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(i) Quantum Computing Explained

(ii) Quantum Computing for Computer Scientists

(iii) Quantum Computing - A beginner's introduction

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Classical computers ⟵ Computers which are based on 1 human architecture

1st 2 lectures are covered in slides, so refer them.

25th Jan

Positive whole nos

$P = \{1, 2, 3, 4 \ldots\}$

Set of natural nos

$N = \{0, 1, 2, \ldots\}$

Set of integers

$Z = \{-\infty, \ldots -1, 0, 1 \ldots \infty\}$

Set of rational nos

$Q = \left\{ \frac{m}{n} \mid m \in Z, n \in N \right\}$

Set of real Number

$R = Q \cup \{ \ldots -\sqrt{2}, \ldots, e, \pi \ldots \frac{e}{\pi} \}$

(Set of rational nos) ∪ (Set of irrational nos)

Set of Complex nos

$x^2 + 1 = 0$

$x = \sqrt{-1} \} \in i$

$i = \sqrt{-1}$

$i^2 = -1$

Def$^n$ → Complex nos are of the form $a + ib$ where $a, b \in R$

$\Rightarrow a + ib \in \mathbb{C}$

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* $(a+ib)+(c+id)$
$= (a+c)+i(b+d)$

* $(a+ib)\times(c+id)$
$= (ac-bd)+i(bc+ad)$

## Fundamental Theorem of Linear Algebra

Polynomial

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

Now if $\underbrace{a_0/a_1/a_2/\ldots a_n}_{\text{atleast one}} \in \mathbb{C}$

Root of eq$^n$ is Cn.

$\therefore \overbrace{(a+ib)}^{c} \in \mathbb{C}$ where $a,b \in \mathbb{R}$

$\Rightarrow c \mapsto (a,b) \Rightarrow c$ is mapped to $(a,b)$

$\Rightarrow a \mapsto (a,0)$

$\Rightarrow b \mapsto (0,b)$

This above concept helps us to ∃

* $(a+ib) * (c+id)$

$\Rightarrow (a,b) * (c,d) = (ac-bd)+i(ad+bc)$

$= (ac-bd,\ ad+bc)$

$\Rightarrow$ ~~(a c~~

* $(a+ib)+(b+id)$

$\Rightarrow (a,b)+(c,d) = (a+c)+i(b+d)$

$= (a+c,\ b+d)$

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eg $i \times i = (0,1) \times (0,1)$
$= (-1, 0)$

$\rightarrow i^2 = -1$

## Properties of complex nos.

① **Commutative** , $c_1, c_2 \in \mathbb{C}$

$\rightarrow c_1 + c_2 = c_2 + c_1$

$\rightarrow c_1 \times c_2 = c_2 \times c_1$

② **Associative** , $c_1, c_2, c_3 \in \mathbb{C}$

$$c_1 + (c_2 + c_3) = (c_1 + c_2) + c_3$$

$$c_1 \times (c_2 \times c_3) = (c_1 \times c_2) \times c_3$$

③ **Distributive**

$$c_1 \times (c_2 + c_3) = (c_1 \times c_2) + (c_1 \times c_3)$$

Derivation

$c_1 = (a_1, b_1) \qquad c_2 = (a_2, b_2) \qquad c_3 = (a_3, b_3)$

LHS $\rightarrow (a_1, b_1) \times ((a_2, b_2) + (a_3, b_3))$

$= (a_1, b_1) \times ((a_2 + a_3), (b_2 + b_3))$

$= (a_1a_2 + a_1a_3 - (b_1b_2 + b_1b_3), a_1b_2 + a_1b_3 + b_1a_2 + b_1a_3)$

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RHS $((a_1,b_1)\times(a_2,b_2)) + ((a_1,b_1)\times(a_3,b_3))$

$\Rightarrow (a_1a_2 - b_1b_2,\ a_1b_2 + b_1a_2) + (a_1a_3 - b_1b_3,\ b_1a_3 + a_1b_3)$

$\Rightarrow (a_1a_2 + a_1a_3 - (b_1b_2 + b_1b_3),\ a_1b_2 + a_1b_3 + b_1a_2 + b_1a_3)$

$\Rightarrow$ LHS = RHS

Complementary operations for '+' & '×' $\Rightarrow$ '−' & '/'

$c_1, c_2 \in \mathbb{C}$

Sub $\rightarrow c_1 - c_2 = (a_1,b_1) - (a_2,b_2)$

$c_1 - c_2 = (a_1 - a_2,\ b_1 - b_2)$

Division $\rightarrow (x,y) = \dfrac{(a_1,b_1)}{(a_2,b_2)}$

$(x,y)\times(a_2,b_2) = (a_1,b_1)$

$(xa_2 - yb_2,\ xb_2 + ya_2) = (a_1,b_1)$

Comparing $\rightarrow (a_1 = xa_2 - yb_2)\ a_2$

$\rightarrow (b_1 = xb_2 + ya_2)\ b_2$

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$\rightarrow a_1 a_2 = x a_2^2 - y a_2 b_2$

$\rightarrow b_1 b_2 = x b_2^2 + y a_2 b_2$

$$a_1 a_2 + b_1 b_2 = x(a_2^2 + b_2^2)$$

$$x = \frac{(a_1 a_2 + b_1 b_2)}{(a_2^2 + b_2^2)}$$

Similarly $\quad y = \dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$

$$\rightarrow \boxed{(x,y) = \frac{(a_1 a_2 + b_1 b_2) + i(a_2 b_1 - a_1 b_2)}{\boxed{(a_2^2 + b_2^2)}}}$$

→ MODULUS OF DENOMINATOR CN

Other way to find

$$\rightarrow \frac{(a_1 + ib_1)}{(a_2 + ib_2)} \times \frac{(a_2 - ib_2)}{(a_2 - ib_2)}$$

$$\boxed{(x,y) = \frac{(a_1 a_2 + b_1 b_2) + i(a_2 b_1 - b_2 a_1)}{a_2^2 + b_2^2}}$$

$a \in \mathbb{C} \rightarrow |a| = \sqrt{a^2} = a$

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## Modulus

* $\exists a \in \mathbb{R} \longrightarrow |a| = \sqrt{a^2}$

$\rightarrow |c| = |a+ib| = \sqrt{a^2+b^2}$

$\rightarrow a = 1-i \Rightarrow |a| = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

* $c_1, c_2 \in \mathbb{C}$

1. $|c_1||c_2| = |c_1 c_2|$

2. $|c_1+c_2| \leq |c_1| + |c_2|$

Proof $\rightarrow$ ~~Squaring LHS & RHS.~~

$\cancel{\rightarrow \text{LHS} = |c_1+c_2|^2 = \left[\sqrt{(c_1+c_2)^2}\right]^2}$

$\Rightarrow$ Prove $|z+w| \leq |z| + |w|$ for $z, w \in \mathbb{C}$

$\longrightarrow$ let $z, w \in \mathbb{C}$

Then $|z+w|^2 = (z+w)(\overline{z+w})$ Since $|u|^2 = u\bar{u}$

$= (z+w)(\bar{z}+\bar{w})$ Since $(\overline{z+w}) = \bar{z}+\bar{w}$

$= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$

$= |z|^2 + z\bar{w} + \bar{z}w + |w|^2$ Since $|u|^2 = u\bar{u}$

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$|z+w|^2 = |z|^2 + \bar{z}\bar{w} + \bar{z}w + |w|^2$ Since $\bar{\bar{u}} = u$

$= |z|^2 + \boxed{\overline{\bar{z}w}} + \boxed{\bar{z}w} + |w|^2$ Since $\overline{\bar{z}w} = z\bar{w}$

$= |z|^2 + 2\,\mathrm{Re}(\bar{z}w) + |w|^2$ Since $2\,\mathrm{Re}(z) = \bar{z}+z$

$\Rightarrow \mathrm{Re}(z) \leq |z| \quad \} \; z = x+iy$

$\longrightarrow \underbrace{\mathrm{Re}(z)} = x = \sqrt{x^2} \leq \sqrt{x^2+y^2} = \underbrace{|z|}$

$\longrightarrow \mathrm{Re}(z) \leq |z|$

$\leq |z|^2 + 2|\bar{z}w| + |w|^2$ Since $\mathrm{Re}(z) \leq |z|$

$= |z|^2 + 2|\bar{z}||w| + |w|^2$ Since $|zw| = |z||w|$

$= |z|^2 + 2|z||w| + |w|^2$ Since $|\bar{z}| = |z|$

$= (|z| + |w|)^2$

$\Rightarrow |z+w|^2 \leq (|z| + |w|)^2$

$\Rightarrow |z+w| \leq |z| + |w|$

* Real nos do not contain Complex nos.
* Real no are subset of Complex nos.

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## Properties

1. '+' is commutative & associative
2. '×' " " " "
3. '−' is defined everywhere
4. '/' is defined everywhere except when divisor is 0.
5. '+' has an identity → (0,0)
6. '×' has an identity → (1,0)

⟹ A set which specifies all of the above properties can be called as the <u>FIELD</u>.

## Algebraically complete

$$a_0x^n + a_1x^{n-1}, a_2x^{n-2} \ldots + a_n = 0$$

→ If we ~~have~~ a polynomial & we are able to find out its all roots, then & all the roots exist in same space i.e → either complex or real or any other, then that statement is called algebraically complete.

→ Complex no space is algebraically complete

eg → $x^2+1 = 0$

$x = \sqrt{-1}$ ⟹ Complex root.

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$a \in R$ , $a \times -1 = -a \in R$

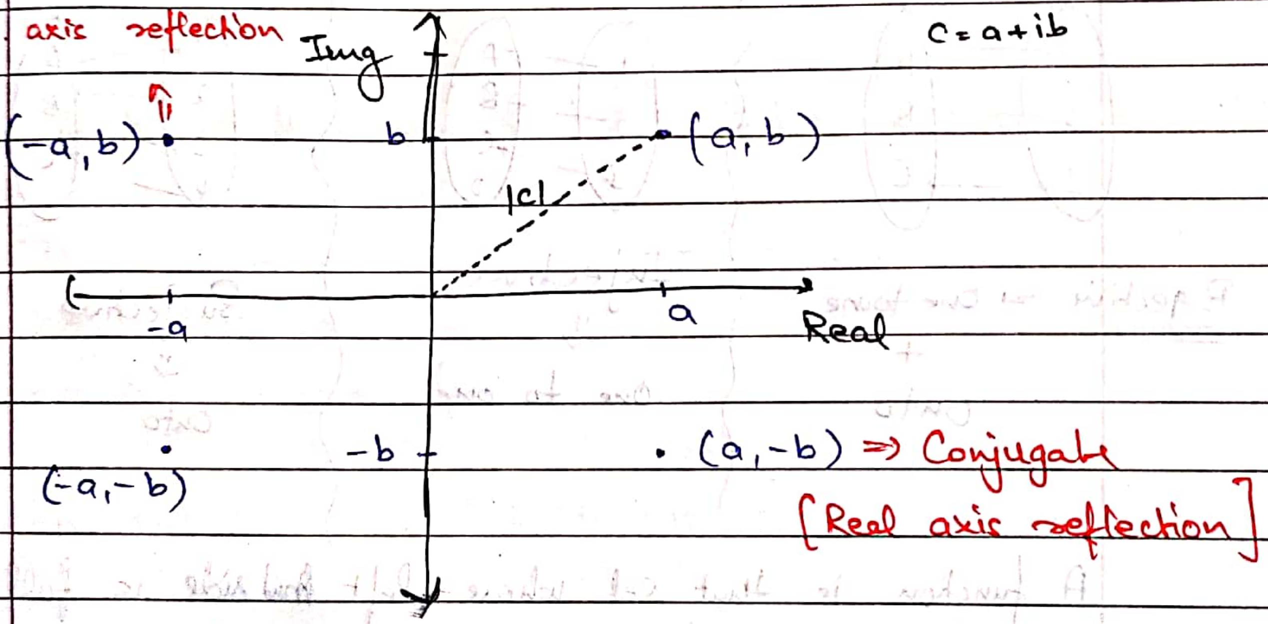
3 ways to change the sign of $a+ib = c$

① Change sign of real part ⇒ $-a+ib$ [Negative of the conjugate of c] $c \mapsto -\bar{c}$

② Change sign of Img part ⇒ $a-ib$ [This is conjugate of $a+ib$] $c \mapsto \bar{c}$

③ $-a-ib$

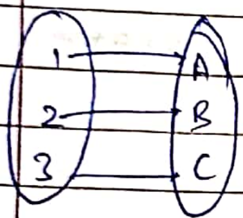
Img axis reflection



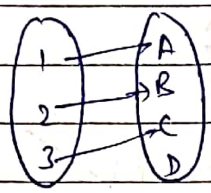

$|c| = \sqrt{a^2+b^2}$ = Modulus of c.

# Properties of Conjugate

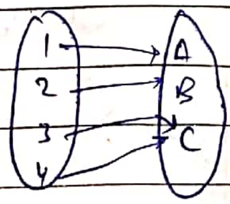
1. $\bar{c}_1 + \bar{c}_2 = \overline{c_1 + c_2}$
2. $\bar{c}_1 \times \bar{c}_2 = \overline{c_1 \times c_2}$
3. $c \mapsto \bar{c}$, the function which maps $c$ to $\bar{c}$, is bijective




Bijective → One to one + onto

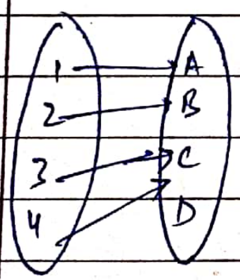



Injective ⇓ One to one

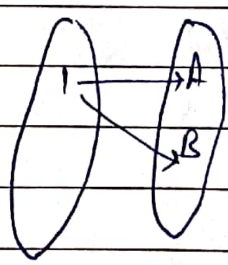



Surjective ⇓ onto

A function is that set whose left hand side is full gone for mapping, now comes injective & surjective for further mapping




Neither injective nor surjective




Not a $f^n$ → a single value has not 2 outputs

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$X: F \to F$ is bijective & it follows addition as well as multiplication, then this property is known as FIELD ISOMORPHISM

→ As on conjugation, we can perform '+' as well as '×', so conjugation is a Field Isomorphism on

$\mathbb{C} \to \mathbb{C}$

Complex number

**Ques** Check whether $(a,b) \mapsto (-a,b)$ is field isomorphism?

→ It is bijective as it is one-one & onto

→ We can perform '+'

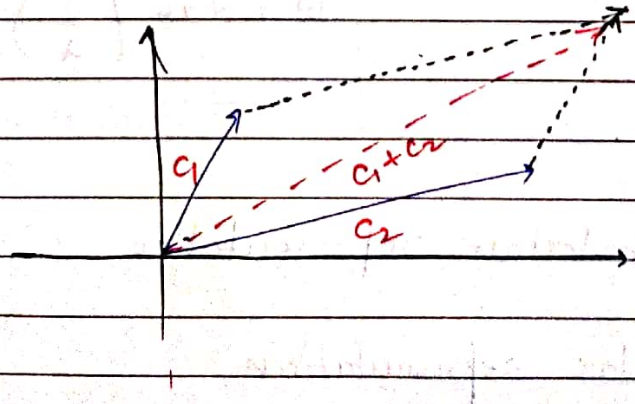
→ We can perform '×'

→ So this is FIELD ISOMORPHISM.

4. $|c|^2 = c \times \bar{c}$

Square of the modulus

5. For addition, we can use parallelogram law

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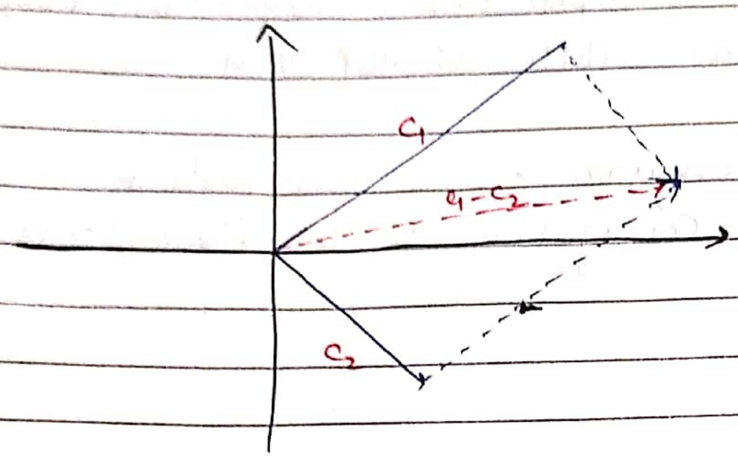
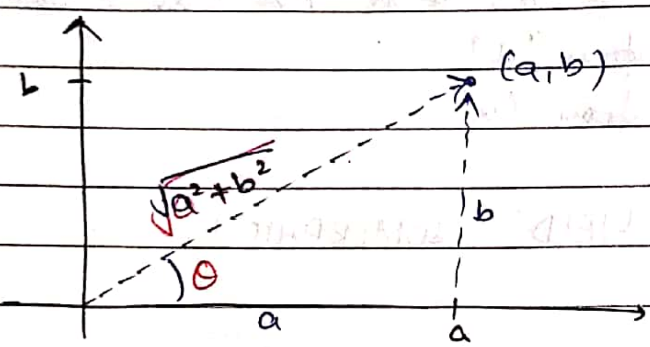
(6) For subtraction also, we can follow parallelogram law




(7) For multiplication
→ we have to find, modulus ($\rho$)
Phase ($\theta$)




Modulus is hypotenuse ⇒ $\boxed{\rho = \sqrt{a^2+b^2}}$

Phase ⇒ $\tan\theta = \frac{b}{a}$

$$\boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

(a, b) → Cartesian representation

($\rho$, $\theta$) → Polar representation

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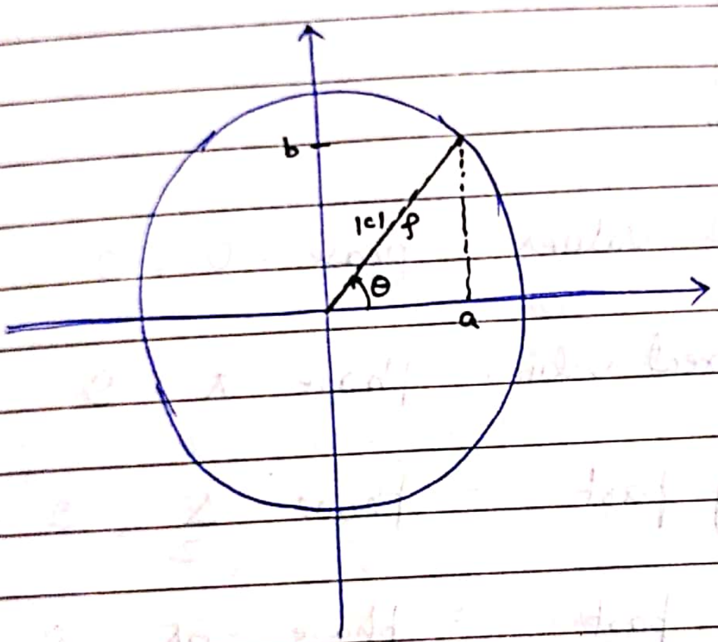
* $b = \rho \sin\theta$

* $a = \rho \cos\theta$

Example → $\rho = \sqrt{1^2+1^2} = \sqrt{2}$

$c = 1+i$ $\quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ = \frac{\pi}{4}$ radians




$\theta$ = Angle or phase

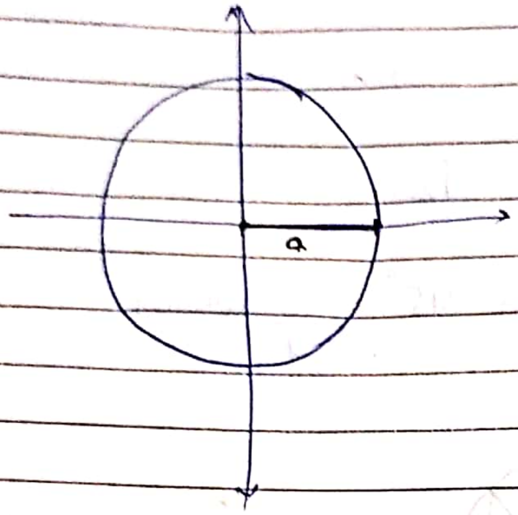
$\rho$ = Magnitude

$\Rightarrow$ $C \equiv$ (Magnitude, Phase)

$= (\rho, \theta)$

$= \left(\sqrt{2}, \frac{\pi}{4}\right)$

**Example 2:** $c = a + ib$ where $b = 0$, convert into polar form




$\rho = a$

$\theta = 0$

For all real values, phase = 0 = $\theta$

For all -ve real values, phase = $\pi$ = $\theta$

For +ve img part = phase = $\frac{\pi}{2}$ = $\theta$

For -ve img part = phase = $\frac{3\pi}{2}$ = $\theta$

★ $\theta$ is the angle from the +ve $x$ axis

$0 \leq \theta < 2\pi$

* $\theta_1 = \theta_2$ iff $\theta_2 = \theta_1 + 2\pi k$, $k \in Z$

**Ques** Tell whether $(2, \underset{\theta_1}{\pi})$ & $(2, \underset{\theta_2}{-\pi})$ are equal or not.

→ $\theta_1 = \theta_2 + 2\pi k$

$\pi = -\pi + 2\pi(1)$

$\pi = \pi$

→ Both are equal

Doubt → In recording you told both are different

* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

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* For multiplication of polar form ↓

→ $(\rho_1, \theta_1) \times (\rho_2, \theta_2) = (\rho_1 * \rho_2, \theta_1 + \theta_2)$

* For division of polar form ↓

→ $c_1 = (\rho_1, \theta_1)$ $\quad$ $c_2 = (\rho_2, \theta_2)$

$$\frac{c_1}{c_2} = \left(\frac{\rho_1}{\rho_2}, \theta_1 - \theta_2\right)$$

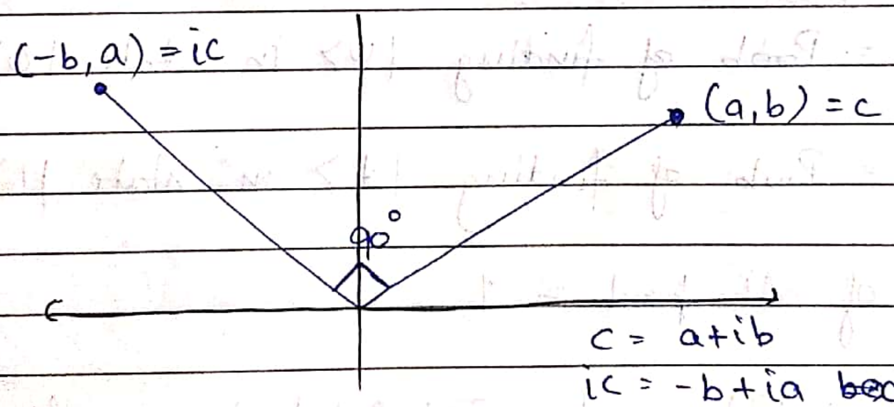
* Power of complex number

→ $c = (\rho, \theta) \in \mathbb{C}$

$$\boxed{c^n = (\rho^n, n\theta)}$$

$$\boxed{c^{\frac{1}{n}} = \left(\rho^{\frac{1}{n}}, \frac{1}{n}\theta\right)} \equiv \boxed{\left(\sqrt[n]{\rho}, \frac{1}{n}(\theta + 2\pi k)\right)}$$

* 




$c = a + ib$

$ic = -b + ia$ because

$i = (1, \frac{\pi}{2})$

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# Qubits

This is the basic unit of computation.

* It has 2 state → $|0\rangle$
→ $|1\rangle$

* '$|\,\rangle$' symbol is known as state / vector / ket

→ Earlier we discussed,
$|0\rangle$ & $|1\rangle$ may be in superposition with each other

⇒ Now qubit has the 3rd state, superposition of $|0\rangle$ & $|1\rangle$

→ $\alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$

* 3rd state = $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Acc. to Quantum mechanics, the modulus of $\alpha$ & $\beta$ is the probability of finding qubit in $|0\rangle$ & $|1\rangle$

Hence we can say,

→ $|\alpha|^2$ = Prob of finding $|\psi\rangle$ in state $|0\rangle$

→ $|\beta|^2$ = Prob of finding $|\psi\rangle$ in state $|1\rangle$

∴ Sum of all prob = 1

→ $|\alpha|^2 + |\beta|^2 = 1$ ⇒ Qubit is normalised

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conjugate $= \bar{c} = c^*$

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Say A gives following states along with their prob.

$\rightarrow$

$$A \rightarrow A_1 \ (p_1), \quad A \rightarrow A_2 \ (p_2), \quad A \rightarrow A_3 \ (p_3), \quad \ldots, \quad A \rightarrow A_n \ (p_n)$$

$$\Rightarrow \sum_{i=1}^{n} p_i = 1$$

* Now $c \times \bar{c} = |c|^2$ OR $c \times c^* = |c|^2$

So $|\alpha|^2 = (\alpha)(\bar{\alpha})$ OR $\alpha \times \alpha^*$

$|\beta|^2 = (\beta)(\bar{\beta})$ OR $\beta \times \beta^*$

Example 1

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

$|\alpha| = \sqrt{\alpha^2} \Rightarrow \alpha$

$\rightarrow$ Prob of finding $|\psi\rangle$ in state $|0\rangle = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$

$\Rightarrow$ Prob of finding $|\psi\rangle$ in state $|1\rangle = \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$

$p_1 + p_2 = 1$

2. $|\psi\rangle$ OR $|\phi\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

$\rightarrow p_1 = \left|\frac{i}{2}\right|^2 = \left(\frac{i}{2}\right)\left(\overline{\frac{i}{2}}\right) = \frac{1}{4}$ OR

OR

$$p_1 = \left[\sqrt{0^2 + \left(\frac{1}{2}\right)^2}\right]^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow p_1$$

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$p_2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$

$\therefore \underline{p_1 + p_2 = 1}$

(3) $|\alpha\rangle = \frac{(1+i)}{\sqrt{3}}|0\rangle - \frac{i}{\sqrt{3}}|1\rangle$

$p_1 = \left|\frac{1+i}{\sqrt{3}}\right|^2 = \left(\frac{1+i}{\sqrt{3}}\right)\left(\frac{1-i}{\sqrt{3}}\right) = \frac{2}{3}$

$p_2 = \left|\frac{-i}{\sqrt{3}}\right|^2 = \left(\frac{-i}{\sqrt{3}}\right)\left(\frac{i}{\sqrt{3}}\right) = \frac{1}{3}$

$p_1 + p_2 = 1$

# Vector Spaces

V is a non-empty space with $u, v \in V$ and a field of scalars s.t. $s \in$ field & the vector space is defined on 2 operations $\Rightarrow$ '+', '*'

(Numbers of field $\Rightarrow$ '+' is commutative & associative
'x' " " " "
'-' is defined everywhere
'/' is defined everywhere except $c_2 = 0$
Identity of addition is (0,0)
Identity of mult. is (1,0)

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## Properties of '+'

1. $\exists\ u, v \in V$ , $u+v \in V$ [Closure property]

If we are performing some operation on elements of a space, then their operation will also lie in the same space.

2. '+' is commutative

$\rightarrow$ $u+v = v+u$

3. $\exists\ 0 \in V$ , s.t. $u+(0) = (0)+u = u$

4. Their exists a additive inverse

$u+(-u) = 0 = (-u)+u$

5. Associative property $\Rightarrow (u+v)+(w) = (u)+(v+w)$

$u, v, w \in V$

## Properties of 'x'

1. $\exists\ \alpha \in V$ , $\alpha v \in V$ $\Rightarrow$ Closure property as the result is in same space

2. Distributive property , $\exists\ u, v$

$\alpha(u+v) = \alpha u + \alpha v$

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(3) $\exists\ \alpha, \beta \in F,\ v \in V$

$\rightarrow\ (\alpha+\beta)v = \alpha v + \beta v$

(4) $\alpha(\beta v) = (\alpha \times \beta)v$

(5) $1 \in F \Rightarrow 1 \cdot u = u$ [Identity property]

* These lecture topics are from
(1) Quantum Computing for Computer scientists → ch-1
(2) Quantum Computing explained → ch-2.

2nd Feb — Denoting a vector space of n-tuples of CN.

$$\mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \cdots \mathbb{C}}_{n\text{ times}} \quad \Big\}\ \text{Complex Vector space}$$

eg $$C^3 = \begin{bmatrix} 6+i \\ i \\ 0 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} = V \Rightarrow \begin{matrix} V[0] = 6+i \\ V[1] = i \\ V[2] = 0 \end{matrix}$$

Operations on Complex vector space $C^n$

(1) Addition $\quad V, W \in \mathbb{C}$

$$V = \begin{bmatrix} 6+i \\ i \\ 0 \end{bmatrix}, \quad W = \begin{bmatrix} i \\ 1+i \\ 2 \end{bmatrix}$$

$V + W \in C^n$

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* Addition is commutative $\Rightarrow$ $v + w = w + v$

* '+' is associative $\Rightarrow$ $(v+w)+x = v+(w+x)$

* $\exists$ a vector $0 \in \mathbb{C}$ s.t. $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad 0[j] = 0$

$$v + 0 = v = 0 + v$$

* $\exists$ an additive inverse, hence for every $v \in V$, we would have $-v \in V$

$$v + (-v) = 0 = (-v) + v$$

There is a $\Rightarrow$ Set $\mathbb{C}^n$ and we can apply addition operation, inverse operation & $\exists 0$ st. '+' is Commutative & associative Then this set is $\boxed{A}$BELIAN GROUP

$\boxed{(+)}$
- $\to$ Addition operation $\to$ Commutative
  - $\to$ Associative
- $\to$ Inverse operation

## ② Multiplication

* $\exists\ c \in \mathbb{C}$ and a vector $v \in V$ (OR) $v \in \mathbb{C}^n$, then $cv \in \mathbb{C}^n$, here c is acting as scalar

eg $c = (a+ib)$ $\quad v = \begin{bmatrix} i \\ 1+i \\ 0 \end{bmatrix}$

$$cv = (a+ib)\begin{bmatrix} i \\ 1+i \\ 0 \end{bmatrix} \in \mathbb{C}^n$$

So;
$(c \times v)[j] = c \times v[j]$

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* $1.v = v \quad\quad 1 \in \mathbb{C}$

* $c_1.(c_2.v) = (c_1 \times c_2)v \quad\quad c_1, c_2 \in \mathbb{C}, \; v \in V, v \in \mathbb{C}$

* $c_3(v+w) = c_3.v + c_3.w \quad\quad v, w \in V, \; c_3 \in \mathbb{C}$

* $(c_1+c_2)v = c_1v + c_2v$

→ If we have ABELIAN GROUP and this group follows all these multiplication properties, then this set along with these operations form Complex Vector Space.

↳ Properties of add. & mul.

Complex vector space → A. C.V.S is a non empty set V with 3 operations ↓

→ Cartesian product.

1) $+ : V \times V \longrightarrow V$

This is cross not multiplication

2) Negation : $V \times V \longrightarrow V$

3) Scalar Multiplication : $\mathbb{C} \times V \longrightarrow V$

Complex Scalar

And in this vector space there exists a vector $0 \in V$

It satisfies the following properties →

1. Commutative $\Rightarrow V + W = W + V$
2. Associative $\Rightarrow V + (W + X) = (V + W) + X$
3. Zero $\Rightarrow V + (0) = (0) + V = V$
4. Add. inverse $\Rightarrow V + (-V) = (-V) + V = 0$

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5. Mul. inverse $\Rightarrow 1 \in V,\ 1 \cdot v = v$
6. $c_1(c_2 \cdot v) = (c_1 \times c_2)v$
7. $c_1(v+w) = c_1 v + c_1 w$
8. $(c_1+c_2)v = c_1 v + c_2 v$

Hence, these above are the 11 (3+8) properties for Complex vector space.

Real numbers are the special case for $\mathbb{C}$

* $\mathbb{R} \subset \mathbb{C}$

$\therefore \mathbb{R} \times V \rightarrow V$

$\Rightarrow \mathbb{R} \times V \subset \mathbb{C} \times V$

n rows 1 col * $\mathbb{C}^n$ is also complex vector space having n rows & each number is complex number

$\mathbb{C}^n$ is also a real vector space. $(\because \mathbb{R} \subset \mathbb{C})$

m rows 1 col * $\mathbb{C}^{m \times n}$ is the set of all $m \times n$ entries & each entry is complex.

$\mathbb{C}^{m \times n}$ follows all above 8 properties

* $A \in \mathbb{C}^{m \times n}$

$\rightarrow A^T[j,k] = A[k,j] \Rightarrow$ Transpose property

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* $\overline{A[j,k]} = \overline{A}[j,k]$ → Conjugate property

Conjugate of a matrix is equivalent to conjugate of each element.

* $(\overline{A})^T = \overline{(A^T)}$ → Adjoint property $(A^\dagger)$

→ Transpose ⇒ $C^{m\times n} \to C^{n\times m}$

– Adjoint ⇒ $C^{m\times n} \to C^{n\times m}$

* Transpose, Conjugate, Adjoint → These follow following properties

1) **Idempotent** ⇒ $(A^T)^T = A$

⇒ $\overline{(\overline{A})} = A$

⇒ $(A^\dagger)^\dagger = A$

2) **Follows Addition** → $(A+B)^T = A^T + B^T$

→ $\overline{A+B} = \overline{A} + \overline{B}$

→ $(A+B)^\dagger = A^\dagger + B^\dagger$

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3) Scalar multi $\Rightarrow (C.A)^T = C(A^T)$

$\Rightarrow \overline{C.A} = \overline{C}\,\overline{A}$ $\quad\Big\}\ C \in \mathbb{C}$

$(C.A)^{\dagger} = \overline{C} A^{\dagger}$

3rd Feb

# Matrix Multiplication

$A \in C^{m\times n}$ , $B \in C^{n\times p}$

$\rightarrow * : C^{m\times n} \underbrace{\times}_{\text{Casterian product}} C^{n\times p} \longrightarrow C^{m\times p}$

(1) $A * B \neq B * A$ (Matrix multiplication is not Commutative except square matrix)

(2) $\exists$ Identity matrix , $I_n = I^{n\times n}$

## Properties

(1) It is not commutative

(2) Associative : $A*(B*C) = A*(B\times C)$

(3) Distributive : $A*(B+C) = (A*B) + (A*C)$
: $(B+C)*A = (B*A) + (C*A)$

(4)

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(4) Matrix mul. respects scalar multi.

$$c(A*B) = (c.A)*B = A*(c.B)$$

(5) $(A*B)^T = B^T A^T$

(6) $\overline{(A*B)} = \overline{A} \times \overline{B}$

(7) $(A*B)^\dagger = B^\dagger \times A^\dagger$ (Because of the transpose)

(8) $I_n \times A = A = A \times I_n$

→ C.V.S with M.M. & satisfying property 2 to 4 and 8 forms complex algebra.

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(LM)

## * Linear Map :

let V and V' be 2 C.V.S , then L.M. from V to V' is a function

$$f : V \to V'$$

$\forall\ v, v_1, v_2 \in V$ and $c \in \mathbb{C}$

$$f(v_1 + v_2) = f(v)$$

$$f(c.v) = c.f(v)$$

## * Isomorphism :

Two C.V.S are isomorphic if there is a one to one and onto on the linear map, then C.V.S are isomorphic.

### Writing vectors in the form of qubits :

$C^n : v_1\ v_2\ v_3 \in C^n$ $|v\rangle$ , $|v_1\rangle$

* $|a\rangle , |b\rangle , |c\rangle \in C^n$

$\longrightarrow$ $|a\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $|b\rangle = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ $|c\rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$a_1, a_2 \ldots a_n$

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* $|\psi\rangle = \underset{\alpha}{\frac{1}{\sqrt{3}}}|0\rangle + \underset{\beta}{\sqrt{\frac{2}{3}}}|1\rangle$

→ Column vector representation of $|\psi\rangle$ is

$$|\psi\rangle = \begin{array}{c} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \end{array} \begin{array}{l} \rightarrow 0^{th} \text{ row} \\ \rightarrow 1^{st} \text{ row} \end{array}$$

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \begin{array}{l} \rightarrow 0^{st} \text{ row} \\ \rightarrow 1^{st} \text{ row} \end{array}$$

* $|\psi\rangle = \left(\frac{i+1}{2}\right)|0\rangle + \left(\frac{5i-1}{\sqrt{3}}\right)|1\rangle$

$\rightarrow |\psi\rangle = \begin{bmatrix} \frac{i+1}{2} \\ \frac{5i-1}{2} \end{bmatrix}$

$|\rangle \Rightarrow$ ket representation $\rightarrow$ Column C.V.S

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Scalar multiplication with qubits

* $\alpha \in \mathbb{C}$, $|a\rangle \in \mathbb{C}^n$ $\equiv$ $\alpha \in \mathbb{C}$, $v \in \mathbb{C}^n$
$\rightarrow \alpha v \in \mathbb{C}^n$

$$\rightarrow \alpha |a\rangle = \alpha \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \end{bmatrix}$$

$$|a\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$\rightarrow$ $\alpha |a\rangle \equiv \alpha v$

This notation is used in quantum Computing

This is used for C.V.S.

# BASIS

Let $V$ be a C.V.S. s.t $v \in V$. Then $v$ is a linear combination of vectors $v_0, v_1, \ldots v_{n-1}$ in $V$ if

$$v = c_0 v_0 + c_1 v_1 + c_2 v_2 + \ldots + c_{n-1} v_{n-1}$$

where $c_0, c_1 \ldots c_{n-1} \in \mathbb{C}$ (or R)

eg → $\alpha \begin{bmatrix} -5 \\ -2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ , $\alpha, \beta \in \mathbb{C}$

→ $[x, y, z]^T$ is a linear combination of $[-5 \ -2 \ 3]^T$ and $[0 \ 1 \ 4]^T$

## Linearly Independent

We will do this topic after this next topic

Basis in terms of quantum comp.

$$|v\rangle = c_0 |v_1\rangle + c_1 |v_2\rangle + \ldots + c_n |v_n\rangle$$

$$|v\rangle = \sum_{n=0}^{n-1} c_i |v_i\rangle$$

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$|0\rangle \qquad |1\rangle$

$\rightarrow \ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Column vector representation of $|\psi\rangle$

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{matrix} \rightarrow |0\rangle \\ \rightarrow |1\rangle \end{matrix}$$

eg $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{2}{\sqrt{3}}|1\rangle$

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

* $v_i \in \mathbb{C}^n$
* $|a\rangle \in \mathbb{C}^n$

Both are equivalent, but the 2nd statement is used in quantum mech.

* $$|a\rangle + |b\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

eg $|a\rangle = \begin{bmatrix} 1+i \\ 1 \\ 3+i \end{bmatrix}$

$|b\rangle = \begin{bmatrix} i \\ 0 \\ 3 \end{bmatrix}$

$\Rightarrow |a\rangle + |b\rangle = \begin{bmatrix} 1+2i \\ 1 \\ 6+i \end{bmatrix}$

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"This question is from ⇒ Matrix Analysis and applied linear algebra"
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Sec- 4.7 — Pg 240

Changing of basis ]

$$v = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix} \in R^3 \qquad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$B = \left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; u_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \; u_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

→ linear superposition of basis ⇒

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = V$$

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = V$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$$C_v \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

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From ⇒ Quantum Computing for computer scientist

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Bases!

Ques $B = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} \in R^3 \longleftrightarrow V = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$

$D = \left\{ \begin{bmatrix} -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \end{bmatrix} \right\}$

→ We have to find $V_D$ from $V_B$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{V_B} = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$$

$$V_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \xrightarrow{P} V_D$$

$P: V_B \longrightarrow V_D$ $\qquad M_{D\leftarrow B} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$

$f^n$

So we will find by ~~Matrix~~ "M - Transition Matrix"

OR

"Change of basis matrix"

$\Rightarrow V_D = M_{D\leftarrow B} * B$

$B = \{u_1, u_2, \dots u_n\}$ $\qquad D = \{v_1, v_2 \dots v_n\}$

$\Rightarrow \alpha_{1i} v_1 + \alpha_{2i} v_2 + \dots \alpha_{ni} v_n = M(u_i)$

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$$\begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix} \begin{bmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{in} \end{bmatrix} = M[u_i]$$

**Ques** $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \in R^3$.

$$D = \left\{ V_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Aim → $(x, y, z) \xrightarrow{P} (x, y, 0)$

→ As we did earlier ;

$\alpha_{1i} V_1 + \alpha_{i2} V_2 + \cdots + \alpha_{in} V_n = M(u_i)$ Solve one by one

$$\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \end{bmatrix} \qquad \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & a_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

1 column Similarly here

$\alpha_{11} V_1 + \alpha_{31} V_2 + \alpha_{21} V_3 = M(u_1)$.

$$\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{31} \\ \alpha_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $\alpha_{11} = -1$
$\alpha_{21} = 0$
$\alpha_{21} + \alpha_{31} = 0 \Rightarrow \alpha_{31} = 0$

$$\Rightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

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2 Column

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \alpha_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \alpha_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \alpha_{32} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

3rd col

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} \alpha_{13} \\ \alpha_{23} \\ \alpha_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{13} \\ \alpha_{23} \\ \alpha_{33} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

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Basis — Same set of vectors used to represent other set of vectors.

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Ques $B = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right\}$ $D = \left\{ \begin{bmatrix} -7 \\ 8 \end{bmatrix} \begin{bmatrix} -5 \\ 7 \end{bmatrix} \right\}$

$V = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$ $V_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\longrightarrow M_{D \leftarrow B} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$

Aim $\Rightarrow V_D = M_{D \leftarrow B} * V_B$

$= \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ -14 \end{bmatrix}$

Ans

$7 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-17) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$

$3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{"}$

$9 \begin{bmatrix} -7 \\ 8 \end{bmatrix} + (-14) \begin{bmatrix} -5 \\ 7 \end{bmatrix} = \text{"}$

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Concept of Basis in Hadamard gate

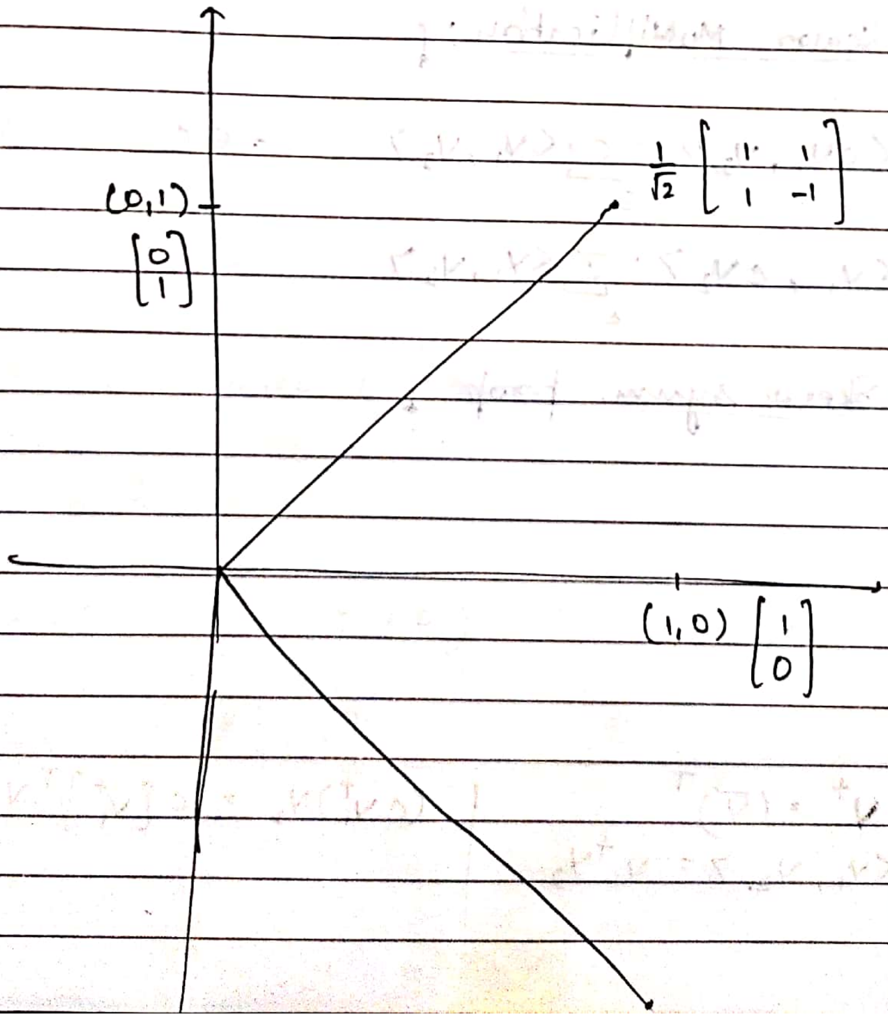
$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ $S' = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{5} \end{bmatrix} \right\}$

$$\boxed{M_{s' \leftarrow s} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H}$$

HADAMARD Matrix

$$\boxed{H * H = I_{2\times2}}$$

$$M_{s \leftarrow s'} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$$



Notation of inner product ① $\langle v_1 , v_2 \rangle \Rightarrow$ Comp. Science
② $\langle v_1 | v_2 \rangle$

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## Inner Product OR Dot Product OR Scalar product

$V$ is a complex vector space (C.V.S)

$\langle \_ , \_ \rangle : V \times V \to C$

**Prop** (i) Non-degeneracy]

$\langle v, v \rangle > 0 \;\to\; \langle v, v \rangle = 0$ iff $\underset{\text{vector}}{v} = 0$

(ii) Addition]

$\langle v_1 , v_2 + v_3 \rangle = \langle v_1 , v_2 \rangle + \langle v_1 , v_3 \rangle$

$\langle v_1 + v_2 , v_3 \rangle = \langle v_1 , v_3 \rangle + \langle v_2 , v_3 \rangle$

(iii) Scalar Multiplication]

$\langle c v_1 , v_2 \rangle = \underbrace{c}_{\bar{c}} \times \langle v_1 , v_2 \rangle \quad c \in C$

$\langle v_1 , c v_2 \rangle = \underbrace{\bar{c}}_{c} \langle v_1 , v_2 \rangle$

(iv) Skew symm. prop]

$A = A^T$

$A^+ = (\bar{A})^T$

* $v^+ = (\bar{v})^T$
* $\langle v_1 , v_2 \rangle = v_1^+ v_2$

$(c v_1^+) v_2 = c [v_1^*]^T v_2$

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(V) Orthogonal ↓

$\langle V_1, V_2 \rangle = 0 \longrightarrow V_1 \perp V_2$

(Complex) Inner Product Space ↓

Str in which set of vector space is defined with inner product.

→ addition, ~~multiplication, nega~~

→ negation

→ Scalar multiplication

⇒ It is a complex vector space with inner product.

* $R^n$ (Real nos.) : $\langle V_1, V_2 \rangle = V_1^T V_2$ $\quad V_1, V_2 \in R^n$

* $C^n$ (Complex nos) : $\langle V_1, V_2 \rangle = V_1^T V_2$

⇒ $A, B \in C^{n \times n}$

$\langle A, B \rangle = \text{Trace}(A^{+} * B)$

⇒ $A, B \in R^{n \times n}$

$\langle A, B \rangle = \text{Trace}(A^T * B)$

☆ $\text{Trace} = \sum_{i=1}^{n} X[i, i]$

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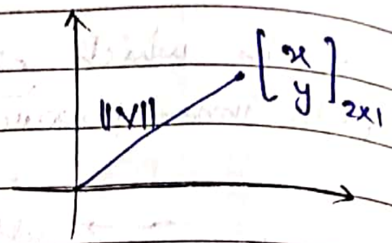
# NORM

Distance of a vector from the origin.

For a C.V.S,

$$\| \ \| : V \to R$$

$$\|V\| = \sqrt{\langle v, v \rangle}$$




**Prop** → Non-degeneracy

(i) $\|V\| > 0$ & $\|V\| = 0$ iff $V = 0$

(ii) Triangle inequality

$$\|V + W\| \leq \|V\| + \|W\|$$

(iii) Scalar Multiplication

$$\|c \cdot V\| = \|c\| \times \|V\|$$ where $c$ is a complex no.

(iv) Distance

For every complex vector space, we can be able to define the distance as the real no.

$$d(\_,\_) : V \times V \to R$$

$$v_1, v_2 \in C^n$$

$$d(v_1, v_2) = \|v_1 - v_2\|$$
$$= \sqrt{\langle v_1 - v_2, v_1 - v_2 \rangle}$$

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Prop:

(i) Non-degeneracy :

$d > 0$ & $d = 0$ iff $v = w$

(ii) Triangle inequality

$d(u,v) \leq d(u,w) + d(w,v)$ $u,v,w \in \mathbb{C}^n$

* Orthogonal basis :

$B = \{u_1, u_2, u_3 \ldots u_{n-1}\}$

$\langle u_i, u_j \rangle = 0 \quad \forall\ i \neq j$

* Orthogonal basis :

$\langle u_i, u_j \rangle = 0$ & $\|u_i\| = 1$

Example

$\overbrace{|u\rangle = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}}^{\text{Column vector}}$ $\overbrace{\langle u| = \begin{bmatrix} 1 & 1-i \end{bmatrix}}^{\text{row vector}}$

* HOW to normalise $|u\rangle$ & $\langle u|$

→ $|\tilde{u}\rangle = \dfrac{|u\rangle}{\|u\|}$ } Column format $\langle \tilde{u}| = \dfrac{|\tilde{u}\rangle}{\|u\|}$ } Row format

→ $\langle \tilde{u} | \tilde{u} \rangle = \left(\dfrac{|\tilde{u}\rangle}{\|u\|}\right)\left(\dfrac{|u\rangle}{\|u\|}\right) = \dfrac{\langle u,u \rangle}{(\|u\|)^2}$

$= \dfrac{\langle u|u \rangle}{\langle u|u \rangle} = 1$

$\|u\| = \sqrt{\langle u,u \rangle}$

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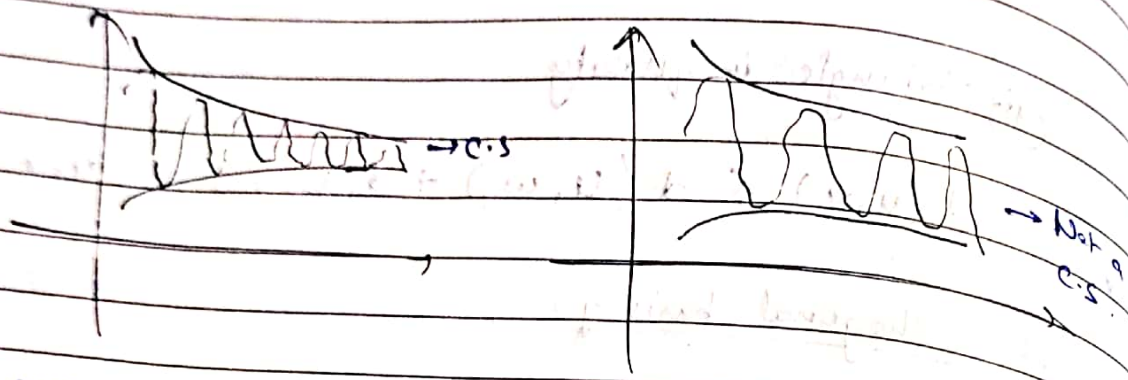
## Cauchy sequence

Elements of cauchy sequence are arbitary close to each other.




Within an I.P. space, V, < ; > with a norm function & the distance $f^n$ $\exists$ $V_0, V_1, V_2 \ldots$ is called a C.S. if for every ~~ass~~ $\epsilon > 0$, there exists an $N_0 \in N$

$\forall n, m \geq N_0 \qquad d(V_m, V_n) < \epsilon$

Complete $\bar{V}$

$$\lim_{n \to \infty} (V_n - \bar{V}) = 0$$

Hilbert space → It is a complex I.P. space which is complete.

Proposition → Every I.P. is a finite dimensional complex vector space which is complete

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Ques $|u\rangle = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

How to normalise $|u\rangle$ ?

→ Any vector is normalised if $\|u\| = \sqrt{\langle u|u\rangle} = 1$

$\langle u| = [1 \quad 1-i]$

$\langle u|u\rangle = [1 \quad 1-i] \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = 1^2 - 1^2 - 1$

$= \underline{\underline{3}}$

Now $\|u\| \neq 1$

So we can normalise $|u\rangle$ by dividing it with its norm

$$|\tilde{u}\rangle = \frac{|u\rangle}{\|u\|} = \frac{|u\rangle}{\sqrt{\langle u|u\rangle}}$$

$\langle u|u\rangle = 1$

Norm is square root of inner product with itself

$$|\tilde{u}\rangle = \frac{|u\rangle}{\|u\|}$$

$$\langle\tilde{u}| = \frac{\langle u|}{\|u\|}$$

$\langle\tilde{u}|\tilde{u}\rangle =$

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Both are orthogonal to each other
& are in normalised form

## Standard Orthonormal basis in $R^2$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\left\{ \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \right\}$$

$\left\{ \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \right\}$ are the standard computational basis in quantum computing.

* $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$\langle 0|0\rangle = \langle 1|1\rangle = 1$

$\langle 0|1\rangle = \langle 1|0\rangle = 0 \qquad |0\rangle \perp |1\rangle$

* $\langle 0| = \begin{bmatrix}1 & 0\end{bmatrix}$ $\quad \langle 1| = \begin{bmatrix}0 & 1\end{bmatrix}$

$\langle 0|0\rangle = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 1^2 + 0^2 = 1$

$\langle 1|1\rangle = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 1$

$\langle 0|0\rangle = \langle 1|1\rangle \Rightarrow |0\rangle$ & $|1\rangle$ are Normalised

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"In $R^n$, there are many other bases ]

So lets take any base as eg:

* $|u\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $|v\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

* $\langle u| = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $\langle v| = \begin{bmatrix} 1 & -1 \end{bmatrix}$

$\Rightarrow \langle u|v\rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0$

$\Rightarrow |u\rangle \perp |v\rangle$

* $\langle u|u\rangle$ or $\langle v|v\rangle \neq 1$

$\rightarrow$ They aren't normalised.

Aim $\rightarrow$ If a base is neighter orthogonal nor normalised how to convert them into orthonormalised.

So concept on next page ]

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# Gram-Schmidt orthogonalisation

* $\langle v_1, v_2, \ldots v_n \rangle$ → They aren't orthonormal
* $\langle w_1, w_2 \ldots w_n \rangle$ →

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle w_1 | v_1 \rangle}{\langle w_1 | w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle w_1 | v_3 \rangle}{\langle w_1 | w_1 \rangle} w_1 - \frac{\langle w_2 | v_3 \rangle}{\langle w_2 | w_2 \rangle} w_2$$

$w.$
$\vdots$

$$w_n = v_n - \frac{\langle w_1 | v_n \rangle}{\langle w_1 | w_1 \rangle} w_1 - \frac{\langle w_2 | v_n \rangle}{\langle w_2 | w_2 \rangle} w_2 - \cdots - \frac{\langle w_{n-1} | v_n \rangle}{\langle w_{n-1} | w_{n-1} \rangle} w_{n-1}$$

Orthonormal → $\left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \ldots, \frac{w_n}{\|w_n\|} \right\}$

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## Dirac Notation
## or
## Bra-Ket Notation

$\phi$

$|\phi\rangle \langle\phi|$

Example $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ is an orthonormal basis in $\mathbb{C}^3$ or a 3-D Hilbert space

$$|\psi\rangle = \frac{1}{\sqrt{5}}|u_1\rangle - i\sqrt{\frac{7}{15}}|u_2\rangle + \frac{1}{\sqrt{3}}|u_3\rangle$$

Check
(a) Is $|\psi\rangle$ normalised? Can be checked by $\langle\psi|\psi\rangle = 1$

→ (b) $\langle\psi| = (|\psi\rangle)^\dagger$

$$= \left(\frac{1}{\sqrt{5}}|u_1\rangle - i\sqrt{\frac{7}{15}}|u_2\rangle + \frac{1}{\sqrt{3}}|u_3\rangle\right)^\dagger$$

$$= \frac{1}{\sqrt{5}}\langle u_1| + i\sqrt{\frac{7}{15}}\langle u_2| + \frac{1}{\sqrt{5}}\langle u_3|$$

As they are orthogonal $\rightarrow \langle u_i|u_i\rangle = 1$

$\langle u_i|u_j\rangle = 0 \quad i \neq j$

We can also do $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$

(b) Prob of finding $|\psi\rangle$ in $|u_1\rangle$ via inner product

$\longrightarrow (\langle u_1|\psi\rangle)^2$

$\langle u_1|u_1\rangle = 1$
$\langle u_1|u_2\rangle = 0$
$\langle u_1|u_3\rangle = 0$

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Operators → It is a mathematical rule

$$V \xrightarrow{\text{operator}} W$$

vector V

vector W

Operator transforms a vector into other vector in same space (Complex vector space)

eg $f(x) = x^2$

$$\frac{d}{dx}(f(x)) = 2x$$

operator

Operators are acting like a function.

⟹ $f(\alpha x) = \alpha f(x)$ ⟶ Linearity property

⟶ Here an operator is linear if $\exists\ \alpha, \beta \in \mathbb{C}$ & $|\psi\rangle, |\psi\rangle \in \mathbb{C}^n$

So $\hat{A}(\alpha|\psi_1\rangle + \beta|\psi\rangle)$

Acting as $f^n$

$\hat{A}(\alpha|\psi_1\rangle) + \hat{A}(\beta|\psi_2\rangle)$

$= \alpha \hat{A}|\psi_1\rangle + \beta \hat{A}|\psi_2\rangle$

Here the simplest operator is the identity operator

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* Identity operator ]

$I|\psi\rangle = |\psi\rangle$

* Zero operator ]

$0|\psi\rangle = 0$

* $\hat{A}|\psi\rangle = |\phi\rangle$

A new vector

* $\langle\psi|\hat{A} = \langle\phi|$

* Pauli operator ] $I, X, Y, Z$

$\sigma_0 \quad \sigma_x \quad \sigma_y \quad \sigma_z$

$\sigma_0 \quad \sigma_1 \quad \sigma_2 \quad \sigma_3$

I

→ $I|0\rangle = |0\rangle$

→ $I|1\rangle = |1\rangle$

X

→ $X|0\rangle = |1\rangle$

→ $X|1\rangle = |0\rangle$

→ Not operator

Y

→ $Y|0\rangle = -i|1\rangle$

→ $Y|1\rangle = i|0\rangle$

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2

$\rightarrow Z|0\rangle = |0\rangle$

$\rightarrow Z|1\rangle = -|1\rangle$

Outer product OR Matrix product of 2 vectors

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$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ $|\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$ $|\chi\rangle = \begin{bmatrix} e \\ f \end{bmatrix}$

Where $a, b, c, \ldots f \in \mathbb{C}$

$a = \alpha_a + i\beta_a$
$b = \_\_\_$
$c = \_\_\_$

* $|\psi\rangle\langle\phi| = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c^* & d^* \end{bmatrix} = \begin{bmatrix} ac^* & ad^* \\ bc^* & bd^* \end{bmatrix}$

* $\left(|\psi\rangle\langle\phi|\right)|\chi\rangle = \underset{2\times2}{\begin{bmatrix} ac^* & ad^* \\ bc^* & bd^* \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} e \\ f \end{bmatrix}}$

$= \underset{2\times1}{\begin{bmatrix} ac^*e + ad^*f \\ bc^*e + bd^*f \end{bmatrix}}$

$= \begin{bmatrix} a(c^*e + d^*f) \\ b(c^*e + d^*f) \end{bmatrix}$

$= \begin{bmatrix} c^*e + d^*f \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$= (c^*e + d^*f)|\psi\rangle = \langle\phi|\chi\rangle|\psi\rangle$

* $\langle\phi|\chi\rangle = \begin{bmatrix} c^* & d^* \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = (c^*e + d^*f)$

→ Proportionality const.

$\rightarrow \left(|\psi\rangle\langle\phi|\right)\left(|\chi\rangle\right) = \langle\phi|\chi\rangle\left(|\psi\rangle\right)$

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So here $(|\psi\rangle\langle\phi|)$ is an operator

$\langle\phi|x\rangle$ is proportionality const as this is a scalar.

$$* \; (|\psi\rangle\langle\phi|)|x\rangle$$

$$\underset{m\times 1}{|\psi\rangle}\;\underbrace{\underset{1\times n}{\langle\phi|}\;\underset{n\times 1}{|x\rangle}}$$

$$\underbrace{1\times n \quad n\times 1}$$

$$= |\psi\rangle\langle\phi|x\rangle$$

$$= \langle\phi|x\rangle|\psi\rangle$$

Example $\hat{A} = |0\rangle\langle 0| - |1\rangle\langle 1|$

$\hat{A}|\psi\rangle = ?$ where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\longrightarrow \hat{A}|\psi\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$

$* \; \left\{ |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}\right.$

$* \; \langle 0|0\rangle = \langle 1|1\rangle = 1$

$* \; \langle 0|1\rangle = \langle 1|0\rangle = 0$

$\Rightarrow \alpha(|0\rangle\langle 0| - |1\rangle\langle 1|)(|0\rangle) + \beta(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle$

$\Rightarrow \alpha(|0\rangle\underbrace{\langle 0|0\rangle}_{1} - |1\rangle\underbrace{\langle 1|0\rangle}_{0}) + \beta(|0\rangle\underbrace{\langle 0|1\rangle}_{0} - |1\rangle\underbrace{\langle 1|1\rangle}_{1})$

$\Rightarrow \alpha|0\rangle - \beta|1\rangle$

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$\Rightarrow \hat{A}(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$

$$\left.\begin{array}{l} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow -|1\rangle \end{array}\right\}$$ And this is the functioning of Pauli - Z operator

Example

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orthonormal basis

# Closure Relation

$S = \{|u_i\rangle\}_{i=1,n}$

It is a basis

$$I = \sum_{i=1}^{n} |u_i\rangle\langle u_i|$$ Cond$^n$ → Basis should be orthonormal

$$|\psi\rangle = \hat{A}|\psi\rangle \quad , \hat{A} = ?$$

$$= I|\psi\rangle$$

$$= \left(\sum_{i=1}^{n} |u_i\rangle\langle u_i|\right)\left(|\psi\rangle\right)$$

$$= \sum_{i=1}^{n} |u_i\rangle \underbrace{\langle u_i|\psi\rangle}_{c_i}$$

$$\boxed{|\psi\rangle = \sum_{i=1}^{n} c_i |u_i\rangle}$$ → Vectors in the basis can be used to generate the other vector using linear comb.

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## Representation of operators using matrices whose dimension is $(n \times n)$ [ONLY FOR ORTHONORMAL BASIS]

$$\hat{A} = (\hat{I}\hat{A})\hat{I}$$

$$= \hat{I}\hat{A}\hat{I}$$

$$= \left(\sum_{i=1}^{n} |u_i\rangle\langle u_i|\right)\hat{A}\left(\sum_{j=1}^{n} |u_j\rangle\langle u_j|\right)$$

$$= \left(\sum_{i=1}^{n} |u_i\rangle\langle u_i|\right)\left(\sum_{j=1}^{n} \hat{A}|u_j\rangle\langle u_j|\right)$$

$$\hat{A} = \sum_{i,j=1}^{n} |u_i\rangle\langle u_i|\hat{A}|u_j\rangle\langle u_j|$$

Now we have basis of vectors $\{|u_i\rangle\}_{i=1:n}$

$\rightarrow$ $n=3$

$\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$

$\rightarrow \hat{I} = \sum_{i=1}^{n} |u_i\rangle\langle u_i|$

$$\boxed{\sum_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n}}$$

If there would be diff basis, like

$\{|v_i\rangle\}_{i=1:n}$

$$\hat{A} = \sum |v_i\rangle\langle v_i|\hat{A}|v_j\rangle\langle v_j|$$

$$\rightarrow \hat{A} = \sum_{i,j} \overset{\text{Vector}}{|u_i\rangle}\,\underbrace{\langle u_i|\overset{\text{Scalar}}{\hat{A}}|u_j\rangle}_{A_{ij}}\,\overset{\text{Vector}}{\langle u_j|}$$

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \cdots & \hat{A}_{1n} \\ \hat{A}_{21} & \hat{A}_{22} & \cdots & & \hat{A}_{2n} \\ \vdots & & & & \\ \hat{A}_{n1} & \hat{A}_{n2} & \cdots & & \hat{A}_{nn} \end{bmatrix}$$

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$$= \begin{bmatrix} \langle u_1|\hat{A}|u_1\rangle & \langle u_1|\hat{A}|u_2\rangle & \cdots & \langle u_1|\hat{A}|u_n\rangle \\ \langle u_2|\hat{A}|u_1\rangle & \langle u_2|\hat{A}|u_2\rangle & \cdots & \langle u_2|\hat{A}|u_n\rangle \\ \vdots & & & \\ \langle u_n|\hat{A}|u_1\rangle & \langle u_n|\hat{A}|u_2\rangle & \cdots & \langle u_n|\hat{A}|u_n\rangle \end{bmatrix}$$

Now we need to deal with 2d vector space

$$A = \begin{bmatrix} \langle 0|A|0\rangle_{A_{11}} & \langle 0|A|1\rangle_{A_{12}} \\ \langle 1|A|0\rangle_{A_{21}} & \langle 1|A|1\rangle_{A_{22}} \end{bmatrix}$$

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$$I = \sum_{i=1} |u_i\rangle\langle u_i|$$

let us take a basis in $\mathbb{C}^2$

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$|u_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ $|u_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\langle u_1| = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\quad$ $\langle u_2| = \begin{bmatrix} 0 & 1 \end{bmatrix}$

$$I = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

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So

$$\hat{A} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \langle u_i | \hat{A} | u_j \rangle \, |u_i\rangle\langle u_j| \right)$$

Suppose (taking the yesterday's ex)

$\mathbb{C}^2 \; \{|u_i\rangle\} = \{|u_j\rangle\}$

$$\hat{A} = \langle u_1|\hat{A}|u_1\rangle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \langle u_1|\hat{A}|u_2\rangle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$+ \langle u_2|\hat{A}|u_1\rangle \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \langle u_2|\hat{A}|u_2\rangle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \langle u_1|\hat{A}|u_1\rangle & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \langle u_1|\hat{A}|u_2\rangle \\ 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 \\ \langle u_2|\hat{A}|u_1\rangle & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \langle u_2|\hat{A}|u_2\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle u_1|\hat{A}|u_1\rangle & \langle u_1|\hat{A}|u_2\rangle \\ \langle u_2|\hat{A}|u_1\rangle & \langle u_2|\hat{A}|u_2\rangle \end{bmatrix}$$

→ In this way, we ~~have~~ can prove the previous $\hat{A}$.

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$$\hat{A} = \begin{bmatrix} \langle 0|\hat{A}|0\rangle & \langle 0|\hat{A}|1\rangle \\ \langle 1|\hat{A}|0\rangle & \langle 1|\hat{A}|1\rangle \end{bmatrix} \text{ in } \mathbb{C}^2$$

There are 4 Pauli operators

- → $\sigma_0$ OR $I$
- → $\sigma_1$ or $\sigma_x$ or $X$
- → $\sigma_2$ or $\sigma_y$ or $Y$
- → $\sigma_3$ or $\sigma_z$ or $Z$

So as we did in lab

qc = QuantumComputing(4)
qc.x(3)

$q_0$ — $|0\rangle$ ——
$q_1$ — $|0\rangle$ ——
$q_2$ — $|0\rangle$ ——
$q_3$ — $|0\rangle$ — $\boxed{X}$ —

$\hat{I}|0\rangle = |0\rangle$
$\hat{I}|1\rangle = |1\rangle$

$$\hat{I} = \begin{bmatrix} \langle 0|\hat{I}|0\rangle & \langle 0|\hat{I}|1\rangle \\ \langle 1|\hat{I}|0\rangle & \langle 1|\hat{I}|1\rangle \end{bmatrix} = \begin{bmatrix} \overset{1}{\langle 0|0\rangle} & \overset{0}{\langle 0|1\rangle} \\ \langle 1|0\rangle & \langle 1|1\rangle \end{bmatrix}$$

$$\boxed{\hat{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|}$$

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As $\hat{X}$ acts as not gate $\Rightarrow$ $\hat{X}|0\rangle = |1\rangle$

$\hat{X}|1\rangle = |0\rangle$

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## Matrix for X operator

$$\hat{X} = \begin{bmatrix} \langle 0|\hat{X}|0\rangle & \langle 0|\hat{X}|1\rangle \\ \langle 1|\hat{X}|0\rangle & \langle 1|\hat{X}|1\rangle \end{bmatrix}$$

$$\hat{X} = \begin{bmatrix} \overbrace{\langle 0|1\rangle}^{0} & \overbrace{\langle 0|0\rangle}^{1} \\ \underbrace{\langle 1|1\rangle}_{1} & \underbrace{\langle 1|0\rangle}_{0} \end{bmatrix}$$

$$\boxed{\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|}$$

## Matrix for Y operator

$\begin{cases} \hat{Y}|0\rangle = -i|1\rangle \\ \hat{Y}|1\rangle = i|0\rangle \end{cases}$

$$\hat{Y} = \begin{bmatrix} \langle 0|\hat{Y}|0\rangle & \langle 0|\hat{Y}|1\rangle \\ \langle 1|\hat{Y}|0\rangle & \langle 1|\hat{Y}|1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0|(-i|1\rangle) & \langle 0|(i|0\rangle) \\ \langle 1|(-i|1\rangle) & \langle 1|(i|0\rangle) \end{bmatrix}$$

$$\boxed{\hat{Y} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0|}$$

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$\hat{z}|0\rangle = |0\rangle$

$\hat{z}|1\rangle = -|1\rangle$

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## Matrix for Z operator

$$\hat{z} = \begin{bmatrix} \langle 0|\hat{z}|0\rangle & \langle 0|\hat{z}|1\rangle \\ \langle 1|\hat{z}|0\rangle & \langle 1|\hat{z}|1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0|0\rangle & -\langle 0|1\rangle \\ \langle 1|0\rangle & -\langle 1|1\rangle \end{bmatrix}$$

$$\hat{z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

eg

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

* $X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

* $X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

# Eigen values & eigen vectors

$A \in C^{n\times n}$ $\quad |v\rangle = C^n \quad c \in \mathbb{C}$

$$\Rightarrow A|v\rangle = \underbrace{c}_{\text{eigen value}}\,\underbrace{|v\rangle}_{\text{eigen vector}}$$

$$\boxed{Av = cv}$$

eg $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \qquad |v\rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A|v\rangle = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \underbrace{2}_{\text{eigen value}}\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\text{eigen vector}}$$

## Properties

$A \in C^{n\times n}$ $\qquad c_0$ = eigen value w.r.t. eigen vector $|v_0\rangle$ and $\exists$ a scalar $c \in \mathbb{C}$

$$A(c|v_0\rangle) = c(A|v_0\rangle)$$

$$= c(c_0|v_0\rangle)$$ because this is the eigen value wrt $A$

$$= \underbrace{c_0}_{\text{e. value}}\underbrace{(c|v_0\rangle)}_{\text{eigen vector}}$$

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* $A \in \mathbb{C}^{n \times n}$, $c|v_0\rangle$ & $c'|v_0\rangle$ are the eigen vectors & $c_0$ is the e.value wrt. both of them.

$$\Rightarrow A(c|v_0\rangle) + A(c'|v_0\rangle)$$
$$= cA|v_0\rangle + c'A|v_0\rangle$$
$$= c\,c_0|v_0\rangle + c'\,c_0|v_0\rangle$$
$$= c_0(c+c')|v_0\rangle$$

* $A \in \mathbb{C}^{n \times n}$, $I_{n \times n}$, $c$

$$\det(A - cI) = 0$$

↓

e.value & then we can find eigen vector

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## Hermitian and Unitary matrices

If $A \in \mathbb{C}^{n \times n}$, then A is hermitian if

$A = A^{\dagger}$ $\left[ A^{\dagger} = (\bar{A})^T \right]$ conjugate

eg → $A = \begin{bmatrix} 7 & 6+30i \\ 6-30i & 3 \end{bmatrix}$

$A^{\dagger} = \begin{bmatrix} 7 & 6+30i \\ 6-30i & 3 \end{bmatrix}$

⇒ Now the matrix is hermitian if $A(i,i) \in \mathbb{R}^n$, i.e.

diagonal elements

→ If A is hermition, then the operator wrt A is called self-adjoint.

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