Discrete Uniform Distribution

A random variable X is said to have a discrete Uniform distribution over the range [1, n], if its pmf is expressed as:

$$P(X = x) = \begin{cases} \frac{1}{n} & \text{for } x = 1, 2, \ldots n \\ 0, & \text{otherwise} \end{cases}$$

Here 'n' is known as parameter of the distribution and lies in the set of all positive integers.

Here $E(X)$ (Expectation) $= \frac{1}{n} \sum_{i=1}^{n} i$

$$= \frac{1}{n} \frac{(n)(n+1)}{2} = \frac{n+1}{2}.$$

$$E(x^2) = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

therefore

$$V(x) = E(x^2) - E(x)^2$$

Moment generating function

$$M_X(t) = E(e^{tX}) = \frac{1}{n}\sum_{x=1}^{n} e^{tx}$$

$$= \frac{e^t(1-e^{nt})}{n(1-e^t)}$$

$$M_X(t) = \frac{1}{n}\sum_{x=1}^{n} e^{tx}.$$

$$M'_x(t)\Big|_{t=0} = \frac{1}{n} \sum_{x=1}^{n} e^{tx} \cdot x \Big|_{t=0}$$

$$= \frac{1}{n} \sum_{x=1}^{n} x \cdot e^{tx} \Big|_{t=0}$$

$$= \frac{1}{n} \sum_{x=1}^{n} x$$

$$= \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)}{2} = \frac{n+1}{2}$$

$$M_x^{2}(t)\Big|_{t=0} = \frac{1}{n}\sum_{x=1}^{n} e^{tx}\cdot x^{2}\Big|_{t=0}$$

$$= \frac{1}{n}\sum_{x=1}^{n} x^{2}\Big|_{t=0}$$

$$= \frac{1}{n}\sum x^{2}$$

$$= \frac{1}{\cancel{n}}\,\frac{\cancel{n}(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

Binomial Distribution

Let us first understand what is a *Bernoulli trial*. A sequence of trials is called a Bernoulli trial if this follows the following characteristics.

(i) Each trial results into one of the only two outcomes. These outcomes are generally called ‘success’ and ‘failure’.

(ii) Probability of success for each trial is same. Let us denote this by p.

(iii) The sequence of trials contains finite number of elements. Let us denote this by *n*.

(iv) The trials are independent with each other.

Binomial distribution

Suppose that a sequence of *n* Bernoulli trials is given. Let *X* be the number of successes in *n* trials. Then *X* is a random variable that can take the values 0, 1, 2, …, *n*. This variable is called the Binomial variable. *X* shall follow the distribution,

$$P(X = x) = {}^{n}c_{x} p^{x} q^{n-x}, \text{ where } q = 1 - p,\ x = 0, 1, 2, \ldots, n.$$

p is the probability of success and

q is the probability of failure

The random variable X is said to have Binomial distribution. This distribution, needs two constants to define it completely. These are n and p. These constants are called the parameters of the distribution. As such, Binomial distribution is a two parameter distribution.

We can also define this distribution as:

A discrete random variable *X* is said to follow Binomial distribution if its *pmf* is of the form,

$$p(x) = {}^{n}c_{x} p^{x} q^{n-x}, \; x = 0, 1, 2, \ldots, n, \text{ where } q = 1 - p \; (> 0).$$

This function is abbreviated as $b(x; n, p)$. One can also show that this probability function satisfies the properties of the *pmf* , i.e., $p(x) \geq 0$ and sum of probabilities over all the values of x is 1.

$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{\infty} {}^{n}C_{x}\, p^{x} q^{n-x}.$$

$$= (q+p)^{n}$$

$$= 1^{n} = 1.$$

So $p(x)$ defines the pmf.

Expected value of Binomial distribution

If X follows Binomial distribution with parameters p and n,

its expected value shall be:

$$E(X) = \sum_{x=0}^{n} x\ {}^{n}c_{x} p^{x} q^{n-x}$$

$$= \sum_{x=1}^{n} x\ \frac{(n)!}{(x)!(n-x)!}\ p^{x} q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{(n)!}{(x-1)!(n-x)!}\ p^{x} q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!(n-x)!} p\ p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$$

$$= np\,(p+q)^{n-1}$$

$$= np$$

As such, expected value of Binomial distribution is np.

Using the similar steps as illustrated in the expectation of a Binomial variable, we can obtain that for a Binomial random variable with parameters *n* and *p*,

$$E(X^2) = n(n-1)p^2 + np.$$

$$Var(X) = \sigma^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np - np^2$$

$$= npq$$

As such, variance of a Binomial variable is *npq*.

$$E(X^2) = \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \left((x-1)x + x \right) \cdot {}^nC_x \, p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} \left\{ x(x-1) + x \right\} \frac{n(n-1)}{x(x-1)} \cdot {}^{n-2}C_{x-2} \, p^x q^{n-x}.$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} \cdot q^{n-x} + np$$

$$= n(n-1) \cdot p^2 \cdot \underset{(q+p)^{n-2}}{1} + n \cdot p$$

$$= n(n-1)p^2 + np$$

Moment generating function for this will be

$$M(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (q + pe^t)^n.$$

Example: Suppose the probability that a chip produced by a machine is defective is 0.2. If 10 such chips are selected at random, what is the probability that not more than one defective chip is found in these chips?

we have,

$n = 10$

p = Probability that chip is defective = 0.2;

So, $q = 0.8$

We are interested in finding $P(X \leq 1) = P(X = 0) + P(X = 1)$. Using the probability mass function of Binomial variable, we can obtain $P(X = 0)$ and $P(X = 1)$.

$$P(X = 0) = {}^{n}C_0 p^0 q^n$$

$$= {}^{10}C_0 (0.2)^0 (0.8)^{10}$$

$$= (0.8)^{10}$$

Similarly, we can obtain that

$P(X = 1) = 10(0.2)(0.8)^9$.

Using these two probabilities, we get the desired probability as 0.3758.

Expected value of the distribution in this example is 2. It gives us the information that on an average, we will get 2 defective chips when we examine 10 chips in independent replications of the experiment.

Example:

A fair die is rolled n times. The probability of obtaining exactly one six is $n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}$.

What is the probability of at least one 6.

Also find the number of trials needed for the probability of at least one 6 to be $\geq \frac{1}{2}$.

Solution :- The probability of obtaining no 6 is

$$= \left(\frac{5}{6}\right)^{n}$$

probability that at least one 6 will be

$$1 - \left(\frac{5}{6}\right)^{n}$$

Therefore number of trials required for this to be $\geq \frac{1}{2}$

$$\Rightarrow \quad 1-\left(\frac{5}{6}\right)^{n} \geq \frac{1}{2}.$$

$$\Rightarrow \quad n \geq \frac{\log 2}{\log 1.2} \simeq 3.8.$$

Hence '4' number of trials will be required.

Poisson Distribution

A random variable X is said to follow Poisson distribution if its *pmf* is given by,

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!},\ x = 0, 1, 2, \ldots;\ \lambda > 0.$$

Let us first show that this is a legitimate *pmf*.

One can note that $\frac{e^{-\lambda}\lambda^{x}}{x!}$ takes non-negative values for $x = 0, 1, 2, \ldots$ and $\lambda > 0$

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1$$

As such, $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x = 0, 1, 2, …; λ > 0 is a legitimate probability mass function. It is worth noting that Poisson distribution involves one parameter.

Expected value of Poisson distribution

If a random variable X follows Poisson distribution with parameter λ, then its expected value can be obtained as:

$$E(X) = \sum_{x=0}^{\infty} x\, e^{-\lambda} \lambda^{x}/(x)!$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x\, \lambda^{x}/(x)!$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \lambda^{x-1}/(x-1)!$$

$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \lambda^{y}/(y)!$$

$$= \lambda e^{-\lambda} e^{\lambda} \qquad = \lambda$$

As such, the expected value of the Poisson distribution is equal to the value of the parameter λ.

Variance of Poisson distribution

If a random variable X follows Poisson distribution with parameter λ, then we can obtain the value of $E(X^2)$ using the similar idea. This will come out that $E(X^2) = \lambda^2 + \lambda$.
As such, $Var(X) = E(X^2) - \ (E(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Expected value and variance of Poisson distribution is same and is equal to the value of the parameter of the distribution

Relationship with Binomial Distribution and Poisson Distribution:

Poisson distribution can be obtained as a limiting form from Binomial distribution. In a Binomial distribution, if the number of trials n tends to infinity and the probability of success p tends to zero such that their product np tends to a finite quantity, say, λ, then the Binomial distribution tends to Poisson distribution with parameter λ.

Proof:

For a Binomial distribution, the *pmf* is given by

$$p(x) = {}^{n}c_{x} p^{x} q^{n-x},\ x = 0, 1, 2, \ldots, n, \text{ where } q = 1 - p\ (> 0)$$

Let us impose the limits on *n* and *p* as above. In that limiting case, this *pmf* shall be:

$$p(x) = \lim_{n \to \infty, p \to 0\ (with\ np = \lambda)} \left(\frac{n(n-1)(n-2)\ldots(n-x+1)}{x!}\right) p^{x} q^{n-x}$$

$$= \lim_{n \to \infty} \left(\frac{n(n-1)(n-2)\ldots(n-x+1)}{x!}\right) \left(\frac{\lambda}{n}\right)^{x} (1 - \frac{\lambda}{n})^{n-x}$$

$$= \lim_{n \to \infty} \left(\frac{1(1-1/n)(1-2/n)\ldots(1-x/n+1/n)}{x!}\right) \lambda^{x} (1 - \frac{\lambda}{n})^{n} / (1 - \frac{\lambda}{n})^{x}$$

$$= \frac{1}{x!} \lambda^{x} e^{-\lambda}$$

As such the *pmf* becomes $p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0, 1, 2, \ldots$ in the limiting case. This is nothing but the *pmf* of the Poisson distribution.

Moment Generating function of Poisson Distribution

$$M(t) = E\left(e^{tx}\right) = e^{-\lambda} \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{\lfloor k}$$

$$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(e^t \cdot \lambda)^k}{\lfloor k}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{-\lambda(1-e^t)}$$

Hence Moment Generating function is

$$e^{-\lambda(1-e^t)}$$

we can find moment about zero in the same way,

$$M'(t) = \frac{d}{dt}\left(e^{-\lambda(1-e^t)}\right)$$

$$= e^{-\lambda(1-e^t)} \cdot \left(-\lambda(0-e^t)\right)$$

$$M'(t)\Big|_{t=0} = E(x) = e^{-\lambda(1-1)} \cdot \left(-\lambda(0-1)\right)$$

$$= \lambda$$

$$M'(t) = \lambda\, e^{t} \cdot e^{-\lambda(1-e^{t})}$$

$$M''(t) = \lambda\left[e^{t} \cdot e^{-\lambda(1-e^{t})} \cdot \left(-\lambda(0-e^{t})\right) + e^{-\lambda(1-e^{t})} \cdot e^{t} \right]$$

$$M''(t)\Big|_{t=0} = \lambda\left[e^{0} \cdot e^{-\lambda(1-1)} \cdot (\lambda) + e^{0} \cdot e^{0} \right]$$

Similarly more moments can be found

Here variance is calculated as

$$V(x) = E(x^2) - (E(x))^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

This is same as we did before.

Example: Suppose the probability that a chip produced by a machine is defective is 0.2. If 10 such chips are selected at random, what is the probability that not more than one defective chip is found in these chips?

Let us here solve this problem using Poisson distribution. We have understood that in a limiting situation Binomial variable converts into Poisson distribution. Assuming those limits, we shall have $\lambda = np = 2$. And using this value, we have the desired probability as,

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= \frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!}$$

$$= 3e^{-2}$$

$$= 0.4060$$

Ex-1

A manufacturer of pins knows that 5% of his product is defective. If he sells pins in boxes of 100 and guarantees that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quantity?

Solution:

$n = 100$;

p = prob. of a defective pin $= 5\% = 0.05$

$\lambda = np = 100 \times .05 = 5$

Let the random variable X denote the number of defective pins in a box of 100.

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{\lfloor x} = \frac{e^{-5} \cdot 5^x}{\lfloor x}, \quad x = 0, 1, 2, \ldots$$

$$P(X > 10) = 1 - P(X \leq 10)$$

$$= 1 - \sum_{x=0}^{10} \frac{e^{-5} \cdot 5^{x}}{\lfloor x}$$

$$= 1 - e^{-5} \sum_{x=0}^{10} \frac{5^{x}}{\lfloor x}$$

A2:

Example :- A car hire firm has two cars, which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson Distribution with mean 1.5. Calculate the proportion of days on which (i) neither car is used

(ii) the proportion of days on which some demand is refused.

Solution:

Let X (R.V) denotes the number of demands for a car on any day

$\lambda = 1.5$

$$P(X = x) = \frac{e^{-1.5} \cdot (1.5)^x}{\lfloor x}, \quad x = 0, 1, 2, \ldots$$

(i) Proportion of days on which neither car is used

$$P(X = 0) = e^{-1.5} = 0.2231$$

(ii) Proportion of days on which some demand is refused

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - P(X = 0) + P(X = 1) + P(X = 2)$$

$$= 1 - e^{1\cdot 5}\left(1 + 1\cdot 5 + \frac{(1\cdot 5)^2}{2!}\right)$$

$$= 0\cdot 19126$$

Example:

In a Poisson frequency Distribution, frequency corresponding to 3 successes is $\frac{2}{3}$ times frequency corresponding to 4 successes. Find the Mean and standard Deviation of the distribution.

Solution: Let X be the random variable following Poisson Distribution

then $f(x) = N . p(x) = N \, P(X=x)$ frequency function

$= N . \dfrac{e^{-\lambda} \lambda^x}{\lfloor x}$; $x = 0, 1, 2, \ldots$

$f(3) = N . \dfrac{e^{-\lambda} \lambda^3}{\lfloor 3}$, $f(4) = N . \dfrac{e^{-\lambda} \lambda^4}{\lfloor 4}$, given that

$$f(3) = \frac{2}{3} f(4)$$

$$\Rightarrow N \cdot \frac{e^{-\lambda} \lambda^3}{\lfloor 3} = \frac{2}{3} \cdot N \cdot \frac{e^{-\lambda} \lambda^4}{\lfloor 4}$$

$$\Rightarrow \lambda = 6$$

$\Rightarrow$ **Standard Deviation** $= \sqrt{\lambda} = \sqrt{6}$] Ans

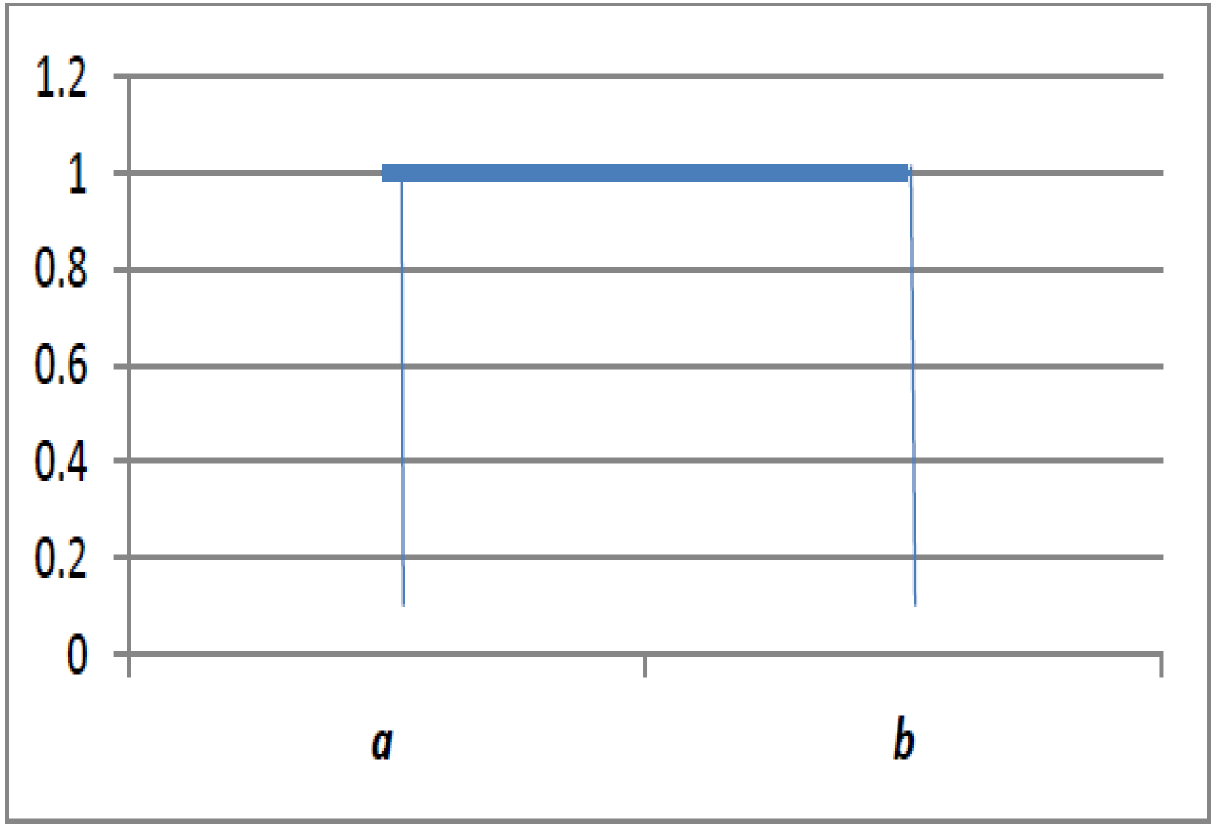
SOME CONTINUOUS DISTRIBUTIONS

Rectangular Distribution (or *Uniform Distribution*)

A continuous random variable X is said to follow uniform distribution if its *pdf* is given by:

$$f(x) = \frac{1}{b-a}, \quad a < x < b;$$
$$= 0, \quad elsewhere.$$

Graph of this *pdf* shall look like as given in Figure



Graph of Uniform Distribution

This distribution is also called Rectangular distribution. To show that this is a legitimate *pdf*. For this, we have to show that

$\int_{-\infty}^{\infty} f(x)dx = 1.$

Here,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{a}^{b} \frac{1}{b-a} dx = \frac{1}{b-a} x|_{a}^{b} =$$

$\frac{b-a}{b-a} = 1.$

Also, $\frac{1}{b-a}$ takes positive values when $a < b$. As such this is a legitimate *pdf*.

Expected value of Uniform distribution

Expected value of a random variable X following uniform distribution shall be:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{a}^{b} x \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)} (b^2 - a^2)$$

$$= \frac{a+b}{2}$$

As such, expected value of uniform distribution is the mid value of the interval on which the distribution is defined.

- Show that variance of uniform distribution is $(b - a)^2/12$.
- Find the moment generating function in this case.

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{(b^3-a^3)}{3(b-a)}$$

$$= \frac{a^2+b^2+ab}{3}$$

∴ Variance $= E(x^2) - (E(x))^2$

$$= \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= (b-a)^2/12$$

Moment Generating function in this case:

$$\underline{M(t)} = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} f(x) \cdot e^{tx}\, dx$$

$$= \int_{a}^{b} \frac{1}{b-a} \cdot e^{tx}\, dx$$

$$= \frac{1}{b-a} \left. \frac{e^{tx}}{t} \right|_{a}^{b}$$

$$= \frac{\left(e^{tb} - e^{ta}\right)}{(b-a) \cdot t}$$

Example: A point is randomly selected from the interval (4,10). What is the probability that this point shall lie in the interval (4.5, 6.5)?

Probability density function, *pdf* of the variable *X* that is randomly distributed in (4, 10) is,

$$f(x) = \frac{1}{6}, \quad 4 < x < 10;$$

$$=0, \quad elsewhere$$

As such desired probability shall be given by,

$$P(4.5 < x < 6.5) = \int_{4.5}^{6.5} \frac{1}{6} dx = \frac{2}{6} = \frac{1}{3}.$$

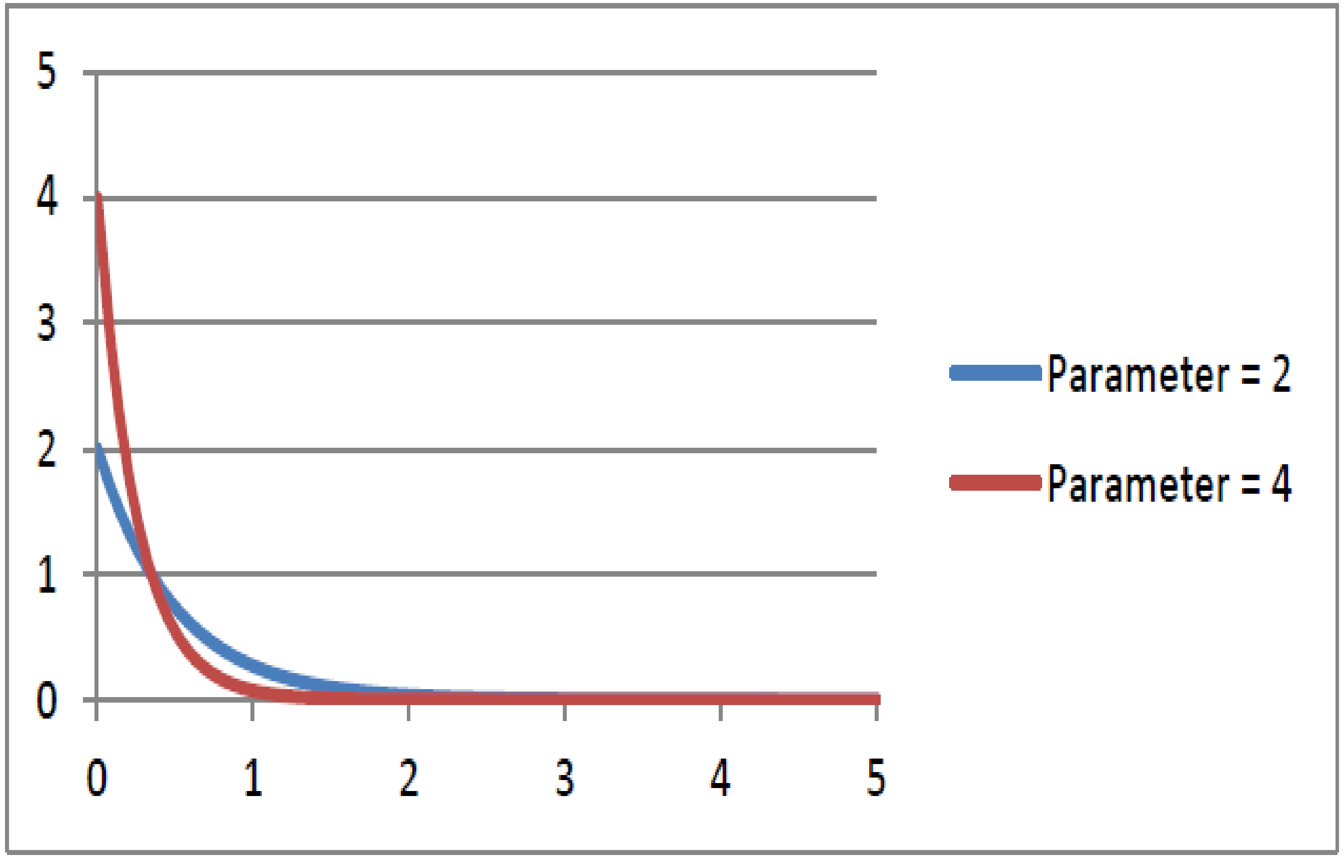
Exponential Distribution

A continuous random variable *X* is said to follow exponential distribution if its *pdf* is given by,

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \ \lambda > 0$$

$$= 0, \text{ otherwise.}$$

Here, λ is the parameter associated with exponential distribution. Graphs of this distribution for λ = 2 and λ = 4 are given in following figure.



Graphs of Exponential Distribution

Expected Value of Exponential Distribution

Expected value of a random variable following exponential distribution can be calculated using the definition as below.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x\lambda e^{-\lambda x} dx,$$ using the definition of exponential *pdf*

$$= \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

$$= \lambda [\frac{x e^{-\lambda x}}{-\lambda}]_0^{\infty} - \lambda \int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda} dx$$

Using the concept of limits, first term in this expression shall be zero and as such the expected value shall be,

$$E(X) = \int_0^\infty e^{-\lambda x} dx$$

$$= [\frac{e^{-\lambda x}}{-\lambda}]_0^\infty$$

$$= 1/\lambda$$

Thus, expected value of a random variable X having exponential distribution with parameter λ is given by $1/\lambda$.

Variance of Exponential Distribution

Variance of exponential distribution can again be obtained by first calculating $E(X^2)$ and then using the relation $Var(X) = E(X^2) - (E(X))^2$. Using the similar steps as illustrated in finding the expected value of an exponential variable, we can obtain that $\mathrm{E}(X^2) = \frac{2}{\lambda^2}$. So variance of exponential variable is,

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Moment generating function of Exponential Distribution:

$$M(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} \cdot \lambda \cdot e^{-\lambda x} \cdot dx$$

$$= \lambda \int_0^{\infty} e^{(t-\lambda)x} \, dx = \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^{\infty}$$

$$= \lambda \left[0 - \frac{1}{t-\lambda} \right] = \frac{\lambda}{(\lambda - t)}$$

$$M(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} \qquad \lambda > 0$$

we can calculate

$$M'(t)\Big|_{t=0} = \frac{d}{dt}\left(1-\frac{t}{\lambda}\right)^{-1}$$

$$= (-1)\left(1-\frac{t}{\lambda}\right)^{-2}\cdot\left(0-\frac{1}{\lambda}\right)\Bigg|_{t=0}$$

$$= (-1)\left(1-0\right)^{-2}\cdot\left(0-\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

$$M''(t)\Big|_{t=0} = \frac{d}{dt}\left(+\frac{1}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2}\right)$$

$$= \frac{1}{\lambda}\left\{(-2)\left(1-\frac{t}{\lambda}\right)^{-3}\left[0-\frac{1}{\lambda}\right)\right]_{t=0}$$

$$= \frac{1}{\lambda}\left[(-2)(1)\left(-\frac{1}{\lambda}\right) = \frac{2}{\lambda^2}\right.$$

Normal Distribution

This distribution is an important distribution owing to the fact that a number of real life variables can be thought of following this distribution

A continuous random variable X is said to follow normal distribution if its *pdf* is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0$$

Symbolically, we write this as $X \sim N(\mu, \sigma^2)$.

First, we show that this function represents a legitimate *pdf*.

For this, we will have to deal with two basic properties of *pdf*.

(i) We can see that $f(x) \geq 0$ for all x.

(ii)We shall show that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let us make the following substitution,

$$z = \frac{x-\mu}{\sigma}$$

$$dz = \frac{dx}{\sigma}$$

So,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}z^2} dz$$

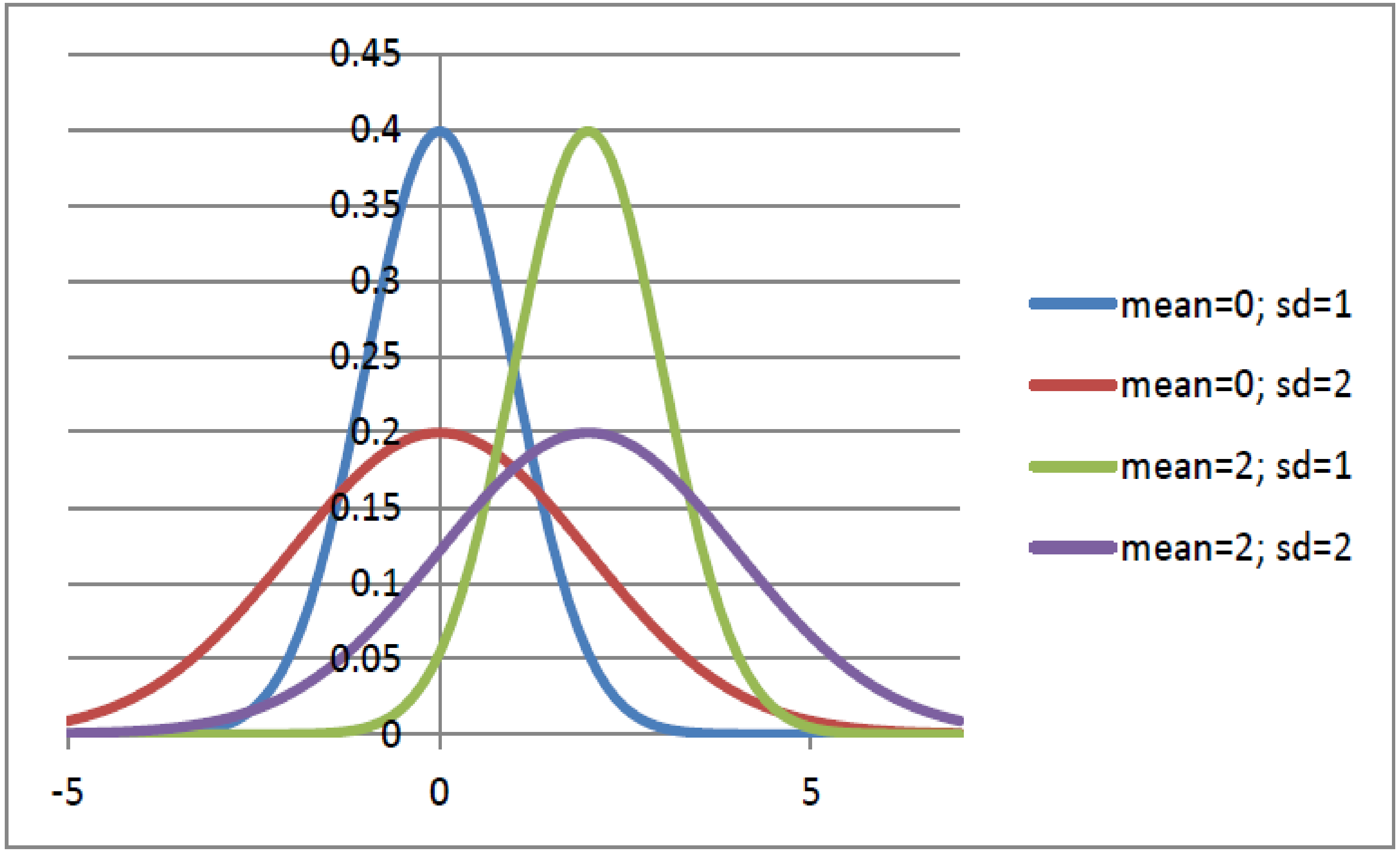
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\frac{-1}{2}z^2} dz$$

= 1

As such, *f*(*x*) represents a legitimate probability density function.

We shall soon show that parameter μ is the expected value (or mean) of normal distribution and σ is the standard deviation (*sd*) of normal distribution.

Following Figure contains graphs of *pdf* of normal distribution for mean = 0, *sd* = 1; mean = 0, *sd* = 2; mean = 2, *sd* = 1 and mean = 2, *sd* = 2.



Graphs of Normal Distribution

Properties of Normal Distribution

(i) The *pdf* $f(x)$ of normal distribution takes maximum value at $x = \mu$.

(ii)The *pdf* $f(x)$ of normal distribution is symmetrical about the line $x = \mu$.

(iii) Points of inflexion of *pdf* of normal distribution are $\mu \pm \sigma$.

(iv)The *pdf* of Normal distribution is a bell-shaped curve.

(v) All moments of odd order of normal distribution are zero.

Expected Value of Normal Distribution

Using definition, expected value of normal distribution can be obtained as below.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let us substitute, $z = \frac{x-\mu}{\sigma}$. This gives $dz = \frac{dx}{\sigma}$ and expression for expected value of *X* becomes,

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{\frac{-z^2}{2}} dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{\frac{-z^2}{2}} dz$$

$$= \mu$$

Here, we can note that the function $g(z) = z e^{\frac{-z^2}{2}}$ is an odd function of z. As such, second term in above expression shall be zero.

Also, $f(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$ is the *pdf* of a normal variable with μ = 0 and σ = 1. As such, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy = 1$. This means that the value of first expression in the above expression shall be μ.

Variance of Normal Distribution

The variance of a Normal distribution with parameters μ and σ may be obtained by a similar integration as above. We can establish that $Var(X) = \sigma^2$.

Standard Normal Distribution

Suppose that $X \sim N(\mu, \sigma^2)$, i.e., X is normally distributed with mean μ, and variance σ^2. Let us transform the variable X to Z where $Z = \frac{X-\mu}{\sigma}$. Then,

$$E(Z) = \frac{E(X)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0 \text{ and}$$

$$V(Z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}V(X-\mu)$$

$$= \frac{1}{\sigma^2}V(X)$$

$$= 1.$$

As such, Z has a mean 0 and standard deviation 1. This can also be shown that Z is also normally distributed. We can thus conclude that if X is a normal variable with mean μ, and variance σ^2, then $Z = \frac{X-\mu}{\sigma}$ is also a normal variate with mean 0 and variance 1. This variable Z is called the standard normal variable. We can see that *pdf* of standard normal variable is $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}, -\infty < z < \infty$.

Note: Study of standard normal distribution is important in order to find the probability that a normal variable lies in a given interval. Have you noticed that the function $e^{\frac{-z^2}{2}}$ *is not integrable and we need to integrate this in order to find the probability of the type P(a < Z < b)? As such, the values of* $\varphi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}} du$, *for some z have been tabulated and one can use these values for finding the probabilities associated with a given normal variable.*

Example: Suppose that X follows normal distribution with mean 4 units and variance 9 units. Find the probability that $P(3 < X < 5)$.

Here, we have to effectively calculate the value of $\int_3^5 \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x-4)^2}{18}} dx$. This is again not possible to calculate this value analytically. We follow the process as outlined below to calculate this probability.

$$P(3 < X < 5) \quad = P(\frac{3-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma})$$

$$= P(\frac{3-4}{3} < Z < \frac{5-4}{3})$$

$$= P(\frac{-1}{3} < Z < \frac{1}{3})$$

$$= P\left(Z < \frac{1}{3}\right) - P(Z < \frac{-1}{3})$$

$$= P\left(Z < \frac{1}{3}\right) - \left(1 - P\left(Z < \frac{1}{3}\right)\right)$$

$$= 2P\left(Z < \frac{1}{3}\right) - 1,$$ (Due to symmetry of the *pdf* of standard normal variate)

The value of $P\left(Z < \frac{1}{3}\right)$ is taken from the table of standard normal variable. One can see that this value is 0.6293 and thus $P(3 < X < 5) = 0.2586$.

Moment Generating function of Normal distribution:

$$M(t) = E(e^{tx})$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2} + x \frac{t\sigma^2 + \mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} \right) dx.$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{\infty} \exp\left[\frac{-(x-\mu-\sigma^2 t)^2}{2\sigma^2} + \mu t + \frac{\sigma^2 t^2}{2} \right] dx.$$

$$e^{\left(\mu t+\frac{\sigma^2 t^2}{2}\right)} \cdot \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{-\left(\frac{(x-\mu-\sigma^2 t)^2}{2\sigma^2}\right)} \cdot dx$$

$$\frac{x-\mu-\sigma^2 t}{\sigma} = z$$

$$\Rightarrow dx = \frac{dz}{\sigma}$$

$$\Rightarrow dx = \sigma\, dz$$

$$e^{\left(\mu t+\frac{\sigma^2 t^2}{2}\right)} \cdot \frac{1}{\sqrt{2\pi}\cdot\cancel{\sigma}}\,\cancel{\sigma}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \cdot dz$$

$$= e^{\mu t+\frac{\sigma^2 t^2}{2}}$$

$$E(X) = M'(t)\Big|_{t=0} = e^{\mu t + \sigma^2 t^2/2} \cdot \left(\mu + \sigma^2 \cdot t\right)\Big|_{t=0}$$

$$= 1 \cdot (\mu + 0) = \mu$$

Mean $= \mu$ for Normal distribution

$$E(X^2) = M''(t)\Big|_{t=0}$$

$$= e^{(\mu t + \sigma^2 t^2/2)} \cdot (0 + \sigma^2) + (\mu + \sigma^2 t)\, e^{(\mu t + \sigma^2 t^2/2)} (\mu + \sigma^2 t)$$

at $t=0$ $\quad 1 \cdot \sigma^2 + (\mu + 0) \cdot 1 (\mu + 0)$

$$= \sigma^2 + \mu^2$$

$\therefore \quad Var(X) = \sigma^2 + \mu^2 - \mu^2$

$$= \sigma^2.$$

$$E(X^3) = M'''(t)\Big|_{t=0}$$

$$\frac{d}{dt}\left[e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \sigma^2 + (\mu + \sigma^2 t)^2 \cdot e^{\left(\mu t + \frac{\sigma^2 t^2}{2}\right)} \right]$$

Can be checked

$$\mu_3 = E(X - E(X))^3$$

$$= E\left(X^3 + (E(X))^3 - 3X^2E(X) + 3XE(X)^2\right)$$

$$= E(X^3) + (E(X))^3 - 3E(X^2)\cdot E(X) + 3(E(X))^3$$

$$= E(X^3) + 4(E(X))^3 - 3\underline{E(X^2)}\,\underline{E(X)}$$

$$+ 4\mu^3 - 3(\sigma^2 + \mu^2)\cdot\mu$$

$\rightarrow$ this will come out as zero.

Example:- X is a normal variate with mean 30 & S.D. 5. find the probability that -

(i) $26 < X < 40$, (ii) $X \geq 45$

(iii) $|X - 30| > 5$

Solution:- Here $\mu = 30$; $\sigma = 5$

(i) when $X = 26$, $Z = \frac{X-\mu}{\sigma} = \frac{26-30}{5} = -0.8$

(ii) when $X = 40$; $Z = \frac{40-30}{5} = 2$

$\therefore P(26 < X < 40) = P(-0.8 \leq Z \leq 2)$

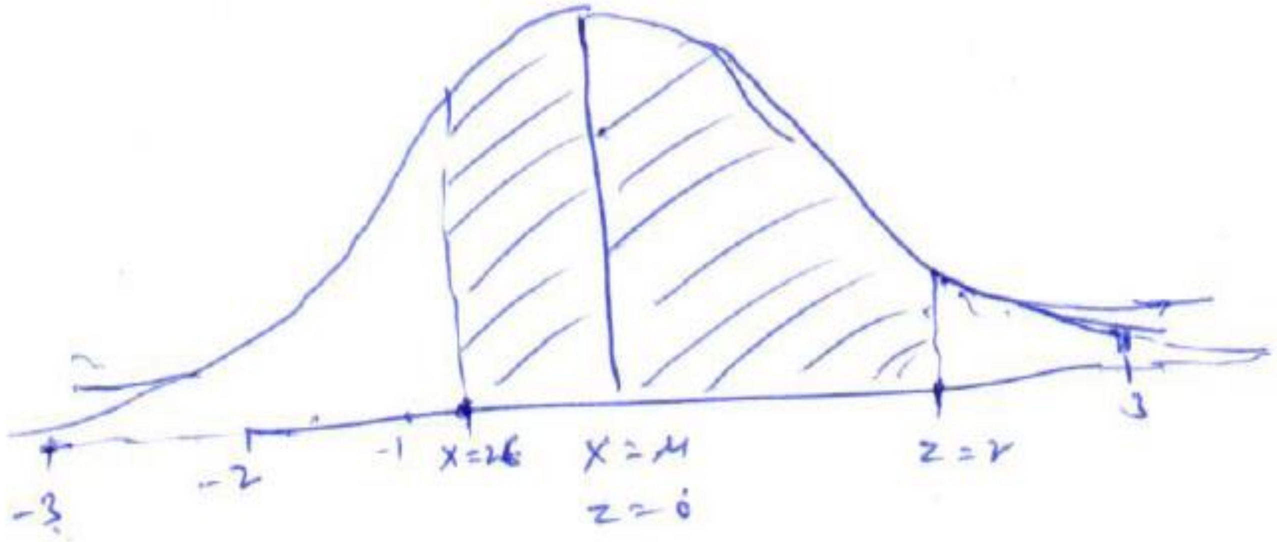
$= P(-0.8 \leq Z \leq 0) + P(0 \leq Z \leq 2)$

$= P(0 \leq Z \leq 0.8) + P(0 \leq Z \leq 2)$

(By Symmetry)

$= 0.2881 + 0.4772$

$= 0.7653$ Ans

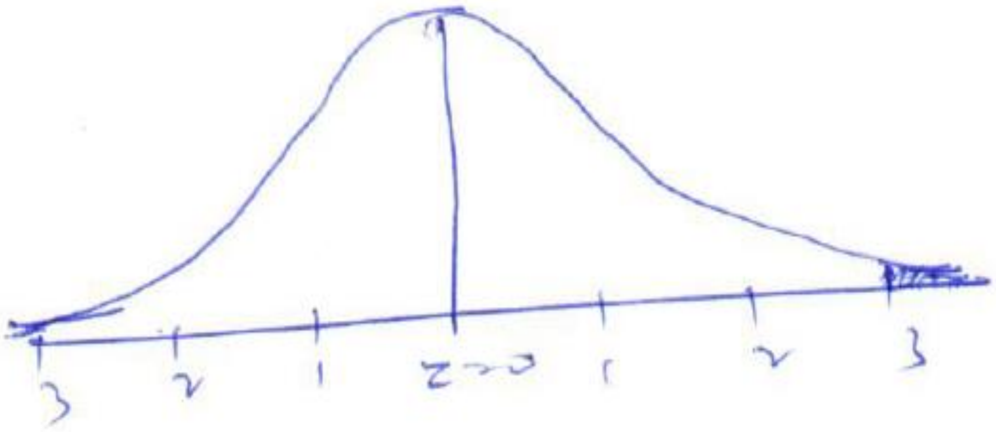


(ii) when $x = 45$; $z = \frac{45-30}{5} = 3$

$\therefore P(X \geq 45) = P(Z \geq 3) = 0.5 - P(0 \leq Z \leq 3)$

$= 0.5 - 0.49865$

$= 0.00135$ Ans.



(iii) $P(|x-30| \leq 5) = P(25 \leq x \leq 35)$

$= P(-1 \leq z \leq 1)$

$= 2P(0 \leq z \leq 1) = 2 \times 0.3413$

$= 0.6826$

$\therefore P(|x-30| > 5) = 1 - P(|x-30| \leq 5)$

$= 1 - 0.6826$

$= 0.3174$ Ans