Lecture 11

Chapter 3: Well-known Random Variables

Math 4441: Probability and Statistics

Reference: Goodman & Yates – Introduction to Probability and Stochastic Process, 3rd Edition

Well-known discrete random variables

- Discrete uniform random variable
- Bernoulli random variable
- Geometric random variable
- Pascal/Negative-binomial random variable
- Poisson random variable
- Hyper geometric random variable
-

Bernoulli random variable:

An experiment with only two outcomes – success and failure - is defined as a Bernoulli experiment (BE)

Even if an experiment has more than two outcomes, it can be converted into a Bernoulli experiment. One of the outcomes is considered as a "success", whereas all the other outcomes are considered as "failure".

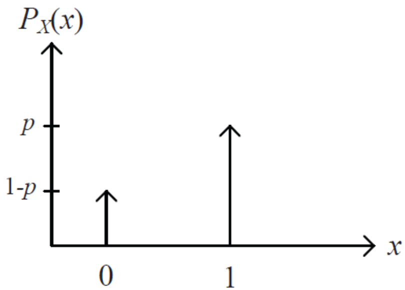
random variable associated with a Bernoulli experiment assumes the value 1 when a success occurs; and assumes a value 0, when a failure occurs, and is defined as a Bernoulli random variable.

the probability that the random variable has a value 1 is the probability that a "success" occurs

the probability that the random variable has a value 0 is the probability that a "failure" occurs

Let p denote the probability that a success occurs in a Bernoulli experiment, where $0 < p < 1$.

$$P_X(x) = \begin{cases} 1-p, & \text{for } x = 0, \\ p, & \text{for } x = 1, \\ 0, & otherwise. \end{cases}$$



$$F_X(x) = \begin{cases} 0, & \text{for } x < 0, \\ (1-p), & \text{for } 0 \leq x < 1, \\ 1, & \text{for } x \geq 1. \end{cases}$$

The Bernoulli random variable is the simplest, but nevertheless an important random variable

- Flip a coin; we may let $\{head\} = \{success\} = s$
- Each computer chip produced is tested; $s = \{the\ chip\ passes\ the\ test\}$
- A packet is transmitted; $s = \{success\} = \{the\ packet\ is\ transmitted\ successfully\}$
-

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{1} x P_X(x) \\
&= 0 \cdot (1-p) + 1 \cdot p \\
&= p.
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[(X - E[X])^2] \\
&= (0-p)^2 \cdot (1-p) + (1-p)^2 \cdot p \\
&= p^2(1-p) + (1-p)^2 p \\
&= p(1-p)(p+1-p) \\
&= p(1-p).
\end{aligned}
$$

Example: Pick a ball randomly from a box with 8 red balls and 7 blue balls.

Bernoulli Trials

Bernoulli experiments can be considered as the building block of many experiments. Such an experiment is often constituted by a number of successive Bernoulli experiments.

A sequence of BEs is called a Bernoulli trials, if

- Each BE is independent from the others
- Probability of success is a constant

1. Sending n data packets successively.
2. Sending a single packet successively until the transmission is successful.

1. Transmit a data packet from one station to another station. The sender continues to transmit the packet again and again until the transmission is successful. Assume the probability of success in each transmission is p, irrespective of the number of transmission already attempted for the packet. Therefore, this constitute a Bernoulli trials. The random variable associated with this Bernoulli trials might be the followings:

   (a) The number of transmission required to transmit the data packet successfully.

   (b) The number of unsuccessful transmissions before the packet is successfully sent.

2. Send n data packets from one station to another. Each of the packets is successfully sent with probability p, and it is dropped with probability $(1-p)$, irrespective of whatever happensed for the earlier packets. Assuming an attempt of successfully sending a packet or dropping it as an Bernoulli experiment, the above example of sending n packets becomes a Bernoulli trials. The following random variables can be defined for this Bernoulli trials:

(a) The random variable X that denotes the number of packets successfully sent.

(b) The random variable Y that denotes the number of packets dropped.

3. Consider the above example of sending n packets from one station to another with a little modification. Instead of sending n packets, the sending station continues to send packets one after another until r packets are successfully sent. The sequence of sending packets now becomes a Bernoulli trials, where the trial ends as soon as r packets are successfully sent. The random variables associated with the Bernoulli trials might include the followings:

   (a) The number of packets the sender attempt to send.

   (b) The number of packets dropped before the sender can send r packets.

In a Bernoulli trials, two quantities are important

- The number of repetitions of the BEs
- The number of success

When one quantity is constant, the other becomes a random variable and vice-versa.

# of repetitions	# of successes	Random variables	Parameters	Distributions

Random Variable: X, PMF: $P_X(x)$

Distributions	Parameters	# of successes	# of failures	# of ways	PMF
Binomial					
Geometric					
Pascal					

# of repetitions

# of successes

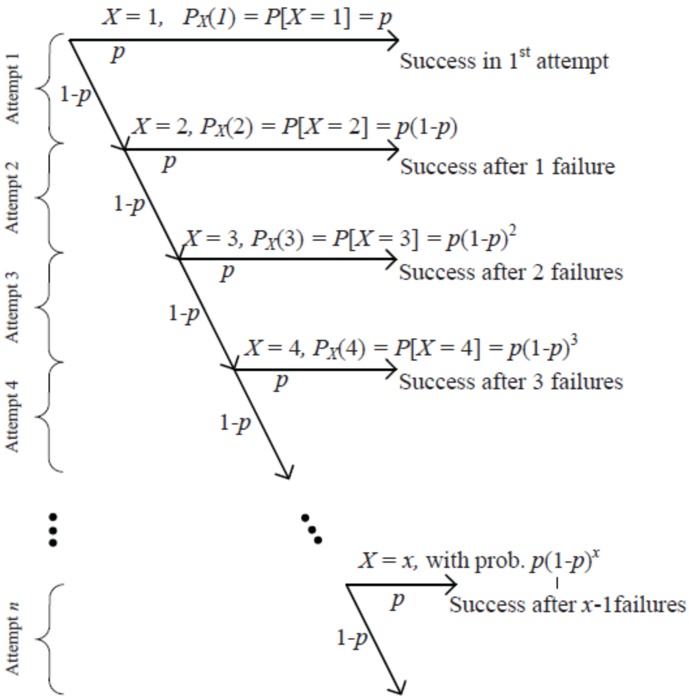
Geometric Distributions

Consider that a station in a wireless network transmits a data packet to another station. After transmitting the packet, the sending station waits for a predefined amount of time to receive feedback (i.e., an ACK) from the receiving station.

Let the random variable X denote the number of transmissions required to send the packet successfully.

$$S_X = \{1, 2, 3, \ldots\}$$

x	1	2	3	4	$\ldots$
$P[X = x]$	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$\ldots$



$X = 1, \ P_X(1) = P[X = 1] = p$
p
Success in 1st attempt
$1\text{-}p$
$X = 2, P_X(2) = P[X = 2] = p(1\text{-}p)$
Success after 1 failure
$X = 3, P_X(3) = P[X = 3] = p(1\text{-}p)^2$
Success after 2 failures
$X = 4, P_X(4) = P[X = 4] = p(1\text{-}p)^3$
Success after 3 failures
$X = x$, with prob. $p(1\text{-}p)^x$
Success after x-1failures
Attempt 1
Attempt 2
Attempt 3
Attempt 4
Attempt n

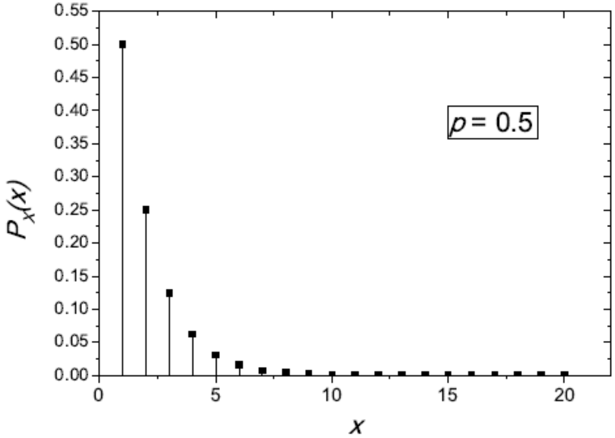
$$P[X = x] = \left(\frac{1}{2}\right)^x \qquad \text{for } x = 1, 2, \ldots$$

$$P[X = x] = 0 \qquad \text{for } x \leq 0.$$

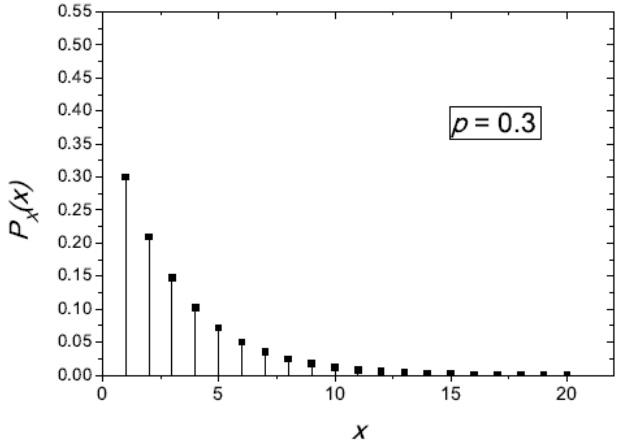
$$P_X(x) = \begin{cases} (\frac{1}{2})^x, & \text{for } x \geq 1; \\ 0, & otherwise \end{cases}$$

1. The trials are independent.
2. Each trial results in only two outcomes, labeled as "success" and "failure".
3. The probability of a "success" in each trial, denoted as p, remains constant for the whole sequence.
4. The sequence of trials ends as soon as the first success occurs.

$$P_X(x) = \begin{cases} p(1-p)^{x-1}, & \text{for } x \geq 1; \\ 0, & otherwise. \end{cases}$$

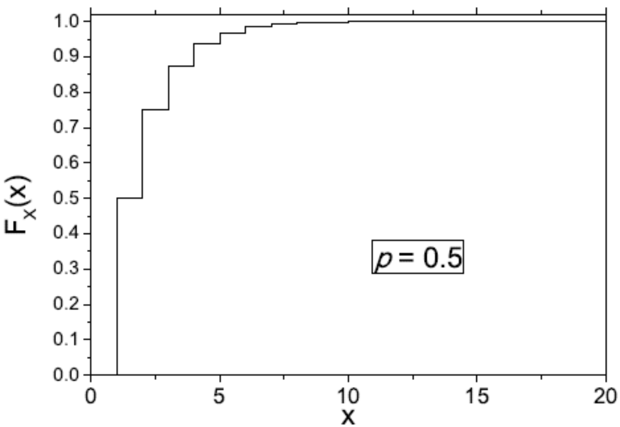


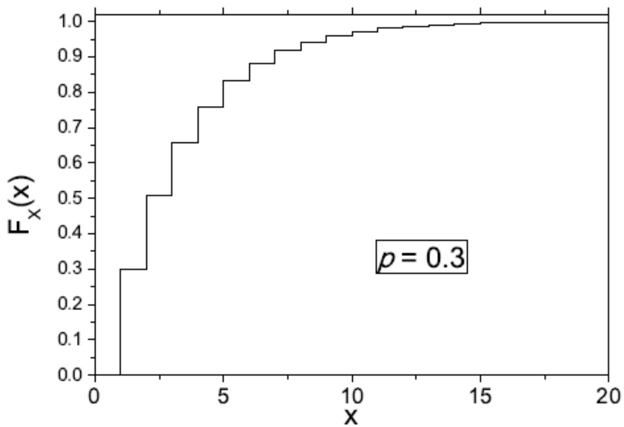
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
5
10
15
20
$P_X(x)$
x
$p = 0.5$



0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
5
10
15
20
$P_X(x)$
x
$p = 0.3$

$$F_X(x) = \begin{cases} 0, & \text{for } x < 1; \\ 1 - (1-p)^x & \text{for } x \geq 1 \end{cases}$$





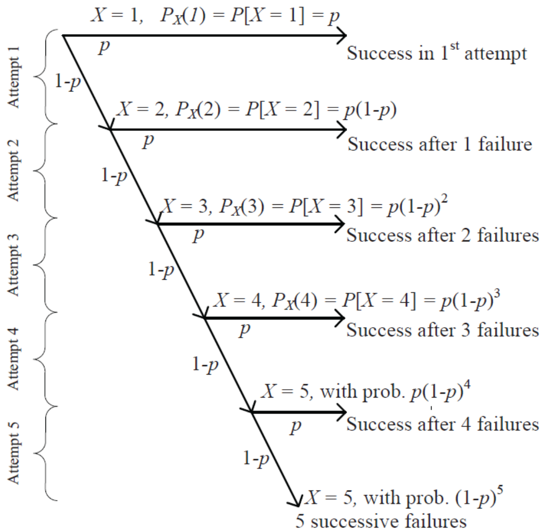
$$
\begin{aligned}
E[X] &= \sum_{x=1}^{\infty} x P_X(x) \\
&= \sum_{x=1}^{\infty} x \cdot p(1-p)^{x-1} \\
&= p \sum_{x=1}^{\infty} \frac{d}{dq}(q^x) \qquad \text{[where } q = 1-p\text{]} \\
&= p \frac{d}{dq}\left(\sum_{x=1}^{\infty} q^x\right) \\
&= p \frac{d}{dq}\left(\frac{q}{1-q}\right) \\
&= \frac{p}{(1-q)^2} = \frac{p}{p^2} \\
&= \frac{1}{p}.
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= E[(X - E[X])^2] \\
&= \frac{(1-p)}{p^2}.
\end{aligned}
$$

Example: A father asks his sons to cut their backyard lawn. Since he does not specify which of the three sons is to do the job, each boy tosses a coin to determine the odd person, who must then cut the lawn. In the case that all three get heads or tails, they continue tossing until they reach a decision. Let p be the probability of heads and $q = 1 - p$, the probability of tails.

a) Find the PMF of the number of rounds (tosses) required.
b) Find the expected number of rounds required.

Truncated Geometric Distribution



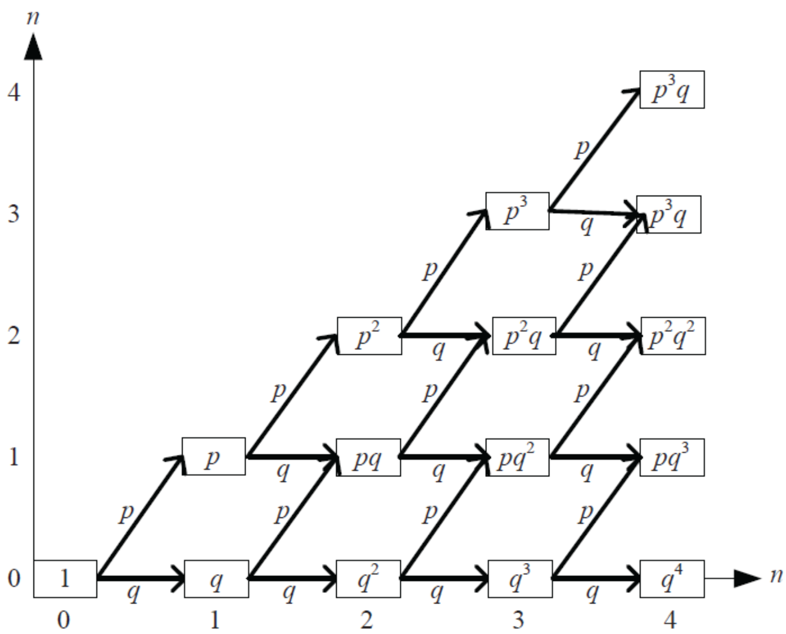
$$P_X(x) = \begin{cases} p(1-p)^{x-1}, & \text{for } x = 1, 2, \ldots, R-1; \\ (1-p)^{R-1}, & \text{for } x = R; \\ 0, & otherwise. \end{cases}$$

Binomial Distribution

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient in the binomial theorem

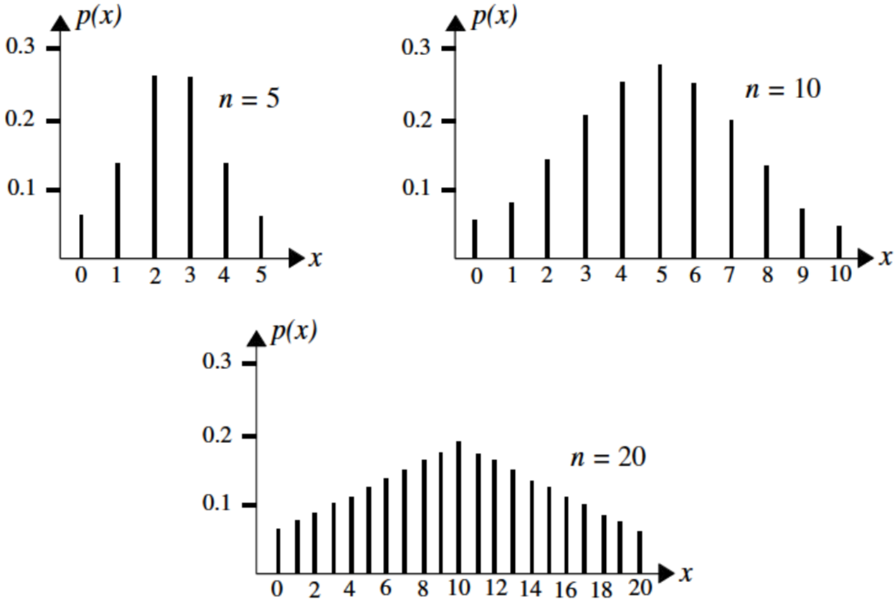
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$


n
4
3
2
1
0
1
q
p
p
pq
p^2
p^3
p^3q
p^2q
pq^2
q^2
q^3
q^4
p^2q^2
pq^3
n
0
1
2
3
4

$$P_X(k) = \begin{cases} \binom{n}{k} p^k q^{n-k}, & \text{for } k = 0, 1, 2, \ldots; \\ 0, & otherwise. \end{cases}$$

$$F_X(x) = \sum_{k=0}^{m \leq x} \binom{n}{k} p^k q^{n-k}$$

$$\boldsymbol{E(X) = np, \quad \mathrm{Var}(X) = np(1-p), \quad \sigma_X = \sqrt{np(1-p)}.}$$


p(x)
0.3
0.2
0.1
n = 5
0 1 2 3 4 5
x
p(x)
0.3
0.2
0.1
n = 10
0 1 2 3 4 5 6 7 8 9 10
x
p(x)
0.3
0.2
0.1
n = 20
0 2 4 6 8 10 12 14 16 18 20
x

$$\frac{P_X(k)}{P_X(k-1)} = \frac{(n-k+1)p}{kq} = 1 + \frac{(n+1)p-k}{kq}.$$

We see from above equation that $P_X(k)$ is greater than $P_X(k-1)$ when $k < (n+1)p$ and is smaller when $k > (n+1)p$. Accordingly, if we define integer k^* by

$$(n+1)p < k^* \leq (n+1)p,$$

Negative Binomial Distribution

In order to determine $P_X(n)$ for this case, let A be the event that the first $n-1$ trials yield exactly $k-1$ successes, regardless of their order, and B the event that a success turns up at the nth trial. Then, owing to independence,

$$P_X(n) = P(A \cap B) = P(A)P(B).$$

Now, $P(A)$ obeys a binomial distribution with parameters $n-1$ and $k-1$, or

$$P(A) = \binom{n-1}{k-1} p^{k-1} q^{n-k}, \qquad n = k, k+1, \ldots,$$

and $P(B)$ is simply

$$P(B) = p.$$

$$P_X(n) = \binom{n-1}{k-1} p^k q^{n-k}, \qquad n = k, k+1, \ldots.$$

Hypergeometric Distribution

In a box, there are r red and g green balls. Pick n balls randomly (a) with replacement and (b) without replacement.

Find the probability that exactly X red balls are picked. ($x \leq r$ and $n \leq r + g$ for without replacement.)

$$P_X(x) = \begin{cases} \frac{\binom{r}{x}\binom{g}{n-x}}{\binom{r+g}{n}}, & x = 0, 1, \dots, \min(r, n) \\ 0, & \text{otherwise.} \end{cases}$$

$$E[X] = \frac{nr}{r+g}$$

$$V[X] = \frac{n(rg)}{(r+g)^2}\left(1 - \frac{n-1}{r+g-1}\right)$$

Poisson Random Variable

$$P_X(x) = \begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$E[X] = \lambda T$

$V[X] = \lambda T$

$$P_X(x) = \begin{cases} (\frac{e^{-\alpha}\alpha^x}{x!}), & \text{for } x = 0, 1, 2, \ldots; \\ 0, & otherwise. \end{cases}$$

Example 2.8. The number of wrong numbered calls that arrive in a certain office has the Poisson distribution with the rate $\lambda = 2$ calls per day. Find the probability that there will be at least 2 wrong calls by tomorrow.

$$\lambda = 4 \text{ calls/day}$$

$$T = 2 \text{ days}$$

$$\alpha = \lambda T = 4 \times 2 = 8 \text{ calls}$$

$$\begin{aligned}
p &= P[X \geq 2] \\
&= 1 - P[X < 2] \\
&= 1 - P[X = 0] - P[X = 1] \\
&= 1 - \frac{e^8 \times 8^0}{0!} - \frac{e^8 \times 8^1}{1!} \\
&= 1 - e^8 - 8e^8 \\
&= 1 - 9e^8.
\end{aligned}$$