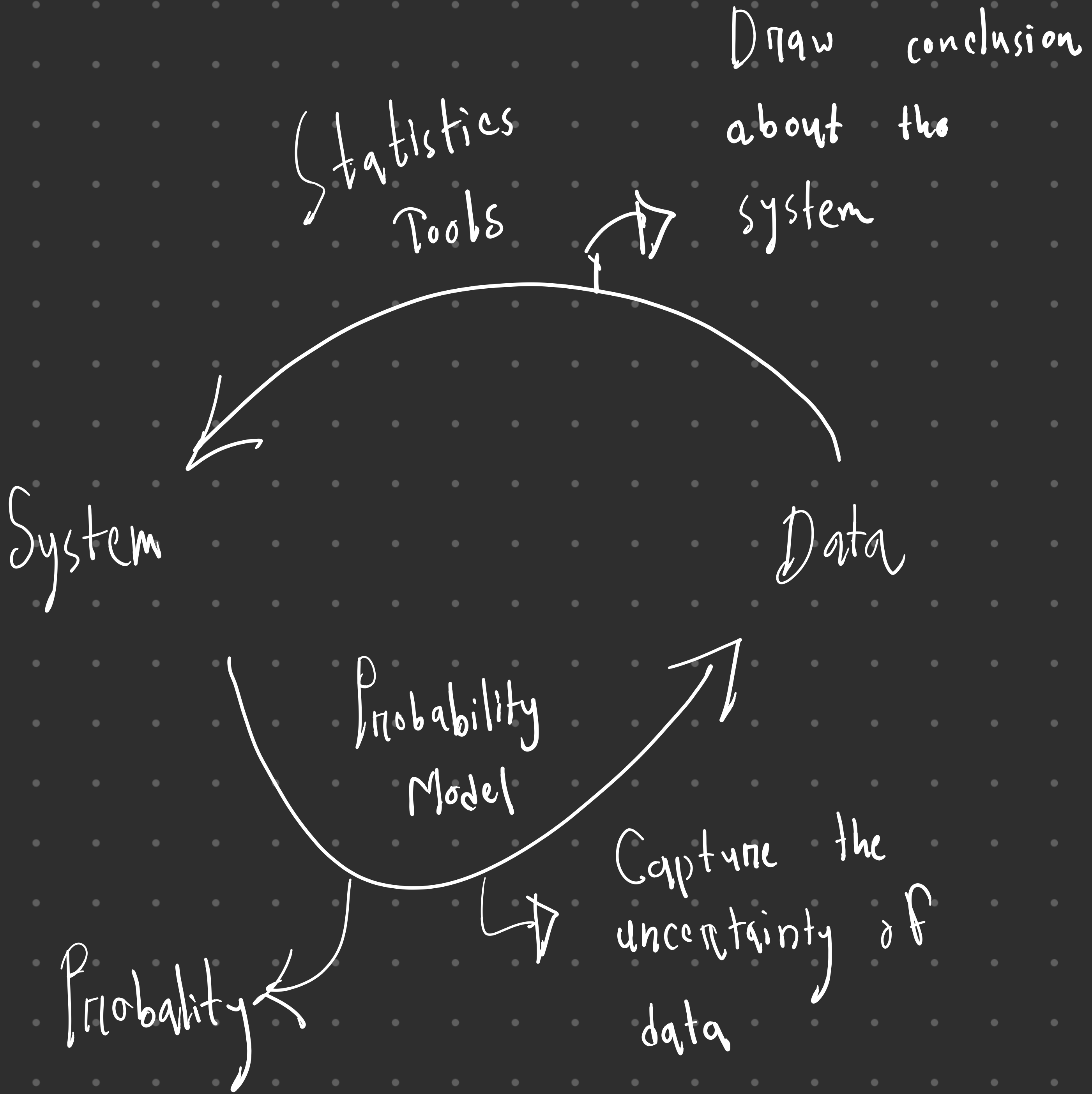
Classroom Code: loyrczjs

Book: Goodman & Yates

Intro to Probability & Scholastic Process, 3rd Edition.



Problem 1 (Disease or Not)

1. The probability that a randomly selected person has a disease is 0.8 percent (8 out of 1,000).
2. The probability that a person with the disease will have a positive test result is 90 percent.
3. The probability that a person with the disease will have a positive test result is 7 percent.

↳ Classic Bayessian Problem.
Will be solved later on.

Problem 2 (Monty Hall Dilemma)

The dilemma is taken from an old American game show called Let's Make a Deal.
The original host of the show, Monty Hall, would select a member of the audience and show that person three large closed doors labeled 1, 2, and 3. Behind one of the doors was a new car. Behind the remaining two doors were joke prizes, such as a live goat.
The contestant was asked to pick a door. Then, Hall would ask that one of the doors the contestant didn't pick be opened, naturally one that didn't have a car behind it. After the audience stopped laughing at whatever joke prize was behind that door, Hall would ask the contestant if they wanted to keep the originally selected door, or if they would rather change their selection to the remaining door. The dilemma is simply that: do they keep their original guess, or do they switch to the remaining door?

↳ Conditional Probability
Will be solved later on.

Problem 3 (Birthdays Problem)

We want to determine the probability that in a class size of r two or more birth dates match. We shall assume that the year consists of 365 days.

Solⁿ: If no two people share a birthday:

The first person can have any of the 365 days → $\frac{365}{365}$

The second person must avoid first's birthday $\rightarrow \frac{364}{365}$

The third person must avoid both $\rightarrow \frac{363}{365}$

The nth person must avoid $\rightarrow \frac{365-n+1}{365}$

So,

$$1-\left(\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}\times\cdots\times\frac{363-n+1}{365}\right)\geq\frac{1}{2}$$

After some calculation,

$\hookrightarrow$ $n = 23$

Probability

↳ A number between zero and one

↳ Represents the likelihood of occurring something in an uncertain phenomenon.

- A probability quantifies the chances by a number.
- Probability is 1 → Guaranteed to occur.

  Probability is 0 → Impossible to occur.

Probability Model → A mathematical description of an uncertain situation.

↳ Quantifies the uncertainty by measurable values.

Experiment

\# of possible result $= 1$ → Deterministic

> 1 → Random

- There are some models that do not capture the entire uncertainty into considerations.

↳ sometimes the possibilities may be ∞.

So we will consider experiments with more than 1 results but the total number will be finite

Random Experiment

↳ Produces exactly one result out of several possible results.

→ Experiment

- Procedure
  - ↓ Real world phenomenon that produces the outcome
- Observation
  - ↓ Expected outcome or more like number of expected outcomes

⊛ Each of the distinct observations of an experiment is an outcome.

Event → A set of outcomes

Example 1.1: For the first example, consider that the procedure is sending three packets of data from one computer to another. The observation here would be the sequence of successful and unsuccessful deliveries. Possible outcomes include DDD, DDF, DFD, etc. where D indicates a successful delivery and F indicates a failed one.

Example 1.2: Consider that the procedure is again that three packets of data are being sent from one computer to another, but the observation is the number of successful deliveries. Possible outcomes include 0, 1, 2 and 3.

Example 1.3: Consider that the procedure is to send a packet of data repeatedly from one computer to another until the packet is delivered successfully and the observation is number of attempts that were required. Here, possible outcomes could be anywhere between 1 and ∞.

Example 1.4: Consider that the procedure is to send packets of data one after another until three packets of data have been sent and the observation is the number of attempts that we needed. Here, the possible outcomes are between 3 and ∞.

✱ Notice that the procedures are similar for all the examples, but these will not be considered the same experiments. Two experiments are only considered to be identical if they have the same procedure and the same observation.

Terminologies

Outcome → An observation

Every possible result of an experiment

Event → A set of one or more outcome from the sample space.

Probability → A measure of the likelihood that a specific event will occur. It ranges from 0 (impossible) to 1 (certain)

Example 1: If you are rolling a dice -

* Rolling a 5 on a dice → outcome
* Rolling a number >3 → $\{4,5,6\}$ → event
* $P(\text{rolling} > 3) \rightarrow \frac{1+1+1}{6} = 0.5$ → probability

Sample Space → Set of all possible outcomes

→ if the outcomes individually collectively satisfy 3 properties.

Properties:

① **Finest Grain:** Include outcomes that are specific and distinct. No outcome should be broken down. For rolling a dice: $S = \{1, 2, 3, 4, 5, 6\}$

This is finely grained because each number is an outcome — cannot be broken further.

② **Mutually Exclusive:** No two outcomes in the sample space can occur at the same time. They must be non-overlapping. When you roll a dice, you can't get both 2 and 5 at once.

So, for any two outcomes A and B, $A \cap B = \phi$.

③ **Collectively Exhaustive:** All possible outcomes of the experiment are included in the sample space. No outcome is left out. For rolling a dice: $S = \{1, 2, 3, 4, 5, 6\}$

Sample Space

Discrete Set

- finite or countably infinite
- Distinct values
- For example:

  Tossing a coin:

  $S = \{Heads, Tails\}$

  Rolling a dice:

  $S = \{1, 2, 3, 4, 5, 6\}$

- Probability assigns to each specific outcome

Continuous Set

- uncountably infinite
- Any values in interval
- For example:

  A person's height:

  $S = [100, 200]$

  Time taken to finish a race:

  $S = (0, \infty)$

- Probability assigns to intervals

Event → An event is a set of outcomes that consist of one or more elements of the sample space.

For a dice rolling: $S = \{1,2,3,4,5,6\}$.

$E_1 = \{\text{outcome is an even number}\} = \{2,4,6\}$ → An event.

$E_2 = \{\text{outcome is an odd number}\} = \{1,3,5\}$ → An event.

Event Space → A set of events that have the following properties:

1. Mutually Exclusive
2. Collectively Exhaustive

Mutually Exclusive: Two events are mutually exclusive, they cannot occur together $E_i \cap E_j = \emptyset,\ i \neq j$

Collectively Exhaustive: Set of events are collectively exhaustive, if they include all possible outcomes.

$\bigcup_{i=1}^{n} E_i = S$.

Example:

🔁 Recap of the Example

✅ Procedure:

Send **3 packets**, each of which may either **succeed (D)** or **fail (F)**.

✅ Observation:

Record the **sequence** of outcomes.

✅ Sample Space S:

This contains all possible outcomes (each is a specific sequence of 3 packets):

$$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$$

This **is** a sample space because:

- It lists **all** possible outcomes (collectively exhaustive).
- Each outcome is **distinct and indivisible** (finest grain).
- Outcomes are **mutually exclusive**.

🟦 Now Define Event Sets Based on Number of Successes:

You group outcomes by how many packets were delivered (D):

- $E_0 = \{FFF\}$
- $E_1 = \{FFD, FDF, DFF\}$
- $E_2 = \{FDD, DFD, DDF\}$
- $E_3 = \{DDD\}$

Let $E = \{E_0, E_1, E_2, E_3\}$

✅ Why is E an **Event Space**?

Because:

- The sets E_0 through E_3 are **mutually exclusive** (no overlapping outcomes).
- Together, they cover all outcomes in S → **collectively exhaustive**.

So E satisfies the two core properties of an **event space**.

🚫 Why is E **Not a Sample Space**?

Because it **fails the "finest grain"** property.

> 📌 The finest grain means each element of the sample space must be **an atomic outcome**—not a set of outcomes.

In E, each element (e.g., E_1) is **not** a single outcome—it's a set of outcomes. So:

- You know that *some* success happened (e.g., one success), but
- You **don't know which specific outcome** occurred (FFD vs FDF vs DFF).

Hence, E **is not a sample space**.

Example 1.6:

Example 1.6: Monitor 3 consecutive phone calls going through a telephone switching office. Classify each one as a voice call (v) if someone is speaking, or a data call (d) if the call is carrying a modem or fax signal. Your observation is a sequence of three letters (each letter is either v or d). For example, two voice calls followed by one data call corresponds to vvd. Write the elements of the following sets:

1. $A_1 =$ {first call is a voice call}
2. $B_1 =$ {first call is a data call}
3. $A_2 =$ {second call is a voice call}
4. $B_2 =$ {second call is a data call}
5. $A_3 =$ {all calls are the same}
6. $B_3 =$ {voice and data alternate}
7. $A_4 =$ {one or more voice calls}
8. $B_4 =$ {two or more data calls}

For each pair of events A_1 and B_1, A_2 and B_2, and so on, identify whether the pair of events is either mutually exclusive or collectively exhaustive or both.

1. A_1 = {first call is a voice call}
   = {vvv, vvd, vdd, vdv}
2. B_1 = {first call is a data call}
   = {ddd, ddv, dvd, dvv}
3. A_2 = {second call is a voice call}
   = {vvv, vvd, dvv, dvd}
4. B_2 = {second call is a data call}
   = {vdv, vdv, ddv, ddd}
5. A_3 = {all calls are the same}
   = {ddd, vvv}
6. B_3 = {voice and data are alternate}
   = {dvd, vdv}
7. A_4 = {one or more voice call}
   = {vvv, vvd, vdd, dvd, dvv, ddv}
8. B_4 = {two or more data calls}
   = {dvd, ddv, ddd, vdd}

Probability of Event Outcomes:

→ How probabilities are assigned to the outcomes and events.

1. Probability Axioms.
   - Rules related to the assignment of probabilities, and every assignment satisfy.
   - Ensures a number between 0 and 1 to each outcome and event.
2. Assignment of Probabilities:
   - Classical Approach
   - Relative Frequency Approach
   - Subjective Approach

Probability Axioms

Axiom 1 (Non-negativity)

↳ If A is an event, then the probability of A satisfies $P[A] \geq 0$.

Axiom 2 (Additivity)

↳ If A_1 and A_2 are two disjoint sets, then

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]$$

Axiom 3 (Normalization)

↳ The probability of all elements in the sample space must be sum to 1. $P[S] = P[\bigcup_{i=1}^{n} \omega_i] = \sum_{i=1}^{n} P[\omega_i] = 1$

Simple Extension of the Axioms

1. If $A_1, A_2, \dots, A_n$ is a disjoint set then
$$P\left[\bigcup_{i=1}^{n} A_i\right] = P[A_1] + P[A_2] + \dots + P[A_n]$$

2. If $A \cap B \neq \phi$, A and B are not mutually exclusive, then $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

3. $P[A^c] = 1 - P[A]$ → complementary probability.

4. If $A \subset B$, then $P[A] \leq P[B]$

Assignment of Probability: Classical Approach

The classical approach to probability is one of the oldest and simplest methods. It applies to experiments where all outcomes are equally likely.

If an experiment has n equally likely outcomes, and an event A contains m of those outcomes, then the probability of event A is:

$$P(A) = \frac{m}{n}$$

m: Number of favorable outcomes
n: Number of total possible outcomes
Assumes: All outcomes are equally likely

Example 1.7: Tossing a fair coin

- Sample Space: $S = \{H, T\}$
- Total outcomes: $n = 2$.
- Event: Getting a head → $m = 1$.

$\therefore P(A) = \frac{1}{2}$.

Example 1.8

Question:

A pair of four-sided fair dice is rolled, and the number of dots on the top face of each die is observed.
Each outcome of this experiment is a **pair of numbers** in the range $\{1, 2, 3, 4\}$.
Thus, the sample space consists of all possible ordered pairs:

$$S = \{(i, j) \mid i \in \{1,2,3,4\},\ j \in \{1,2,3,4\}\}$$

There are a total of $4 \times 4 = 16$ possible outcomes.
Since the dice are fair and all outcomes are equally likely, each outcome has a probability of:

$$P(\text{each outcome}) = \frac{1}{16}$$

Now, suppose we want to find the probability of the event:

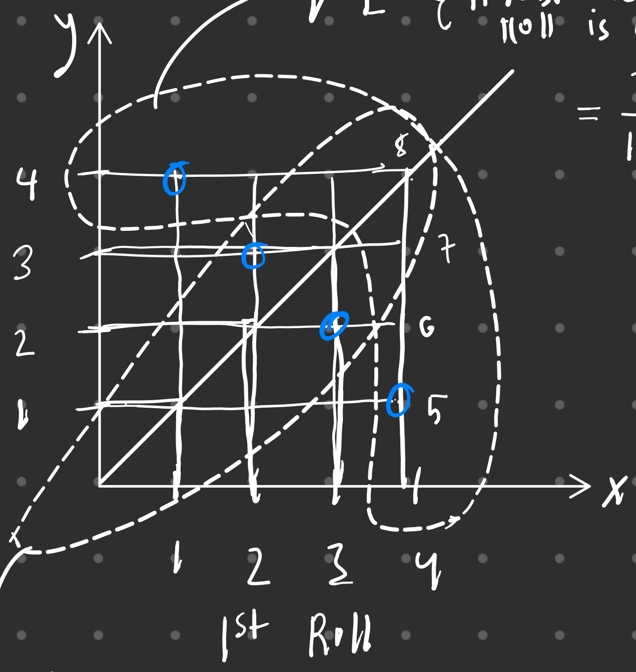
$$A = \{\text{sum of the two dice is } 5\}$$

$A = \{$sum of the two dice is $5\}$

$$= \frac{4}{16} = \frac{1}{4}.$$

$E = \{$at least one roll is a $4\}$

$$= \frac{7}{16}.$$



$E = \{$the first roll is equal to the second$\}$

Probability $= \frac{4}{16}$

Assignment of Probability: Relatively Frequency Approach

This approach is used when the outcomes in the sample space are not equally likely.

Steps:
a. The outcomes may have different probabilities.
b. Repeat the experiment a large number of times (say, n) such that $n \to \infty$
c. Count the number of times a particular outcome ω occurs, denoted by n (sub w) .

Then, the probability of occurrence of outcome ω is defined as:

$$P[\omega] = \lim_{n \to \infty} \frac{n_\omega}{n}$$

Similarly, to calculate the probability of an event A, count the number of times event A occurs, denoted by n(sub A), over n repetitions.

$$P[A] = \lim_{n \to \infty} \frac{n_A}{n}.$$

Example: Coin Toss Experiment (Uneven Coin)

Suppose you have a biased coin — that is, the probability of heads is not necessarily equal to the probability of tails.

You **don't know** the actual probabilities of heads (H) or tails (T). So, you decide to estimate them experimentally using the **relative frequency approach**.

Step-by-step:

1. You toss the coin $n = 1000$ times.
2. You observe the results and find:
   - Number of times **Heads** appeared: $n_H = 620$
   - Number of times **Tails** appeared: $n_T = 380$

Now, using the relative frequency formula:

$$P[\text{Heads}] \approx \frac{n_H}{n} = \frac{620}{1000} = 0.62$$

$$P[\text{Tails}] \approx \frac{n_T}{n} = \frac{380}{1000} = 0.38$$

As you increase the number of trials (say to 10,000 or more), these estimates become more accurate, approaching the **true probability**.

Conclusion:

You didn't assume any theoretical or classical probabilities. Instead, you estimated the probabilities **based on actual observed frequencies** — that's the **relative frequency approach**.

Assignment of Probability : Subjective Approach

The Subjective Approach to probability is based on personal judgment, opinion, or belief, rather than mathematical calculation or experimental data. It's often used when:

There is no historical data available.

Events are unique or rare, so frequencies are meaningless.

Probabilities are based on expert intuition or opinion.

Key Features:
a. Probabilities are assigned by engaging subject experts
Experts use their knowledge, experience, and intuition to assess how likely an event is to happen. This is common in areas like weather forecasting, financial risk, sports, or politics.

b. Manually assign the probabilities
The assigned values are not derived mathematically but are chosen manually based on reasoning, experience, or insight.

c. Vary from expert to expert
Since these probabilities are based on opinions, different experts may assign different probabilities to the same event.

Example 1: Weather Forecasting
A meteorologist might say:

“I believe there is a 70% chance of rain tomorrow.”

This is not based purely on historical data or simulations. It might involve interpreting satellite images, pressure systems, wind patterns, and personal experience. Another meteorologist might say it's 60%.

Example 2: Stock Market Crash
An economist might say:

“There’s a 20% chance the stock market will crash this year.”

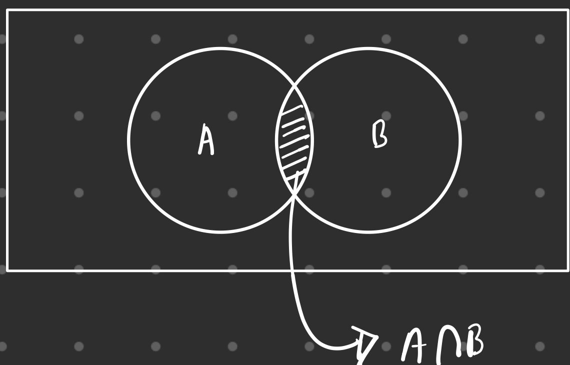
This estimate is based on their interpretation of economic indicators, policies, and historical patterns. Other economists may disagree and give different estimates.

Joint Probability

↪ Probability of two (or more) events happening at the same time. If A and B are two events, then the joint probability of A and B is written as: $P(A\cap B)$.

✻ If events A and B are mutually exclusive, they two cannot occur at the same time, $P[AB] = 0$, because $A\cap B = \phi$.

✻ $P[A\cup B] = P[A] + P[B] - P[A\cap B]$



↪ Calculation:

1. Classical Approach: Find $A\cap B$

$= \{\text{elements common in both events}\}$

$$\therefore P[A\cap B] = P[\cup_{\omega_i \in A\cap B} \omega_i] = \sum_{\omega_i \in A\cap B} P[\omega_i]$$

2. Relative Frequency Approach: If $n_{A,B}$ is the number of times events A and B occur simultaneously in n repitition then, $P[A,B] = \lim_{n\to\infty} \frac{n_{A,B}}{n}$.

Example 1.7

🎴 Given:
-> A standard deck has 52 cards.
-> There are 4 suits: ♠️ Spades, ❤️ Hearts, ♦️ Diamonds, ♣️ Clubs.
-> Each suit has 13 cards: Ace, 2–10, Jack, Queen, King.
-> ❤️ Hearts and ♦️ Diamonds are red suits → total 26 red cards.
-> ♠️ Spades and ♣️ Clubs are black suits → total 26 black cards.
-> Face cards: Jack, Queen, King (3 in each suit × 4 suits = 12 face cards).
-> Number cards: 2 to 10 (9 in each suit × 4 suits = 36 number cards).

Now, solve the questions:
1. Event A = {red card selected}
2. Event B = {number card selected}
3. Event C = {heart selected}

Sol^n:

1. $P[A] = \dfrac{\text{red cards}}{\text{total cards}} = \dfrac{13+13}{52} = \dfrac{1}{2}$.

2. $P[B] = \dfrac{\text{number cards}}{\text{total cards}} = \dfrac{10}{13}$ (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10)

3. $P[C] = \dfrac{\text{heart cards}}{\text{total cards}} = \dfrac{13}{52} = \dfrac{1}{4}$

then → red and number cards $P[AB] = \dfrac{1}{2} \times \dfrac{40}{52} = \dfrac{5}{13}$

→ red and heart cards $P[CA] = \dfrac{1}{2} \times \dfrac{13}{52} = \dfrac{1}{8}$

→ number and heart cards $P[BC] = \dfrac{40}{52} \times \dfrac{13}{52} = \dfrac{5}{26}$

Example 1.10

Roll a six-sided fair die and observe the number of dots on the top side S = {1, 2, 3, 4, 5, 6}.

Define two events:
A = {Outcome is square of an integer} = {1, 4}
B = {Outcome is an even number} = {2, 4, 6}

Sol^n:

$$P[A] = \frac{2}{6} = \frac{1}{3}$$

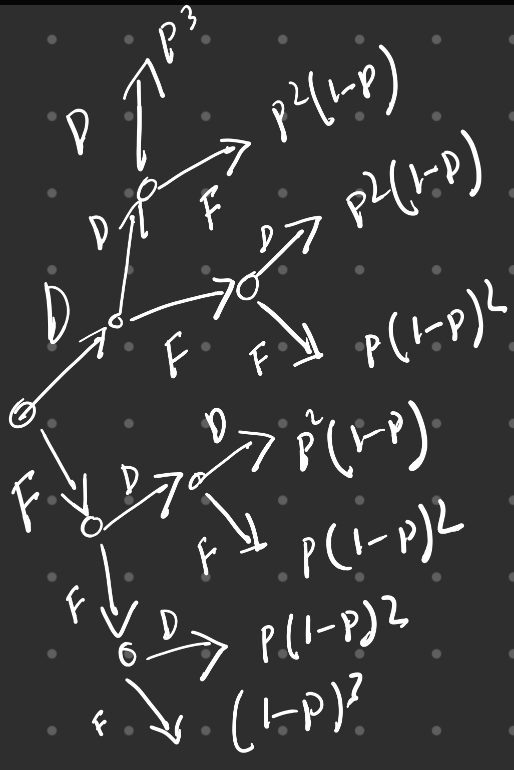
$$P[B] = \frac{3}{6} = \frac{1}{2}$$

$$P[AB] = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

Example 1.11 (Experiment 1.2)

- **Procedure**: Send 3 packets from a sender to a receiver.
- **Observation:** Sequence of successes (D for delivered) and failures (F for failed) of the deliveries.
- **Sample Space** (S): $\{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$.
    - Each outcome represents the sequence of results for the 3 packets.
    - There are $2^3 = 8$ possible sequences, as each packet can either succeed or fail.
- **Interpretation**: This is a model for the order of successes and failures over a fixed number of trials (3 packets). Each sequence is equally likely if the delivery success probability is constant (e.g., in a Bernoulli trial with p as the success probability).

$$P[FFF] = (1-p)^3$$



✲ Write the rest of the diagrams by yourself.

Example 1.12 (Experiment 1.3)

- **Procedure**: Send 3 packets from a sender to a receiver.
- **Observation**: Number of successes.
- **Sample Space** (S): $\{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$.
  - The sample space lists all possible sequences, but the observation focuses on counting successes (D's) in each sequence.
  - Number of successes: 0 (FFF), 1 (FFD, FDF, DFF), 2 (FDD, DFD, DDF), 3 (DDD).
- **Interpretation**: This differs from Example 1.11 in intent. While the sample space is the same (all sequences), the focus is on the total number of deliveries rather than the specific order. The distinct outcomes based on success count are:
  - 0 successes: 1 outcome (FFF).
  - 1 success: 3 outcomes (FFD, FDF, DFF).
  - 2 successes: 3 outcomes (FDD, DFD, DDF).
  - 3 successes: 1 outcome (DDD). This suggests a binomial distribution if each packet's delivery is independent with a fixed probability.

Example 1.13 (Experiment 1.4)

- **Procedure**: Keep sending packets until 1 packet is delivered.
- **Observation**: Number of attempts.
- **Sample Space** (S): $\{D, FD, FFD, FFFD, \ldots\}$.
  - Each outcome represents the number of attempts until the first success (D).
  - The sequence ends with D, preceded by zero or more failures (F).
  - The sample space is countably infinite, as the number of failures before the first success can be 0, 1, 2, ...
- **Interpretation**: This models a geometric distribution, where the number of trials until the first success is observed. For example:
  - D: 1 attempt (success on first try).
  - FD: 2 attempts (failure, then success).
  - FFD: 3 attempts (two failures, then success), and so on.

Example 1.14 (Experiment 1.5)

- **Procedure**: Keep sending packets until 3 packets are delivered.
- **Observation**: Number of attempts.
- **Sample Space** (S): $\{DDD, FDDD, DFDD, DDFD, FFDDD, \ldots\}$.
  - Each outcome represents a sequence ending with exactly 3 D's (deliveries), with zero or more F's (failures) before reaching the third D.
  - The sample space is countably infinite, as the number of failures before the third success can vary.
- **Interpretation**: This models a negative binomial distribution, where the number of trials until the r-th success (here, $r = 3$) is observed. For example:
  - DDD: 3 attempts (all successes).
  - $FDDD$: 4 attempts (one failure, then three successes).
  - $DFDD$: 4 attempts (one failure, three successes in a different order), and so on.