FUNCTIONS OF A RANDOM VARIABLE

Suppose that *X* is the random variable representing the radius of a disc and this variable follows the *pdf* *f*(*x*). Let *A* represents the area of this disc. We can see that if *X* is a random variable so should be *A*. Here *A* is a function of *X* and this is of the form of $A = \pi X^2$. We can argue that since *A* is a function of *X*, we should somehow be able to find the *pdf* of *A*, say, *g*(*x*) from the given *pdf* of *X*.

So, we will be dealing with the problem of finding the probability distribution of $Y = \varphi(X)$, when the probability distribution of X is given.

We will consider the two cases:

(i) when X is a discrete variable and

(ii) when X is a continuous variable.

Case of a Discrete Variable

Suppose that X follows the distribution:

$$P(X = x_i) = p(x_i), \quad i = 1, 2, 3, \dots$$

Let us be given a function of X, say, $Y = \varphi(X)$. Then, probability function of Y is defined as:

$$P(Y = y_i) = p(y_i) = p\big(\varphi(x_i)\big) \; such \; that \; y_i = \varphi(x_i),$$

$$i = 1, 2, 3, \dots$$

Example: Let us be given the *pmf* of a random variable X as:

$X = x_i$	-1	0	1
p_i	¼	1/3	5/12

Then *pmf* of $Y = \varphi(X) = X^2$ shall be:

$Y = y_i$	0	1
p_i	1/3	¼+5/12 = 2/3

(ii) Case of a Continuous Variable

Let X be a continuous random variable and let $Y = \varphi(X)$ be a function of X, then Y is also a random variable. Let us also assume that function $\varphi(X)$ is differentiable for all values of X and that $\varphi(X)$ is either monotonically increasing or monotonically decreasing function of X, then the *pdf* of Y can be obtained with the help of following result.

Result:

Let the *pdf* of a random variable X be given by $f(x)$, then the *pdf* of $Y = \varphi(X)$ is given by:

$$g(y) = f\left(\varphi^{-1}(y)\right)\left|\frac{dx}{dy}\right|.$$

The domain of definition of Y is determined from the domain of definition of X.

Example: Let us take the random variable X that follows the *pdf*,

$$f(x) = 2x, 0 < x < 1;$$

$$= 0, elsewhere.$$

Let us obtain the *pdf* of $Y = \varphi(X) = 3X + 1$.

Here, $y = 3x + 1$ gives

$$x = \frac{y-1}{3} = \varphi^{-1}(y)$$

And this relation gives

$$\frac{dx}{dy} = 1/3.$$

Using the result given above, we have,

$$g(y) = f\left(\varphi^{-1}(y)\right)|\frac{dx}{dy}|$$

$$or,\ \ g(y) = \left(2.\frac{y-1}{3}\right)1/3$$

$$or,\ \ g(y) = \frac{2}{9}(y-1)$$

We can also find the domain of definition of Y using the arguments.

Since $0 < x < 1$, we have $3 * 0 + 1 < 3x + 1 < 3 * 1 + 1$.

As such $1 < y < 4$. This gives us complete definition of *pdf* of Y as,

$$g(y) = \frac{2}{9}(y - 1), 1 < y < 4; 0, elsewhere.$$

Alternative Method (Using cumulative distribution function)

Let us take that cumulative distribution function of Y as $G(y)$ and that of X as $F(x)$.

Then $G(y)$ given by,

$$G(y) = P(Y \leq y)$$

$$= P(3X + 1 \leq y)$$

$$= P(X \leq \frac{y-1}{3})$$

$$= F(\frac{y-1}{3})$$

Using the relation, $f(x) = F'(x)$, we get

$$g(y) = G'(y)$$

$$= \frac{d}{dy} F(\frac{y-1}{3})$$

$$= f\left(\frac{y-1}{3}\right) * \frac{1}{3}$$

$$= 2\left(\frac{y-1}{3}\right) * \frac{1}{3}$$

$$= \frac{2}{9}(y-1)$$

Example: A random variable X follows the *pdf*, $f(x) = \frac{1}{4}, -2 < x < 2;$

$= 0, elsewhere.$

Find the *pdf* of $Y = X^2$.

Solution:

Note that $Y = X^2$ is not a monotonic function on the given interval. As such, we will have to follow the alternative method in order to find the *pdf* of Y.

Let us again take that cumulative distribution function of Y as $G(y)$ and that of X as $F(x)$. Then $G(y)$ given by:

$$G(y) = P(Y \leq y)$$

$$= P(X^2 \leq y)$$

$$= P(-\sqrt{Y} \leq X \leq \sqrt{Y})$$

$$= P\left(X \leq \sqrt{y}\right) - P(X \leq -\sqrt{y})$$

$$= F\left(\sqrt{y}\right) - F(-\sqrt{y})$$

This gives,

$$g(y) = \frac{d}{dy}\{F(\sqrt{y}) - F(-\sqrt{y})\}$$

$$= f(\sqrt{y})\frac{d}{dy}\sqrt{y} - f(-\sqrt{y})\frac{d}{dy}(-\sqrt{y})$$

$$= f(\sqrt{y})\frac{1}{2\sqrt{y}} + f(-\sqrt{y})\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\{f(\sqrt{y}) + f(-\sqrt{y})\}$$

$$= \frac{1}{2\sqrt{y}}(\frac{1}{4} + \frac{1}{4})$$

$$= \frac{1}{4\sqrt{y}}$$

Thus the *pdf* of Y shall be:

$$g(y) = \frac{1}{4\sqrt{y}}, \qquad 0 < y < 4;$$

$$= 0, \quad elsewhere.$$

Example :-

Let X be an RV with PMF

$P(X=-2)=\frac{1}{5}$, $P(X=-1)=\frac{1}{6}$

$P(X=0)=\frac{1}{5}$; $P(X=1)=\frac{1}{15}$

$P(X=2)=\frac{11}{30}$

Let $Y=X^2$, find p.d.f of Y.

$$y_i = 0, 1, 4$$

$$P[Y = y] = \begin{cases} \frac{1}{5} & ; \quad y = 0 \\ \frac{1}{6} + \frac{1}{15} = \frac{7}{30} & ; \quad y = 1 \\ \frac{1}{5} + \frac{11}{30} = \frac{17}{30} & , \quad y = 4 \end{cases}$$

Example

let x be a R.V. with pmf

$$P(X=k) = \begin{cases} \dfrac{e^{-\lambda} \cdot \lambda^k}{k!} & k = 0,1,2,\dots \quad \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

let $Y = X^2 + 3$; find pdf of Y.

$$y = x^2 + 3$$

$$\Rightarrow \quad x = \sqrt{y-3}$$

$A = \{0, 1, 2, 3, \ldots\}$; $B = \cancel{\{0, 1, 4, 9, \ldots\}}$

$= \{3, 4, 7, 12, \ldots\}$

inverse map is

$x = \sqrt{y-3}$; and since there are no negative values in 'A' we have

$$P\{Y=y\} = P(X=\sqrt{y-3}) = \frac{e^{-\lambda}\cdot\lambda^{\sqrt{y-3}}}{\sqrt{y-3}\,!}\ ; y \in B$$

$$\& \ P\{Y=y\} = 0, \quad \text{elsewhere}$$

Example:- Let X be a random variable with pdf

$$f(x) = 1, \quad 0 < x < 1$$
$$= 0 \quad \text{otherwise.}$$

Let $Y = e^{x}$; find the p.d.f of Y.

Solution: Here

$X = \log Y$, we have $= \phi^{-1}(y)$

as $y = e^x$ is monotonically increasing and differentiable

$$g(y) = f(\phi^{-1}(y)) \left| \frac{dx}{dy} \right|$$

$$\Rightarrow g(y) = 1 \cdot \left| \frac{1}{y} \right| , \quad 0 < \log y < 1.$$

$$\therefore g(y) = \begin{cases} \frac{1}{y} ; & 1 < y < e \\ 0 ; & \text{otherwise} \end{cases}$$

suppose $y = -2\log x$, $\Rightarrow$ $x = e^{-y/2}$

then

$$h(y) = \left| -\frac{1}{2} e^{-y/2} \right| \cdot 1 \quad , \quad 0 < e^{-y/2} < 1$$

or $h(y) = \frac{1}{2} e^{-y/2}$, $0 < y < \infty$

$= 0$, otherwise

CHARACTERISTICS OF PROBABILITY DISTRIBUTIONS

This is important to know two things about a probability distribution. These are: (i) where is the distribution located and (ii) how is the distribution varying with respect to a central location. These two things give us two important characteristics of a distribution, namely, average of the distribution and variance of the distribution.

Expectation

Definition 1: Let X be a random variable taking values x_1, x_2, …, x_n with probabilities p_1, p_2, …, p_n, respectively, i.e., $P(X = x_i) = p_i$, $i = 1, 2, \ldots, n$. Then the mathematical expectation (or expectation) of X is defined as

$$\mathrm{E}(X) = \sum_{i=1}^{n} x_i p_i \,.$$

Example:

Let X denotes the outcome of a single throw of a die. The *pmf* for X can be given as:

$X = x_i$	1	2	3	4	5	6
p_i	1/6	1/6	1/6	1/6	1/6	1/6

Then, $E(X) = 1.1/6 + 2.1/6 + 3.1/6 + 4.1/6 + 5.1/6 + 6.1/6 = 21/6 = 3.5.$

Definition 2:

Let *X* be a random variable with *pdf* *f(x)*, then E(*X*) is defined as

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \text{ if } \int_{-\infty}^{\infty} |x|\, f(x)\, dx < \infty.$$

Example: Let us again consider the *pdf*, $f(x) = 2x$, $0 < x < 1$; 0, elsewhere. Expected value of the random variable following this *pdf* is:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{1} x\, 2x\, dx = 2/3.$$

Using these two definitions of expectation, we can define the r^{th} moment of a random variable X about a point a.

Properties of Expectation

(i) If X is a random variable and $g(X)$ is a function of X, then

E[g(X)] = $\sum g(x_i)p_i$, if X is a discrete random variable and

E[g(X)] = $\int_{-\infty}^{\infty} g(x)f(x)dx$, if X is a continuous random variable.

(i) E(c) = c, where c is a constant.

(ii)E(cX) = cE(X).

(iii)E(a + bX) = a + bE(X), a and b are two constants.

(iv)E(X + Y) = E(X) + E(Y), X and Y being two random variables.

(v)E(XY) = E(X).E(Y), if X and Y are two independent random variables.

Definition 3: The r^{th} moment of a random variable X about a point a is denoted by μ_r' and is defined as:

$$\mu_r' = E(X - a)^r$$

If we take $a = 0$, this is the moment about origin and it will thus be:

$$\mu_r' = E(X^r)$$

There are some moments that are of special interest. These are defined by considering point a as the expected value of X and these moments are called the central moments. We define μ_r as:

$$\mu_r = E(X - \mu)^r \text{ where } \mu = E(X).$$

When r = 0, we have μ_0 = 1; and for r = 1, we have μ_1 = 0.

When r = 2, we have $\mu_2 = E(X - \mu)^2$. This moment is the measure of dispersion and is called the variance of random variable X, denoted by σ^2. Positive square root of variance is called standard deviation of random variable X. As such standard deviation (*sd*) of a random variable is denoted by σ.

When r = 3, we have $\mu_3 = E(X - \mu)^3$. This moment is the measure of asymmetry i.e. skewness.

When r = 4, we have $\mu_4 = E(X - \mathrm{\mu})^4$. This moment is the measure of kurtosis, i.e., whether the values are concentrated around the mean or are scattered. This measure defines the peakedness of the curve.

The Markov's Inequality

Requirement

Distribution is not completely defined, and we want to find the probabilities of extreme events.

Statement**:** If $X \geq 0$ and $a > 0$, then $P(X \geq a) \leq E(X)/a$

Suppose $X \geq 0$ is continuous (Similar arguments can be given for discrete variable)

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{\infty} x f(x)\,dx \qquad \text{(Why???)}$$

$$\geq \int_{a}^{\infty} x f(x)\,dx$$

$$\geq \int_{a}^{\infty} a f(x)\,dx$$

$$= a\,\mathrm{P}(X \geq a)$$

OR,

$$\mathrm{P}(X \geq a) \leq \mathrm{E}(X)/a$$

This is the Markov's Inequality.

Example 1:

Consider that X is exponential with parameter k.

Its pdf is given by:

$$f(x) = ke^{-kx}, x \geq 0;$$

$$= 0, elsewhere; \qquad k > 0.$$

Its expected value E(X) = 1/k.

$$E(X) = \int_0^{\infty} x \cdot k \cdot e^{-kx} \cdot dx \qquad \because E(x) = \int_{-\infty}^{\infty} x \cdot f(x) dx.$$

$$= k \left\{ x \cdot \frac{e^{-kx}}{-k} - \int 1 \cdot \frac{e^{-kx}}{-k} \right\}$$

$$= -1 \left\{ \frac{x}{e^{kx}} + \frac{e^{-kx}}{k} \right\}_0^{\infty}$$

$$= \frac{1}{k}.$$

So for k = 1, E(X) = 1.

As such Markov's inequality suggests that:

$$P(X \geq a) \leq 1 / a$$

What is $P(X \geq a)$? This is e^{-a}, using CDF of exponential distribution,

$$P(X \geqslant a) = \int_a^\infty e^{-1 \cdot x} \cdot dx = -e^{-x}\Big|_a^\infty$$

$$= e^{-a}.$$

and we know that $e^{-a} < 1/a$. So, the inequality holds good.

Example 2:

Consider that X is uniform on the interval [-4, 4].

Its pdf is given by:

$$f(x) = \frac{1}{8}, -4 < x < 4; 0, elsewhere$$

Using Markov's inequality we have

$$\mathrm{P}(X \geq 3) \leq \mathrm{P}(|X| \geq 3) \leq \frac{E(|X|)}{3} = 2/3.$$

$$E(|X|) = 2\int_0^4 x \cdot \frac{1}{8} \cdot dx = 2 \cdot \frac{x^2}{2 \cdot 8}\Big|_0^4$$

$$= 2$$

So $P(X \geq 3) \leq 2/3$.

Now $P(X \geq 3) = \int_3^\infty \frac{1}{8} \cdot dx = \int_3^4 \frac{1}{8}\, dx = \frac{1}{8}$.

So the Markov inequality holds here too.

We can have one more idea here in this example

P($X \geq 3$) = 0.5 * P($|X| \geq 3$) $\leq$ 0.5 * $\frac{E(|X|)}{3}$ = 1/3

The Chebyshev's Inequality

Requirements

- Random variable X, with finite mean μ and variance σ^2.
- If the variance is small, then X is unlikely to be too far from the mean.

Statement: $$\boldsymbol{P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}}$$

This can be proved using Markov's inequality.

We have:

$$\boldsymbol{P(|X-\mu| \geq c) = P\left((X-\mu)^2 \geq c^2\right) \leq \frac{E((X-\mu)^2}{c^2} = \frac{\sigma^2}{c^2}}$$

Illustration:

If we take $c = k\sigma$, the Chebyshev's inequality reduces to:

$$\boldsymbol{P(|X-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}}$$

Now, if we take *k* = 3, we have:

$$\boldsymbol{P(|X-\mu| \geq 3\sigma) \leq \frac{1}{9}}$$

This means for all the distributions, the probability that the RV deviates from the mean by more than 3σ, is less than 1/9.

Chebyshev Inequality is stronger and more informative.

Example: Let

$$P(X = 0) = 1 - \frac{1}{k^2}$$

$$P(X = \mp 1) = \frac{1}{2k^2}$$

here
$E(X) = 0$; $E(x^2) = \frac{1}{k^2}$; $\sigma = \frac{1}{k}$

$$P\{|X| \geq k\sigma\} = P[|X| \geq 1] = \frac{1}{k^2}.$$

Chebyshev's inequality holds.