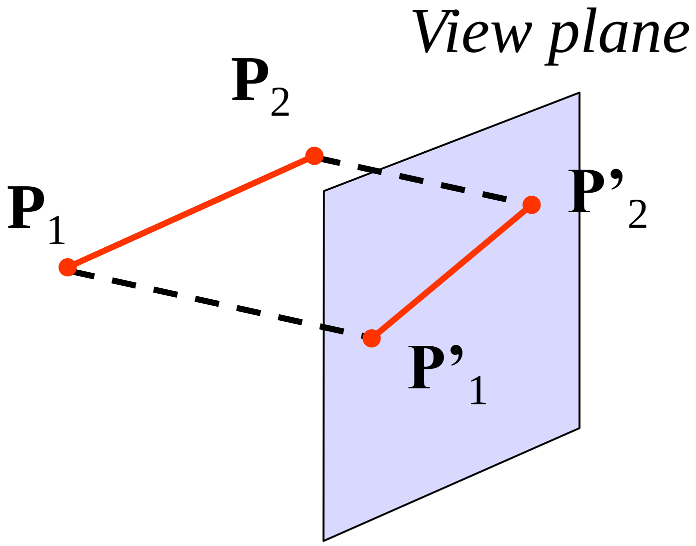
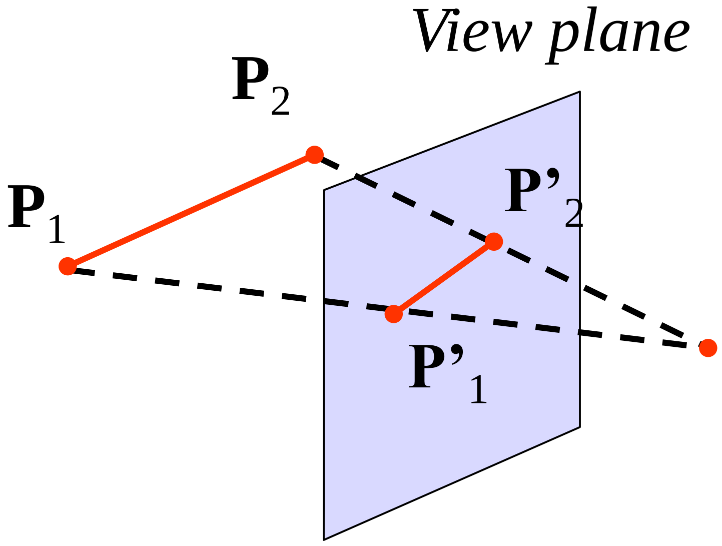
Projection transformations

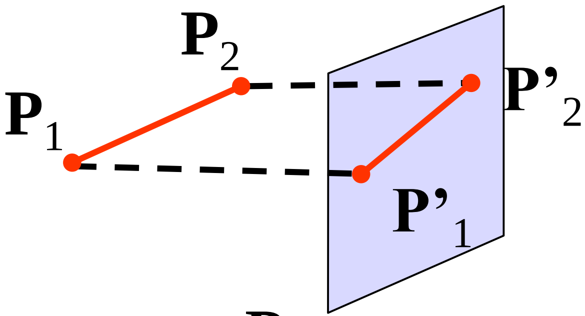


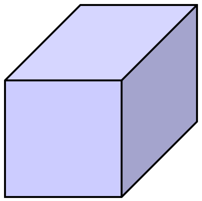
Parallel projection



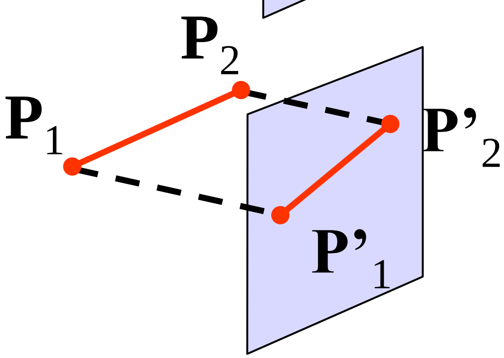
Perspective projection

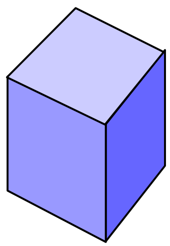
Orthogonal projections 1



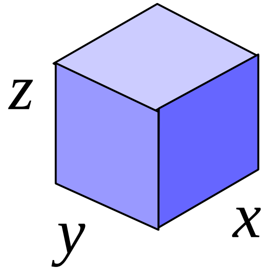


Parallell projection:
Projection lines are parallel



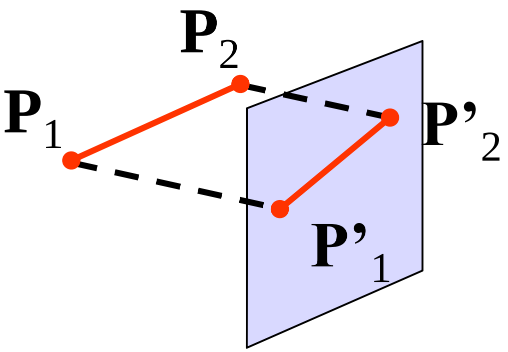


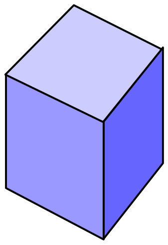
Orthogonal projection:
Projection lines are parallel and perpendicular to projection plane



Isometric projection:
Projection lines are parallel, perpendicular to projection plane, and have the same angle with axes.

Orthogonal projections 2





Orthogonal projection:

Projection of (x, y, z) (from view coordinates to projection coordinates):

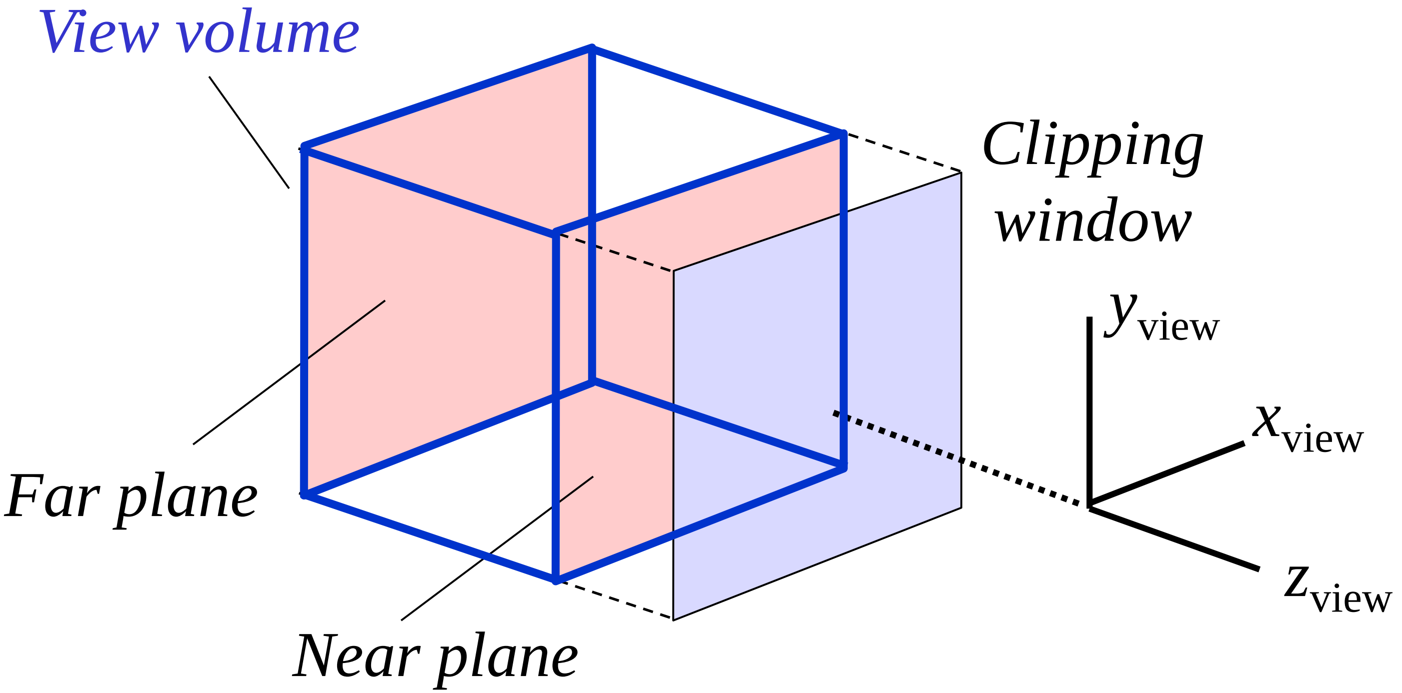
$$x_p = x$$

$$y_p = y$$

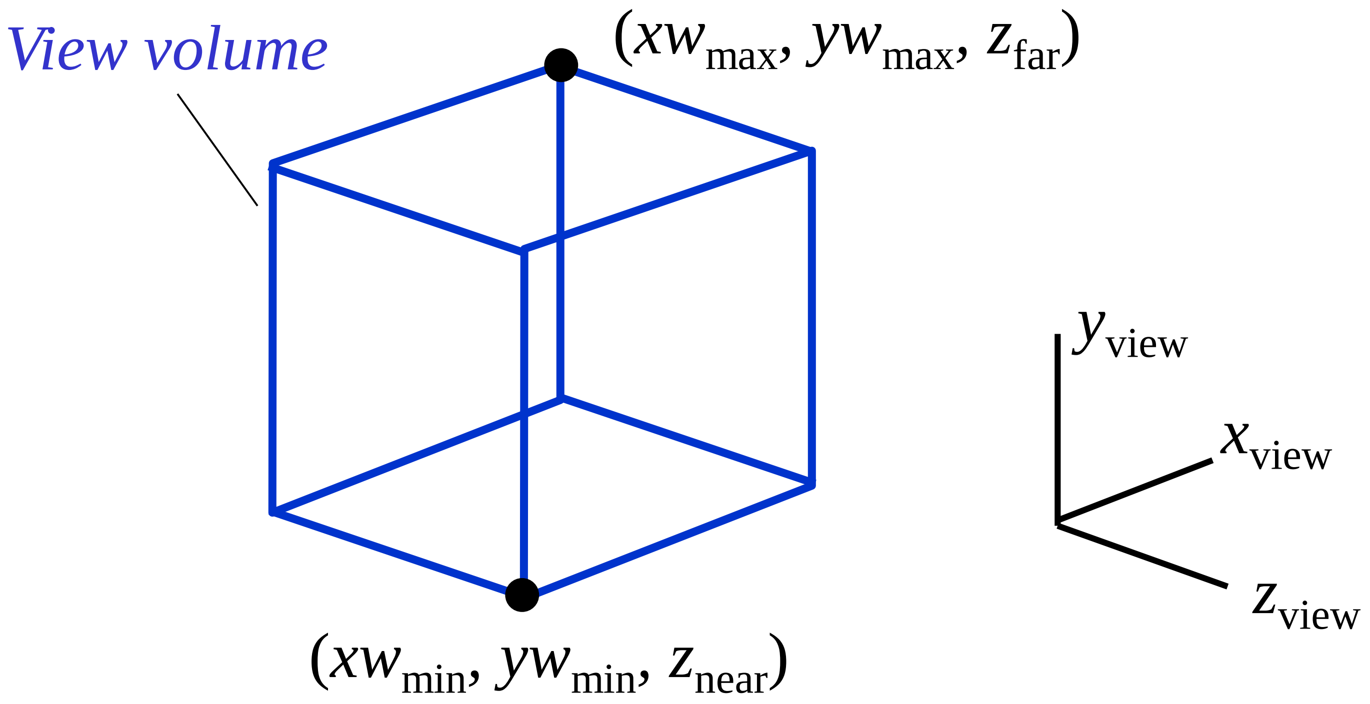
$$z_p = z$$

Trivial!

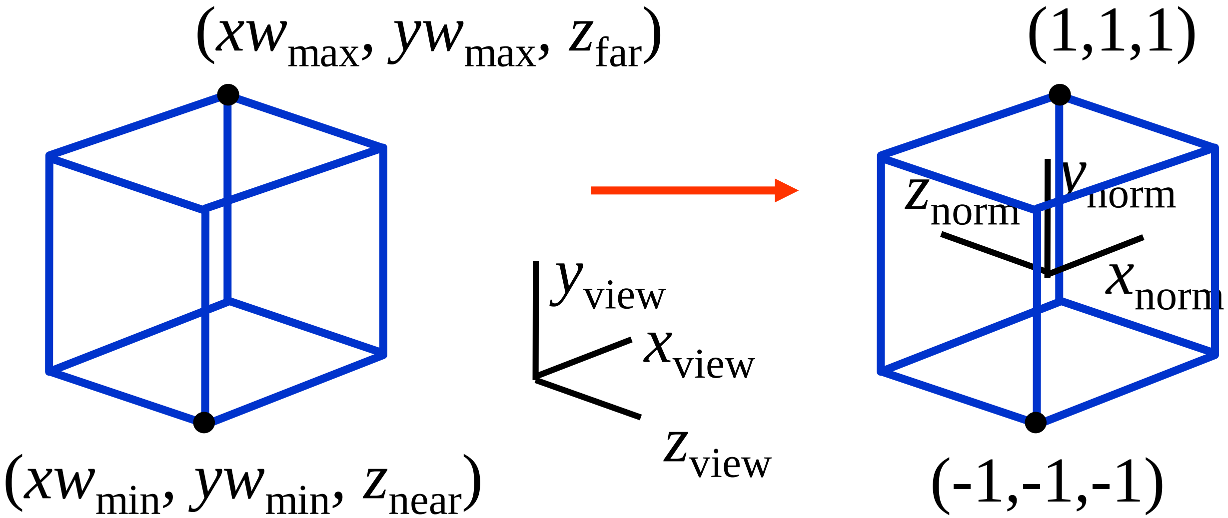
Orthogonal projections 3



Orthogonal projections 4



Orthogonal projections 4



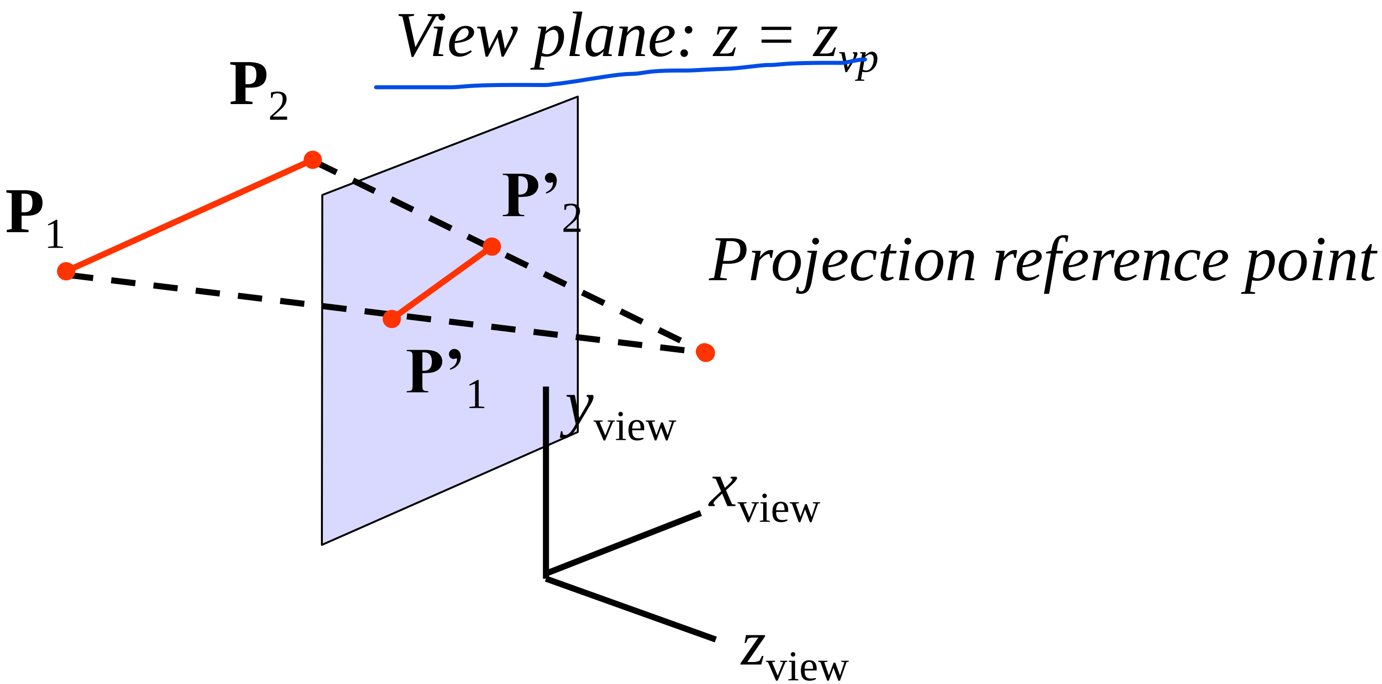
View volume → Normalized View volume

Translation

Scaling

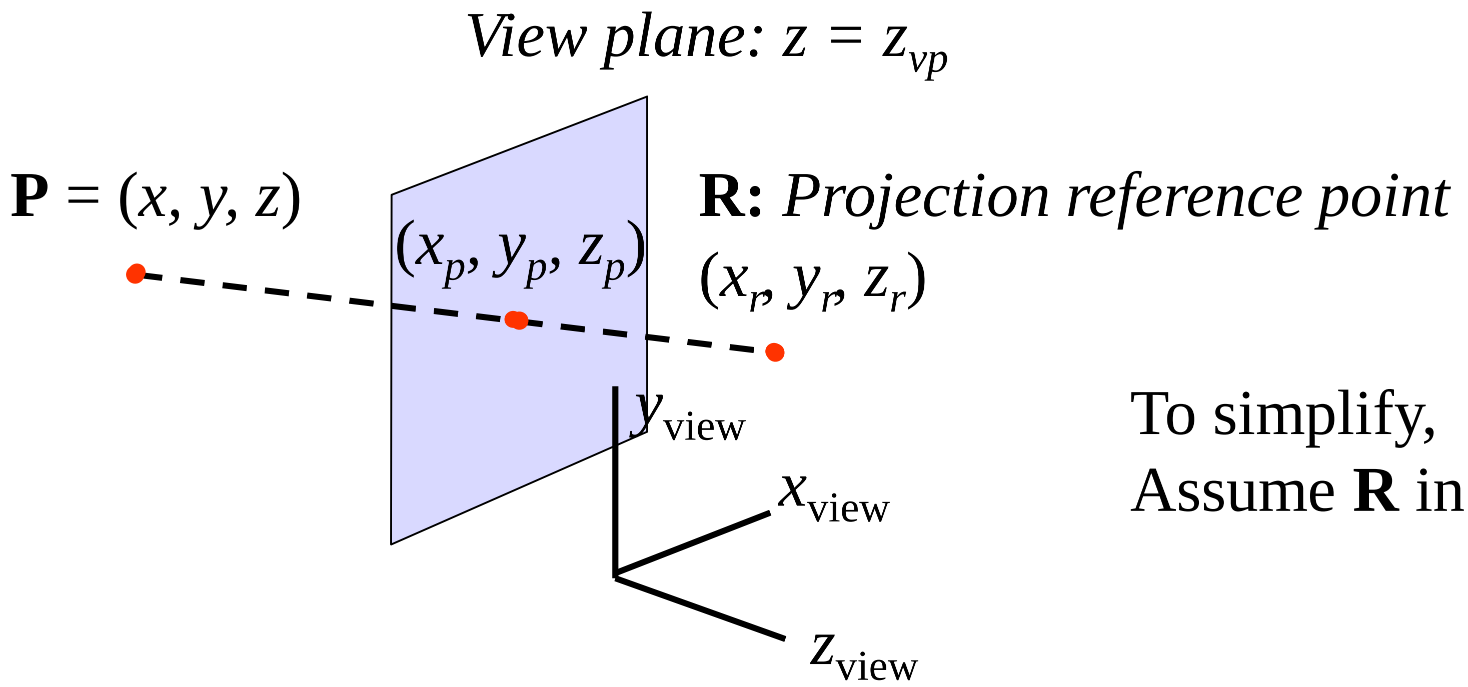
From right- to left handed

Perspective projection 1



View plane: orthogonal to z_{view} axis.

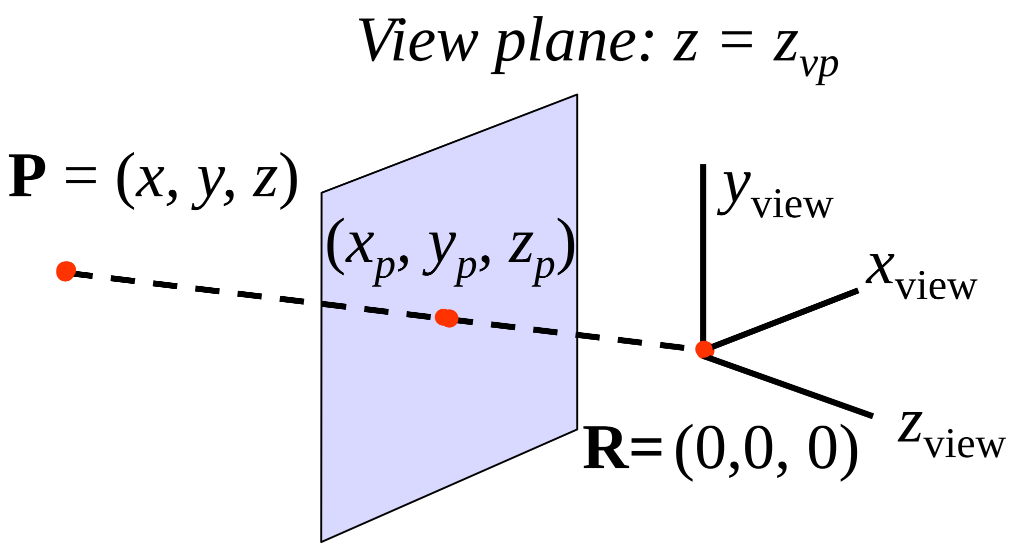
Perspective projection 2



To simplify,
Assume **R** in origin

Question: What is the projection of **P** on the view plane?

Perspective projection 3



Line from $\mathbf{R}$ (origin) to $\mathbf{P}$:

$\mathbf{X}' = u\mathbf{P}$, with $0 \le u \le 1$,

or $x' = ux$; $y' = uy$; and $z' = uz$.

At crossing with plane:

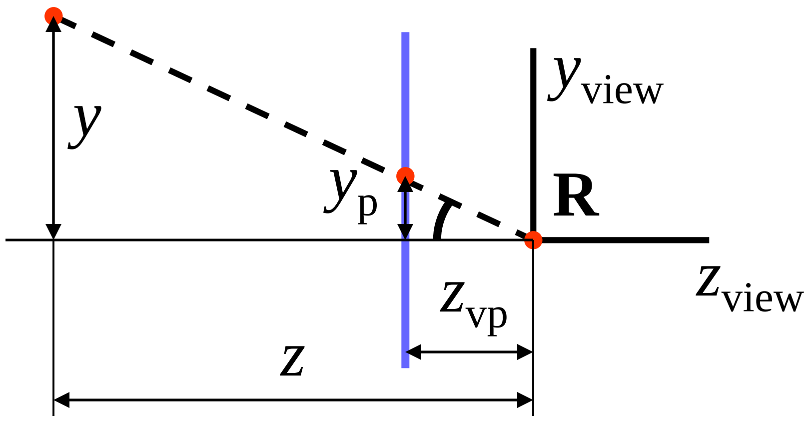
$z' = z_{vp}$ hence $u = \dfrac{z_{vp}}{z}$.

Substituti on gives

$$x_p = \frac{z_{vp}}{z} x \text{ and } y_p = \frac{z_{vp}}{z} y$$

Perspective projection 4

$\mathbf{P} = (x, y, z)$ *View plane*



Viewed from the side

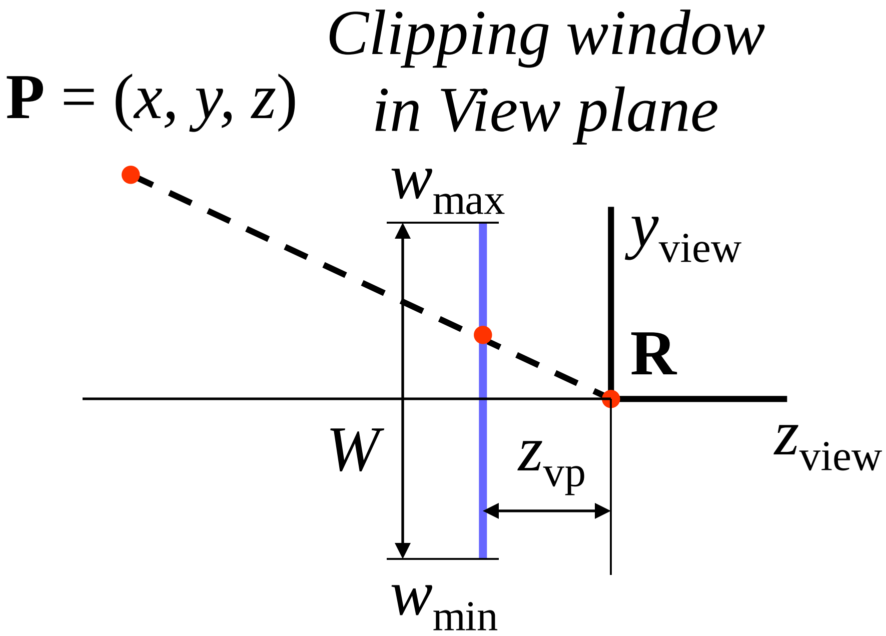
We can see that :

$$\frac{y}{z} = \frac{y_p}{z_{vp}}$$

hence

$$y_p = \frac{z_{vp}}{z} y$$

Perspective projection 5



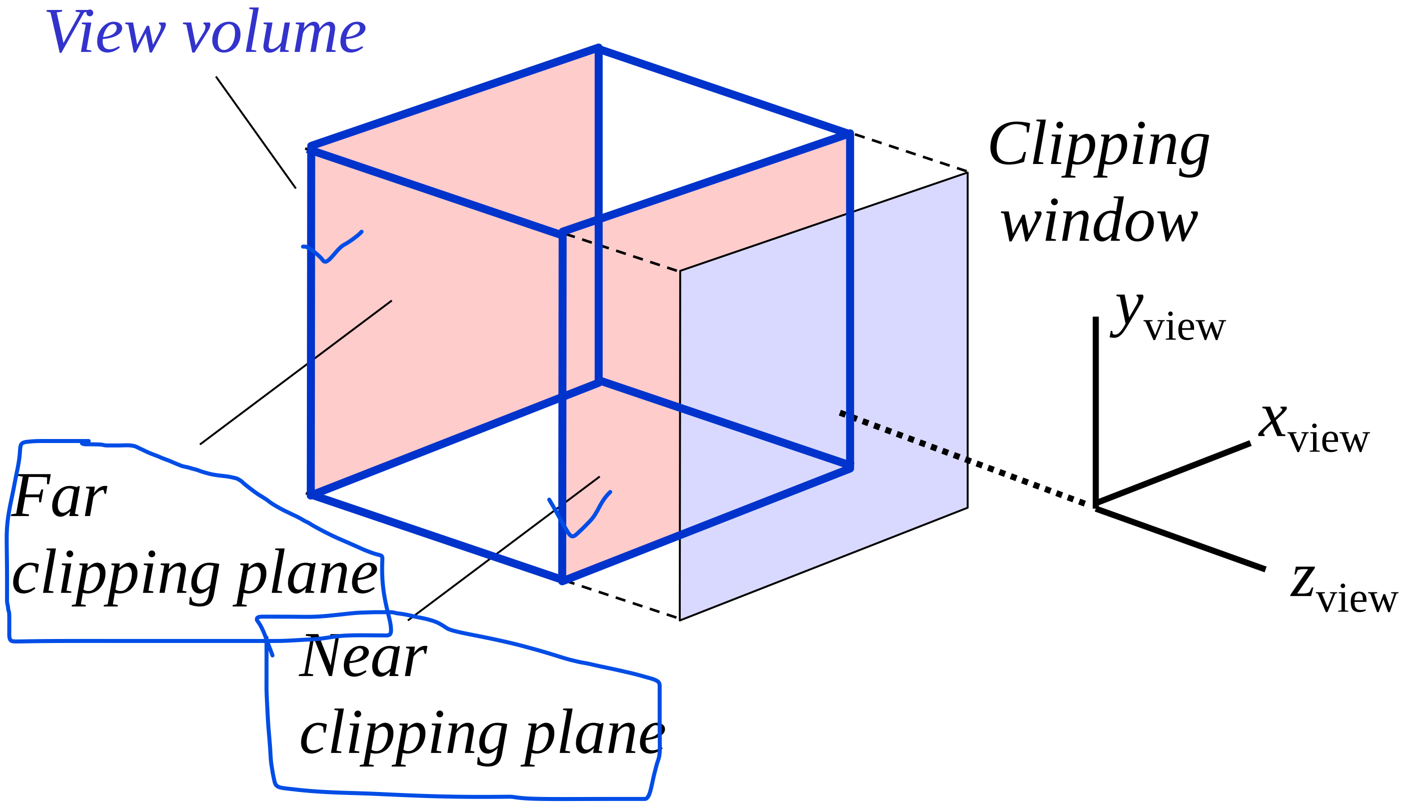
Viewed from the side

Ratio between

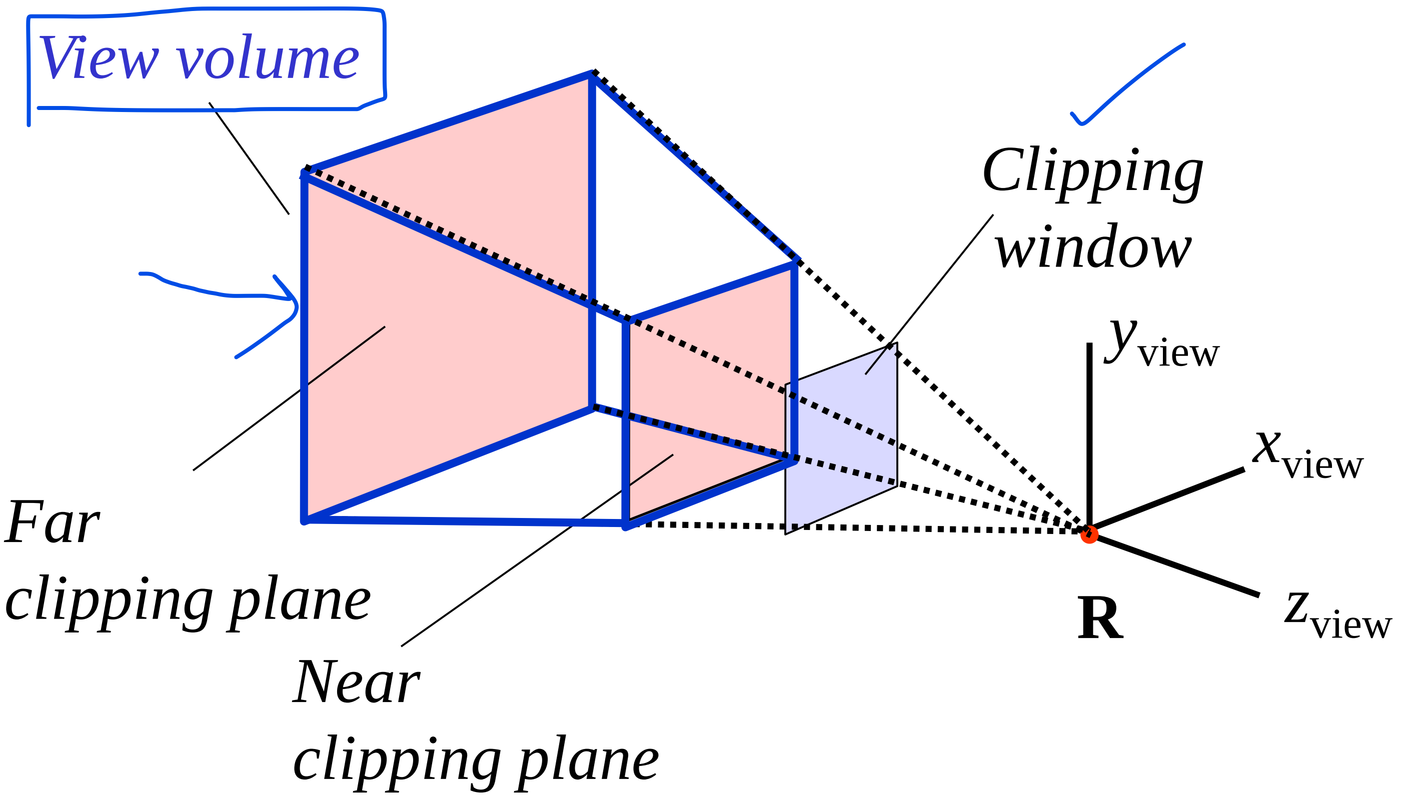
$W = w_{max} - w_{min}$ and z_{vp}

determines strenght perspective

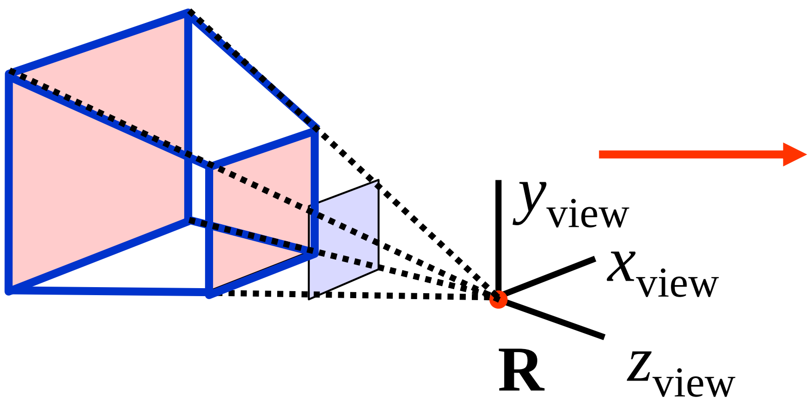
View volume orthogonal…

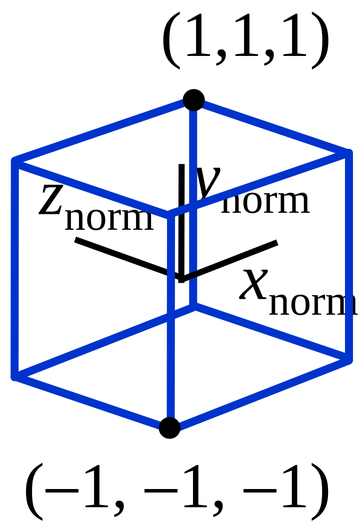


View volume perspective



To Normalized Coordinates…

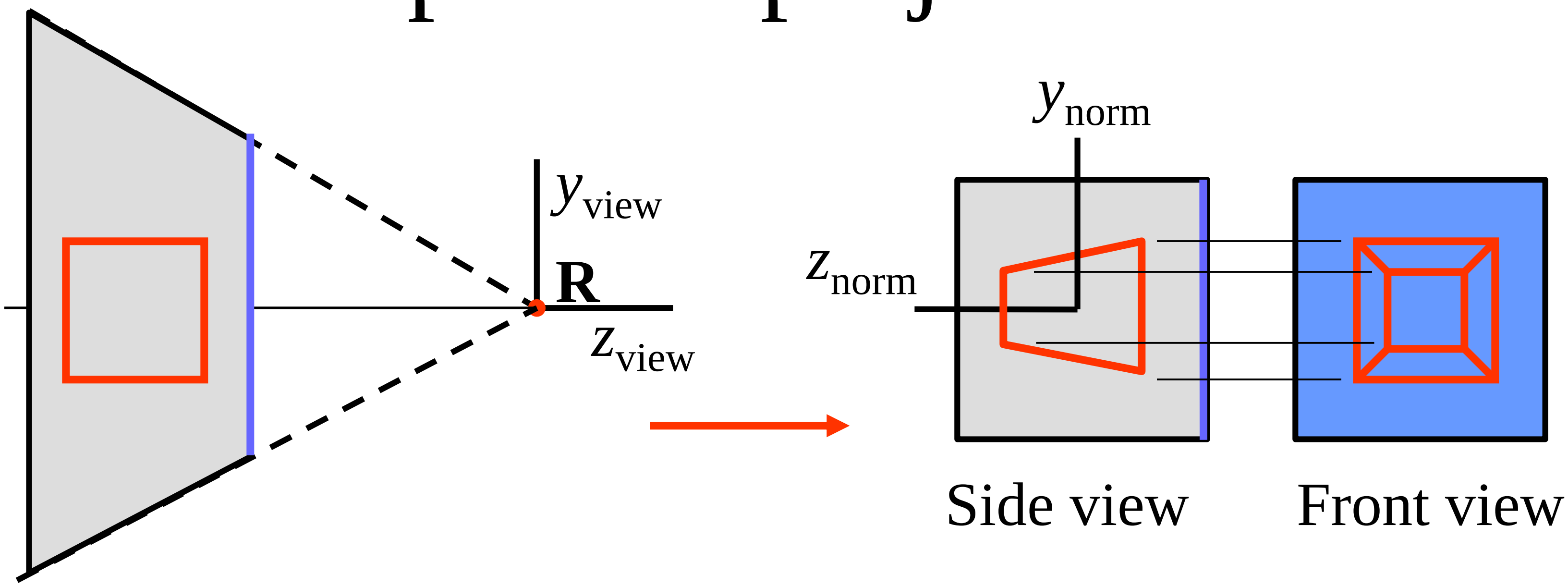




Rectangular frustum
View Volume

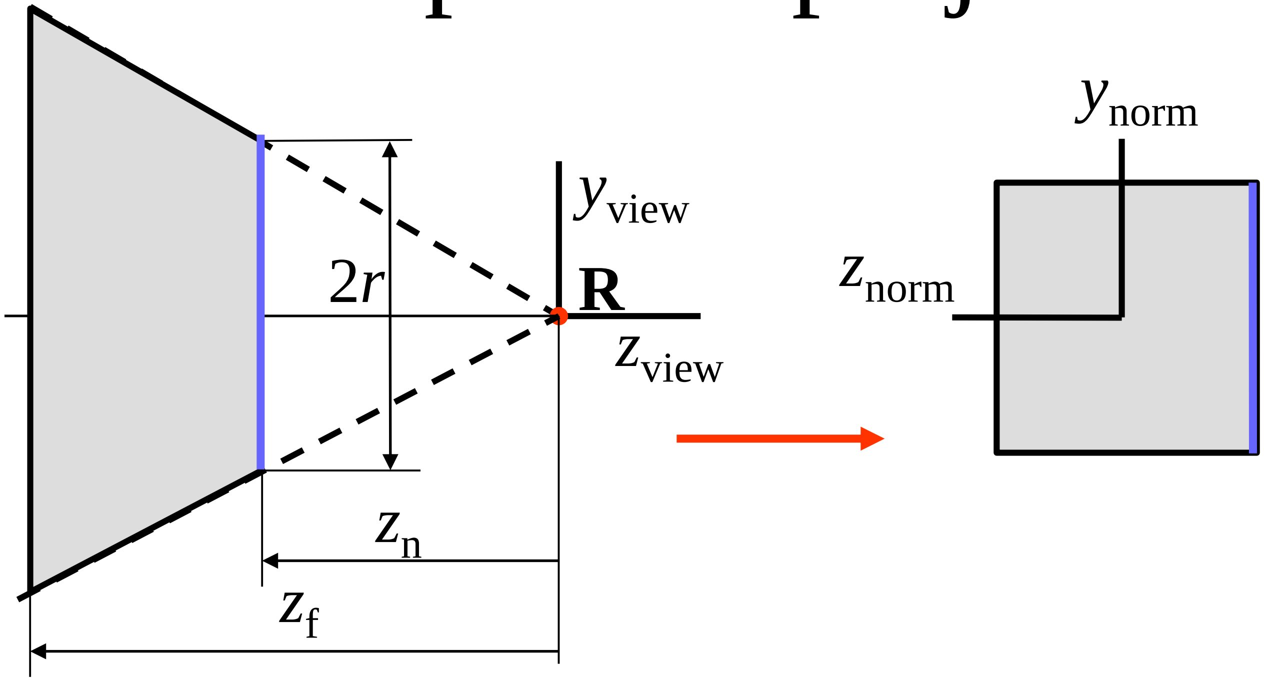
Normalized View volume

Perspective projection 6



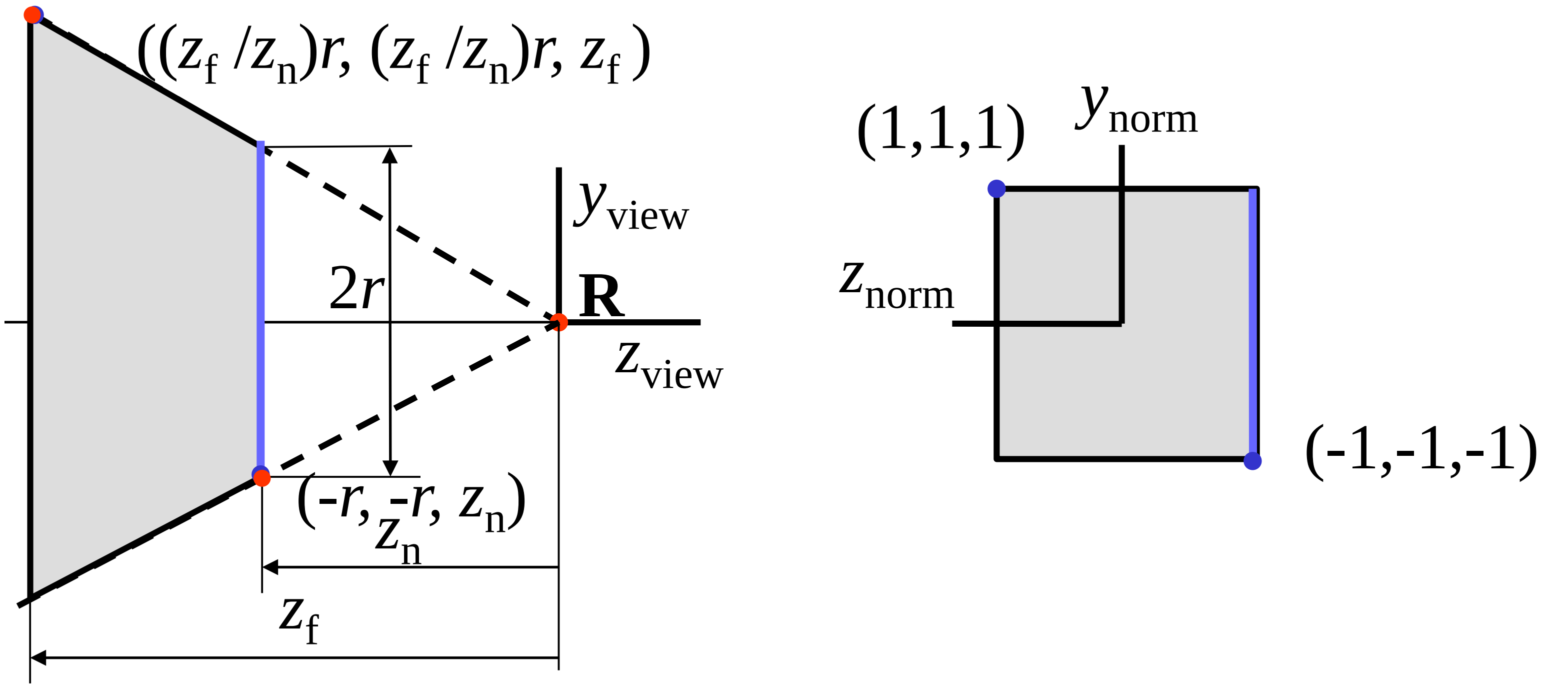
Perspective transformation:
Distort space, such that perpendicular projection gives an image in perspective.

Perspective projection 7

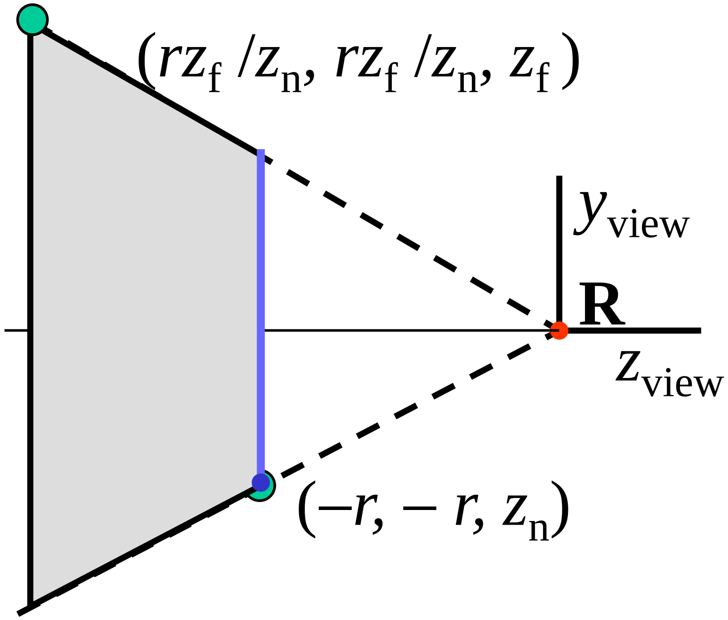


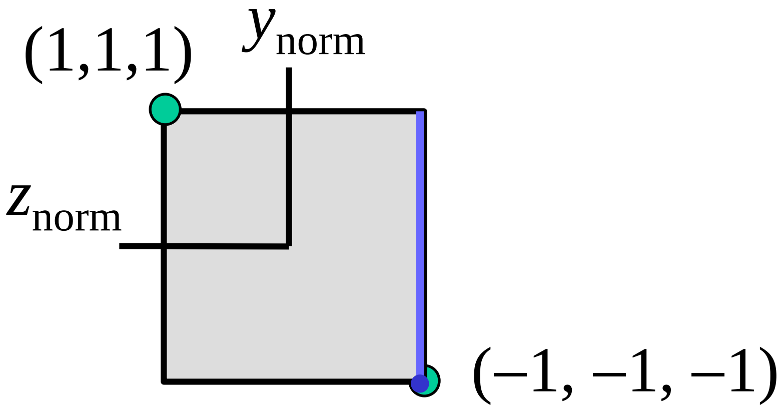
Simplest case:
Square window,
clipping plane coincides with view plane: $z_n = z_{vp}$

Perspective projection 8



Perspective projection 9





Earlier: $x_p = \dfrac{z_{vp}}{z} x, \;\; y_p = \dfrac{z_{vp}}{z} y$

How to put this transformation in the pipeline?
How to process division by z?

Homogeneous coordinates (reprise)

- Add extra coordinate:

  $\mathbf{P} = (p_x , p_y , p_z , \ p_h)$ or

  $\mathbf{x} = (x, y, z, h)$

- Cartesian coordinates: divide by h

  $\mathbf{x} = (x/h, y/h, z/h)$

- Points: ~~$h = 1$ (temporary…)~~ *perspective:* $h = -z$!

Homogeneous coordinates (reprise)

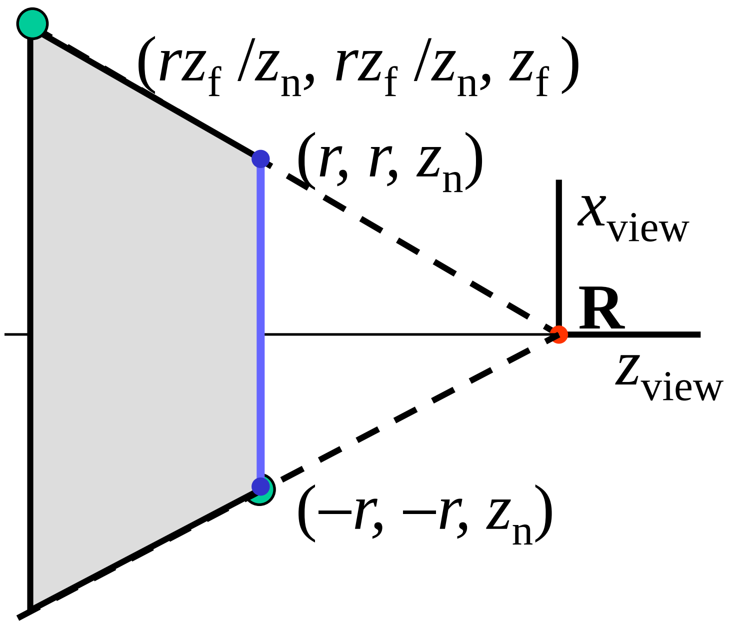
Perspective transformation can be described by:

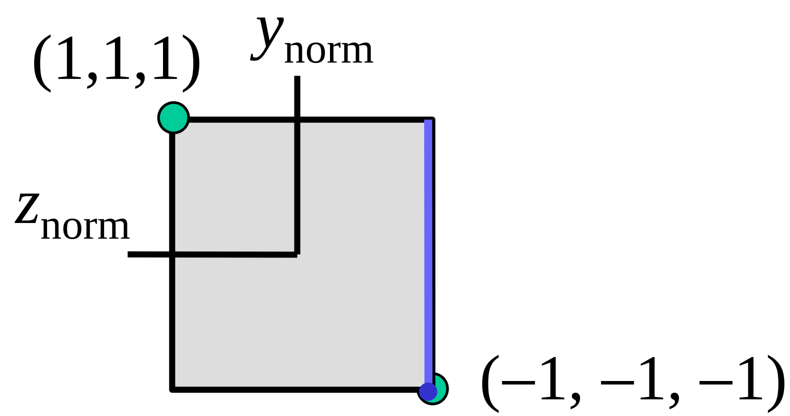
$$\begin{pmatrix} x_h \\ y_h \\ z_h \\ h \end{pmatrix} = \begin{pmatrix} s_{xx} & s_{xy} & s_{xz} & t_x \\ s_{yx} & s_{yy} & s_{yz} & t_y \\ s_{zx} & s_{zy} & s_{zz} & t_z \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_v \\ y_v \\ z_v \\ 1 \end{pmatrix}$$

such that projected coordinates are given by:

$$\begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} x_h / h \\ y_h / h \\ z_h / h \end{pmatrix}.$$

Perspective projection 10





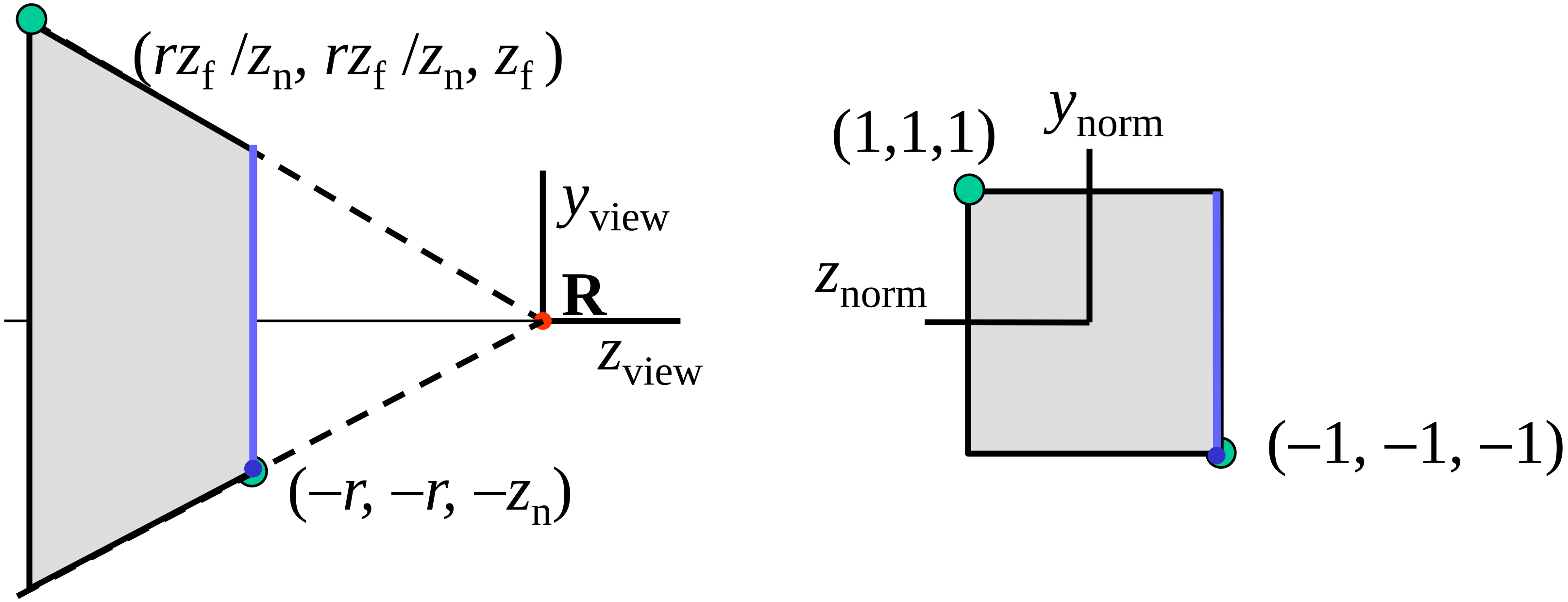
First x. Generic form is $x_p = (s_{xx}x + s_{xy}y + s_{xz}z + t_x)/-z$.

If $x = r$ and $z = z_n$, then $x_p = 1$.

If $x = rz_f / z_n$ and $z = z_f$, then also $x_p = 1$.

Elaboration gives : $s_{xx} = -z_n / r, s_{xy} = s_{xz} = t_x = 0$.

Perspective projection 11

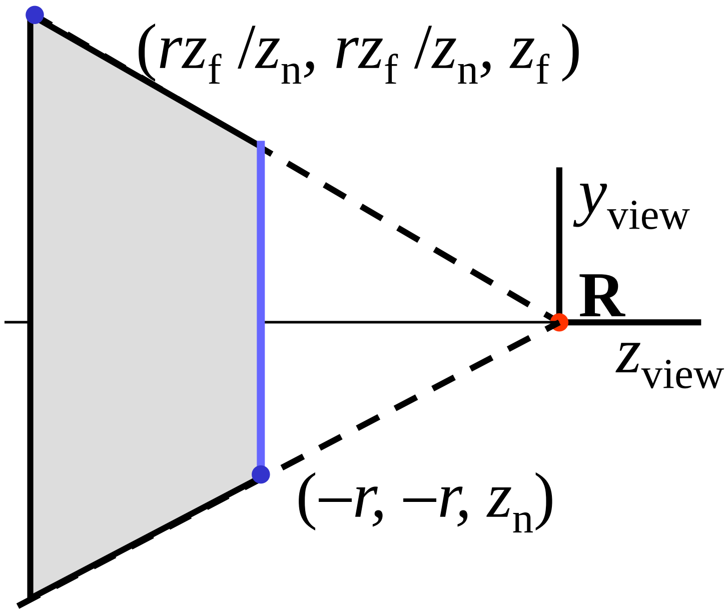


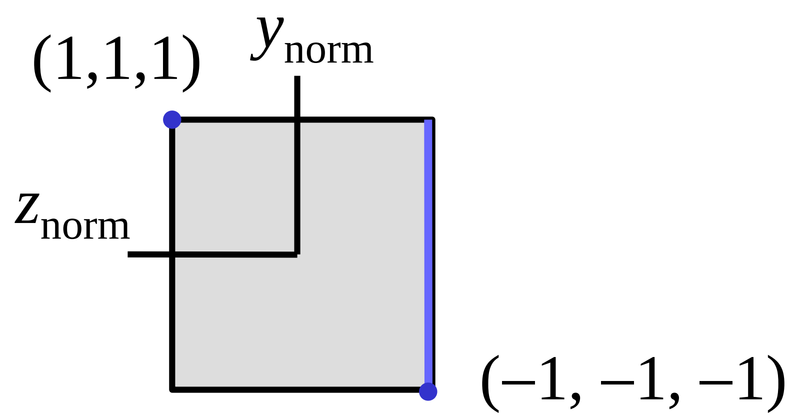
Next the y. Same as x, gives :

$$y_p = (-z_n / r) y / -z.$$

Or : $s_{yy} = (-z_n / r), s_{yx} = s_{yz} = t_y = 0.$

Perspective projection 12





Finally : z. Generic form is : $z_p = (s_{zx}x + s_{zy}y + s_{zz}z + t_z)/-z.$

If $z = z_n$, then $z_p = -1.$

If $z = z_f$, then $z_f = 1.$ Elaboration gives

$$s_{zz} = \frac{z_n + z_f}{z_n - z_f}, t_z = \frac{-2z_n z_f}{z_n - z_f}, s_{zx} = s_{zy} = 0.$$

Perspective projection 13

Perspective transformation can hence be described by:

$$\begin{pmatrix} x_h \\ y_h \\ z_h \\ h \end{pmatrix} = \begin{pmatrix} -z_n/r & 0 & 0 & 0 \\ 0 & -z_n/r & 0 & 0 \\ 0 & 0 & \dfrac{z_n+z_f}{z_n-z_f} & \dfrac{-2z_n z_f}{z_n-z_f} \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_v \\ y_v \\ z_v \\ 1 \end{pmatrix}$$

where the projected coordinates follow from:

$$\begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} = \begin{pmatrix} x_v/h \\ y_v/h \\ z_v/h \end{pmatrix}.$$