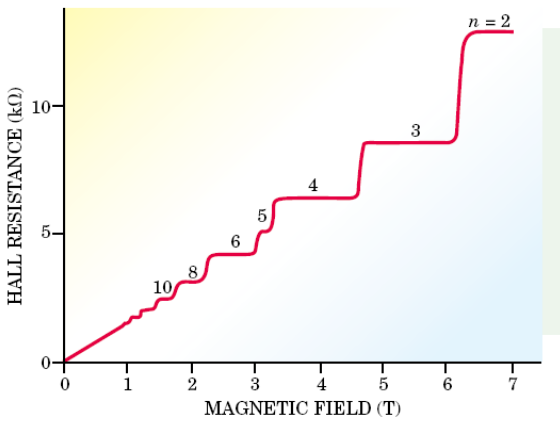


FIG. 2. This figure shows the change of transversal resistance with respect to perpendicular magnetic field strength.

in SI units. Another academic field that greatly benefits from the discovery of QHE is metrology. Since QHE gives us a highly precise method of determing resistance, we can use $\frac{e^2}{h} = 1/(25812.807\Omega)^{10}$ as a standard for measuring resistance. Klitzing was awarded Nobel prize for physics in 1985 for his remarkable contribution to both theoretical and applied physics.

## III. DERIVATION OF TKNN FORMULA

Electromagnetic field is required for the emergence of QHE. According to Maxwell's equations, $\mathbf{B} = \nabla\times\mathbf{A}, \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$. Here we assume there is no scalar potential for the electric field. The presense of vector potential $\mathbf{A}$ will modify Schrödinger's equation in the usual way, that is, $\boldsymbol{p} \to \boldsymbol{p} + e\mathbf{A}$. Thus, Schrodinger's equation coupled to electromagnetic field is:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}(\boldsymbol{p} + e\mathbf{A})|\psi(t)\rangle. \tag{1}$$

If we write the Hamiltonian in momentum space, then

$$\mathcal{H}(\boldsymbol{p} + e\mathbf{A}) \to \mathcal{H}(\boldsymbol{k} + \frac{e}{\hbar}\mathbf{A}). \tag{2}$$

Now we can expand the Hamiltonian in terms of vector potential $\mathbf{A}$ to linear order:

$$\mathcal{H}(\boldsymbol{k} + \frac{e}{\hbar}\mathbf{A}) \simeq \mathcal{H}(\boldsymbol{k}) + \frac{e}{\hbar}\mathbf{A}\cdot\frac{\partial\mathcal{H}}{\partial\boldsymbol{k}}. \tag{3}$$

Now we can identify the vector that couples to vector potential $\mathbf{A}$ as the current induced by external electromagnetic field, that is, we define the current vector as:

$$\boldsymbol{j} = \frac{e}{\hbar}\frac{\partial\mathcal{H}}{\partial\boldsymbol{k}}. \tag{4}$$

The actual measured current would be the expectation value of $\boldsymbol{j}$. Assume the system is in state $|\psi_n(t)\rangle$, then the expectation value of $\boldsymbol{j}$ is $\langle\psi_n(t)|\boldsymbol{j}|\psi_n(t)\rangle$. Here, $|\psi_n(t)\rangle$ satisfies Schrodinger's equation with the Hamiltonian expanded to first order:

$$i\hbar\frac{\partial}{\partial t}|\psi_n(t)\rangle = (\mathcal{H}(\boldsymbol{k}) + \frac{e}{\hbar}\mathbf{A}\cdot\frac{\partial\mathcal{H}}{\partial\boldsymbol{k}})|\psi_n(t)\rangle. \tag{5}$$

We denote the free part of the Hamiltonian as $\mathcal{H}_0 = \mathcal{H}(\boldsymbol{k})$, and the interaction part of the Hamiltonian as $\mathcal{H}_I = \frac{e}{\hbar}\mathbf{A}\cdot\frac{\partial\mathcal{H}}{\partial\boldsymbol{k}}$. Then the above equation can be written as:

$$i\hbar\frac{\partial}{\partial t}|\psi_n(t)\rangle = (\mathcal{H}_0 + \mathcal{H}_I)|\psi_n(t)\rangle. \tag{6}$$

Now we assume that the wave function $|\psi_n(t)\rangle$ can be written as $|\psi_n(t)\rangle = e^{-i\mathcal{H}_0 t/\hbar}|\phi_n(t)\rangle$, then the above equation simplifies into the form below:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\phi_n(t)\rangle &= e^{i\mathcal{H}_0 t/\hbar}\mathcal{H}_I e^{-i\mathcal{H}_0 t/\hbar}|\phi_n(t)\rangle \\ &\doteq \tilde{\mathcal{H}}_I(t)|\phi_n(t)\rangle. \end{aligned} \tag{7}$$

An iterative solution to the above equation is

$$|\phi_n(t)\rangle = |\phi_n(-\infty)\rangle - \frac{i}{\hbar}\int_{-\infty}^{t} dt'\tilde{\mathcal{H}}_I(t')|\phi_n(t')\rangle. \tag{8}$$

Here we have assumed that we are adiabatically turning on the interaction term from $t = -\infty$ to current time $t$. Therefore when $t = -\infty$ there is no interaction and thus $|\phi_n(-\infty)\rangle$ is an eigenstate of the free Hamiltonian, which will be denoted as $|u_n\rangle$ from now on. To lowest order perturbation. we can approximate $|\phi_n(t)\rangle$ as

$$|\phi_n(t)\rangle = |u_n\rangle - \frac{i}{\hbar}\int_{-\infty}^{t} dt'\tilde{\mathcal{H}}_I(t')|u_n\rangle. \tag{9}$$

Denoting $S(t,-\infty) \doteq -\frac{i}{\hbar}\int_{-\infty}^{t} dt' \tilde{\mathcal{H}}_I(t')$, we can write the wave function as

$$
\begin{aligned}
|\psi_n(t)\rangle &= e^{-i\mathcal{H}_0 t/\hbar}|\phi_n(t)\rangle \\
&= e^{-i\mathcal{H}_0 t/\hbar}(|u_n\rangle + S(t,-\infty)|u_n\rangle).
\end{aligned} \tag{10}
$$

The expectation value of current vector is

$$
\begin{aligned}
\langle \boldsymbol{j}(t)\rangle_n &= \langle \psi_n(t)|\boldsymbol{j}|\psi_n(t)\rangle \\
&= \langle \phi_n(t)|\boldsymbol{j}(t)|\phi_n(t)\rangle \\
&\simeq \frac{i}{\hbar}\int_{-\infty}^{t} dt' \langle u_n|[\tilde{\mathcal{H}}_I(t'), \boldsymbol{j}(t)]|u_n\rangle.
\end{aligned} \tag{11}
$$

In the above equation, we have assumed that $\langle u_n|\boldsymbol{j}(t)|u_n\rangle = 0$ which is reasonable because we will generally not expect to observe spontaneous current flow in the absense of external field. The total current at zero temperature would be the sum of all current expectation values for states whose energy is below Fermi energy, that is,

$$
\langle j_\alpha(t)\rangle = \frac{i}{\hbar}\sum_{\epsilon_n<\epsilon_F}\int_{-\infty}^{t} dt' \langle u_n|[\tilde{\mathcal{H}}_I(t'), \boldsymbol{j}(t)]|u_n\rangle \tag{12}
$$

Fourier transforming the current, we have

$$
\begin{aligned}
\langle j_\alpha(\omega)\rangle &= \int_{-\infty}^{\infty} dt e^{i\omega t}\langle j_\alpha(t)\rangle \\
&= \sum_{m,\epsilon_n<\epsilon_F} \frac{\langle \mathcal{H}_I(\omega)\rangle_{nm}\langle j_\alpha\rangle_{mn}}{\epsilon_n-\epsilon_m-\hbar\omega-i\xi} \\
&\quad + \frac{\langle \mathcal{H}_I(\omega)\rangle_{mn}\langle j_\alpha\rangle_{nm}}{\epsilon_n-\epsilon_m+\hbar\omega+i\xi}
\end{aligned} \tag{13}
$$

We have already known that the interaction part of the Hamiltonian is $\mathcal{H}_I = \mathbf{A}(t)\cdot\boldsymbol{j}$, thus the fourier transform of $\mathcal{H}_I$ is

$$
\begin{aligned}
\mathcal{H}_I(\omega) &= \mathbf{A}(\omega)\cdot\boldsymbol{j} \\
&= \frac{1}{i\omega}\mathbf{E}(\omega)\cdot\boldsymbol{j} \\
&= \frac{1}{i\omega}E_\beta(\omega)j_\beta
\end{aligned} \tag{14}
$$

Plugging the component expression for $\mathcal{H}_I(\omega)$ into the equation for current, we get

$$
\langle j_\alpha(\omega)\rangle = \frac{K_{\alpha\beta}(\omega)}{i\omega}E_\beta, \tag{15}
$$

where

$$
\begin{aligned}
K_{\alpha\beta}(\omega) &= \sum_{m,\epsilon_n<\epsilon_F} \frac{\langle j_\beta\rangle_{nm}\langle j_\alpha\rangle_{mn}}{\epsilon_n-\epsilon_m-\hbar\omega-i\xi} \\
&\quad + \frac{\langle j_\beta\rangle_{mn}\langle j_\alpha\rangle_{nm}}{\epsilon_n-\epsilon_m+\hbar\omega+i\xi}
\end{aligned} \tag{16}
$$

We are now interested in the direct current limit, that is, we should take $\omega \to 0$ while calculating the conductivity. When $\omega \to 0$, we have

$$
\begin{aligned}
\lim_{\omega\to 0} K_{\alpha\beta} &= \sum_{m,\epsilon_n<\epsilon_F} \frac{\langle j_\beta\rangle_{nm}\langle j_\alpha\rangle_{mn}}{\epsilon_n-\epsilon_m} \\
&\quad + i\pi\langle j_\beta\rangle_{nm}\langle j_\alpha\rangle_{mn}\delta(\epsilon_n-\epsilon_m) \\
&\quad + \frac{\langle j_\beta\rangle_{mn}\langle j_\alpha\rangle_{nm}}{\epsilon_n-\epsilon_m} \\
&\quad - i\pi\langle j_\beta\rangle_{mn}\langle j_\alpha\rangle_{nm}\delta(\epsilon_n-\epsilon_m) \\
&= \sum_{m,\epsilon_n<\epsilon_F} \frac{\langle j_\beta\rangle_{nm}\langle j_\alpha\rangle_{mn} + (n\leftrightarrow m)}{\epsilon_n-\epsilon_m}.
\end{aligned} \tag{17}
$$

We see that in the direct current limit, $K_{\alpha\beta}$ is a pure real number. Since we are only interested in the real part of the conductivity, we wish to isolate the imaginary part of $K_{\alpha\beta}$, which, in the $\omega \to 0$, is

$$
\mathrm{Im}K_{\alpha\beta}(\omega\to 0) = \lim_{\omega\to 0} K_{\alpha\beta}(\omega) - K_{\alpha\beta}(0). \tag{18}
$$

Therefore, real part of the conductivity in direct current limit is

$$
\begin{aligned}
\sigma_{\alpha\beta} &= \lim_{\omega\to 0}\frac{K_{\alpha\beta}(\omega)-K_{\alpha\beta}(0)}{i\omega} \\
&= \lim_{\omega\to 0}\frac{\partial K_{\alpha\beta}}{i\partial\omega} \\
&= i\hbar\sum_{m,\epsilon_n<\epsilon_F}\frac{\langle j_\alpha\rangle_{nm}\langle j_\beta\rangle_{mn} - (n\leftrightarrow m)}{(\epsilon_n-\epsilon_m)^2}
\end{aligned} \tag{19}
$$

As has already been noted, $j_\alpha = \frac{e}{\hbar}\frac{\partial\mathcal{H}}{\partial k_\alpha}$, therefore we have

$$
\langle j_\alpha\rangle_{nm} = \frac{e}{\hbar}\langle u_n|\frac{\partial\mathcal{H}}{\partial k_\alpha}|u_m\rangle. \tag{20}
$$

Since $|u_m\rangle$ is the eigenstate of free Hamiltonian, we have $\mathcal{H}_0|u_m\rangle = \epsilon_m|u_m\rangle$. Diffrentiating both sides with respect to $k_\alpha$, and then taking inner product with $\langle u_n|$, we obtain

$$
\langle j_\alpha\rangle_{nm} = \frac{e}{\hbar}(\epsilon_m-\epsilon_n)\langle u_n|\frac{\partial u_m}{\partial k_\alpha}\rangle + \frac{e}{\hbar}\delta_{nm}\frac{\partial\epsilon_m}{\partial k_\alpha}. \tag{21}
$$

Plugging $\langle j_\alpha\rangle_{nm}$ back to the equation for $\sigma_{\alpha\beta}$ and notice that $n \neq m$, we have

$$
\begin{aligned}
\sigma_{\alpha\beta} &= \frac{ie^2}{\hbar}\sum_{m,\epsilon_n<\epsilon_F}\langle u_m|\frac{\partial u_n}{\partial k_\alpha}\rangle\langle u_n|\frac{\partial u_m}{\partial k_\beta}\rangle \\
&\quad - \langle u_n|\frac{\partial u_m}{\partial k_\alpha}\rangle\langle u_m|\frac{\partial u_n}{\partial k_\beta}\rangle.
\end{aligned} \tag{22}
$$

Because of orthgonality of eigenstates, we have $\langle u_m|u_n\rangle = \delta_{mn}$. After differentiation with respect to $k_\alpha$, we have

$$
\langle\frac{\partial u_m}{\partial k_\alpha}|u_n\rangle + \langle u_m|\frac{\partial u_n}{\partial k_\alpha}\rangle = 0. \tag{23}
$$

As a result of this, the equation for $\sigma_{\alpha\beta}$ can be further simplified as

$$
\sigma_{\alpha\beta} = \frac{ie^2}{\hbar}\sum_{\epsilon_n<\epsilon_F}\langle\frac{\partial u_n}{\partial k_\alpha}|\frac{\partial u_n}{\partial k_\beta}\rangle - \langle\frac{\partial u_n}{\partial k_\beta}|\frac{\partial u_n}{\partial k_\alpha}\rangle. \tag{24}
$$

Here use has been made of the completeness relation $\sum_m |u_m\rangle\langle u_m| = 1$. Equation (24) is the famous TKNN formula for Hall conductivity. To get the Hall conductivity $\sigma_{xy}$ that was measured in experiment, we have to integrate over all possible $k_x$ and $k_y$. That is,

$$\sigma_{xy} = A\frac{ie^2}{\hbar}\sum_{\epsilon_n<\epsilon_F}\int\frac{dk_x dk_y}{(2\pi)^2}\mathcal{F}_{xy}, \tag{25}$$

where $A$ is the area of the device, and $\mathcal{F}_{xy}$ is

$$\mathcal{F}_{xy} = \langle\frac{\partial u_n}{\partial k_x}|\frac{\partial u_n}{\partial k_y}\rangle - \langle\frac{\partial u_n}{\partial k_y}|\frac{\partial u_n}{\partial k_x}\rangle. \tag{26}$$

This is the final result for the derivation of the formula for Hall conductivity. In the next section, we will resort to topological argument to show that the Hall conductivity is actually an integer multiple of $\frac{e^2}{h}$.

## IV. HALL CONDUCTIVITY AND FIRST CHERN NUMBER

To associate Hall conductivity with Chern number, we first consider a simple case, where the system is fully described by a Hamiltonian $\mathcal{H}(\alpha)$ that depends on an array of parameters $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$. Assume that the system is in an eigenstate $|\phi_\alpha\rangle$, which, as we naturally expect, also depends on parameter $\alpha$ and satisfies Schrodinger equation $\mathcal{H}(\alpha)|\phi_\alpha\rangle = \epsilon_\alpha|\phi_\alpha\rangle$. Now we adiabatically change the parameter $\alpha$ along a closed path in parameter space such that $\alpha(t=0) = \alpha(t=T)$, and at any instant during the time interval $0 \leq t \leq T$ the state $|\phi_{\alpha_t}\rangle$ is always an eigenstate of the Hamiltonian $\mathcal{H}(\alpha_t)$. At the end of the path, $\alpha$ returns to its initial value, so does the Hamiltonian whose invariance after the adiabatic process dictates that $|\phi_{\alpha_T}\rangle$ should be equivalent to the initial state $|\phi_{\alpha_0}\rangle$. Here by equivalent we mean that these two states can differ by at most up to a phase factor $\gamma$, and the main task of Berry's investigation[5] is just to identify this phase factor. The evolution of the state in time will generate a dynamical phase factor automatically. Therefore Berry phase should be identified as the part of the phase that is not due to dynamics. We can write the evolving state as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t d\tau\epsilon_{\alpha_\tau}}e^{i\gamma_{\alpha_t}}|\phi_{\alpha_t}\rangle. \tag{27}$$

Inserting this into time-dependent Schrodinger's equation yields Berry's equation for quantal phase factor $\gamma$:

$$\frac{d\gamma_\alpha}{dt} = i\langle\phi_\alpha|\frac{\partial\phi_\alpha}{\partial\alpha}\rangle\frac{d\alpha}{dt}. \tag{28}$$

Solution of the above equation gives us Berry phase:

$$\gamma_\alpha = i\oint_C\langle\phi_\alpha|\frac{\partial\phi_\alpha}{\partial\alpha}\rangle d\alpha. \tag{29}$$

We can reformulate Berry's result in the language of geometry as follows[11]. Using differential forms, equation (28) can be re-written as

$$d\gamma_\alpha = i\langle\phi_\alpha|\frac{\partial\phi_\alpha}{\partial\alpha}\rangle d\alpha \doteq i\omega. \tag{30}$$

In the above equation, we have defined in parameter space the connection 1-form $\omega$ as

$$\omega = \langle\phi_\alpha|\frac{\partial\phi_\alpha}{\partial\alpha}\rangle d\alpha = \langle\phi_\alpha|d\phi_\alpha\rangle. \tag{31}$$

We now define the covariant exterior differentiation of state $|\phi_\alpha\rangle$ along the closed path $C$ as

$$\nabla|\phi_\alpha\rangle = |\phi_\alpha\rangle\omega. \tag{32}$$

In the language of geometry, Berry's original equation (28) can be re-interpreted as the condition for realizing parallell displacement of $e^{\gamma_\alpha}|\phi_\alpha\rangle$ along the closed path $C$, that is,

$$\begin{aligned}\nabla(e^{i\gamma_\alpha}|\phi_\alpha\rangle) &= \nabla e^{i\gamma_\alpha}|\phi_\alpha\rangle + e^{i\gamma_\alpha}\nabla|\phi_\alpha\rangle \\ &= e^{i\gamma_\alpha}id\gamma_\alpha|\phi_\alpha\rangle + e^{i\gamma_\alpha}|\phi_\alpha\rangle\omega \\ &= e^{i\gamma_\alpha}|\phi_\alpha\rangle(id\gamma_\alpha + \omega) \\ &= 0\end{aligned} \tag{33}$$

The equivalence of equations (30) and (33) is obvious.

We now show that Berry phase is proportional to the integral of curvature in parameter space. According to Cartan, the curvature 2-form $\theta$ of connection $\omega$ is

$$\theta = d\omega + \omega\wedge\omega. \tag{34}$$

From equation (31), we find our curvatue as

$$\theta = d\omega = \langle d\phi_\alpha|d\phi_\alpha\rangle, \tag{35}$$

since $\omega\wedge\omega = 0$ in this case.

It is a remarkable fact that although connection 1-form $\omega$ depends on the local choice of phase for states, the curvature does not. Actually, had we made a phase transformation for state $|\phi_\alpha\rangle$, that is, $|\phi_\alpha\rangle \to e^{i\Lambda(\alpha)}|\phi_\alpha\rangle$, $\omega$ would transform as

$$\omega \to \omega' = \omega + id\Lambda. \tag{36}$$

However, curvature $\theta$ is invariant under this change of phase, as is clear that

$$\theta' = d\omega' = d\omega = \theta, \tag{37}$$

thanks to Poincare lemma: $d^2 = 0$. The invariance of $\theta$ under local change of phase signifies that the Berry curvature is an intrinsic quantity and thus should be a physical observable.

Using diffrential forms and Stokes theorem, equation (29) can be converted into an integral over a surface $S$

that spans the closed path $C$, that is,

$$\gamma_\alpha = i \oint_C \omega \tag{38}$$
$$= i \int\int_S d\omega = i \int\int_S \theta$$
$$= i \int\int_S \langle d\phi_\alpha | d\phi_\alpha \rangle .$$

In the case of quantum Hall effect, all we have to do is to replace $\alpha$ with $\boldsymbol{k} = (k_x, k_y)$ and then write out $d|\phi_\alpha\rangle$ as

$$|d\phi_{\boldsymbol{k}}\rangle = |\frac{\partial \phi_{\boldsymbol{k}}}{\partial k_x}\rangle dk_x + |\frac{\partial \phi_{\boldsymbol{k}}}{\partial k_y}\rangle dk_y. \tag{39}$$

Consequently, curvature $\theta$ is

$$\theta = \langle d\phi_{\boldsymbol{k}} | d\phi_{\boldsymbol{k}} \rangle \tag{40}$$
$$= (\langle \frac{\partial \phi_{\boldsymbol{k}}}{\partial k_x} | \frac{\partial \phi_{\boldsymbol{k}}}{\partial k_y} \rangle - \langle \frac{\partial \phi_{\boldsymbol{k}}}{\partial k_y} | \frac{\partial \phi_{\boldsymbol{k}}}{\partial k_x} \rangle) dk_x \wedge dk_y.$$

In terms of curvature, equation (25) for Hall conductivity can be transformed to

$$\sigma_{xy} = A \frac{e^2}{h} \frac{i}{2\pi} \sum_{\epsilon_n < \epsilon_F} \int\int \theta_n. \tag{41}$$

In the calculation of quantum Hall conductivity, we have used periodic boundary condition, and thus the surface $S$ over which we integrate the curvature should be considered as a closed torus $T^2$. According to Gauss-Bonnet-Poincare-Chern theorem,

$$\frac{i}{2\pi} \int \theta = C_1 = integer, \tag{42}$$

where $C_1$ is the first Chern number which must be an integer. Therefore, we finally proved the quantization of Hall conductivity, that is,

$$\frac{\sigma_{xy}}{A} = n \frac{e^2}{h}. \tag{43}$$

## V. EFFECTIVE ACTION FOR QUANTUM HALL EFFECT

In the previous sections, we have already proved the TKNN formula for Hall conductivity. After averaging in momentum space, we can rewrite Equation (24) as follows:

$$\sigma_{\alpha\beta} = A \frac{ie^2}{\hbar} \sum_{\epsilon_n < \epsilon_F} \int \frac{dk_x dk_y}{(2\pi)^2} \mathcal{F}_{\alpha\beta}, \tag{44}$$

where

$$\mathcal{F}_{\alpha\beta} = \langle \frac{\partial u_n}{\partial k_\alpha} | \frac{\partial u_n}{\partial k_\beta} \rangle - \langle \frac{\partial u_n}{\partial k_\beta} | \frac{\partial u_n}{\partial k_\alpha} \rangle. \tag{45}$$

It is clear that in the case where $\alpha = \beta$, the integrand in equation (44) would vanish identically, thus the diagonal part of the conductivity tensor $\sigma_{\alpha\beta}$ is always equal to 0. This reminds us of the completely anti-symmetric tensor $\epsilon^{ij}$ in two dimensional space, whose definition is $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{ij} = 0$ otherwise. Using this notation, we can write the relation between Hall current and external field succintly as:

$$j_\alpha = \sigma_{\alpha\beta} E_\beta \tag{46}$$
$$= \sigma_H \epsilon^{\alpha\beta} E_\beta, \alpha, \beta = 1, 2$$

where $\sigma_H = An\frac{e^2}{h}$ is the Hall conductivity, as already proved in equation(43).According to Maxwell equation, electric field can be experssed in terms of vector potential, that is,

$$E_\beta = -\frac{\partial A_\beta}{\partial t} \doteq -\partial_0 A_\beta, \tag{47}$$

where we have used the convention that $x = (x^0, \boldsymbol{x}) = (x^0, x^1, x^2) = (t, x, y)$. As a result of this, equation (46) can be recast into a form that is convenient for later study:

$$j^i = -\sigma_H \epsilon^{ij} \partial_0 A_j. \tag{48}$$

From the charge conservation equation

$$\partial_0 \rho + \nabla \cdot \boldsymbol{j} = 0, \tag{49}$$

we have

$$\partial_0 \rho = -\partial_i j^i \tag{50}$$
$$= -\sigma_H \epsilon^{ij} \partial_i E_j$$
$$= -\sigma_H (\nabla \times \mathbf{E})_z$$

Plugging Maxwell equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ into the above equation, we have

$$\partial_0 \rho = \sigma_H \partial_0 \mathbf{B}_z. \tag{51}$$

That is, $\partial_0(\rho - \sigma_H B_z) = 0$, from which we have

$$\rho(B) - \rho_0 = \sigma_H B. \tag{52}$$

Here, we have used the fact that $B = B_z$ since magnetic field lies parallel to z axis in our case. Now assume that $\rho_0 = 0$, that is, when there is no applied magnetic field, there should be no charge accumulation on the ground state, then equation (52) becomes

$$\rho(B) = \sigma_H (\partial_1 A_2 - \partial_2 A_1). \tag{53}$$

Denoting $j = (j^0, \boldsymbol{j}) = (\rho, j^1, j^2)$, we can combine equations (48) and (53) into a much more compact form as:

$$j^\mu = \sigma_H \epsilon^{\mu\nu\tau} \partial_\nu A_\tau, \mu, \nu, \tau = 0, 1, 2 \tag{54}$$

where we have used the completely anti-symmtric tensor $\epsilon^{\mu\nu\tau}$ in (2+1) dimensional space, and defined the generalized vector potential $A_\mu = (A_0, \boldsymbol{A}) = (\phi, A_1, A_2)$. We also have made the assumption that the scalar potential $\phi = 0$, an assumption that we have already used in section III for the derivation of TKNN formula.

Having botained the general formula for the relation between current density $j^\mu$ and generalized vector potential $A_\mu$, we are now ready to write down the effective action for this system. According to the definition of current density in field theory, current vector can be obtained by taking the variation of effetive action with respect to vector potentional, that is,

$$j^\mu = \frac{\delta S_{eff}}{\delta A_\mu}. \tag{55}$$

It can be shown that an effective action like

$$\begin{aligned} S_{eff} &= \frac{\sigma_H}{2} \int\int d^2x dt \mathcal{L} \\ &= \frac{\sigma_H}{2} \int\int d^2x dt \epsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau \end{aligned} \tag{56}$$

will fulfill the job. To see this, take variation of the effective action $S_{eff}$, and we get

$$\delta S_{eff} = \frac{\sigma_H}{2} \int\int d^2x dt \delta\mathcal{L}, \tag{57}$$

where

$$\begin{aligned} \delta\mathcal{L} &= \delta(\epsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau) \\ &= \epsilon^{\mu\nu\tau} \delta A_\mu \partial_\nu A_\tau + \epsilon^{\mu\nu\tau} A_\mu \partial_\nu \delta A_\tau \end{aligned} \tag{58}$$

The second term in the second line of equation (58) can be rewritten as

$$\begin{aligned} \epsilon^{\mu\nu\tau} A_\mu \partial_\nu \delta A_\tau &= \partial_\nu(\epsilon^{\mu\nu\tau} A_\mu \delta A_\tau) - \epsilon^{\mu\nu\tau} \partial_\nu A_\mu \delta A_\tau \\ &= \partial_\nu(\epsilon^{\mu\nu\tau} A_\mu \delta A_\tau) - \epsilon^{\tau\nu\mu} \partial_\nu A_\tau \delta A_\mu. \end{aligned} \tag{59}$$

Thus we obtain the variation of Lagrangian $\mathcal{L}$, that is,

$$\delta\mathcal{L} = 2\epsilon^{\mu\nu\tau} \partial_\nu A_\tau \delta A_\mu + \partial_\nu(\epsilon^{\mu\nu\tau} A_\mu \delta A_\tau). \tag{60}$$

Plugging the above equation back into equation (57), we finally get

$$\delta S_{eff} = \sigma_H \int\int d^2x dt \epsilon^{\mu\nu\tau} \partial_\nu A_\tau \delta A_\mu, \tag{61}$$

where we have already discarded the boundary term $\partial_\nu(\epsilon^{\mu\nu\tau} A_\mu \delta A_\tau)$. Thus we finally proved the desired result, that is,

$$j^\mu = \frac{\delta S_{eff}}{\delta A_\mu} = \sigma_H \epsilon^{\mu\nu\tau} \partial_\nu A_\tau. \tag{62}$$

As we have known in electrodynamics, physical quantities that are described by vector potential should have instrinsic meaning, that is, these physical quantities should be indepdent of the gauge we choose for the vector potential. The effective action $S_{eff}$ from which the physical relationship between current vector and external field is derived should thus be gauge invariant and its gauge invariance can be proved as follows.

The external field that we are applying is the electromagnetic field, which is an Abelian $U(1)$ field, therefore under a gauge transformation the vector potential transforms as

$$A_\mu \to A'_\mu = A_\mu + i\partial_\mu \Lambda, \tag{63}$$

where $\Lambda$ is a scalar function. With such a transformation,

$$\begin{aligned} \mathcal{L} \to\ & \epsilon^{\mu\nu\tau}(A_\mu + i\partial_\mu\Lambda)(\partial_\nu A_\tau + i\partial_\nu\partial_\tau\Lambda) \\ =\ & \epsilon^{\mu\nu\tau}(A_\mu \partial_\nu A_\tau + i\partial_\mu\Lambda\partial_\nu A_\tau) \\ =\ & \epsilon^{\mu\nu\tau}(A_\mu \partial_\nu A_\tau + i\partial_\nu(\partial_\mu\Lambda A_\tau) - i\partial_\mu\partial_\nu\Lambda A_\tau) \\ =\ & \epsilon^{\mu\nu\tau}(A_\mu \partial_\nu A_\tau + i\partial_\nu(\partial_\mu\Lambda A_\tau)), \end{aligned} \tag{64}$$

and the effective action transforms as

$$\begin{aligned} S_{eff} &\to \frac{\sigma_H}{2} \int\int d^2x dt \epsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau \\ &= S_{eff}. \end{aligned} \tag{65}$$

Therefore, if we can discard the boundary term in the effective action integral, then $S_{eff}$ is gauge invariant, just as we expected.

If we now define connection 1-form $\omega^1 = A_\mu dx^\mu$ in (2+1) diemensional spacetime, then we have

$$\begin{aligned} \omega^1 \wedge d\omega^1 &= A_\mu \partial_\nu A_\tau dx^\mu \wedge dx^\nu \wedge dx^\tau \\ &= \epsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau dx^1 \wedge dx^2 \wedge dx^0 \end{aligned} \tag{66}$$

In differential geometry, Chern-Simons 3-form is defined as[11,12]

$$\mathcal{C}^3 = \mathrm{Tr}(\omega^1 \wedge d\omega^1 + \frac{2}{3}\omega^1 \wedge \omega^1 \wedge \omega^1). \tag{67}$$

In Abelian $U(1)$ theory, which is just the case we are studying, connection 1-form $\omega^1$ is a commutative $1 \times 1$ matrix, and Chern-Simons 3-form in case reduces simply to

$$\mathcal{C}^3 = \omega^1 \wedge d\omega^1, \tag{68}$$

which, as has already been proved in equation (66), is nothing but the Lagrangian we got in equation (56). As a result of this, the effective field theory for QHE was named Chern-Simons theory.

## VI. CONCLUSION

In this paper, a concise description of expertimental setup for QHE was given, and then TKNN formula for Hall conductivity was derived from basic principles of quantum mechanics. Relationship was established between Berry curvature in parameter space and TKNN formula. It was shown that quantization of Hall conductivity is due to the fact that the first Chern number for a closed manifold has to be an integer, which makes QHE an interesting example of application of topology and geometry to condensed matter physics. Finally, the effective field theory for QHE was derived and was shown to be gauge invariant, and this effetive field theory is precisely the Chern-Simons theory in (2+1) dimensions.

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