3^{rd} postulate of QM and examples on Tensor Product, Measurement and Density operator

3^{rd} Postulate of QM

- The possible measurement results w.r.t. a dynamical variable A are its eigenvalues λ_i.
- Let $\{|u_i\rangle\}_{i=1:n}$ be the set of orthonormal basis vectors. Using SDT, we can write operator A with the help of its eigenvalues and projection operators $P_i = |u_i\rangle\langle u_i|$ as

$$A = \sum_i \lambda_i P_i.$$

- The probability of obtaining measurement result λ_i is $Pr(\lambda_i) = \langle\psi|P_i|\psi\rangle = Tr(P_i|\psi\rangle\langle\psi|)$.
- The probability amplitude $c_i = \langle u_i|\psi\rangle$ gives the probability of obtaining measurement result λ_i as

$$Pr(\lambda_i) = \frac{|c_i|^2}{\langle\psi|\psi\rangle}.$$

3^{rd} Postulate of QM

- A measurement result causes the collapse of wave function as $|u_i\rangle$ such that the state of system in post-measurement is

$$\psi \xrightarrow{measurement} \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}.$$

Example

A system is in the state

$$|\psi\rangle = \frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle$$

where $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle, |u_5\rangle\}$ are a complete and orthonormal set of vectors. Each $|u_i\rangle$ is an eigenstate of the system's Hamiltonian corresponding to the possible measurement result $H|u_n\rangle = n\varepsilon|u_n\rangle$, where $n = 1, 2, 3, 4, 5$.

(a) Describe the set of projection operators corresponding to the possible measurement results.

(b) Determine the probability of obtaining each measurement result. What is the state of the system after measurement if we measure the energy to be 3ε?

(c) What is the average energy of the system?

Example 1

(a) The possible measurement results are $\varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon$, and 5ε. These measurement results correspond to the basis states $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$, $|u_4\rangle$, and $|u_5\rangle$, respectively. Hence the projection operators corresponding to each measurement result are

$$P_1 = |u_1\rangle\langle u_1|$$

$$P_2 = |u_2\rangle\langle u_2|$$

$$P_3 = |u_3\rangle\langle u_3|$$

$$P_4 = |u_4\rangle\langle u_4|$$

$$P_5 = |u_5\rangle\langle u_5|$$

Since the $|u_i\rangle$'s are a set of orthnormal basis vectors, the completeness relation is satisfied and

$$\sum_i P_i = I$$

Example 1

check and see if the state is normalized. This is done by calculating

$$\sum_{i=1}^{5} |c_i|^2$$

and seeing if the result is 1. We have

$$\begin{aligned}
\sum_{i=1}^{5} |c_i|^2 &= \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\frac{1}{\sqrt{19}}\right|^2 + \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\sqrt{\frac{6}{19}}\right|^2 \\
&= \frac{4}{19} + \frac{4}{19} + \frac{1}{19} + \frac{4}{19} + \frac{6}{19} \\
&= \frac{19}{19} = 1
\end{aligned}$$

The state is normalized, so we can proceed. Before doing so, recall that the fact that the basis states are orthonormal means that

$$\langle u_i | u_j \rangle = \delta_{ij}$$

So, in the first case, applying the Born rule we have

$$\Pr(\varepsilon) = |\langle u_2|\psi\rangle|^2 = \left|\langle u_1| \left(\frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle \right)\right|^2$$

Figure: Part(b)

Example 1

To calculate the remaining probabilities, let's use the projection operators

We find that

$$P_3|\psi\rangle = (|u_3\rangle\langle u_3|)\left(|\psi\rangle = \frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle\right)$$

$$= |u_3\rangle\left(\frac{1}{\sqrt{19}}\langle u_3|u_3\rangle\right) = \frac{1}{\sqrt{19}}|u_3\rangle$$

Therefore

$$\Pr(3\varepsilon) = \langle\psi|P_3|\psi\rangle$$

$$= \left(\frac{2}{\sqrt{19}}\langle u_1| + \frac{2}{\sqrt{19}}\langle u_2| + \frac{1}{\sqrt{19}}\langle u_3| + \frac{2}{\sqrt{19}}\langle u_4| + \sqrt{\frac{6}{19}}\langle u_5|\right)\left(\frac{1}{\sqrt{19}}|u_3\rangle\right)$$

Figure: Part(b)

Example 1

(c)

$$\langle H\rangle = \sum_{i=1}^{5} E_i \langle\psi|P_i|\psi\rangle = \varepsilon\langle\psi|P_1|\psi\rangle + 2\varepsilon\langle\psi|P_2|\psi\rangle + 3\varepsilon\langle\psi|P_3|\psi\rangle$$

$$+ 4\varepsilon\langle\psi|P_4|\psi\rangle + 5\varepsilon\langle\psi|P_5|\psi\rangle$$

$$= \varepsilon\frac{4}{19} + 2\varepsilon\frac{4}{19} + 3\varepsilon\frac{1}{19} + 4\varepsilon\frac{4}{19} + 5\varepsilon\frac{6}{19}$$

$$= \frac{61}{19}\varepsilon$$

Figure: Part(c)

Example 2

A qubit is in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$$

A measurement with respect to Y is made. Given that the eigenvalues of the Y matrix are ± 1, determine the probability that the measurement result is $+1$ and the probability that the measurement result is -1.

Example 2

First we verify that the state is normalized

$$\begin{aligned}\langle\psi|\psi\rangle &= \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{1}{2}\langle 1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right) \\ &= \frac{3}{4}\langle 0|0\rangle - \frac{\sqrt{3}}{4}\langle 1|0\rangle - \frac{\sqrt{3}}{4}\langle 0|1\rangle + \frac{1}{4}\langle 1|1\rangle \\ &= \frac{3}{4} + \frac{1}{4} = 1\end{aligned}$$

Since $\langle\psi|\psi\rangle = 1$ the state is normalized. Recall that $Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. You need to show that the eigenvectors of the Y matrix are

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

corresponding to the eigenvalues ± 1, respectively. The dual vectors in each case, found by computing the transpose of each vector can taking the complex conjugate of each element, are

$$\langle u_1| = (|u_1\rangle)^\dagger = \frac{1}{\sqrt{2}}(1 \quad -i), \quad \langle u_2| = (|u_2\rangle)^\dagger = \frac{1}{\sqrt{2}}(1 \quad i)$$

The projection operators corresponding to each possible measurement result are

$$\begin{aligned}P_{+1} &= |u_1\rangle\langle u_1| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}(1 \quad -i) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \\ P_{-1} &= |u_2\rangle\langle u_2| = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}(1 \quad i) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\end{aligned}$$

Example 2

Writing the state $|\psi\rangle$ as a column vector, we have

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}$$

Hence

$$P_{+1}|\psi\rangle = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}\sqrt{3}+i\\-1+i\sqrt{3}\end{pmatrix}$$

$$P_{-1}|\psi\rangle = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}\sqrt{3}-i\\-1-i\sqrt{3}\end{pmatrix}$$

Now, if a measurement is made of the Y observable, the probability of finding $+1$ is

$$\Pr(+1) = \langle\psi|P_{+1}|\psi\rangle = \frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\frac{1}{4}\begin{pmatrix}\sqrt{3}+i\\-1+i\sqrt{3}\end{pmatrix}$$

$$= \frac{1}{8}(3+i\sqrt{3}+1-i\sqrt{3}) = \frac{1}{8}(3+1) = \frac{1}{2}$$

Similarly find

$$\Pr(-1) = \langle\psi|P_{-1}|\psi\rangle = \frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\frac{1}{4}\begin{pmatrix}\sqrt{3}-i\\-1-i\sqrt{3}\end{pmatrix}$$

Tensor Products: Revisited

Suppose that A is a projection operator in H_1 where $A = |0\rangle\langle 0|$ and B is a projection operator in H_2 where $B = |1\rangle\langle 1|$. Find $A \otimes B|\psi\rangle$ where

$$|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

Solution

Using what we know about the action of tensor products of operators, we write

$$A \otimes B|\psi\rangle = A \otimes B\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}[(A|0\rangle)(B|1\rangle) + (A|1\rangle)(B|0\rangle)]$$

Now

$$A|0\rangle = (|0\rangle\langle 0|)|0\rangle = |0\rangle\langle 0|0\rangle = |0\rangle$$

$$A|1\rangle = (|0\rangle\langle 0|)|1\rangle = |0\rangle\langle 0|1\rangle = 0$$

$$B|0\rangle = (|1\rangle\langle 1|)|0\rangle = |0\rangle\langle 1|0\rangle = 0$$

$$B|1\rangle = (|1\rangle\langle 1|)|1\rangle = |1\rangle\langle 1|1\rangle = |1\rangle$$

Therefore we find that

$$A \otimes B|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|1\rangle$$

Tensor Products: Revisited

Suppose

$$|\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

Describe the action of $X \otimes I$ on the state.

Solution

The action of $X \otimes I$ is as follows:

$$X \otimes I|\psi\rangle = X \otimes I\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}[(X|0\rangle)|0\rangle - (X|1\rangle)|1\rangle]$$

$$= \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

Measurement on Composite Systems: Example

Describe the action of the operators $P_0 \otimes I$ and $I \otimes P_1$ on the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Measurement on Composite Systems: Example

Solution

The first operator, $P_0 \otimes I$, tells us to apply the projection operator, $P_0 = |0\rangle\langle 0|$, to the *first qubit* and to leave the second qubit alone. The result is

$$P_0 \otimes I|\psi\rangle = \frac{1}{\sqrt{2}}[(|0\rangle\langle 0|0\rangle) \otimes |1\rangle - (|0\rangle\langle 0|1\rangle \otimes |0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

Interestingly, applying a projective measurement to the first qubit causes the second qubit to assume a definite state. As we will see in the next chapter, this is a property of entangled systems. Apparently it doesn't matter if the qubits are spatially separated for a collapse of the system to occur.

To find the properly normalized state of the system after measurement we use (6.20). We have

$$\langle\psi|P_0 \otimes I|\psi\rangle = \left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right)\frac{|01\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle\langle 1|1\rangle - \langle 1|0\rangle\langle 0|1\rangle}{2} = \frac{1}{2}$$

The state after measurement is

$$|\psi'\rangle = \frac{P_0 \otimes I|\psi\rangle}{\sqrt{\langle\psi|P_0 \otimes I|\psi\rangle}} = \frac{|01\rangle/\sqrt{2}}{(1/\sqrt{2})} = |01\rangle$$

As can be seen, while applying (6.20) in the single qubit case can seem like overkill, in this case this allows us to quickly write down the properly normalized state after measurement.

Measurement on Composite Systems: Example

The second operator, $I \otimes P_1$, tells us to leave the *first qubit alone* and to apply the projection operator $P_1 = |1\rangle\langle 1|$ to the second qubit. This gives

$$I \otimes P_1|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes (|1\rangle\langle 1|1\rangle) - |1\rangle \otimes (|1\rangle\langle 1|0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

We have therefore the same state, but this time doing the projective measurement represented by $P_1 = |1\rangle\langle 1|$ the second qubit has forced the first qubit into the state $|0\rangle$. Let's redo the calculation using matrices. The operator is

$$I \otimes P_1 = \begin{pmatrix} 1 \cdot P_1 & 0 \cdot P_1 \\ 0 \cdot P_1 & 1 \cdot P_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then we have

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So the state of the system prior to measurement is

Measurement on Composite Systems: Example

A system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

(a) What is the probability that measurement finds the system in the state $|\phi\rangle = |01\rangle$?

(b) What is the probability that measurement finds the first qubit in the state $|0\rangle$? What is the state of the system after measurement?

Measurement on Composite Systems: Example

(a) Given that the system is in the state $|\psi\rangle$, the probability of finding it in the state $|\phi\rangle = |01\rangle$ is calculated using the Born rule, which is $\Pr = |\langle\phi|\psi\rangle|^2$. Since $\langle 0|1\rangle = \langle 1|0\rangle = 0$, we have

$$\begin{aligned}
\langle\phi|\psi\rangle &= \langle 01| \left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle \right) \\
&= \frac{1}{\sqrt{8}}\langle 0|0\rangle\langle 1|0\rangle + \sqrt{\frac{3}{8}}\langle 0|0\rangle\langle 1|1\rangle + \frac{1}{2}\langle 0|1\rangle\langle 1|0\rangle + \frac{1}{2}\langle 0|1\rangle\langle 1|1\rangle \\
&= \sqrt{\frac{3}{8}}
\end{aligned}$$

Therefore the probability is

$$\Pr = |\langle\phi|\psi\rangle|^2 = \frac{3}{8}$$

Measurement on Composite Systems: Example

(b) To find the probability that measurement finds the first qubit in the state $|0\rangle$, we can apply $P_0 \otimes I = |0\rangle\langle 0| \otimes I$ to the state. So the projection operator P_0 is applied to the first qubit and the identity operator to the second qubit, leaving the second qubit unchanged.

Measurement on Composite Systems: Example

(b) To find the probability that measurement finds the first qubit in the state $|0\rangle$, we can apply $P_0 \otimes I = |0\rangle\langle 0| \otimes I$ to the state. So the projection operator P_0 is applied to the first qubit and the identity operator to the second qubit, leaving the second qubit unchanged. This obtains

$$\begin{aligned}
P_0 \otimes I|\psi\rangle &= (|0\rangle\langle 0| \otimes I)\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle\right) \\
&= \frac{1}{\sqrt{8}}|0\rangle\langle 0|0\rangle \otimes |0\rangle + \sqrt{\frac{3}{8}}|0\rangle\langle 0|0\rangle \otimes |1\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle\langle 0|1\rangle \otimes |1\rangle \\
&= \frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle
\end{aligned}$$

The probability of obtaining this result is

$$\begin{aligned}
\Pr &= \langle\psi|P_0 \otimes I|\psi\rangle \\
&= \left(\frac{1}{\sqrt{8}}\langle 00| + \sqrt{\frac{3}{8}}\langle 01| + \frac{1}{2}\langle 10| + \frac{1}{2}\langle 11|\right)\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle\right) \\
&= \frac{1}{8} + \frac{3}{8} = \frac{1}{2}
\end{aligned}$$

The state of the system after measurement using (6.20) is found to be

$$\begin{aligned}
|\psi'\rangle &= \frac{\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle}{\sqrt{\langle\psi|P_0 \otimes I|\psi\rangle}} = \sqrt{2}\left(\frac{1}{\sqrt{8}}|00\rangle + \sqrt{\frac{3}{8}}|01\rangle\right) \\
&= \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}|01\rangle
\end{aligned}$$

Measurement on Composite Systems: Example

A three-qubit system is in the state

$$|\psi\rangle = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle + \frac{i}{2}|111\rangle$$

(a) Is the state normalized? What is the probability that the system is found in the state $|000\rangle$ if all 3 qubits are measured?

(b) What is the probability that a measurement on the first qubit only gives 0? What is the postmeasurement state of the system?

Measurement on Composite Systems: Example

(b) The probability that a measurement on the first qubit is zero can be found by acting on the state with the operator $P_0 \otimes I \otimes I$ and computing $\langle \psi | P_0 \otimes I \otimes I | \psi \rangle$. This will project onto the $|0\rangle$ state for the first qubit while leaving the second and third qubits

alone. We find that

$$
\begin{aligned}
P_0 \otimes I \otimes I|\psi\rangle &= \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)(|0\rangle\langle 0| \otimes I \otimes I)|000\rangle + \frac{1}{\sqrt{2}}(|0\rangle\langle 0| \otimes I \otimes I)|001\rangle \\
&\quad + \frac{1}{\sqrt{10}}(|0\rangle\langle 0| \otimes I \otimes I)|011\rangle + \frac{i}{2}(|0\rangle\langle 0| \otimes I \otimes I)|111\rangle \\
&= \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle
\end{aligned}
$$

The last term vanishes, since $\langle 0|1\rangle = 0$ So

$$
\frac{i}{2}(|0\rangle\langle 0| \otimes I \otimes I)|111\rangle = \frac{i}{2}(|0\rangle\langle 0|1\rangle) \otimes |1\rangle \otimes |1\rangle = 0
$$

Hence the probability that measurement on the first qubit finds 0 is

$$
\langle \psi | P_0 \otimes I \otimes I | \psi \rangle = \left|\frac{\sqrt{2}+i}{\sqrt{20}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{10}}\right|^2 = \frac{3}{20} + \frac{1}{2} + \frac{1}{10} = \frac{3}{4}
$$

The postmeasurement state is

$$
|\psi'\rangle = \frac{P_0 \otimes I \otimes I|\psi\rangle}{\sqrt{\langle \psi | P_0 \otimes I \otimes I | \psi \rangle}} = \sqrt{\frac{4}{3}}\left(\left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle\right)
$$

Some concepts

We know that $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Also,

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

such that

$$|+\rangle + |-\rangle = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sqrt{2}|0\rangle,$$

Hence,

$$|0\rangle = \frac{1}{\sqrt{2}} \Big(|+\rangle + |-\rangle \Big).$$

Some concepts

We know that $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Also,

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly,

$$|+\rangle - |-\rangle = \sqrt{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sqrt{2}|1\rangle,$$

Hence,

$$|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - |-\rangle$$

GHZ state

It is the three qubit state given by

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \tag{1}$$

GHZ state

It is the three qubit state given by

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \tag{2}$$

Example 6.7

A system is in the GHZ state where

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

Suppose that an observable $A = \sigma_x \otimes \sigma_y \otimes \sigma_z$. If $|\pm\rangle = |0\rangle \pm |1\rangle/\sqrt{2}$, what is the probability that measurement finds the system in the state $|+++\rangle$ and in the state $|---\rangle$ when the system is in the state $A|\psi\rangle$? What is the expectation value of A?

Measurement on Composite Systems: Example

Recall the action of the Pauli operators:

$$\sigma_x|0\rangle = |1\rangle, \qquad \sigma_x|1\rangle = |0\rangle$$
$$\sigma_y|0\rangle = -i|1\rangle, \quad \sigma_y|1\rangle = i|0\rangle$$
$$\sigma_z|0\rangle = |0\rangle, \qquad \sigma_z|1\rangle = -|1\rangle$$

Write

$$\begin{aligned} A|\psi\rangle &= (\sigma_x \otimes \sigma_y \otimes \sigma_z)\left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right) \\ &= -\frac{i}{\sqrt{2}}(|110\rangle + |001\rangle) \end{aligned}$$

Example having GHZ state

Recall the action of the Pauli operators:

$$\sigma_x|0\rangle = |1\rangle, \qquad \sigma_x|1\rangle = |0\rangle$$
$$\sigma_y|0\rangle = -i|1\rangle, \quad \sigma_y|1\rangle = i|0\rangle$$
$$\sigma_z|0\rangle = |0\rangle, \qquad \sigma_z|1\rangle = -|1\rangle$$

Write

$$\begin{aligned} A|\psi\rangle &= (\sigma_x \otimes \sigma_y \otimes \sigma_z)\left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right) \\ &= -\frac{i}{\sqrt{2}}(|110\rangle + |001\rangle) \end{aligned}$$

Then rewrite this in the $|\pm\rangle = |0\rangle \pm |1\rangle/\sqrt{2}$ basis. For the first term,

$$\begin{aligned} |110\rangle &= \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) \\ &= \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|++\rangle + |+-\rangle - |-+\rangle - |--\rangle}{2}\right) \\ &= \frac{1}{2\sqrt{2}}(|+++\rangle + |++-\rangle - |+-+\rangle - |+--\rangle - |-++\rangle \\ &\quad - |-+-\rangle + |--+\rangle + |---\rangle) \end{aligned}$$

Similarly for the second term,

$$|001\rangle = \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)$$

Example having GHZ state

Therefore in the $|\pm\rangle$ basis the state is

$$A|\psi\rangle = -\frac{i}{2}(|{+}{+}{+}\rangle - |{+}{-}{-}\rangle - |{-}{+}{-}\rangle + |{-}{-}{+}\rangle)$$

The probability that the system is found in the $|{-}{-}{-}\rangle$ state is zero, while the probability it is found in the $|{+}{+}{+}\rangle$ state is

$$|\langle + + +|A|\psi\rangle|^2 = \left|-\frac{i}{2}\right|^2 = \frac{1}{4}$$

In the $|\pm\rangle$ basis the initial state is

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(|{+}{+}{+}\rangle + |{+}{-}{-}\rangle + |{-}{+}{-}\rangle + |{-}{-}{+}\rangle)$$

The expectation value is

$$\langle\psi|A|\psi\rangle = \frac{1}{2\sqrt{2}}(\langle{+}{+}{+}| + \langle{+}{-}{-}| + \langle{-}{+}{-}| + \langle{-}{-}{+}|)$$

Example

A two qubit system is in the state

$$|\phi\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$$

A Y gate is applied to the first qubit. After this is done, what are the possible measurement results if both qubits are measured, and what are the respective probabilities of each measurement result?

Example

The action of the Y gate on the computational basis states is

$$Y|0\rangle = i|1\rangle, \quad Y|1\rangle = -i|0\rangle$$

Hence

$$Y \otimes I|\phi\rangle = \frac{\sqrt{3}}{2}(Y \otimes I)|00\rangle + \frac{1}{2}(Y \otimes I)|11\rangle = i\frac{\sqrt{3}}{2}|10\rangle - \frac{i}{2}|01\rangle$$

If both qubits are measured, the possible measurement results are 10 and 01. The probability of finding 10 is

$$\left|i\frac{\sqrt{3}}{2}\right|^2 = \left(i\frac{\sqrt{3}}{2}\right)\left(-i\frac{\sqrt{3}}{2}\right) = \frac{3}{4}$$

The probability of finding 01 is

$$\left|\frac{i}{2}\right|^2 = \left(\frac{i}{2}\right)\left(-\frac{i}{2}\right) = \frac{1}{4}$$

Example

Does the matrix

$$\rho = \begin{pmatrix} \frac{1}{4} & \frac{1-i}{4} \\ \frac{1-i}{4} & \frac{3}{4} \end{pmatrix}$$

represent a density operator?

Example

Does the matrix

$$\rho = \begin{pmatrix} \frac{1}{4} & \frac{1-i}{4} \\ \frac{1-i}{4} & \frac{3}{4} \end{pmatrix}$$

represent a density operator?

We see that the matrix has unit trace

$$Tr(\rho) = \frac{1}{4} + \frac{3}{4} = 1$$

So it looks like it might be a valid density operator. However,

$$\rho^\dagger = \begin{pmatrix} \frac{1}{4} & \frac{1+i}{4} \\ \frac{1+i}{4} & \frac{3}{4} \end{pmatrix} \neq \rho$$

Since the matrix is not Hermitian, it cannot represent a density operator.

Example

A system is found to be in the state

$$|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$$

(a) Write down the density operator for this state.

(b) Write down the matrix representation of the density operator in the $\{|0\rangle, |1\rangle\}$ basis. Verify that $Tr(\rho) = 1$, and show this is a pure state.

(c) A measurement of Z is made. Calculate the probability that the system is found in the state $|0\rangle$ and the probability that the system is found in the state $|1\rangle$.

(d) Find $\langle X \rangle$.

Example

(a) To write down the density operator, first we construct the dual vector $\langle\psi|$. This can be done by inspection

$$\langle\psi| = \frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|$$

The density operator is

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)\left(\frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|\right) \\ &= \frac{1}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|\end{aligned}$$

(b) The matrix representation of the density operator in the $\{|0\rangle, |1\rangle\}$ basis is found by writing

$$[\rho] = \begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}$$

For the state given in this problem, we have

$$\begin{aligned}\langle 0|\rho|0\rangle &= \frac{1}{5} \\ \langle 0|\rho|1\rangle &= \frac{2}{5} = \langle 1|\rho|0\rangle \\ \langle 1|\rho|1\rangle &= \frac{4}{5}\end{aligned}$$

Example

So in the $\{|0\rangle, |1\rangle\}$ basis the density matrix is

$$\rho = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

The trace is just the sum of the diagonal elements. In this case

$$Tr(\rho) = \frac{1}{5} + \frac{4}{5} = 1$$

To determine whether or not this is a pure state, we need to determine if $Tr(\rho^2) = 1$. Now

$$\rho^2 = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{25} + \frac{4}{25} & \frac{2}{25} + \frac{8}{25} \\ \frac{2}{25} + \frac{8}{25} & \frac{4}{25} + \frac{16}{25} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{25} & \frac{10}{25} \\ \frac{10}{25} & \frac{20}{25} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \rho$$

Example

(c) In this simple example we can see by inspection that the probability that the system is in state $|0\rangle$ is 1/5 while the probability that the system is found in state $|1\rangle$ is 4/5. Let's see if we can verify this using the density operator formalism. First, we write down the projection operators in matrix form. The measurement operator that corresponds to the measurement result $|0\rangle$ is $P_0 = |0\rangle\langle 0|$ while the measurement operator $P_1 = |1\rangle\langle 1|$. The matrix representation in the given basis is

$$P_0 = \begin{pmatrix} \langle 0|P_0|0\rangle & \langle 0|P_0|1\rangle \\ \langle 1|P_0|0\rangle & \langle 1|P_0|1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_1 = \begin{pmatrix} \langle 0|P_1|0\rangle & \langle 0|P_1|1\rangle \\ \langle 1|P_1|0\rangle & \langle 1|P_1|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The probability of finding the system in state $|0\rangle$ is

$$p(0) = Tr(P_0\rho) = Tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}\right] = Tr\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 0 \end{pmatrix} = \frac{1}{5}$$

The probability of finding the system in state $|1\rangle$ is

$$p(1) = Tr(P_1\rho) = Tr\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}\right] = Tr\begin{pmatrix} 0 & 0 \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \frac{4}{5}$$

Example

(d) We can find the expectation value of X by calculating $Tr(X\rho)$. First, let's do the matrix multiplication:

$$X\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

The trace is the sum of the diagonal elements of this matrix

$$\langle X \rangle = Tr(X\rho) = Tr \begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}$$

Example

Recalling the basis states

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Suppose that a statistical mixture has 75% in the $|+\rangle$ state and 25% in the $|-\rangle$ state. A member of the ensemble is drawn. What are the probabilities of finding it in the $|0\rangle$ state and $|1\rangle$ state, respectively? Show that it is a mixed state and that this calculation doesn't depend on the basis we use. Contrast the statistical mixture with 75% of the systems in the $|+\rangle$ state and 25% of the systems in the $|-\rangle$ state, with the pure state described by

$$|\psi\rangle = \sqrt{\frac{3}{4}}|+\rangle + \sqrt{\frac{1}{4}}|-\rangle$$

Write down the probability that a measurement of the pure state $|\psi\rangle$ finds the system in $|0\rangle$.

Example

The density operator for the ensemble is given by

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

Now

$$|+\rangle\langle+| = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right) = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$|-\rangle\langle-| = \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\right) = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)$$

So we find that

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

Example

The density operator for the ensemble is given by

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

Now

$$|+\rangle\langle+| = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right) = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$|-\rangle\langle-| = \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\right) = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)$$

So we find that

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

The respective probabilities are

$$\begin{aligned}
p(0) &= Tr(\rho|0\rangle\langle 0|) \\
&= \langle 0|\rho|0\rangle \\
&= \langle 0|\left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right)|0\rangle = \frac{1}{2} \\
p(1) &= Tr(\rho|1\rangle\langle 1|) \\
&= \langle 1|\rho|1\rangle \\
&= \langle 1|\left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right)|1\rangle = \frac{1}{2}
\end{aligned}$$

Example

In the $\{|+\rangle, |-\rangle\}$ basis the density matrix is

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Notice that the trace of the density matrix is one, as it should be. Also note that the off-diagonal terms are zero. Squaring, we find that

$$\rho^2 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{9}{16} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$$

$$\Rightarrow Tr(\rho^2) = \frac{9}{16} + \frac{1}{16} = \frac{10}{16} = \frac{5}{8}$$

Example

Let's look at the matrix in the $\{|0\rangle, |1\rangle\}$ basis. We find that

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

So

$$\rho^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{16} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{16} \end{pmatrix}$$

$$\Rightarrow Tr(\rho^2) = \frac{5}{16} + \frac{5}{16} = \frac{10}{16} = \frac{5}{8}$$

Once again, notice that $Tr(\rho^2) = 5/8 < 1$, confirming that this is a mixed state.

Example

For the last part of this problem, recall that for the statistical mixture the probability of obtaining measurement result 0 was $p_m(0) = 1/2$, where we used the subscript m to remind ourselves this is the probability given the mixed state. Now we consider the pure state:

$$|\psi\rangle = \sqrt{\frac{3}{4}}|+\rangle + \sqrt{\frac{1}{4}}|-\rangle$$

This state has a superficial resemblance to the statistical mixture with 75% of the systems in the $|+\rangle$ state and 25% of the systems in the $|-\rangle$ state, since a measurement in the $\{|+\rangle, |-\rangle\}$ basis gives $|+\rangle$ with a probability $p(+) = 3/4 = 0.75$ and $|-\rangle$ with a probability $p(-) = 1/4 = 0.25$. For example, look at the density operator for the mixed state:

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$$

The probability of finding $|+\rangle$ is

$$p_m(+) = Tr(|+\rangle\langle+|\rho) = \langle+|\rho|+\rangle = \langle+|\left(\frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|\right)|+\rangle = \frac{3}{4}$$

However, it turns out that if we instead consider measurements with respect to the $\{|0\rangle, |1\rangle\}$ basis, we will find dramatically different results. Let's rewrite the given pure state in that basis. We find that

$$|\psi\rangle = \sqrt{\frac{3}{4}}|+\rangle + \sqrt{\frac{1}{4}}|-\rangle = \sqrt{\frac{3}{4}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \sqrt{\frac{1}{4}}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$