Lecture 09 - 10

Chapter 02: Random Variables

Math 4441: Probability and Statistics

Reference: Goodman & Yates – Introduction to Probability and Stochastic Process, 3rd Edition

Expected Values

Example: Suppose you play a game, where in each play you lose \$1 with probability 0.60 and you win \$1, \$2 and \$3 with probabilities 0.30, 0.08 and 0.02, respectively. Find your net gain in each play of the game.

- If you play only a few times, your gain might depend on your luck
- If you play a large number of times – your luck no longer determines the total gain – rather it will be multiple of average gain

Say, you play the game n times, approximately in $0.6 \times n$ games you will loose \$1 per game and will win \$1, \$2 and \$3 in $0.3 \times n$, $0.08 \times n$ and $0.02 \times n$ games, respectively.

Your total gain is

$$(0.6) \times n \cdot (-1) + (0.3) \times n \cdot (1) + (0.08) \times n \cdot (2) + (0.02) \times n \cdot (3) = (-0.08) \times n.$$

Average loss of \$0.08 per game

The more you play, the less luck interferes and your loss comes to \$0.08 per game

- Let X denote the gain in one play
- Then -0.08 is the expected value of X
- $E[X], \mu_X,$ or μ is the expected value of X

x	-1	1	2	3	
$P_X(x)$	0.60	0.30	0.08	0.02	1.00
$x \cdot P_X(x)$	-0.60	0.30	0.16	0.06	-0.08

$$P_X(x) = \begin{cases} 0.60, & x = -1 \\ 0.30, & x = 1 \\ 0.08, & x = 2 \\ 0.02, & x = 3 \\ 0, & o/w \end{cases}$$

$$-1 \times\ 0.6n + 1 \times 0.3n + 2 \times\ 0.08n + 3 \times 0.02n = -0.08n$$

Or, $-1 \times\ 0.6 + 1 \times 0.3 + 2 \times\ 0.08 + 3 \times 0.02 = -0.08$

Or, $-1 \times\ P_X(-1) + 1 \times P_X(1) + 2 \times P_X(2) + 3 \times P_X(3) = -0.08$

$$E[X] = \sum_{x \in S_X} x \,.\, P_X(x)$$

Definition (Expected Value): The *expected value* of a discrete random variable X with the set of possible values S_X and probability mass function $P_X(x)$ is defined by

$$E[X] = \sum_{x \in S_X} x P_X(x)$$

$E[X]$ exists if this sum converges

The expected value of a random variable X is also called the **mean**, or the **mathematical expectation**, or simply the **expectation** of X. It is denoted by $E[X]$, μ_X, or simply μ.

Example: Three packets are sent from one computer to another. Each packet is successfully sent with a probability of 0.8 irrespective of the others. Each successfully sent packet results in a reward of 1 unit whereas 1 unit penalty (negative reward) is imposed if none of the packets is sent successfully. Find the expected net reward.

Let X is the net reward. X is a discrete random variable with values

$$S_X = \{-1, 1, 2, 3\}$$

$$P[X = -1] = 0.008$$

$$P[X = 1] = 0.096$$

$$P[X = 2] = 0.384$$

$$P[X = 3] = 0.512$$

$$E[X] = -1 \times 0.008 + 1 \times 0.096 + 2 \times 0.384 + 3 \times 0.512$$

Example: A college mathematics department sends 8 to 12 professors to the

annual meeting of the American Mathematical Society, which lasts five days. The hotel at which the conference is held offers a bargain rate of a dollars per day per person if reservations are made 45 or more days in advance, but charges a cancellation fee of 2a dollars per person. The department is not certain how many professors will go. However, from past experience it is known that the probability of the attendance of i professors is 1/5 for i = 8, 9, 10, 11, and 12. If the regular rate of the hotel is 2a dollars per day per person, should the department make any reservations? If so, how many?

$$E(X_8) = (40a)\frac{1}{5} + (50a)\frac{1}{5} + (60a)\frac{1}{5} + (70a)\frac{1}{5} + (80a)\frac{1}{5} = 60a$$

$$E(X_9) = (42a)\frac{1}{5} + (45a)\frac{1}{5} + (55a)\frac{1}{5} + (65a)\frac{1}{5} + (75a)\frac{1}{5} = 56.4a,$$

$$E(X_{10}) = (44a)\frac{1}{5} + (47a)\frac{1}{5} + (50a)\frac{1}{5} + (60a)\frac{1}{5} + (70a)\frac{1}{5} = 54.2a,$$

$$E(X_{11}) = (46a)\frac{1}{5} + (49a)\frac{1}{5} + (52a)\frac{1}{5} + (55a)\frac{1}{5} + (65a)\frac{1}{5} = 53.4a,$$

$$E(X_{12}) = (48a)\frac{1}{5} + (51a)\frac{1}{5} + (54a)\frac{1}{5} + (57a)\frac{1}{5} + (60a)\frac{1}{5} = 54a.$$

X_{11} has the smallest expected value. Therefore, making 11 reservation is the most reasonable policy

Derived Random Variable:

If x is a sample value of a random variable X, and y is a sample value of a random variable Y. If we have a function $Y = g(X)$ and we get the values of Y from the values of X. Because we obtain *Y* from another random variable, we refer to *Y* as a **derived random variable**.

Definition 2.4 (Derived Random Variable). If X and Y are two random variables, and each sample value y of Y is a mathematical function $g(x)$ of X, then the relationship of these two random variables is given by

$$\mathrm{Y} = \mathrm{g}(\mathrm{X}),$$

and the random variable Y is called a derived random variable.

What will be the PMF of Y?

For a discrete random variable Y, if there exists another random variable X, where $Y = g(X)$, then the PMF of X is

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

Expected Value of a derived random variable

$$E[X+Y]=?$$

$$E[aX+b]=?$$

n-th Moment of a Random Variable

Variance: Variance measures the average magnitude of the fluctuations of a random variable from its expectations.

$$E\big[X - E[X]\big] = E[X - \mu] = E[X] - \mu = E[X] - E[X] = 0.$$

Hence, $E[X - E[X]]$ is not an appropriate measure for the variance.

if we consider $E(|X - E[X]|)$ instead, the problem of negative and positive deviations canceling each other disappears.

mathematically, $E(|X - E[X]|)$ is difficult to handle

The quantity $E[(X - E[X])^2]$ is used to measure the average amount of fluctuations, and is called the variance of X

The square root of $E[(X - E[X])^2]$ is called the **standard deviation** of X.

Definition 2.7. Let X be a discrete random variable with a set of possible values S_X, probability mass function $P_X(x)$, and $E(X) = \mu$. Then σ_X and $Var[X])$, called the **standard deviation** and the **variance** of X, respectively, and are defined by

$$\sigma_X = \sqrt{E[(X-\mu)^2]},$$

and

$$Var[X] = E[(X-\mu)^2].$$

Note that by this definition, we have

$$Var[X] = E[(X-\mu)^2] = \sum_{x \in S_X} (x-\mu)^2 P_X(x).$$

$$Var[X] = E[X^2] - \big(E[X]\big)^2.$$

Proof.

$$\begin{aligned} Var[X] &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2 \\ &= E[X^2] - \big(E[X]\big)^2. \end{aligned}$$

$Var[X] = 0$ *if and only if X is a constant with probability 1.*

$$Var[aX + b] = a^2 Var[X]$$

$$P_X(x) = \begin{cases} 0.60, & x = -1 \\ 0.30, & x = 1 \\ 0.08, & x = 2 \\ 0.02, & x = 3 \\ 0, & o/w \end{cases}$$

$$E[X] = \mu = 0.08$$

$$V[X] = [-1 - (-0.08) \times 0.60 + (1 - 0.08) \times 0.30 + (2 - 0.08) \times 0.08 + (3 - .08) \times 0.02$$

n-th Central Moment

Expected Value and Variance for Continuous Random Variables

$$E[X] = \int_{-\infty}^{+\infty} x . f_X(x) dx$$

$$V[X] = \int_{-\infty}^{+\infty} (x - \mu)^2 . f_X(x) dx$$

Example: The time elapsed, in minutes, between the placement of an order of pizza and its delivery is random with pdf

$$f(x) = \begin{cases} 1/15 & \text{if } 25 < x < 40 \\ 0 & \text{otherwise.} \end{cases}$$

$$E(X) = \int_{25}^{40} x \frac{1}{15} dx = 32.5,$$

$$E(X^2) = \int_{25}^{40} x^2 \frac{1}{15} dx = 1075.$$

$$V[X] = 1075 - (32.5)^2 = 18.75$$

$$\sigma_X = \sqrt{V[X]} = 4.33$$

Sketch the graph of the function

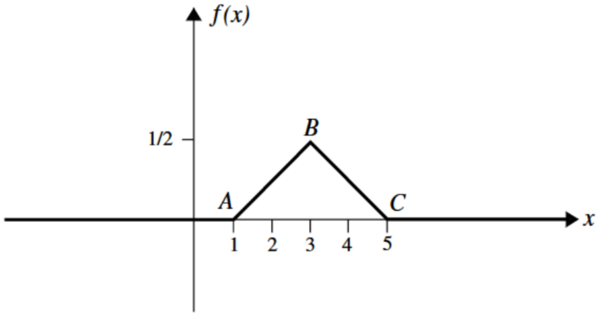
$$f(x) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{4}|x-3| & 1 \le x \le 5 \\ 0 & \text{otherwise,} \end{cases}$$

and show that it is the probability density function of a random variable X.

Find F, the distribution function of X, and show that it is continuous.

Find $E[X]$ and $V[X]$.

$$f(x) = \begin{cases} 0 & x < 1 \\ \dfrac{1}{2} + \dfrac{1}{4}(x-3) & 1 \le x < 3 \\ \dfrac{1}{2} - \dfrac{1}{4}(x-3) & 3 \le x < 5 \\ 0 & x \ge 5. \end{cases}$$



For $t < 1$,

$$F(t) = \int_{-\infty}^{t} 0\,dx = 0;$$

for $1 \le t < 3$,

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{1}^{t} \left[\frac{1}{2} + \frac{1}{4}(x-3)\right] dx = \frac{1}{8}t^2 - \frac{1}{4}t + \frac{1}{8};$$

for $3 \le t < 5$,

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f(x)\,dx = \int_{1}^{3} \left[\frac{1}{2} + \frac{1}{4}(x-3)\right] dx + \int_{3}^{t} \left[\frac{1}{2} - \frac{1}{4}(x-3)\right] dx \\ &= \frac{1}{2} + \left(-\frac{1}{8}t^2 + \frac{5}{4}t - \frac{21}{8}\right) = -\frac{1}{8}t^2 + \frac{5}{4}t - \frac{17}{8}; \end{aligned}$$

and for $t \ge 5$,

$$\begin{aligned} F(t) &= \int_{-\infty}^{t} f(x)\,dx = \int_{1}^{3} \left[\frac{1}{2} + \frac{1}{4}(x-3)\right] dx + \int_{3}^{5} \left[\frac{1}{2} - \frac{1}{4}(x-3)\right] dx \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$