Example 2.1: **(Example 1.12)**

Procedure: Send 3 packets from a sender to a receiver.

Observation: Number of successes.

Convert into number

$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$

Probabilities of Outcomes $P[FFF] = (1-p)^3$ $P[FFD] = p(1-p)^2$ $P[FDF] = p(1-p)^2$ $P[FDD] = p^2(1-p)$ $P[DFF] = p(1-p)^2$ $P[DFD] = p^2(1-p)$ $P[DDF] = p^2(1-p)$ $P[FDD] = p^3$	$P[\text{number of success is } 0] = (1-p)^3$ $P[\text{number of success is } 1] = 3p(1-p)^2$ $P[\text{number of success is } 2] = 3p^2(1-p)$ $P[\text{number of success is } 3] = p^3$

1. Not a number → number
2. It is number → assign it
3 Numbers → another

$p \triangleq$ probability that a single packet is delivered

Each of the deliveries is independent of the others

Probability Models

$\frac{X}{\text{R.V}}$ $\frac{x}{\text{value}}$

Random Variable:

Random variables express the outcome of an experiment by real numbers

- It is a function that generates values (numbers) on demand
- The values generated are random (Not kwon which one will appear)
  - Set of Possible values is known
- Values are related to the events of the experiment
  - Each value has its own chances of appearing
  - But it needs to be related to one event
- Function converts the events into real numbers

event → 0
event → 1
→ 2
→ 3

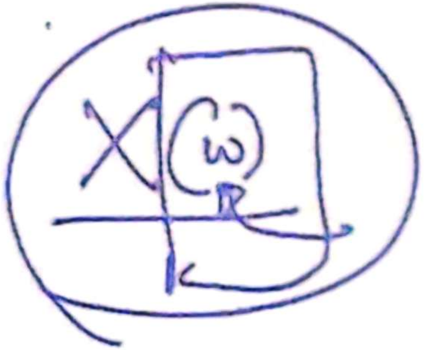
Distribution Function:

A distribution function represents a collection of probabilities

- Each probability is related to a real number, x
- Represents the chance of occurring the event represented by x

Probability Models by Random Variables

$X_1, X_2, \ldots$



X

X, Y, Z

Chapter 1

Random Experiment → Probability (Simple) Model → Sample Space (S), Prob. of Event P[.] → Random Variable, X: S -> R → Real Numbers S_x (S_X), {X <= x} is an event → Distribution Function → P[X=x], P[X<=x]

ω

P[•]

$P_X(x)$

$F_X(x)$

* Prob. that X has a value x

* Prob. that RV X will generate a value x.

$P[X = x]$

$x \in R$

$\{X \leq x\}$

$(-\infty, x]$

<u>Events associated with a random variable are</u>:

- The random variable has a specific value
  - $\{X = x\}$ or more specifically $\{X(\omega) = x\}$
- The random variable has a value which is less than or equal to a specific value:
  - $\{X \leq x\}$ or $X(\omega) \leq x\}$
- The random variable has value which is greater than a specific value
  - $\{X \geq x\}$ or $\{X(\omega) \geq x\}$

$X > x$

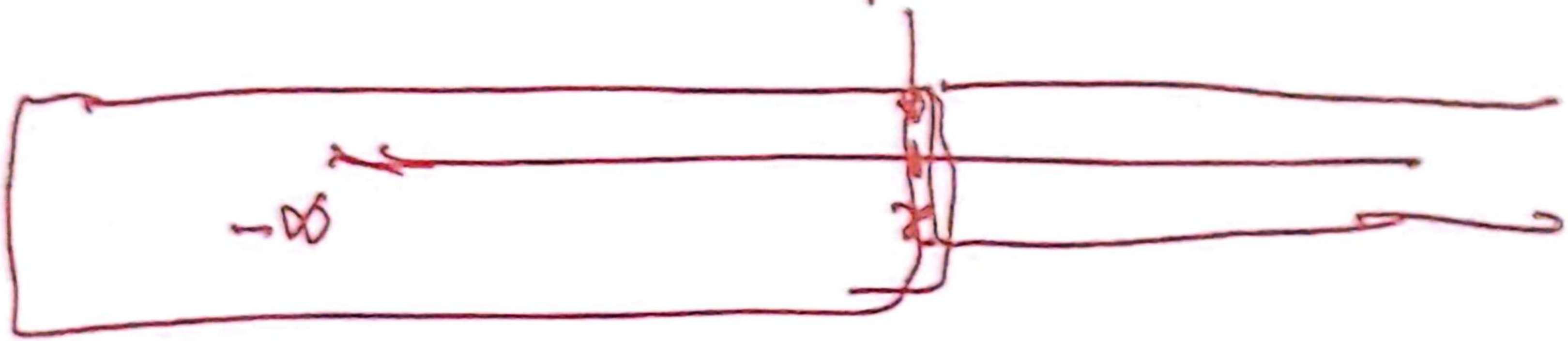
$P[X=1]$

$P[X \leq 2] =$

$P[X=0] +$

$P[X=1] +$

$P[X=2]$



PMF $\quad P_X(x) = P[X=x]$

Possible distribution functions are:

RV

- Probability Mass function (PMF): The probability that a random variable X has a specific value x $\quad P[X=x]$
  - $P[X = x]$

- Cumulative distribution function (CDF): The probability that a random variable has a value which is less than or equal to a specific value x
  - $P[X \leq x]$

- Complementary cumulative distribution function (CCDF): The probability that a random variable has a value which is greater than a specific value x
  - $P[X > x]$

- Complementary cumulative distribution function (CCDF): The probability that a random variable has a value which is greater than a specific value x
  - $P[X > x]$

$-\infty$ x

CGPA(5) =

$P[X \leq 2] = \frac{7}{8}$

$= P[X=0] + P[X=1] + P[X=2]$

$\frac{1}{8}$ $\frac{3}{8}$ $\frac{3}{8}$

$S_X = \{0, 1, 2, 3\}$

Example 2.1: (Continued)

<u>Procedure</u>: Send 3 packets from a sender to a receiver.

<u>Observation</u>: Number of successes.

$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$

Procedure: Send 3 packets from a sender to a receiver.

Observation: Number of successes.

Chap # 1

$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$

$E = \{E_0, E_1, E_2, E_3\}$

$P[X = 1.5]$

$P[X = x]$

$P[X \le x]$

$P[X > x]$

$X \triangleq$ Random variable that counts the number of successes

$S_X = \{0, 1, 2, 3\}$

$p = \frac{1}{2}$

	E_0	E_1	E_2	E_3
S	FFF	FFD FDF DFF	FDD DFD DDF	DDD
x	0	1	2	3
$P[X = x]$	$\boxed{P[X=0] = \frac{1}{8}}$	$P[X=1] = \frac{3}{8}$	$P[X=2] = \frac{3}{8}$	$P[X=3] = \frac{1}{8}$
$P[X \le x]$	$P[X \le 0] = \frac{1}{8}$	$P[X \le 1] = \frac{1}{2}$	$P[X \le 2] = \frac{7}{8}$	$P[X \le 3] = 1$

$P[X \le 5000] = 1$ $\quad P[X \le 1.5] = \frac{1}{2}$

Probability Mass Function (PMF): $P_X(x)$

$P_X(x) = P(X = x)$

Probability Mass Function (PMF): $\boxed{P_X(x)}$

If the set of possible values of X, $S_X = \{x_1, x_2, \ldots, x_n\}$, then

1. $P_X(x_i) = 0$, if $x_i \notin S_X$ $\quad i =$
2. $\boxed{P_X(x_i) = P[X = x_i]}$ and hence, $P_X(x_i) > 0$, for $i = 1, 2, \ldots, n$
3. $\boxed{\sum_i^n P_X(x_i) = 1}$

$$P_X(x) = \begin{cases} \frac{1}{8}, & x = 0, 3 \\ \frac{3}{8}, & x = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

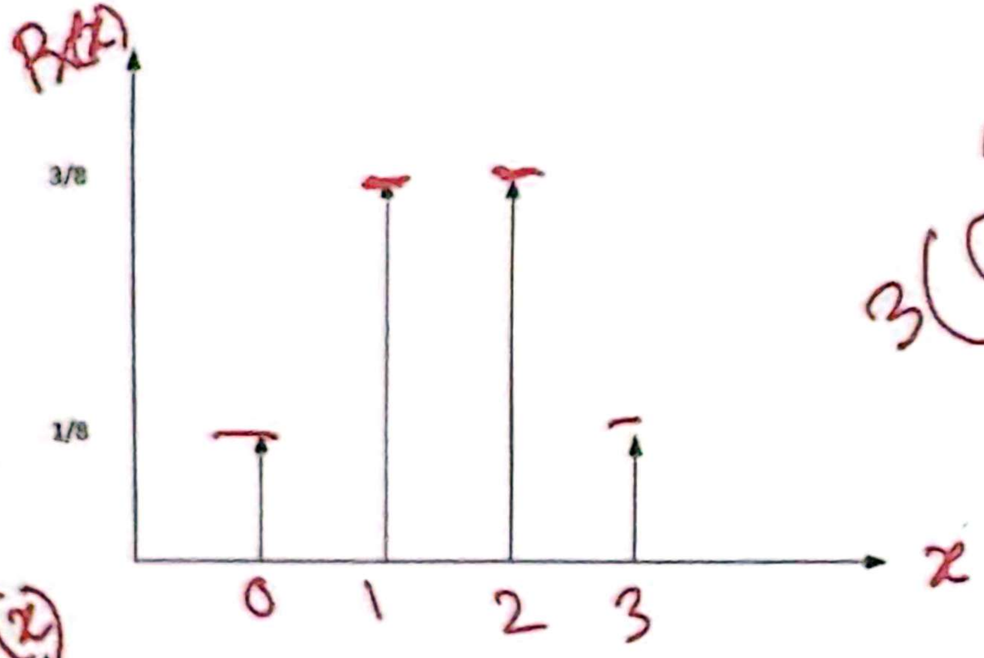
p

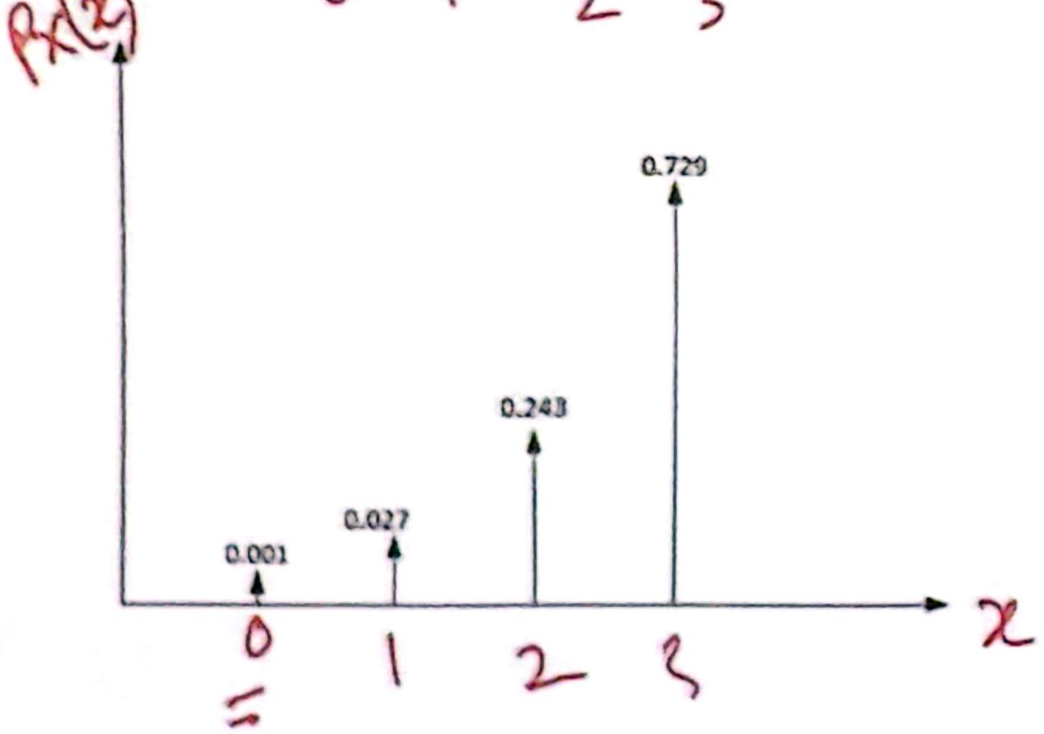
Example 2.1 (Continued)

$$P_X(x) = \begin{cases} (1-p)^3, & x = 0 \\ 3p(1-p)^2, & x = 1 \\ 3p^2(1-p), & x = 2 \\ p^3, & x = 3 \\ 0, & \text{otherwise} \end{cases} \qquad P_X(x) = \begin{cases} 1/8, & x = 0 \\ 3/8, & x = 1 \\ 3/8, & x = 2 \\ 1/8, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

For $p = \frac{1}{2}$,

Graphical Representation of the PMF





$0{\cdot}1 \times {\cdot}1 \times {\cdot}1 = {\cdot}001$

$3(0{\cdot}1 \times {\cdot}1 \times {\cdot}9) = {\cdot}009 \times 3 = {\cdot}027$

$p = {\cdot}9$

S_x Prob. x

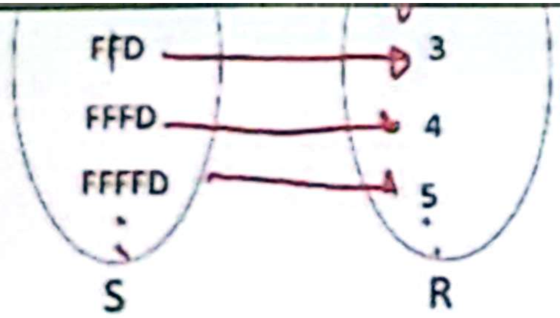
$$P_X(x) = \begin{cases} {\cdot}001, & x = 0 \\ {\cdot}027, & x = 1 \\ {\cdot}243, & x = 2 \\ {\cdot}729, & x = 3 \\ 0, & o/w \end{cases}$$

${\cdot}9 \times {\cdot}9 \times {\cdot}1$

$= {\cdot}081$

${\cdot}9 \times {\cdot}9 \times {\cdot}9$

$=$



$p(1-p)^2$

$P_X(x) = \{ 0 ?$

$X: S \to R$

	E_1	E_2	E_3	E_4	E_5
S	D	FD	FFD	$FFFD$	$FFFFD$
x	1	2	3	4	5
$P[X = x]$	p	$p(1-p)$	$p(1-p)^2$	$p(1-p)^3$	$p(1-p)^4$
$P[X \leq x]$					

x

$p(1-p)^{x-1}$

	2	2	3	2	2
$P[X \le x]$					

$$P_X(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, \ldots \\ 0, & o/w \end{cases}$$

$$P_X(x) = \begin{cases} \frac{1}{2}, & x = 1 \\ \left(\frac{1}{2}\right)^2, & x = 2 \\ \left(\frac{1}{2}\right)^3, & x = 3 \\ \vdots & \\ 0. & o/w \end{cases}$$

$$\boxed{F_X(x) = 1 - (1-p)^x \quad x \geq 1}$$

Example 2.1 (Continued)

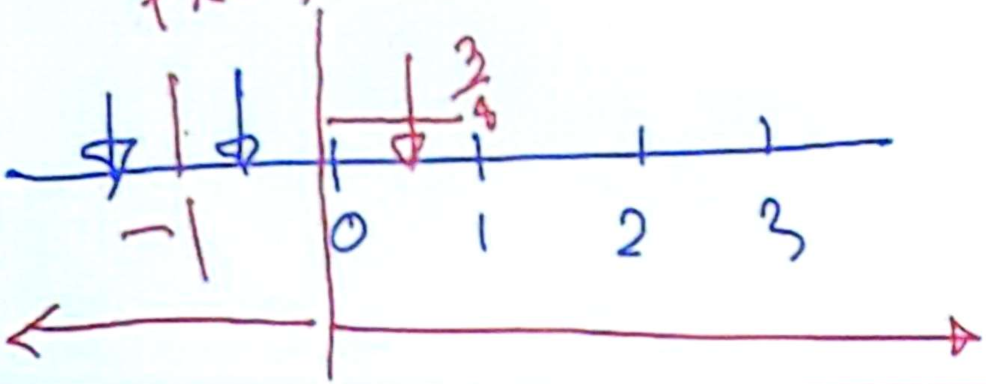
$$P_X(x) = \begin{cases} \quad , & x = 0 \\ \quad , & x = 1 \\ \quad , & x = 2 \\ \quad , & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$P_X(x) = \begin{cases} \frac{1}{8} \, , & x = 0 \\ \quad , & x = 1 \\ \quad , & x = 2 \\ \quad , & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

For $p = \frac{1}{2}$,

$F_X(.5)$

$F_X(-1)$



$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{7}{8}, & 2 \le x < 3 \end{cases}$$

Example 2.1 (Continued)

$$P_X(x) = \begin{cases} \quad , & x = 0 \\ \quad , & x = 1 \\ \quad , & x = 2 \\ \quad , & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

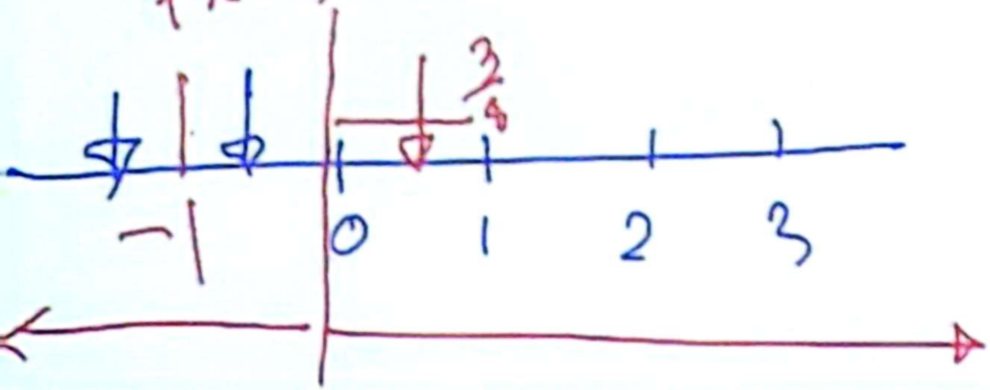
$$P_X(x) = \begin{cases} \frac{1}{8}, & x = 0 \\ \quad , & x = 1 \\ \quad , & x = 2 \\ \quad , & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

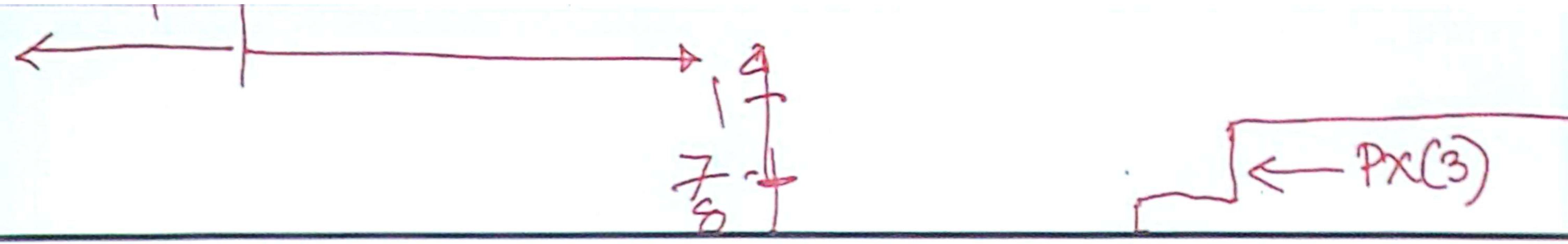
For $p = \frac{1}{2}$,

$F_X(.5)$

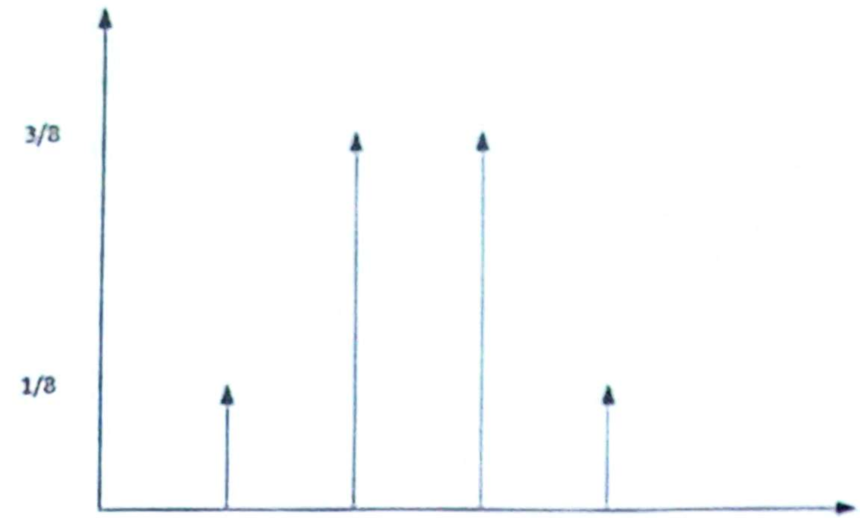
$F_X(-1)$

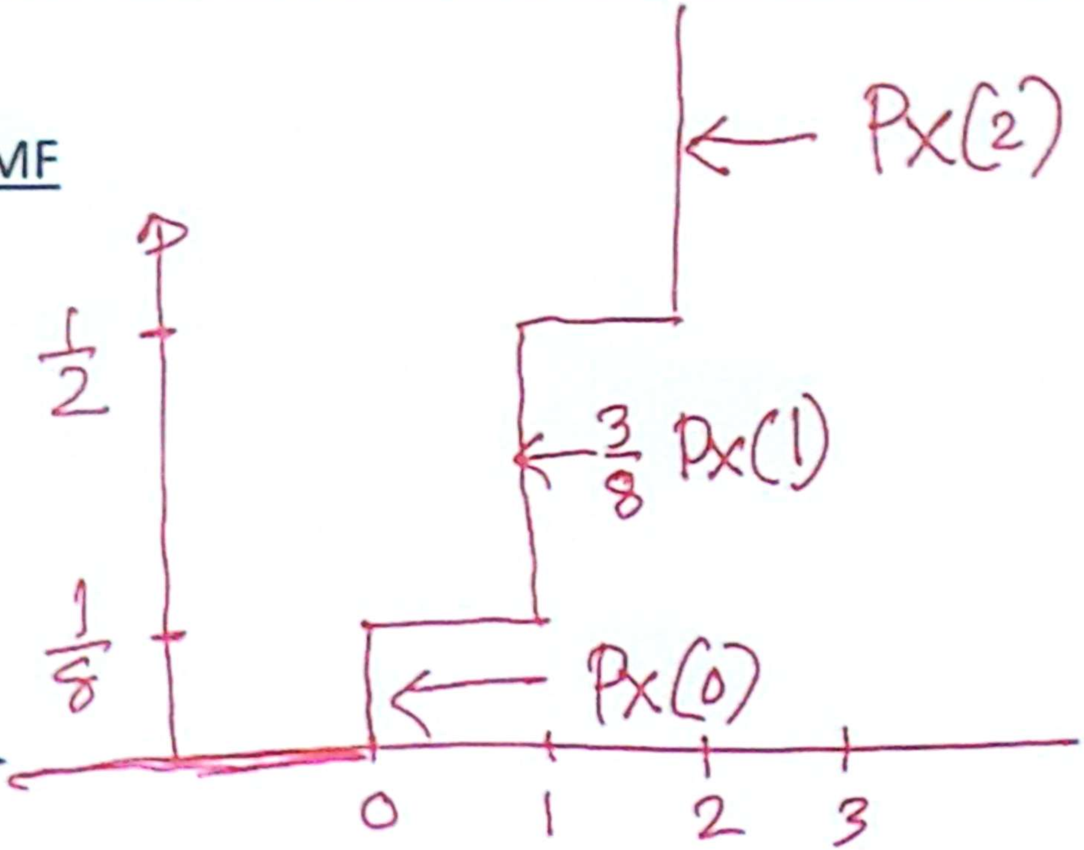
$$F_X(x) = \begin{cases} \boxed{0, \quad x < 0} \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

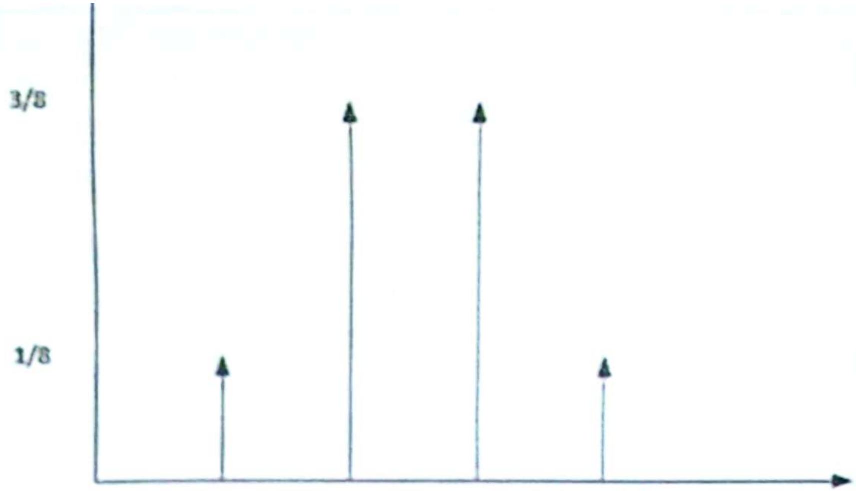


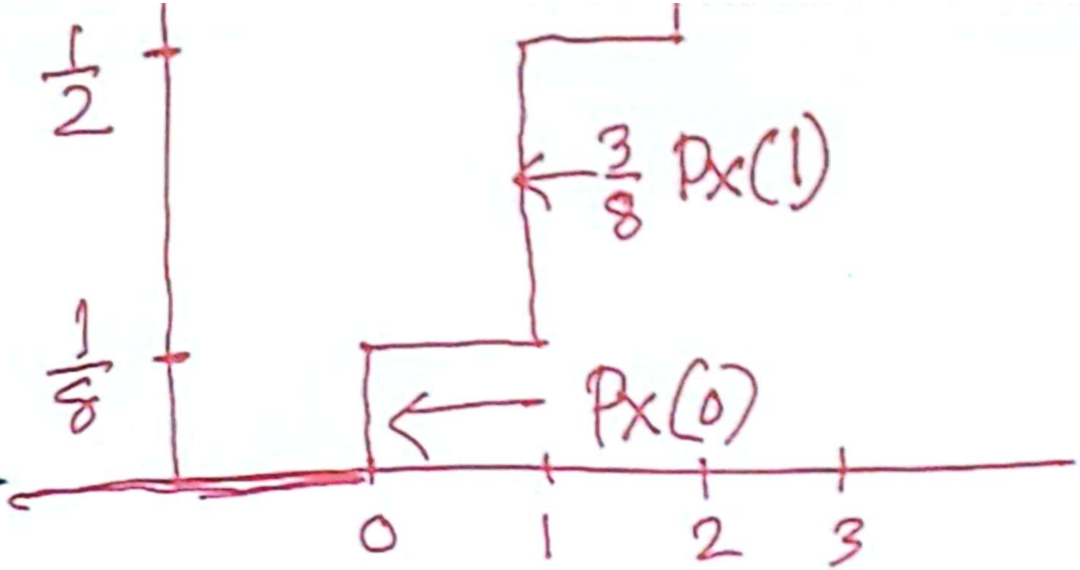


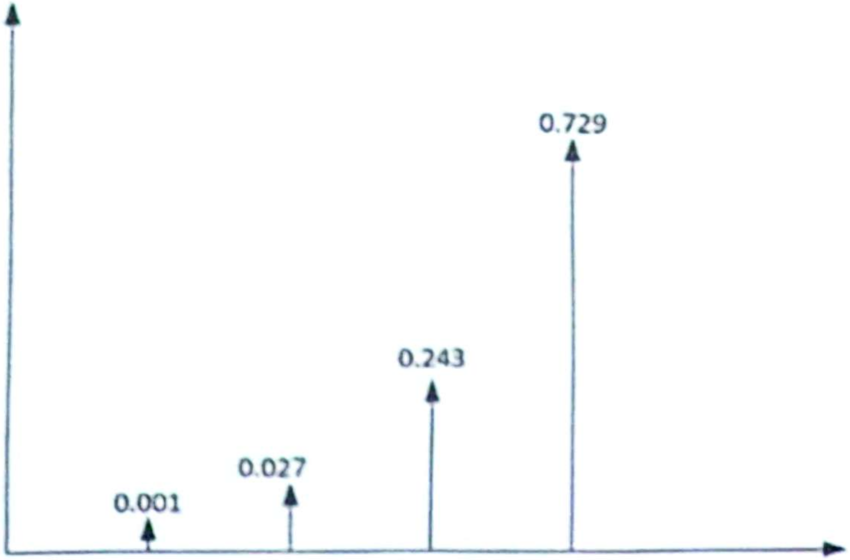
Graphical Representation of the PMF





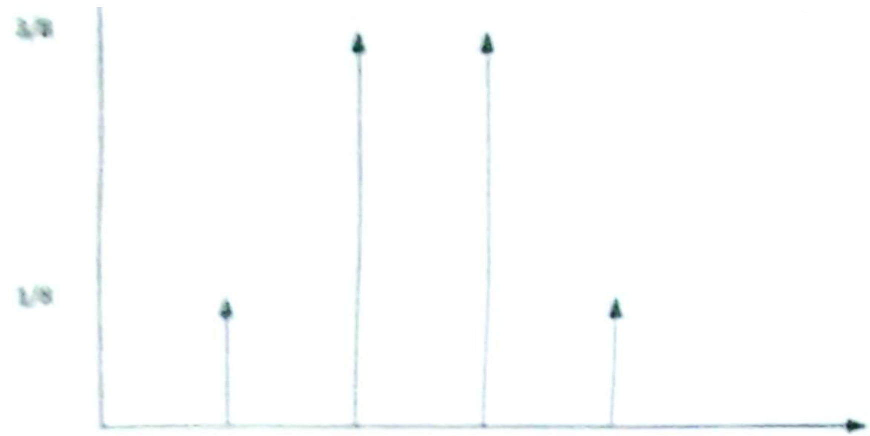


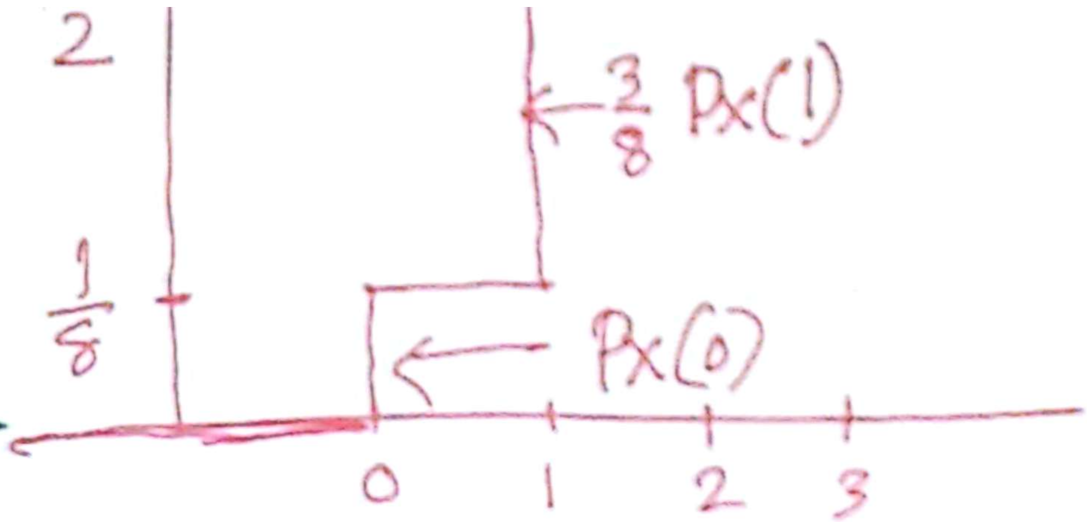


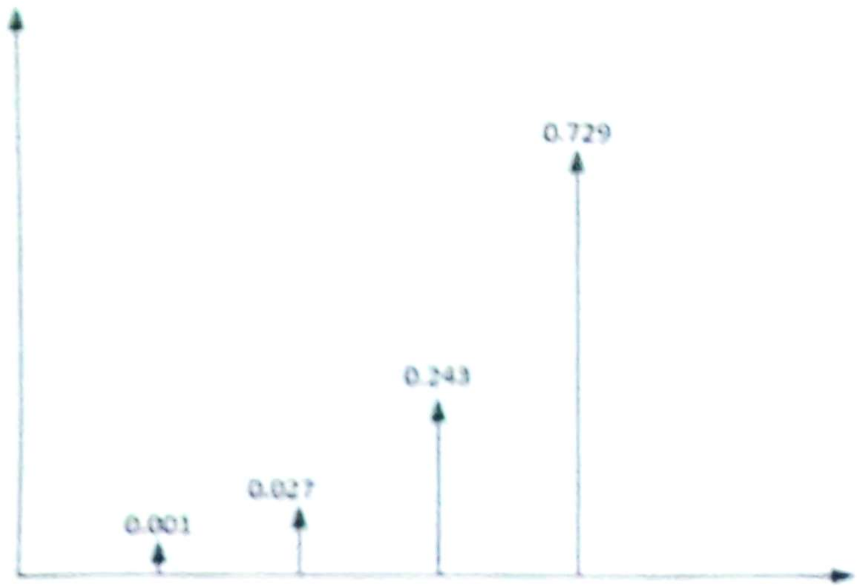


$$F_X(x) = \sum_{i=0}^{3} P_X(i)$$

$$F_X(0) = P_X(0)$$
$$F_X(1) = P_X(0) + P_X(1)$$
$$F_X(2) = P_X(0) + P_X(1) + P_X(2)$$







$$F_X(x) = \sum_{i=0}^{x} P_X(i)$$

$$F_X(0) = P_X(0)$$

$$F_X(1) = P_X(0) + P_X(1)$$

$$F_X(2) = P_X(0) + P_X(1) + P_X(2)$$

$$F_X(x) = \begin{cases} \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{7}{8} & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

$$P_X(1) = F_X(1) - F_X(0) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$P_X(x) = F_X(x) - F_X(x-1)$$

$$P_X(3) = F_X(3) - F_X(2) = 1 - \frac{7}{8} = \frac{1}{8}$$

$$P_X(2) = F_X(2) - F_X(1) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{7}{8} & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

$$= \frac{1}{8} - 0 = \frac{1}{8}$$

$$P_X(0) = F_X(0) - F_X(-1)$$

$$P_X(1) = F_X(1) - F_X(0) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$P_X(x) = F_X(x) - F_X(x-1)$$

$$P_X(3) = F_X(3) - F_X(2) = 1 - \frac{7}{8} = \frac{1}{8}$$

$$P_X(2) = F_X(2) - F_X(1) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$$

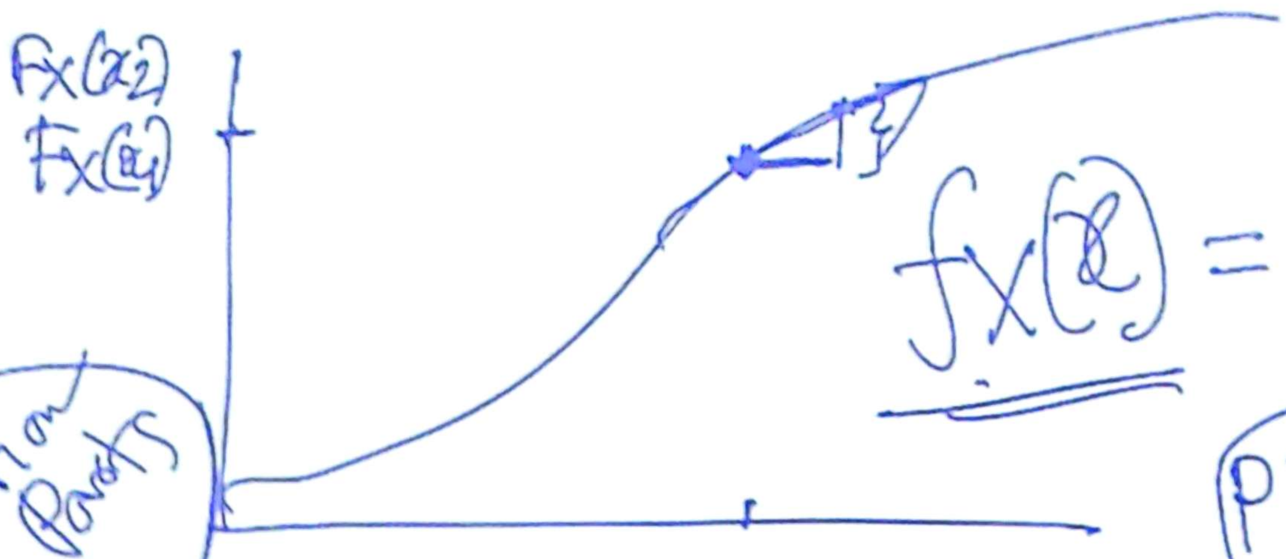
- Set of possible values of a random variable is uncountable or denumerable
- Set of values are represented by a range of values or by an interval
    - The delay of a packet to reach the destination from the source.
- The probability that a continuous random variable has a specific value is ?

$$P[X=x]=0$$

$$\frac{1}{n}$$

$$n=\infty$$

Prob. density



Integration by Parts

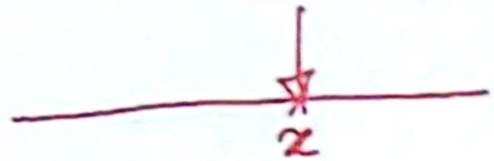
$$f_X(x) = \frac{d}{dx} F_X(x)$$

PDF

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

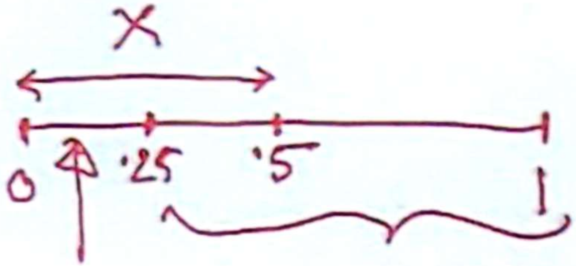
- The probability of a continuous random variable for a specific value is not defined.
- However, probability for a range of values, an interval, is well defined
  - $P[a < X \le b]$ is well defined
- All points in [0,1] are equally likely to be selected as point C $= P[X \le x]$
- Assume $a = 0$ and $c = 0.5$, intuitively we can say that 50% of the time point C will be selected in the interval $[0, 0.5]$

$P[-\infty < X \le x]$



- Probability that X has a value between 0 and 0.5 is
  $P[a < X \le b] = P[0 < X \le 0.5] = \frac{0.5}{1.0} = 0.5$
- Further assume that $\boxed{a = \infty}$ and $\boxed{b = x}$ then

$$P[a \le X \le b] = P[-\infty < X \le x] = P[X \le x] = F_X(x)$$



$P[0 \le X \le .5] = 0.5$

$P[\underbrace{0.25 < X \le .5}_{a \qquad b}] = \frac{1}{4}$

Cumulative Distribution Function (CDF) of Continuous Random Variables

$P[X \leq .5] = .5$

The cumulative distribution function of a continuous random variable X is denoted as $F_X(x)$, and is defined as

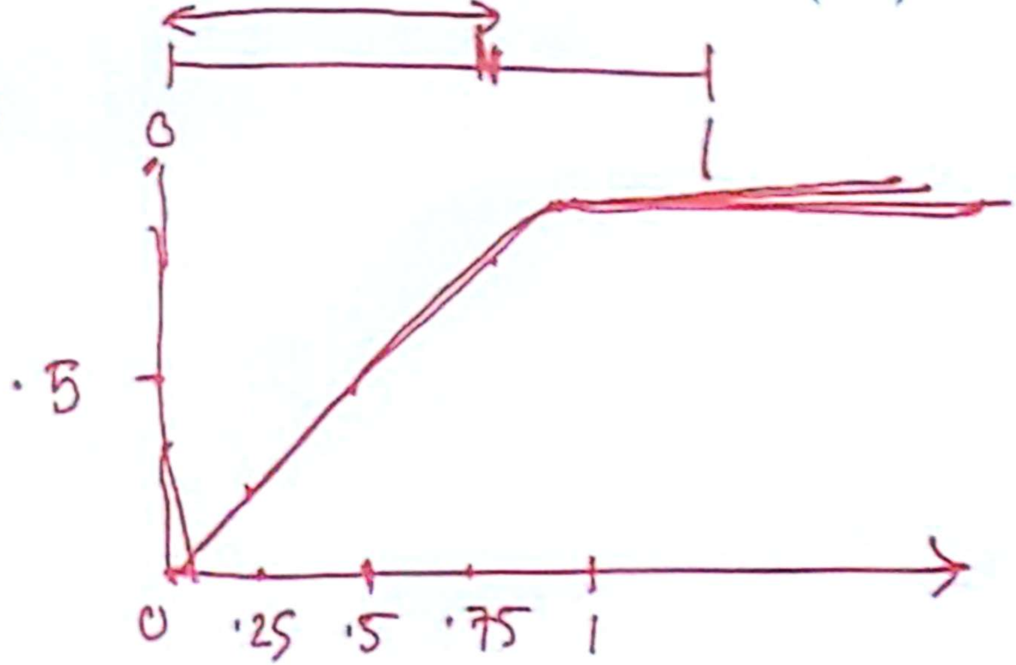
$P[X \leq .25]$

$$\boxed{F_X(x) = P[X \leq x].} \quad = .25 \tag{3.9}$$

X

The CDF of the example Random variable is

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0; \\ x, & \text{for } 0 \leq x < 1; \\ 1, & \text{for } x \geq 1. \end{cases}$$



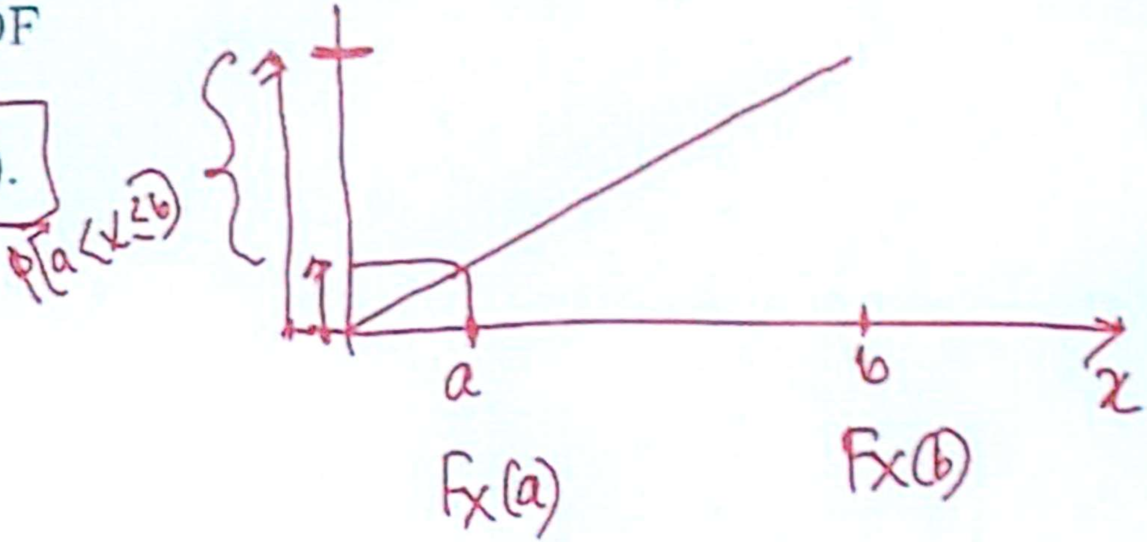
$$F_X(-\infty) = P[X \le -\infty] = 0.$$

2. The CDF of a continuous random variable for a value of $+\infty$ is one, i.e.,

$$F_X(+\infty) = P[X \le +\infty] = 1.$$

3. The probability that a continuous random variable for a value within the interval $(a, b]$ can be given in terms of its CDF

$$\boxed{P[a < X \le b] = F_X(b) - F_X(a).}$$



4. For continuous random variables, following four possibilities are equal due to the fact that $P[X = x] = 0$.

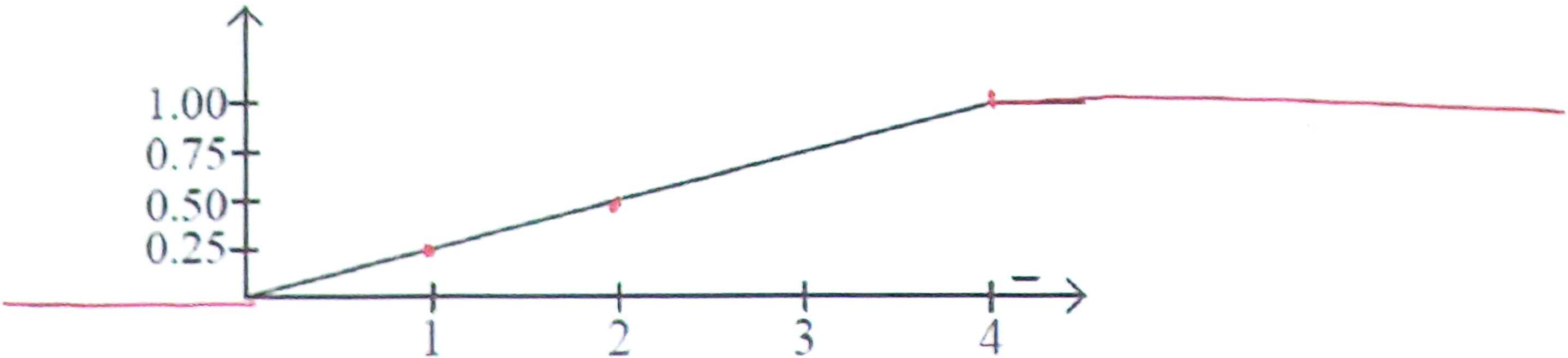
$$\boxed{P[a \leq X \leq b] = P[a \leq X \leq b] = P[a < X < b] = P[a \leq X \leq b].}$$

5. For continuous random variables, if $a < b$, then $F_X(a) < F_X(b)$.

6. The CDF of a continuous random variable is non-decreasing and continuous function of x.

$$P[a \leq X \leq b] = P[a < X \leq b] + \underbrace{P[X = a]}_{0}$$

$$P[a \leq X \leq b] = P[a < X \leq b]$$


1.00
0.75
0.50
0.25
1
2
3
4

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0; \\ \frac{x}{4}, & \text{for } 0 \le x < 4; \\ 1, & \text{for } x \ge 4. \end{cases}$$

a) Draw the CDF curve.

b) Find the values of $F_X(-1)$, $F_X(1)$, $P[2 < X \le 3]$ and $F_X(1.5)$.

$F_X(2) = \frac{2}{2}$

$F_X(-1) = 0$

$F_X(1)$

$P[2 < X \le 3] = F_X(3) - F_X(2)$

$= \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$

$F_X(1.5) = \frac{1.5}{4}$

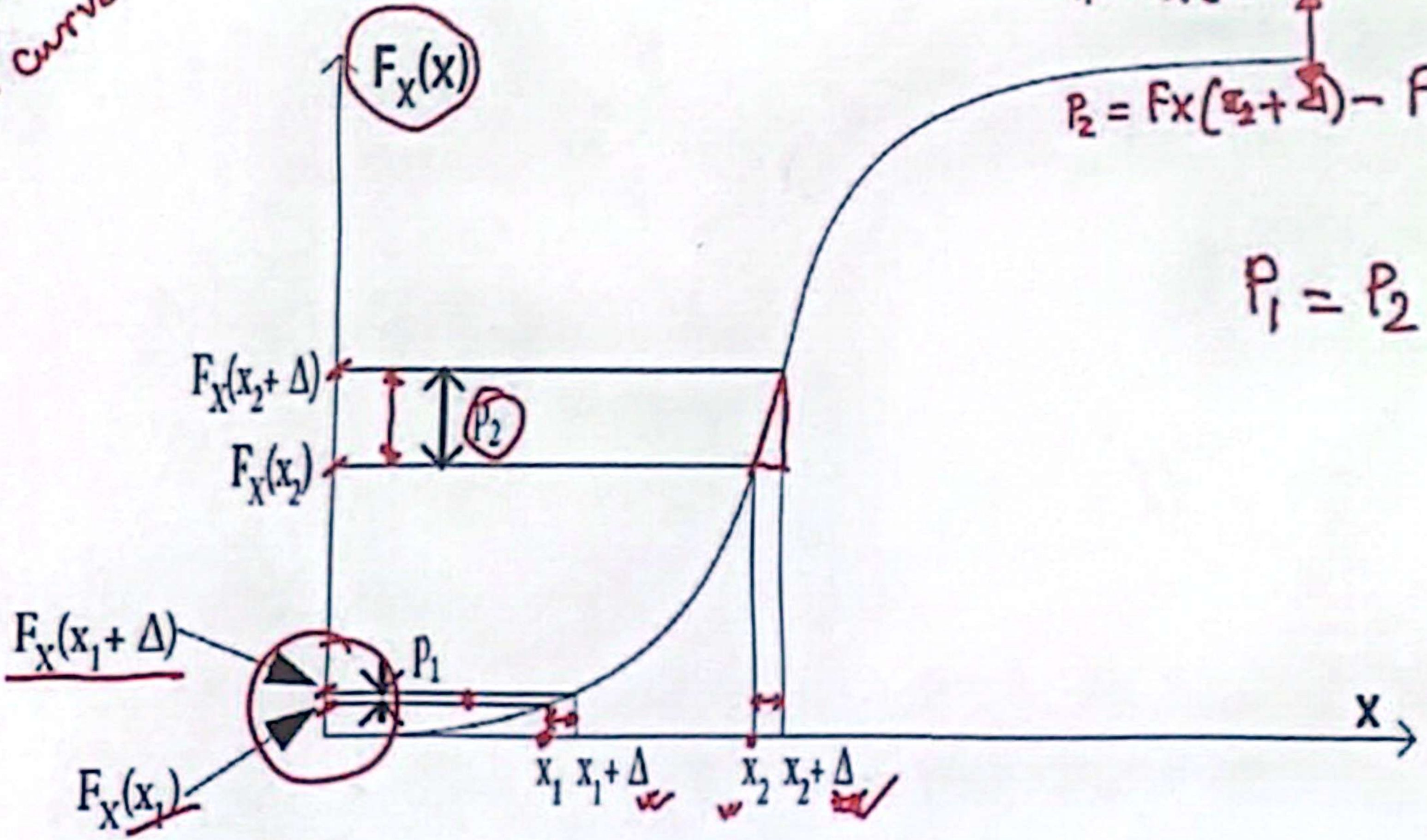
? $F_X(x)$

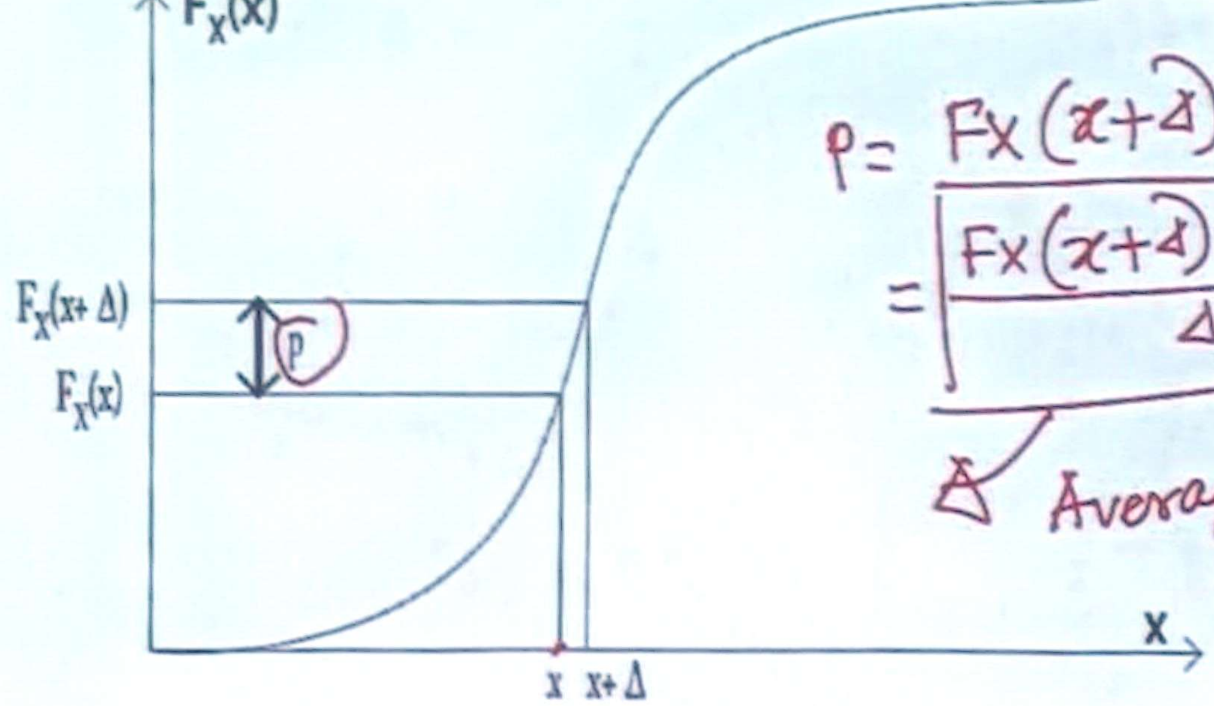
Slope of the CDF curve

$P_1 = F_X(x_1+\Delta) - F_X(x_1)$

$P_2 = F_X(x_2+\Delta) - F_X(x_2)$

$P_1 = P_2$



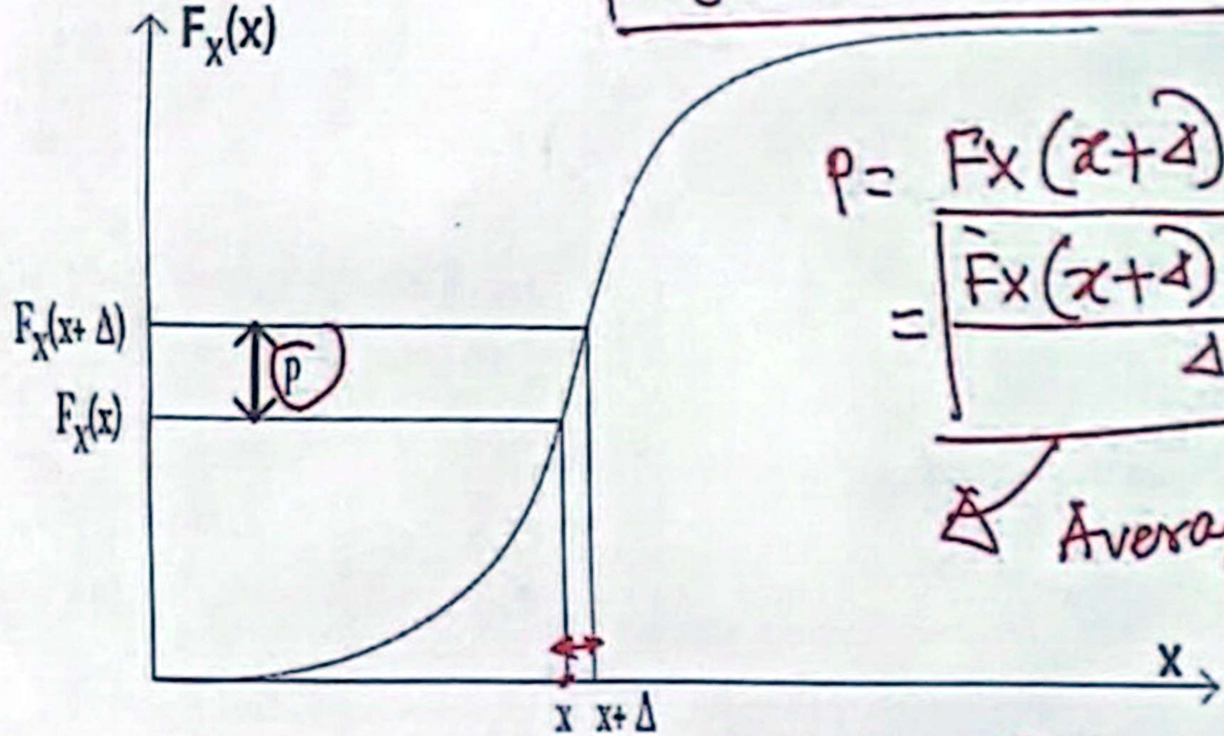


$$P = F_X(x+\Delta) - F_X(x) = \boxed{\frac{F_X(x+\Delta) - F_X(x)}{\Delta}} \cdot \Delta$$

Δ Average Prob. in the length Δ

$\boxed{\text{Avg. prob. per unit length}}$

$$\frac{P}{\Delta}$$



$$P = F_X(x+\Delta) - F_X(x)$$

$$= \boxed{\frac{F_X(x+\Delta) - F_X(x)}{\Delta}} \cdot \Delta$$

Average Prob. in the length Δ

↪ Average Prob. near x

Density: measure of amount of mass in a given space (volume)

Probability Density: Measure of the amount of probability per unit length

$$f_X(x) = \boxed{\lim_{\Delta \to 0} \frac{F_X(x+\Delta) - F_X(x)}{\Delta}}$$

Prob. density at x

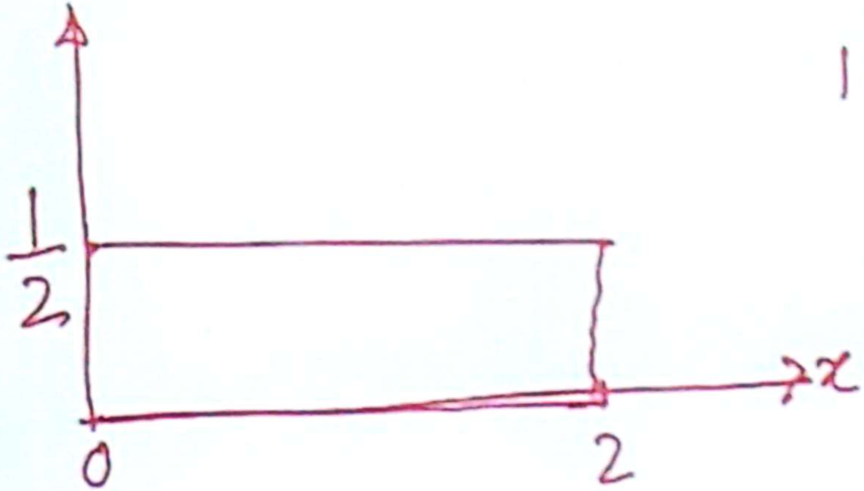
$$f_X(x) = \boxed{\lim_{\Delta \to 0} \frac{F_X(x+\Delta) - F_X(x)}{\Delta}}$$

Prob. density at x

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Continuous

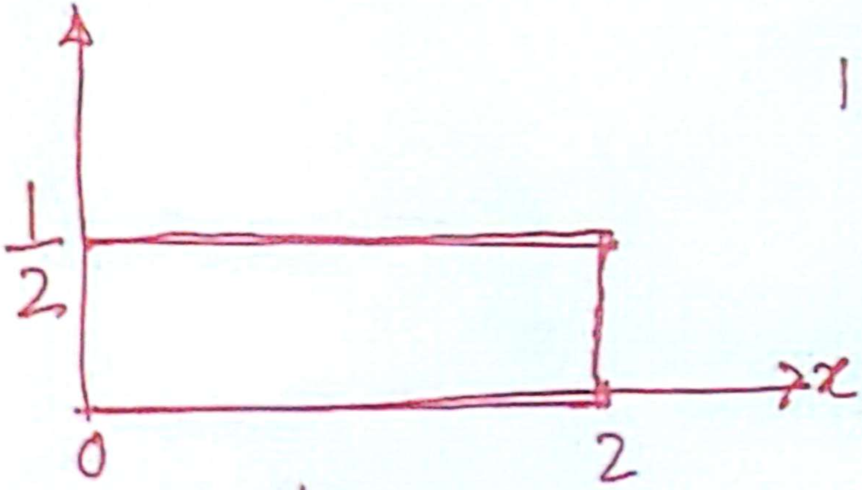
$$f_X(x) = \frac{d}{dx} F_X(x)$$



$$f_X(x) = \frac{d}{dx}F_X(x)$$

$$f_X(x) = \frac{d}{dx}F_X(x)$$

Continuous

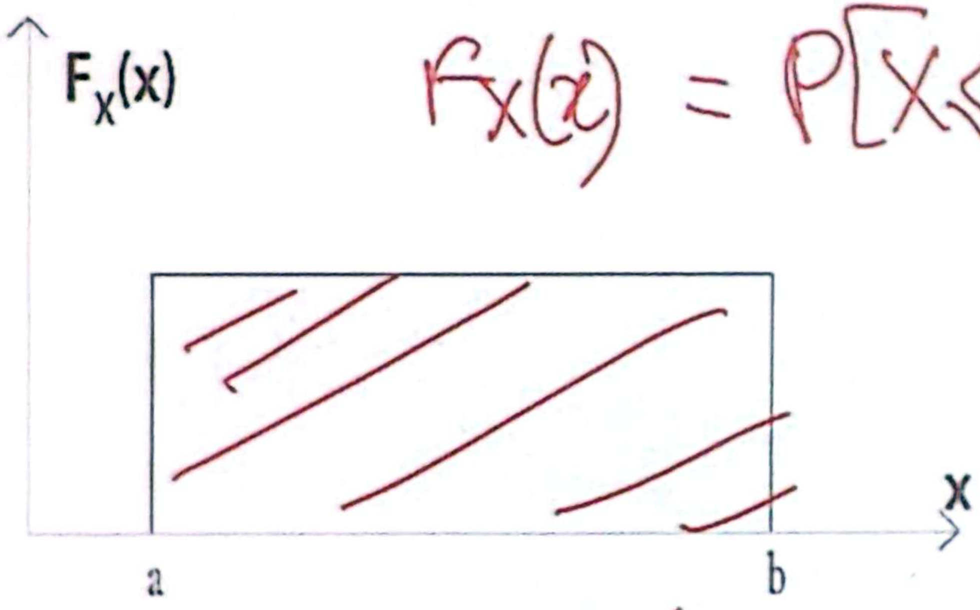


1. for

$$\int_a^b f_X(x)\,dx = \underline{P[a < X \le b]} \rightarrow \text{Area under the curve between a to b}$$

Area Under the Curve (AUC)

$$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(x)\,dx$$



$$\boxed{P[a < X \le b]} = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a)$$

$$\boxed{\int_{-\infty}^{+\infty} f_X(x)\,dx = 1}$$

Condition to be pdf function

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Chapter 02 v02.pdf Chapter 02 v3.pdf Chapter 02 v01.pdf Chapter 02 v03.pdf Chapter 02 v03 (1)

$$f_X(x) = \frac{1}{4}$$

$$F_X(x) = \int_{-\infty}^{+\infty} f_X(x)\,dx$$

$$= \int_0^x \frac{1}{4}\,dx = \frac{x}{4}\Big|_0^x = \frac{x}{4}$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$
\begin{aligned}
f_X(x) &= \frac{d}{dx}F_X(x) \\
&= \frac{d}{dx}\frac{x}{4} \\
&= \frac{1}{4}.
\end{aligned}
$$

$$
f_X(x) = \begin{cases} \frac{1}{4}, & \text{for } 0 \le x \le 4; \\ 0, & \text{ohterwise.} \end{cases}
$$

$$P[2 < X < 3] = \int_2^3 \frac{1}{4}\,dx = \frac{x}{4}\Big|_{x=2}^{x=3} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$$

$$= F_X(3) - F_X(2) = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$$

Example 2.13: The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} cxe^{-x/2}, & \text{for } x \geq 0; \\ 0, & \text{ohterwise.} \end{cases}$$

$cx\bar{e}^{-x/2}, \quad x \geqslant 0$

$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

1. Find the value of the constant c.
2. Find the CDF of the random variable X.
3. Find the probability $P[2 \leq X \leq 5]$ from the PDF of X.
4. Find the probability $P[2 \leq X \leq 5]$ from the CDF of X.

$$\int_{-\infty}^{+\infty} f_X(x)dx = \int_0^\infty cxe^{-x/2}dx$$

$$= cx(-2)e^{-x/2}\Big|_{x=0}^{x=\infty} - \int_0^\infty 1.(-2)e^{-x/2}dx$$

$$= 0 + 2c\int_0^\infty e^{-x/2}dx$$

$$= 2c(-2)e^{-x/2}\Big|_{x=0}^{x=\infty}$$

$$= 0 + 4c = 4c$$

$4c = 1$ and $c = \frac{1}{4}$.