Continuous Random Variable (CRV)

A continuous random variable is a type of random variable that can take any real value within a certain range or interval. Unlike discrete random variables, which take countable values (like 1, 2, 3...), continuous variables have uncountably infinite possibilities.

↳ So, For a random variable X with its range S_X consists of one or more interval:

if, X is continuous random variable:

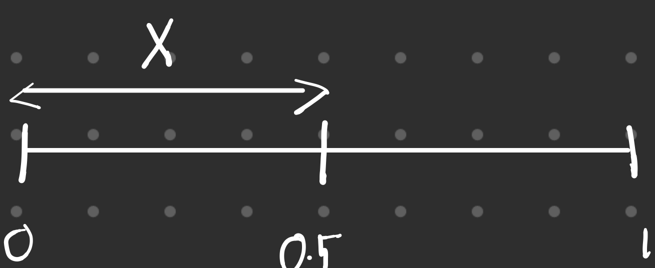
then, $S_X \rightarrow$ contains infinite values

$$\therefore P[X = x] = \frac{1}{n} = \frac{1}{\infty} = 0$$

So, the probability of each individual is zero. (interesting...)

CDF of CRV

↳ $F_X(x) = P[X \leq x]$



$P[X \leq 0.5] = 0.5$

$P[0 \leq x \leq 0.5] = 0.5$

$P[0.25 < x \leq 0.5] = 0.25$

For any random variable X,

i) $F_X(x_1) \le F_X(x_2)$ for $x_1 < x_2$

ii) $F_X(+\infty) = 1$, $F_X(-\infty) = 0$

iii) $P[x_1 < X < x_2] = F_X(x_2) - F_X(x_1)$

$P[a \le X \le b] = P[a < X \le b] = P[a \le X < b] = P[a < X < b]$

How?

$P[a \le X \le b] \geq P[a < X \le b]$

$\Rightarrow P[a \le X \le b] = P[a < X \le b] + \underbrace{P[X = a]}_{0}$ $\hookrightarrow \frac{1}{\infty} = 0$

$\Rightarrow P[a \le X \le b] = P[a < X \le b]$

↳ works other combinations too.

Example 2.10:

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x}{4}, & \text{for } 0 \le x < 4 \\ 1, & \text{for } x \geq 4 \end{cases}$$

i) Draw the CDF curve.

ii) Find $P[2 < x \le 3]$, $F_X(1.5)$

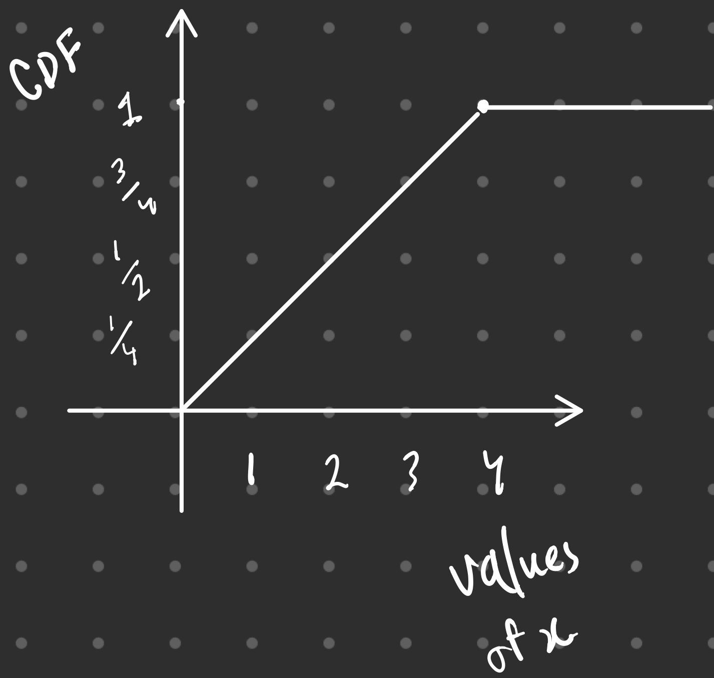
Sol^n:

$P[2 < x \leq 3]$

$= F_X(3) - F_X(2)$

$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$.

$F_X(1.5) = \frac{1.5}{4} = \frac{3}{8}$.



Probability Density Function (PDF)

Probability Density → Measure of the amount of probability per unit length.

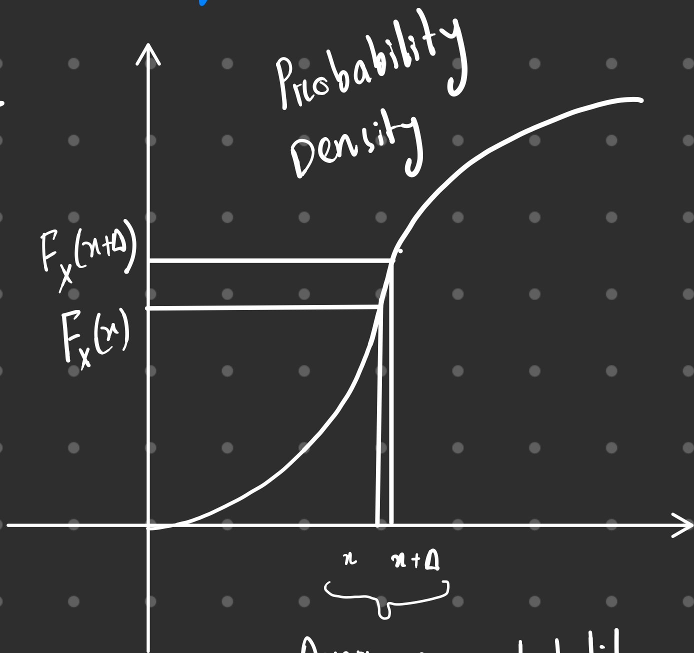
↳ $f_X(x) = \lim_{\Delta \to 0} \frac{F_X(x+\Delta) - F_X(x)}{\Delta}$

PDF ↙

$= \left(\frac{P}{\Delta}\right)$

↳ $f_X(x) = \frac{d}{dx} F_X(x)$

↳ $F_X(x) = \int f_X(x)$



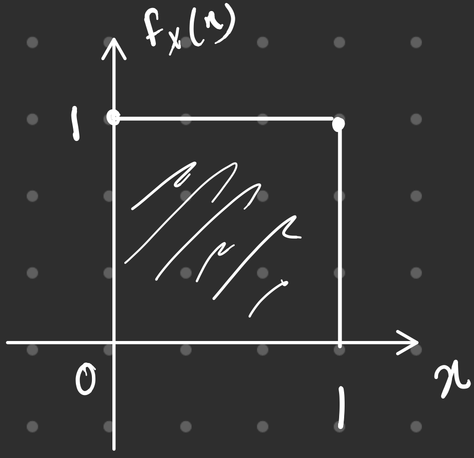
Average probability near x

$P = F_X(x+\Delta) - F_X(x)$

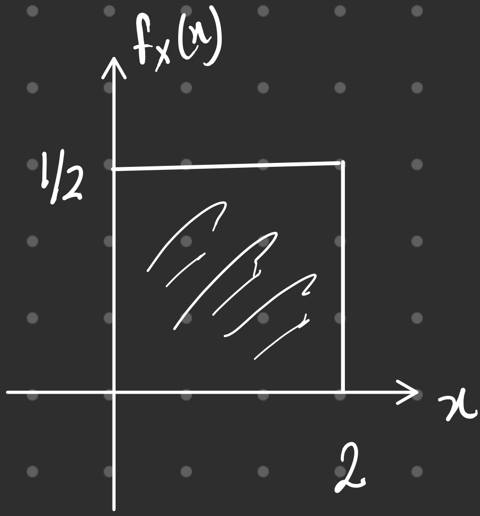
$= \frac{F_X(x+\Delta) - F_X(x)}{\Delta} \cdot \Delta$

The PDF, denoted by $f_X(x)$, satisfies:

i) Non-negative: $f_X(x) \geq 0$



ii) Total Area = 1 : $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$



iii) Interval Probability:

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

Example 2.11:

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x}{4}, & \text{for } 0 \leq x < 4 \\ 1, & \text{for } x \geq 4 \end{cases}$$

i) Find the PDF from this and draw the curve.

ii) Find $P[2 < x \leq 3]$ using PDF.

Soln:

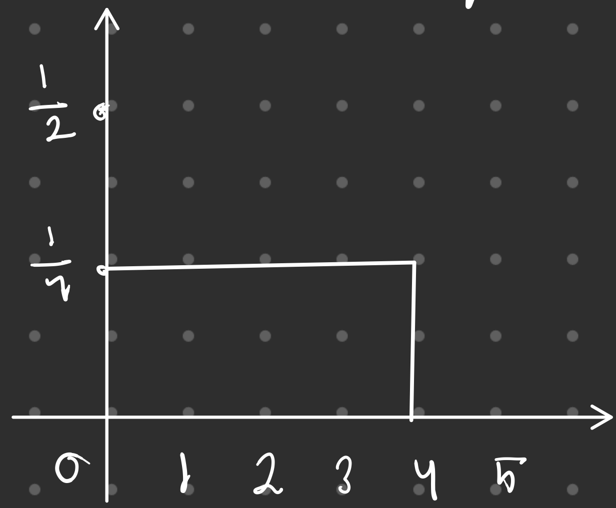
$$\frac{d}{dx}(0) = 0$$

$$\frac{d}{dx}\frac{x}{4} = \frac{1}{4}$$

$$\frac{d}{dx}1 = 0$$

$$\rightsquigarrow f_X(x) = \begin{cases} \frac{1}{4}, & 0 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

$$P[2 < x \le 3] = \int_2^3 \frac{1}{4}\,dx = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$



Example 2.13: The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} cxe^{-x/2}, & \text{for } x \ge 0; \\ 0, & \text{ohterwise.} \end{cases}$$

1. Find the value of the constant c.
2. Find the CDF of the random variable X.
3. Find the probability $P[2 \le X \le 5]$ from the PDF of X.
4. Find the probability $P[2 \le X \le 5]$ from the CDF of X.

Soln:

① $$\int_{-\infty}^{+\infty} f(x)\,dx = \int^{+\infty} cxe^{-x/2}\,dx$$

$$= cx\int_0^{+\infty} e^{-x/2}\,dx - c\int_0^{+\infty} 1\left(\int_0^{+\infty} e^{-x/2}\,dx\right)dx$$

$$= \left[cx(-2)e^{-x/2} \right]_0^{+\infty} + 2c\int_0^{\infty} e^{-x/2}\,dx$$

$$= (0-0) + 2c\left[-2e^{-x/2}\right]_0^{\infty}$$

$$= 2c\left[0-(-2)\right] = 4c$$

Here, $4c = 1 \Rightarrow c = \frac{1}{4}$.

2 Given, $f_X(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & \text{for } x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$

CDF: $F_X(x) = \int_0^x f_X(x)\,dx$

$$= \int_0^x \frac{1}{4} x e^{-x/2}\,dx$$

$$= \frac{1}{4}\left[x\int_0^x e^{-x/2}\,dx - \int_0^x 1\left(\int_0^x e^{-x/2}\,dx\right)dx\right]$$

$$= \frac{1}{4}\left[x\cdot(-2)e^{-x/2}\right]_0^x -$$

$$\frac{1}{4}\int_0^x (-2)\left(e^{-x/2}-1\right)dx$$

$$= \frac{1}{2}x\left(e^{-x/2}-1\right) - \frac{1}{4}\int_0^x -2e^{-x/2} - \frac{1}{4}\int_0^x +2\,dx$$

$$= -\frac{1}{2}x\left(e^{-x/2}-1\right) + \left[2(-2)e^{-x/2}\right]_0^x - \frac{1}{2}x$$

$$= -\frac{1}{2}xe^{-x/2} + \frac{1}{2}x - e^{-x/2} + 1 - \frac{1}{2}x$$

$$= 1 - e^{-x/2} - \frac{1}{2}xe^{-x/2}$$

$$\therefore F_X(x) = \begin{cases} 1-e^{-x/2} - \frac{1}{2}xe^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

③ $P[2 \leq X \leq 5] = \int_2^5 \frac{1}{4}xe^{-x/2}\,dx$

$$= \left[1-e^{-x/2} - \frac{1}{2}xe^{-x/2}\right]_2^5 = e^{-4} - e^{-10} = 0.01827$$

④ $P[2 \leq X \leq 5] = F_X(5) - F_X(2)$

$$= e^{-4} - e^{-10} = 0.01827.$$

(*) Expected Value are well discussed in Discrete Random Variable

As for, Continuous Random Variable,

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx$$

$$V[X] = \int_{-\infty}^{+\infty} (x-\mu)^2 \cdot f_X(x)\, dx$$

Example:

Example: The time elapsed, in minutes, between the placement of an order of pizza and its delivery is random with pdf

$$f(x) = \begin{cases} 1/15 & \text{if } 25 < x < 40 \\ 0 & \text{otherwise.} \end{cases}$$

Soln:

Here, $E[X] = \int_{25}^{40} x \frac{1}{15} dx = 32.5$

$$E[X^2] = \int_{25}^{40} x^2 \frac{1}{15} dx = 1075$$

$$\therefore V[X] = 1075 - (32.5)^2 = 18.75$$

$$\sigma_x = \sqrt{V[X]} = 4.33$$

Example from slide:

Sketch the graph of the function

$$f(x) = \begin{cases} \frac{1}{2} - \frac{1}{4}|x-3| & 1 \le x \le 5 \\ 0 & \text{otherwise,} \end{cases}$$

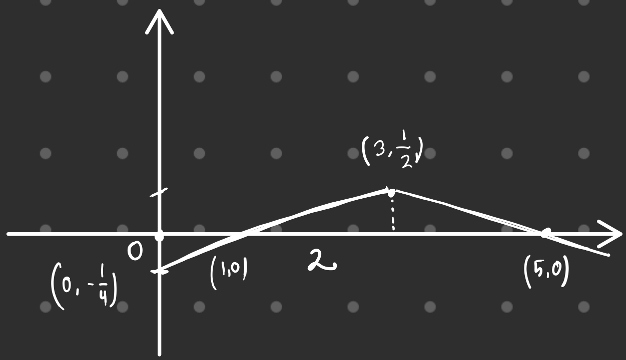
and show that it is the probability density function of a random variable X.

Find F, the distribution function of X, and show that it is continuous.

Find $E[X]$ and $V[X]$.

Sol^n:

Graph:



To check the validity of a function to be PDF, it must be $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

Here, $\int_1^5 f(x)\,dx$

$= \int_1^5 \frac{1}{2} - \frac{1}{4}|x-3|\,dx = 1$

So, it is a valid PDF.

Here, $f(x) = \frac{1}{2} - \frac{1}{4}|x-3|$, a piecewise linear.

So, $f(x) = \begin{cases} 0, & \text{if } x < 1 \\ \frac{1}{2} + \frac{1}{4}(x-3), & 1 \le x < 3 \\ \frac{1}{2} - \frac{1}{4}(x-3), & 3 \le x < 5 \\ 0, & x \ge 5 \end{cases}$

$f(x) = 0$?

$0 = \frac{1}{2} - \frac{1}{4}|x-3|$

$\frac{1}{4}|x-3| = \frac{1}{2}$

$\Rightarrow |x-3| = 2$

$\Rightarrow x-3 = 2 \qquad -x+3 = 2$

$\Rightarrow x = 5 \qquad \Rightarrow x = 1$

$x = 0$?

$f(x) = \frac{1}{2} - \frac{1}{4}|-3|$

$= \frac{1}{2} - \frac{3}{4} = -\frac{1}{4}$

$x = 3$?

$f(x) = \frac{1}{2}$

For, $x<1 \rightarrow F(x) = \int_{\infty}^{x} 0\,dx = 0$

For, $1 \leq x < 3 \rightarrow \int_{-\infty}^{x} f(x)dx = \int_{1}^{x} \left[\frac{1}{2} + \frac{1}{4}(x-3)\right]dx$

$$\rightarrow F(x) = \frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{8}$$

For, $3 \leq x < 5 \rightarrow \int_{-\infty}^{x} f(x)dx = \int_{1}^{3} f(x)dx + \int_{3}^{x} f(x)dx$

$$= F(3) + \int_{3}^{x} \left[\frac{1}{2} - \frac{1}{4}(x-3)\right]dx$$

$$= \frac{1}{2} + \left(-\frac{1}{8}x^2 + \frac{5}{4}x - \frac{21}{8}\right)$$

$$= -\frac{1}{8}x^2 + \frac{5}{4}x - \frac{17}{8}$$

For, $t \geq 5 \rightarrow F(x) = 1$ (Beyond the interval)

$$\therefore F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{8} & 1 \leq x \leq 3 \\ -\frac{1}{8}x^2 + \frac{5}{4}x - \frac{17}{8} & 3 < x \leq 5 \\ 1 & x > 5 \end{cases}$$

Now, $E[X] = \int_{1}^{5} x f(x)dx = 3$

$$V[X] = E[X^2] - (E[X])^2 = \int_{1}^{5} x^2 f(x)dx - 3^2 = \frac{2}{3}.$$