# Quantum Measurement Theory

# Introduction

WHAT IS MEASUREMENT?

- It helps in RETRIEVING information from the COMUTATIONAL SYSTEM.

But it alters the state of the system in an IRREVERSIBLE WAY.

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \longrightarrow \begin{bmatrix}1\\ 0\end{bmatrix}$$


$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \xrightarrow{M} \begin{bmatrix}1\\ 0\end{bmatrix} \text{ or } \begin{bmatrix}0\\ 1\end{bmatrix}$$

# Loosing Quantum Superposition

Suppose $\exists$ a qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

If we measure this QUBIT, then

$$|\psi\rangle \xrightarrow{\text{Measuring}} |0\rangle$$

or

$$|\psi\rangle \xrightarrow{\text{Measuring}} |1\rangle$$

Hence, $|\psi\rangle \longmapsto |0\rangle$ or $|\psi\rangle \longmapsto |1\rangle$.

But now the SUPERPOSITIONED STATE $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ will be LOST. ie We can never DETERMINE what is $\alpha$ or $\beta$.

# Closed System vs Open System

This MEASUREMENT can be done with a MEASURING device KNOWN as ANCILLA.

s.t.

It involves SOME type of interaction with LARGER ENVIRONMENT in which the QUANTUM SYSTEM (that is being measured) exists.

This TYPE of SYSTEM is known as OPEN SYSTEM.

# Closed System vs Open System

Time Evolution of a CLOSED SYSTEM is

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$U = e^{-iHt/\hbar}$$

$$\Rightarrow \quad |\psi(t)\rangle = \underbrace{e^{-iHt/\hbar}}_{\text{Unitary EVOLUTION operator}} |\psi(0)\rangle$$

SHOWS how the Quantum STATES evolve with TIME.

$\therefore$ STATE OF the system at time t is

$$\rho_t = U \rho_0 U^\dagger$$

Now, Dynamics of Quantum System is TRACE - PRESERVING.

# Trace preserving vs Trace decreasing

Why?

$Tr(\rho_0) = 1$ at some initial time $t_0$

Then, after evolution of the system at time $t$ $(> t_0)$,

$Tr(\rho_t) = 1$

$$|\psi(0)\rangle \longrightarrow |\psi(1)\rangle \longrightarrow |\psi(2)\rangle \cdots\cdots \rightarrow |\psi(t)\rangle$$

# Trace preserving vs Trace decreasing

But MEASUREMENT is TRACE-DECREASING Quantum Operations.

Suppose $M_m$ is a MEASUREMENT OPERATOR s.t. it TRANSFORMS a density operator $\rho$ to $\rho'$ as

$$\rho' = M_m \rho M_m^\dagger$$

Now $Tr(\rho') \leq 1.$

# Code where measurement destroys superposition

```
# code without measurement
qc = QuantumCircuit(1)
initial_state = [1/sqrt(2), 1j/sqrt(2)]
qc.initialize(initial_state, 0)
qc.save_statevector()
qobj = assemble(qc)
state = sim.run(qobj).result().get_statevector()#pri
qobj = assemble(qc)
results = sim.run(qobj).result().get_counts()
plot_histogram(results)

#code with measurement
qc = QuantumCircuit(1)
initial_state = [1/sqrt(2), 1j/sqrt(2)]
qc.initialize(initial_state, 0)
qc.measure_all()
qc.save_statevector()
qc.draw()
qobj = assemble(qc)
state = sim.run(qobj).result().get_statevector()
print("State of Measured Qubit = " + str(state))
result = sim.run(qobj).result().get_counts()
plot_histogram(result)
```

(a)

Figure: Python code using qiskit

# Projective Measurements

- It is the oldest measurment model proposed by Von Neumann and hence also known as Von Neumann measurements.

# Projective Measurements

We know that a system may exist in various mutually exclusive states. For example,

- State of a qubit in $\mathbb{C}^2$ is
$$\{|0\rangle, |1\rangle\},$$
- State of an atom
$$\{|g\rangle, |e\rangle\},$$
where $|g\rangle$ represents the grounded state while $|e\rangle$ represnts the excited state of an atom,
- Locating the position of a particle
$$\{x_1, x_2\}.$$

# Projective Measurements

Projective Measurements is based on the following idea

> What is the state of a system in a given scenario w.r.t. a given set of mutually exclusive possible states?

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> What is the state of a system in a given scenario w.r.t. a given set of mutually exclusive possible states?

The examples of such mutually exclusive possible states are being given in previous slide!

# Projective Measurements

These possibilities are described with the help of

*Projection Operators*

# Projection operators: Revisited

- Projection Operator $P$ is Hermitian i.e.

$$P = P^{\dagger}.$$

- Equal to its square i.e.

$$P = P^2.$$

# A few more number of their properties

Suppose $\exists$ projection operators $P_1$ and $P_2$. Then,

- $P_1 \perp P_2, \quad \text{if} \quad P_1 P_2 = 0,$
  or we can say that both are orthogonal to each other if their product is zero.

  It further implies that, for a state $|\psi\rangle$, we can say

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  $$P_1 P_2 |\psi\rangle = 0.$$

- $P_1 P_2$ is also projection operator if both commute. If we have a set of such operators as

  $$\{P_1, P_2, \ldots, P_n\},$$

  then the mutually orthogonal relation in between them can be represented as

  $$P_i P_j = \delta_{ij} P_i,$$

  where $\delta_{ij} = 0$ (or orthogonal) if $i \neq j$, otherwise it is 1 (or $P_i$ is equal to its square).

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  where $\delta_{ij} = 0$ (or orthogonal) if $i \neq j$, otherwise it is 1 (or $P_i$ is equal to its square).

- $P_1 + P_2$ is a projection operator iff they commute.

# Projection operators and measurements

- If we are determining the measurement results on a system (with mutually exclusive states like $\{|0\rangle, |1\rangle\}$ for a qubit), each of these results corresponds to an orthogonal projection operator $P_i$ that is being applied on state space ($|\psi(t)\rangle$) of a system i.e.

$$P_i|\psi(t)\rangle,$$

  or we can write in short as

$$P_i|\psi\rangle.$$

# Projective measurements

- A complete set of orthogonal projection operators is being defined as

$$\boxed{\sum_i P_i = I.}$$

# Projective measurements

- A complete set of orthogonal projection operators is being defined as

$$\boxed{\sum_i P_i = I.} \tag{1}$$

- This complete set of the orthogonal projection operators

$$\{P_1, P_2, \ldots, P_n\},$$

corresponds to the set of mutually exclusive measurement results such as

$$\{M_1, M_2, \ldots, M_m\},$$

Equation (1) is equivalent to summation of probablities values as one.

# Projective measurements (contd.)

- If $dim(\mathbb{H}) = n$ and the number of projection operators is $m$, then
$$\boxed{m \leq n}.$$

# Projective measurements (contd.)

- If $dim(\mathbb{H}) = n$ and the number of projection operators is $m$, then
$$\boxed{m \leq n}.$$
- Suppose that $m = n$ and the system is in some state $|\psi\rangle$. Then, the probability of finding an $i$-th outcome w.r.t. a measurement is

$$\begin{aligned} Pr(i) &= |P_i|\psi\rangle|^2, \\ &= (P_i|\psi\rangle)^\dagger(P_i|\psi\rangle), \\ &= \langle\psi|P_i^2|\psi\rangle, \\ &= \langle\psi|P_i|\psi\rangle, && (4) \\ &= Tr(P_i|\psi\rangle\langle\psi|). && (5) \end{aligned}$$

# Born Rule

Let $\{|u_i\rangle\}_{i=1:n}$ be the set of orthonormal basis vectors and each vector is also the eigenvector associated with an operator A w.r.t. an eigenvalue $\{\lambda_i\}_{i=1:n}$. Then, according to SDT, we have

$$A = \sum_{i=1}^{n} \lambda_i |u_i\rangle\langle u_i| = \sum_{i=1}^{n} \lambda_i P_i,$$

where projection operator $P_i$ is in corresponding to the measurement outcome $\lambda_i$.

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where projection operator $P_i$ is in corresponding to the measurement outcome $\lambda_i$.
Then

$$\boxed{Pr(i) = \begin{cases} |\langle u_i|\psi\rangle|^2, & \text{if } \psi \text{ is normalized,} \\ \frac{|\langle u_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}, & \text{otherwise,} \end{cases}}$$

which is also known as Born rule.

# Degenerate Eigenvalues

It should be recalled that if there are two or more linearly independent eigenvectors with the same eigenvalue, that eigenvalue is said to be degenerate. For instance,

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

would be having eigenvalue as 3 for

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

# Born Rule (contd.)

If $\exists$ a $\lambda_i$ which is degenerated and there are $m$ number of eigenvectors w.r.t. such degenerated eigenvalue, then the probability is the summation over all eigenvectors that correspond to $\lambda_i$ i.e.

$$\boxed{Pr(i) = \begin{cases} \sum_{j=1}^{m} |\langle u_j|\psi\rangle|^2, & \text{if } \psi \text{ is normalized,} \\ \\ \frac{\sum_{j=1}^{m} |\langle u_j|\psi\rangle|^2}{\langle\psi|\psi\rangle}, & \text{otherwise,} \end{cases}}$$

where $|u_j\rangle$ is the $j$-th instance of eigenvector w.r.t. degenerated eigenvalue $\lambda_i$.

# Collapse of wave function $|\psi(t)\rangle$

If the state of system before measurement is $|\psi(t)\rangle$ (or just $|\psi\rangle$) such that it can be written as the linear combination of orthnonormal basis vectors $|u_i\rangle$

$$|\psi\rangle = \sum_{i=1}^{n} c_i |u_i\rangle, \tag{6}$$

where $c_i$ is the complex coefficient.

## Collapse of wave function $|\psi(t)\rangle$

If the state of system before measurement is $|\psi(t)\rangle$ (or just $|\psi\rangle$) such that it can be written as the linear combination of orthnonormal basis vectors $|u_i\rangle$

$$|\psi\rangle = \sum_{i=1}^{n} c_i |u_i\rangle, \tag{8}$$

where $c_i$ is the complex coefficient.
Then, after the measurement, it collapses to that basis state corresponding to which we have obtained the measurement result.
Then, the measured state is

$$\boxed{|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}}, \tag{9}$$

where $\sqrt{\langle\psi|P_i|\psi\rangle}$ ensures that $|\psi'\rangle$ is normalized.

# Expectation of operator $A$

Since we know that

$$\begin{aligned} A &= \sum_{i=1}^{n} \lambda_i |u_i\rangle\langle u_i|, \\ &= \sum_{i=1}^{n} \lambda_i P_i, \end{aligned} \tag{10}$$

and measurement result w.r.t. all the projections is equivalent to summation of probabilities to be one. Then,

$$\boxed{\langle A\rangle = \sum_i \lambda_i \langle\psi|P_i|\psi\rangle}. \tag{11}$$

# Example

A system is in the state

$$|\psi\rangle = \frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle$$

where $\{|u_1\rangle, |u_2\rangle, |u_3\rangle, |u_4\rangle, |u_5\rangle\}$ are a complete and orthonormal set of vectors. Each $|u_i\rangle$ is an eigenstate of the system's Hamiltonian corresponding to the possible measurement result $H|u_n\rangle = n\varepsilon|u_n\rangle$, where $n = 1, 2, 3, 4, 5$.

(a) Describe the set of projection operators corresponding to the possible measurement results.

(b) Determine the probability of obtaining each measurement result. What is the state of the system after measurement if we measure the energy to be $3\varepsilon$?

(c) What is the average energy of the system?

# Example 1

(a) The possible measurement results are $\varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon$, and $5\varepsilon$. These measurement results correspond to the basis states $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$, $|u_4\rangle$, and $|u_5\rangle$, respectively. Hence the projection operators corresponding to each measurement result are

$$P_1 = |u_1\rangle\langle u_1|$$

$$P_2 = |u_2\rangle\langle u_2|$$

$$P_3 = |u_3\rangle\langle u_3|$$

$$P_4 = |u_4\rangle\langle u_4|$$

$$P_5 = |u_5\rangle\langle u_5|$$

Since the $|u_i\rangle$'s are a set of orthnormal basis vectors, the completeness relation is satisfied and

$$\sum_i P_i = I$$

# Example 1

check and see if the state is normalized. This is done by calculating

$$\sum_{i=1}^{5} |c_i|^2$$

and seeing if the result is 1. We have

$$\begin{aligned}
\sum_{i=1}^{5} |c_i|^2 &= \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\frac{1}{\sqrt{19}}\right|^2 + \left|\frac{2}{\sqrt{19}}\right|^2 + \left|\sqrt{\frac{6}{19}}\right|^2 \\
&= \frac{4}{19} + \frac{4}{19} + \frac{1}{19} + \frac{4}{19} + \frac{6}{19} \\
&= \frac{19}{19} = 1
\end{aligned}$$

The state is normalized, so we can proceed. Before doing so, recall that the fact that the basis states are orthonormal means that

$$\langle u_i | u_j \rangle = \delta_{ij}$$

So, in the first case, applying the Born rule we have

$$\Pr(\varepsilon) = |\langle u_2 | \psi \rangle|^2 = \left| \langle u_1 | \left( \frac{2}{\sqrt{19}} |u_1\rangle + \frac{2}{\sqrt{19}} |u_2\rangle + \frac{1}{\sqrt{19}} |u_3\rangle + \frac{2}{\sqrt{19}} |u_4\rangle + \sqrt{\frac{6}{19}} |u_5\rangle \right) \right|^2$$

Figure: Part(b)

# Example 1

To calculate the remaining probabilities, let's use the projection operators
We find that

$$P_3|\psi\rangle = (|u_3\rangle\langle u_3|)\left(|\psi\rangle = \frac{2}{\sqrt{19}}|u_1\rangle + \frac{2}{\sqrt{19}}|u_2\rangle + \frac{1}{\sqrt{19}}|u_3\rangle + \frac{2}{\sqrt{19}}|u_4\rangle + \sqrt{\frac{6}{19}}|u_5\rangle\right)$$

$$= |u_3\rangle\left(\frac{1}{\sqrt{19}}\langle u_3|u_3\rangle\right) = \frac{1}{\sqrt{19}}|u_3\rangle$$

Therefore

$$\Pr(3\varepsilon) = \langle\psi|P_3|\psi\rangle$$

$$= \left(\frac{2}{\sqrt{19}}\langle u_1| + \frac{2}{\sqrt{19}}\langle u_2| + \frac{1}{\sqrt{19}}\langle u_3| + \frac{2}{\sqrt{19}}\langle u_4| + \sqrt{\frac{6}{19}}\langle u_5|\right)\left(\frac{1}{\sqrt{19}}|u_3\rangle\right)$$

Figure: Part(b)

# Example 1

**(c)**

$$
\begin{aligned}
\langle H\rangle &= \sum_{i=1}^{5} E_i\langle\psi|P_i|\psi\rangle = \varepsilon\langle\psi|P_1|\psi\rangle + 2\varepsilon\langle\psi|P_2|\psi\rangle + 3\varepsilon\langle\psi|P_3|\psi\rangle \\
&\quad + 4\varepsilon\langle\psi|P_4|\psi\rangle + 5\varepsilon\langle\psi|P_5|\psi\rangle \\
&= \varepsilon\frac{4}{19} + 2\varepsilon\frac{4}{19} + 3\varepsilon\frac{1}{19} + 4\varepsilon\frac{4}{19} + 5\varepsilon\frac{6}{19} \\
&= \frac{61}{19}\varepsilon
\end{aligned}
$$

Figure: Part(c)

# Example 2

A qubit is in the state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$$

A measurement with respect to $Y$ is made. Given that the eigenvalues of the $Y$ matrix are $\pm 1$, determine the probability that the measurement result is $+1$ and the probability that the measurement result is $-1$.

# Example 2

First we verify that the state is normalized

$$\begin{aligned}\langle\psi|\psi\rangle &= \left(\frac{\sqrt{3}}{2}\langle 0| - \frac{1}{2}\langle 1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)\\ &= \frac{3}{4}\langle 0|0\rangle - \frac{\sqrt{3}}{4}\langle 1|0\rangle - \frac{\sqrt{3}}{4}\langle 0|1\rangle + \frac{1}{4}\langle 1|1\rangle\\ &= \frac{3}{4} + \frac{1}{4} = 1\end{aligned}$$

Since $\langle\psi|\psi\rangle = 1$ the state is normalized. Recall that $Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. You need to show that the eigenvectors of the $Y$ matrix are

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |u_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

corresponding to the eigenvalues $\pm 1$, respectively. The dual vectors in each case, found by computing the transpose of each vector can taking the complex conjugate of each element, are

$$\langle u_1| = (|u_1\rangle)^\dagger = \frac{1}{\sqrt{2}}(1 \quad -i), \quad \langle u_2| = (|u_2\rangle)^\dagger = \frac{1}{\sqrt{2}}(1 \quad i)$$

The projection operators corresponding to each possible measurement result are

$$\begin{aligned}P_{+1} &= |u_1\rangle\langle u_1| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}(1 \quad -i) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\\ P_{-1} &= |u_2\rangle\langle u_2| = \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}(1 \quad i) = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\end{aligned}$$

# Example 2

Writing the state $|\psi\rangle$ as a column vector, we have

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}$$

Hence

$$P_{+1}|\psi\rangle = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}\sqrt{3}+i\\-1+i\sqrt{3}\end{pmatrix}$$

$$P_{-1}|\psi\rangle = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}\sqrt{3}-i\\-1-i\sqrt{3}\end{pmatrix}$$

Now, if a measurement is made of the $Y$ observable, the probability of finding $+1$ is

$$\Pr(+1) = \langle\psi|P_{+1}|\psi\rangle = \frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\end{pmatrix}\frac{1}{4}\begin{pmatrix}\sqrt{3}+i\\-1+i\sqrt{3}\end{pmatrix}$$

$$= \frac{1}{8}(3+i\sqrt{3}+1-i\sqrt{3}) = \frac{1}{8}(3+1) = \frac{1}{2}$$

Similarly find

$$\Pr(-1) = \langle\psi|P_{-1}|\psi\rangle = \frac{1}{2}(\sqrt{3} \quad -1)\frac{1}{4}\begin{pmatrix}\sqrt{3}-i\\-1-i\sqrt{3}\end{pmatrix}$$

# Tensor Products: Revisited

Suppose that $A$ is a projection operator in $H_1$ where $A = |0\rangle\langle 0|$ and $B$ is a projection operator in $H_2$ where $B = |1\rangle\langle 1|$. Find $A \otimes B|\psi\rangle$ where

$$|\psi\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

## Solution

Using what we know about the action of tensor products of operators, we write

$$A \otimes B|\psi\rangle = A \otimes B\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}[(A|0\rangle)(B|1\rangle) + (A|1\rangle)(B|0\rangle)]$$

Now

$$A|0\rangle = (|0\rangle\langle 0|)|0\rangle = |0\rangle\langle 0|0\rangle = |0\rangle$$

$$A|1\rangle = (|0\rangle\langle 0|)|1\rangle = |0\rangle\langle 0|1\rangle = 0$$

$$B|0\rangle = (|1\rangle\langle 1|)|0\rangle = |0\rangle\langle 1|0\rangle = 0$$

$$B|1\rangle = (|1\rangle\langle 1|)|1\rangle = |1\rangle\langle 1|1\rangle = |1\rangle$$

Therefore we find that

$$A \otimes B|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|1\rangle$$

# Tensor Products: Revisited

Suppose

$$|\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

Describe the action of $X \otimes I$ on the state.

---

## Solution

The action of $X \otimes I$ is as follows:

$$X \otimes I|\psi\rangle = X \otimes I\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}[(X|0\rangle)|0\rangle - (X|1\rangle)|1\rangle]$$

$$= \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

# Measurement on Composite Systems: Example

Describe the action of the operators $P_0 \otimes I$ and $I \otimes P_1$ on the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

# Measurement on Composite Systems: Example

## Solution

The first operator, $P_0 \otimes I$, tells us to apply the projection operator, $P_0 = |0\rangle\langle 0|$, to the *first qubit* and to leave the second qubit alone. The result is

$$P_0 \otimes I|\psi\rangle = \frac{1}{\sqrt{2}}[(|0\rangle\langle 0|0\rangle) \otimes |1\rangle - (|0\rangle\langle 0|1\rangle \otimes |0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

Interestingly, applying a projective measurement to the first qubit causes the second qubit to assume a definite state. As we will see in the next chapter, this is a property of entangled systems. Apparently it doesn't matter if the qubits are spatially separated for a collapse of the system to occur.

To find the properly normalized state of the system after measurement we use (6.20). We have

$$\langle\psi|P_0 \otimes I|\psi\rangle = \left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right)\frac{|01\rangle}{\sqrt{2}} = \frac{\langle 0|0\rangle\langle 1|1\rangle - \langle 1|0\rangle\langle 0|1\rangle}{2} = \frac{1}{2}$$

The state after measurement is

$$|\psi'\rangle = \frac{P_0 \otimes I|\psi\rangle}{\sqrt{\langle\psi|P_0 \otimes I|\psi\rangle}} = \frac{|01\rangle/\sqrt{2}}{(1/\sqrt{2})} = |01\rangle$$

As can be seen, while applying (6.20) in the single qubit case can seem like overkill, in this case this allows us to quickly write down the properly normalized state after measurement.

# Measurement on Composite Systems: Example

The second operator, $I \otimes P_1$, tells us to leave the *first qubit alone* and to apply the projection operator $P_1 = |1\rangle\langle 1|$ to the second qubit. This gives

$$I \otimes P_1|\psi\rangle = \frac{1}{\sqrt{2}}[|0\rangle \otimes (|1\rangle\langle 1|1\rangle) - |1\rangle \otimes (|1\rangle\langle 1|0\rangle)] = \frac{|01\rangle}{\sqrt{2}}$$

We have therefore the same state, but this time doing the projective measurement represented by $P_1 = |1\rangle\langle 1|$ the second qubit has forced the first qubit into the state $|0\rangle$. Let's redo the calculation using matrices. The operator is

$$I \otimes P_1 = \begin{pmatrix} 1 \cdot P_1 & 0 \cdot P_1 \\ 0 \cdot P_1 & 1 \cdot P_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then we have

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$