Conditional Probability

↳ The probability of an event A occurring given that another event B has already occurred.

↳ Formula: $$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$P(A\cap B)$ → Probability that both A and B occur

$P(B)$ → Probability of B (>0)

$P(A|B)$ → Probability of A given B has occured.

Example 1: What is the probability that a selected card is a queen given that it is a face card?

Sol^n: Face cards = $3\times 4 = 12$ (Joker, Queen, King)

Total Queens = 4

$$\therefore P(Q|F) = \frac{P(Q\cap F)}{P(F)} = \frac{4/52}{12/52} = \frac{1}{3}.$$

⊛ Conditional Probability focuses only on the subset where the condition is true.

⊛ It reduces the sample space to the given condition.

Example 1.16: A box contains 8 red balls and 5 blue balls. Balls are mixed together.

Pick a ball randomly, observe the color, and return the ball. Pick the ball again, observe the color and return the ball.

$B \triangleq$ Event that the first ball is red
$A \triangleq$ Event that the second ball is red.

Solⁿ: $P(B) = \frac{8}{13}$ $P(A) = \frac{8}{13}$

Example 1.17: A box contains 8 red balls and 5 blue balls. Balls are mixed together.
Pick a ball randomly, observe the color, and keep it. Pick a ball again, observe the color and keep it as well.

$B \triangleq$ Event that the first ball is red
$A \triangleq$ Event that the second ball is red

Solⁿ: $P(B) = \frac{8}{13}$ $P(A) = \frac{7}{12}$.

Law of Multiplication (Product Rule)

↳ Defines the joint probability of occuring two events simultaneously.

↳ For dependent events:

$$\boxed{\begin{aligned} P(A \cap B) &= P(A) \cdot P(B|A) \\ &= P(B) \cdot P(A|B) \end{aligned}}$$

* This is when the occurence of A affects the probability of B.

* As explained, $P(B|A)$ is the conditional probability of B given A.

Example 1.18: A box contains 8 red balls and 5 blue balls. Balls are mixed together.
Pick a ball randomly, observe the color, and keep it. Pick a ball again, observe the color and keep it as well.

$R_i \triangleq$ Event that the i-th ball is red

$B_i \triangleq$ Event that the i-th ball is blue

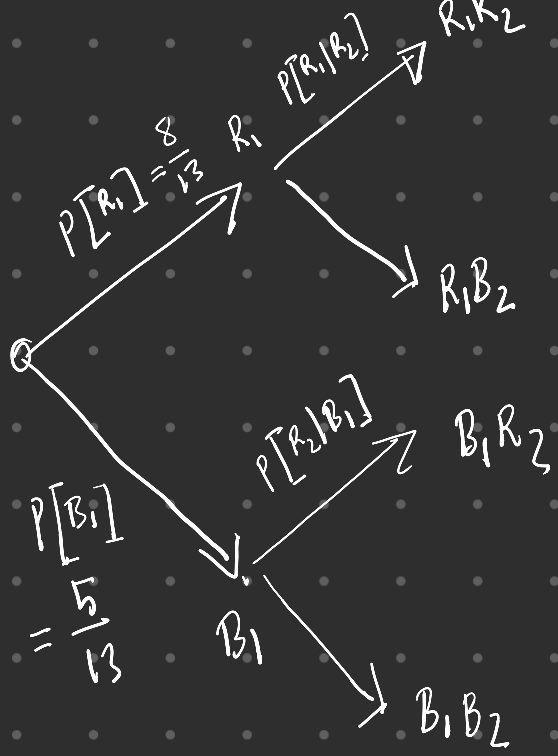
$$P[R_1R_2] = P[R_1]\,P[R_2|R_1]$$

$$= \frac{8}{13} \times \frac{7}{12} = \frac{14}{39}$$

$$P[R_1B_2] = P[R_1]\,P[B_2|R_1]$$

$$= \frac{8}{13} \times \frac{5}{12}$$

$$= \frac{10}{39}$$



The Chain Rule

↳ The joint probability of occurring more than two events can be found by extending the product rule.

$$\boxed{P[ABC] = P(A) \cdot P(B|A) \cdot P(C|AB)}$$

$$\circ \xrightarrow{P[A]} A \xrightarrow{P[B|A]} AB \xrightarrow{P[C|AB]} ABC$$

$$P[A_1 A_2 \ldots A_n] = P(A_1) \cdot P(A_2|A_1)\, P(A_3|A_1 A_2) \ldots \ldots \; P(A_n|A_1 A_2 \ldots A_n).$$

Example 1.19: A box contains 8 red balls and 5 blue balls. Balls are mixed together.
Pick a ball randomly, observe the color, and keep it. Now assume you pick 4 balls successively without replacement. Find the probability that the sequence of colors of the picked balls is Red, Blue, Red, and Blue.

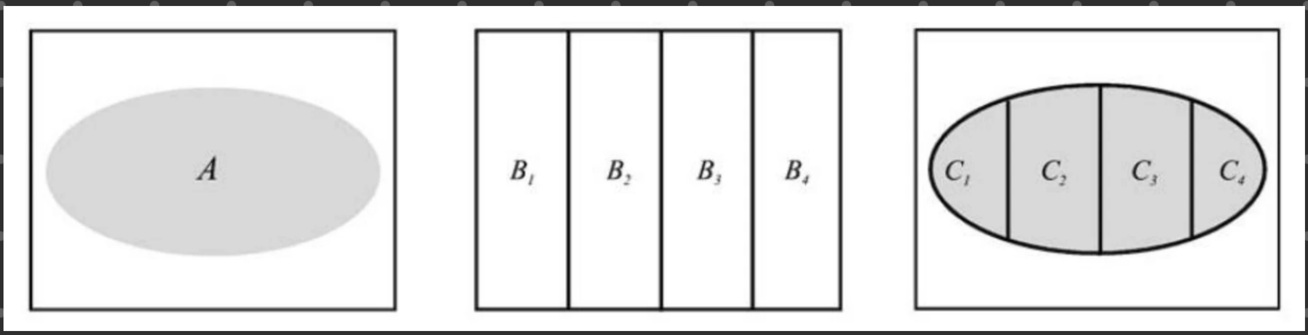
Sol^n:

$$P(R_1 B_2 R_3 B_4) = \frac{8}{13} \times \frac{5}{12} \times \frac{7}{11} \times \frac{4}{10}$$

$$= \frac{28}{429}$$

(*) For an event space $B = \{B_1, B_2, \ldots, B_n\}$ and any event A in the sample space. Let $C_i = A \cap B_i$, for $i = 1, 2, \ldots, n$.

For $i \neq j$, the events C_i and C_j are mutually exclusive and $A = C_1 \cup C_2 \cup \ldots \cup C_n$. All these are disjoint sets.



Law of Total Probability

↳ Computes the probability of an event by considering all the different ways that event can happen, through mutually exclusive and exhaustive events.

(*) Let B be an event with $P[B] \geq 0$ and $P[B^c] \geq 0$. then for an event A

$$P[A] = P[A|B]\,P[B] + P[A|B^c]\,P[B^c]$$

Let $B = \{B_1, B_2, \ldots, B_n\}$ an event space.

then for event A :-

$$P[A] = P[A|B_1]\,P[B_1] + P[A|B_2]\,P[B_2] + \cdots + P[A|B_n]\,P[B_n]$$

$$= \sum_{i=1}^{n} P[A|B_i]\,P[B_i].$$

Example 1.20: A box contains 8 red balls and 5 blue balls. Balls are mixed together.
Pick two balls randomly without replacement and observe their color.
Find the probability that the second ball is red.

Soln:

$$P(R_2) = P(R_2|R_1)\,P(R_1) + P(R_2|B_1)\,P(B_1)$$

$$= \frac{7}{12} \times \frac{8}{13} + \frac{8}{12} \times \frac{5}{13}$$

$$= \frac{8}{13}.$$

Bayes Theorem

↳ A way to update probabilities based on new evidence

It helps us find the probability of a cause given an outcome

So, if $P(A|B) = \frac{P(A\cap B)}{P(B)}$, then,

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

We know,
For dependent events:

$$P(A\cap B) = P(A) \cdot P(B|A)$$
$$= P(B) \cdot P(A|B)$$

So, $P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$

↳ $P(B|A) = \frac{P(A|B)\, P(B)}{P(A)}$

Example 1.21 You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). Suppose you play a game against a randomly chosen opponent, and you win. What is the probability that you had an opponent of type 1?

Soln: Given $P(T1) = \frac{1}{2}$, $P(win|T1) = 0.3$

$P(T2) = \frac{1}{4}$, $P(win|T2) = 0.4$

$P(T3) = \frac{1}{4}$, $P(win|T3) = 0.5$

From Bayes Theorem, $P(T1|win) = \frac{P(win|T1)\, P(T1)}{P(win)}$

$$\hookrightarrow P(T_1 | \text{win}) = \frac{0.3 \times 0.5}{0.375}$$

$$\Rightarrow 0.4$$

$$= \frac{2}{5}$$

From law of total probability.

$$P(\text{win}) = P(\text{win} | T_1) \cdot P(T_1) + P(\text{win} | T_2) \cdot P(T_2) + P(\text{win} | T_3) \cdot P(T_3)$$

$$= (0.3)(0.5) + (0.4)(0.25) + (0.5)(0.25)$$

$$= 0.375$$

Extra Example:

Problem Statement:
Suppose a certain type of cancer affects 1% of the population. A test has the following properties:

If a person has cancer, the test is positive with probability 0.99 (this is called sensitivity).

If a person does not have cancer, the test is positive with probability 0.05 (this is called false positive rate).

Now, suppose a person tests positive.
What is the probability that they actually have cancer?

Sol^n: Probability of,

a person having cancer $= P(C) = 0.01$

a person not having cancer $= P(C^c) =$

positive test, having cancer $= P(T^+ | C) = 0.99$

positive test, not having cancer $= P(T^+ | C^c) = 0.05$

Probability of actually having cancer when they tests positive $= P(C|T^+)$

From Bayes Theorem, $P(C|T^+)$

$$= \frac{P(T^+|C)\ P(C)}{P(T^+)}$$

$$= \frac{0.99 \times 0.01}{0.0594}$$

$$= \frac{1}{6}$$

$P(T^+)$

$= P(T^+|C)\,P(C) + P(T^+|C^c)\,P(C^c)$

$= (0.99)(0.01) + (0.05)(0.99)$

$= 0.0594$

Independence

→ Two events A and B are said to be independent if the occurrence of one does not affect the probability of the other. So Events A and B are independent iff

$P(A \cap B) = P(A) \cdot P(B)$ and also,

$P(A|B) = P(A)$ or $P(B|A) = P(B)$.

→ $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$A \perp B \triangleq A$ is independent of B.

$A \perp B = B \perp A$.

(*) Two events are independent if their joint probability is the product of their individual probabilities.

Basic Example: Toss two fair coins. Let,

A: First coin is heads $\Rightarrow P(A) = \frac{1}{2}$

B: Second coin is heads $\Rightarrow P(B) = \frac{1}{2}$

Now, $P(A \cap B) = P(\text{both heads}) = \frac{1}{4}$.

$P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, $\quad \therefore P(A \cap B) = P(A) \cdot P(B)$

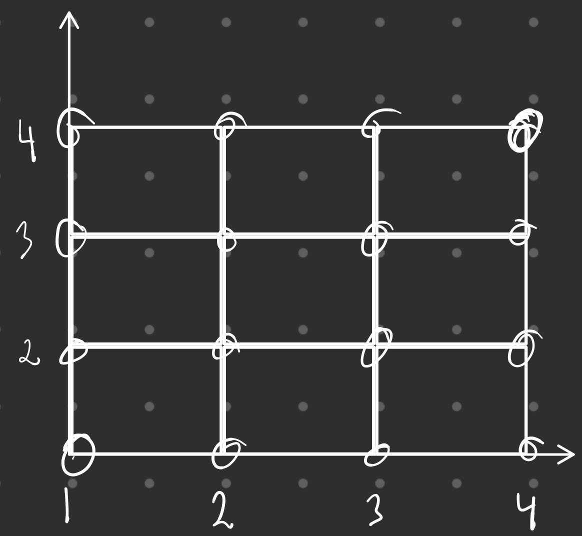
Example 1.22 Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

1. Are the events Ai = {1st roll results in i}, Bj = {2nd roll results in j}, independent?
2. Are the events A = {1st roll is a 1} , B = {sum of the two rolls is a 5}, independent?
3. Are the events A = {maximum of the two rolls is 2}, B = {minimum of the two rolls is 2}, independent?

Soln:

1. For any outcome, there are $4 \times 4 = 16$ outcomes.

$\therefore P(\text{each outcome}) = \frac{1}{16}$



Now, if, $A_i = \{\text{1st roll results in } i\}$

where $i \in \{1, 2, 3, 4\}$.

if, $B_j = \{2^{nd} \text{ roll results in } j\}$

where, $j \in \{1, 2, 3, 4\}$

So, if $i=1 \Rightarrow A_1 = \{(1,1), (1,2), (1,3), (1,4)\}$

$i=2 \Rightarrow A_2 = \{(2,1), (2,2), (2,3), (2,4)\}$

$i=3 \Rightarrow A_3 = \{(3,1), (3,2), (3,3), (3,4)\}$

$i=4 \Rightarrow A_4 = \{(4,1), (4,2), (4,3), (4,4)\}$

So, if $j=1 \Rightarrow B_1 = \{(1,1), (2,1), (3,1), (4,1)\}$

$j=2 \Rightarrow B_2 = \{(1,2), (2,2), (3,2), (4,2)\}$

$j=3 \Rightarrow B_3 = \{(1,3), (2,3), (3,3), (4,3)\}$

$j=4 \Rightarrow B_4 = \{(1,4), (2,4), (3,4), (4,4)\}$

$\therefore P(A_i) =$ There are four outcomes where the first roll is i

$$= \frac{4}{16} = \frac{1}{4}.$$

$P(B_j) =$ There are four outcomes where the second roll is j

$$= \frac{4}{16} = \frac{1}{4}.$$

$\therefore P(A_i \cap B_j) = P(\text{each outcome}) = \frac{1}{16}$

$P(A_i) \cdot P(B_j) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. $\therefore P(A_i \cap B_j) = P(A_i) P(B_j)$

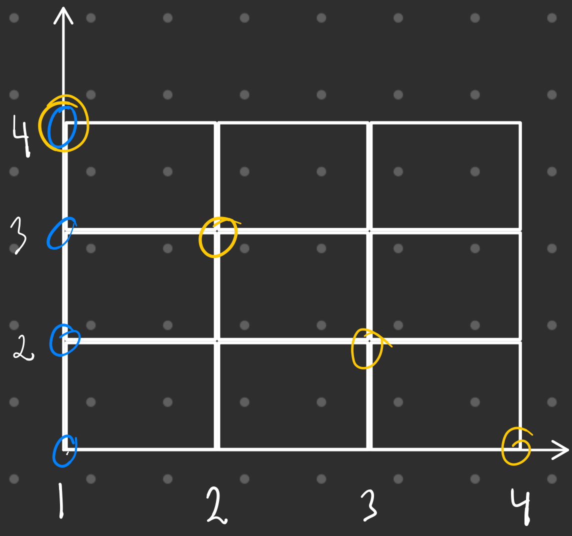
$\hookrightarrow$ They are independent.

② $A = \{ 1^{st} \text{ roll is a } 1 \}$

$= \{ (1,1), (1,2), (1,3), (1,4) \}$

$B = \{ \text{sum of the two roll is a } 5 \}$

$= \{ (1,4), (2,3), (3,2), (4,1) \}$



$\therefore P(A) = \frac{4}{16} = \frac{1}{4}$, $P(B) = \frac{4}{16} = \frac{1}{4}$

Here, $P(A \cap B) = \frac{1}{16}$ [$\because (1,4) \Rightarrow 1$ outcome is common]

Now, $P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

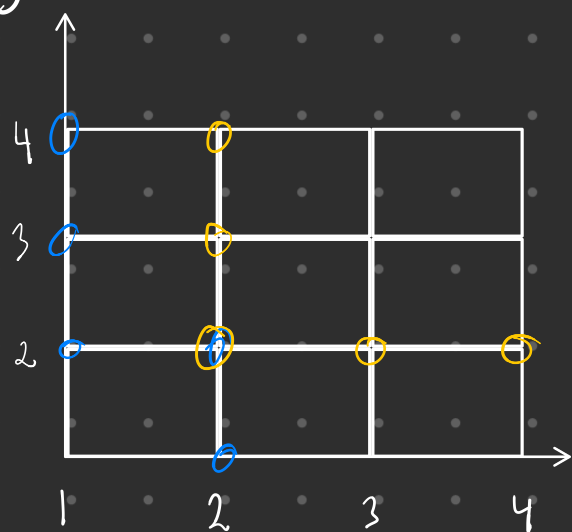
$\therefore P(A \cap B) = P(A) \cdot P(B) \Rightarrow A$ and B are independent.

③ $A = \{ \text{maximum of two rolls is } 2 \}$

$= \{ (1,2), (2,1), (2,2) \}$

$B = \{ \text{minimum of the two rolls is } 2 \}$

$= \{ (2,2), (2,3), (3,2), (2,4), (4,2) \}$



$\therefore P(A) = \frac{3}{16}$, $P(B) = \frac{5}{16}$

Here, $P(A \cap B) = \frac{1}{16}$ [$\because (2,2) \Rightarrow 1$ outcome in common]

$$P(A)\cdot P(B) = \frac{3}{16} \times \frac{5}{16} = \frac{15}{256}$$

$\therefore P(A\cap B) \neq P(A)\,P(B) \nrightarrow$ the events A and B are not independent.

If A is a set of events, $A = \{A_1, A_2, \ldots, A_n\}$ are independent, if for every subset of $A_i \subset A$.

So, for every subset $A_i \subset A$ with k element,

$k = 2, 3, \ldots, n.$

$$P[A_{i_1}, A_{i_2}, A_{i_3}, \ldots, A_{i_k}] = P[A_{i_1}]\,P[A_{i_2}] \ldots P[A_{i_k}]$$

So, if a set of events consists of A, B and C then, A, B and C are independent, when.

$$P[ABC] = P[A]\,P[B]\,P[C]$$

Conditional Probability and Regular Probability

1. Conditional probabilities behave like regular probabilities:

- They **obey the same 3 axioms** of probability:
  1. Non-negativity: $P(A|B) \geq 0$
  2. Normalization: $P(S|B) = 1$ (the probability of the sample space, given B, is 1)
  3. Additivity: If $A_1, A_2, \ldots$ are disjoint, then

$$P(A_1 \cup A_2 \cup \ldots |B) = P(A_1|B) + P(A_2|B) + \ldots$$

2. Conditional probability can itself be conditioned:

This means you can compute conditional probabilities like:

- $P(A|B)$: the probability of A given B
- $P(A|B \cap C)$: the probability of A given both B and C occurred
- Or more generally:

$$P(A|B, C) = P(A|B \cap C)$$

The expression asked on the slide is:

$$\boxed{P(A|BC) = \frac{P(A \cap B \cap C)}{P(B \cap C)}}$$

This uses the **definition of conditional probability**:

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)} \quad (\text{if } P(Y) > 0)$$

🔹 1. Conditional Product Rule

The **product rule** lets us break down a joint probability into a sequence of conditional probabilities. The conditional version does this **given an event** B:

Formula:

$$\Pr(A_1 \cap A_2 \cap \cdots \cap A_n \mid B) = \Pr(A_1 \mid B) \cdot \Pr(A_2 \mid A_1 \cap B) \cdot \ldots \cdot \Pr(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1} \cap B)$$

What it means:

It decomposes a **chain of dependent events** conditioned on B into a **series of simpler conditional probabilities**.

✅ **Use case:**

Useful in Bayesian networks, Markov models, or any complex model where events depend on a shared condition or context B.

🔹 2. Conditional Sum Rule (Conditional Law of Total Probability)

This is an extension of the **law of total probability**, conditioned on another event C.

Formula:

$$\Pr(A \mid C) = \sum_{j=1}^{k} \Pr(B_j \mid C) \cdot \Pr(A \mid B_j \cap C)$$

Where:

- $B_1, B_2, \ldots, B_k$ form a **partition** of the sample space.
- You're essentially **averaging** $\Pr(A \mid B_j \cap C)$ weighted by $\Pr(B_j \mid C)$.

✅ **Use case:**

Suppose you're trying to compute the conditional probability of an event A, but A's behavior depends on which of the B_j's occurred — this rule helps **break it into manageable components**.

What Is Conditional Independence?

We say that events $A_1, A_2, \ldots, A_k$ are **conditionally independent given** another event B, if:

For **any subset** $A_{i_1}, A_{i_2}, \ldots, A_{i_j}$ of these events (where $j \geq 2$),

$$\Pr(A_{i_1} \cap \cdots \cap A_{i_j} \mid B) = \Pr(A_{i_1} \mid B) \cdot \cdots \cdot \Pr(A_{i_j} \mid B)$$

Special Case: Two Events

Let's say we only have events A_1, A_2 and B. Then:

> A_1 and A_2 are **conditionally independent given** B if:

$$\Pr(A_2 \mid A_1, B) = \Pr(A_2 \mid B)$$

This says that knowing A_1, **in addition to knowing** B, **does not change** the probability of A_2.

Contrast with Unconditional Independence

Without conditioning:

$$\Pr(A_1 \cap A_2) = \Pr(A_1) \cdot \Pr(A_2)$$

With conditioning on B:

$$\Pr(A_1 \cap A_2 \mid B) = \Pr(A_1 \mid B) \cdot \Pr(A_2 \mid B)$$

These are **different concepts**. Two events can be dependent in general, but independent **given some information**.

Conditional Bayes Theorem

⇨ Computes the posterior probability of an event A given both an event C and some background condition B.

So, if A, C and B are events such that $P(C|B) > 0$, then the conditional Bayes Theorem states that:

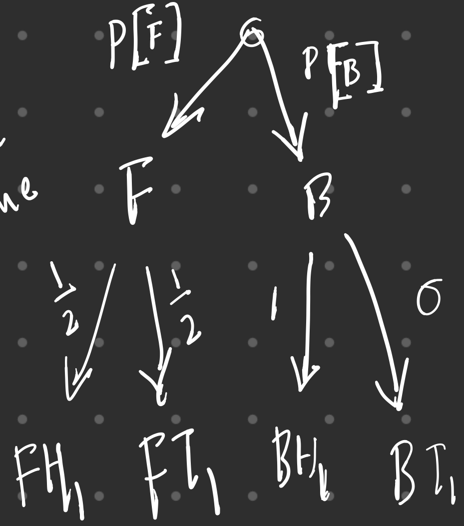
$$P(A|CB) = \frac{P(C|AB)\, P(A|B)}{P(C|B)}$$

Also, $P(C|AB) = \dfrac{P(A|CB)\, P(C|AB)}{P(A|B)}$

Example 1.24:

1. Suppose a box contains one fair and one double headed coin. Suppose a coin is selected randomly and a head is obtained. Find the probability that the selected coin is the fair coin. P[F|H1] = ?
2. Suppose the same coin is tossed again and that the two tosses are conditionally independent given both F and B. Suppose further that another head is obtained. Find the probability that the selected coin is fair. P[F|H1H2] = ?

Soln: ① Events,



$F \triangleq \{\text{selected coin is fair}\}$

The double Headed one

$B \triangleq \{\text{selected coin is biased}\}$

$H_1 \triangleq \{\text{first toss produces head}\}$

$$\therefore P[F|H_1] = \frac{P[H_1|F]\,P[F]}{P[H_1]}$$

$$= \frac{1/2 \times 1/2}{3/4}$$

$$= \frac{1}{3}.$$

$P[H_1] =$

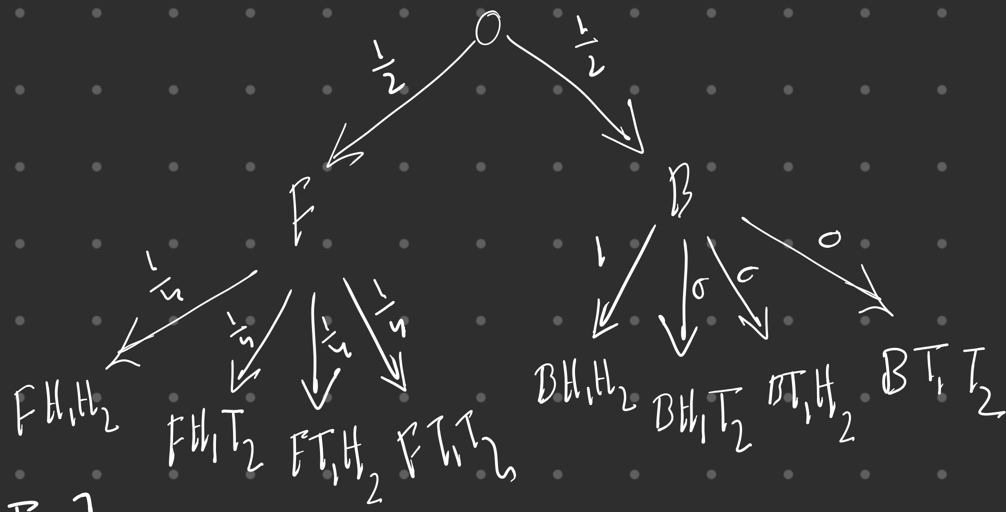
$P[H_1|F]\,P[F] +$

$P[H_1|B]\,P[B]$

$$= \frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}$$

$$= \frac{3}{4}$$

2



$$P[F \mid H_1H_2] = \frac{P[H_1H_2 \mid F]\; P[F]}{P[H_1H_2]}$$

$$= \frac{1/4 \times 1/2}{5/8} = \frac{1}{5}.$$

$$P[H_1H_2]$$

$$= P[H_1H_2 \mid F]\,P[F] + P[H_1H_2 \mid B]\,P[B]$$

$$= \frac{1}{4} \times \frac{1}{2} + 1 \times \frac{1}{2}$$

$$= \frac{5}{8}$$

Back to that problem,

Problem 1 (Disease or Not)

1. The probability that a randomly selected person has a disease is 0.8 percent (8 out of 1,000).
2. The probability that a person with the disease will have a positive test result is 90 percent.
3. The probability that a person with the disease will have a positive test result is 7 percent.

Soln:

Events: $D = \{\text{Person has the disease}\}$

$D^c = \{\text{Person does not have the disease}\}$

$T^+ = \{\text{Person Tests Positive}\}$

Given, $P(D) = 0.008$, $P(T^+|D) = 0.9$. $P(T^+|D^c) = 0.07$.

$$\therefore P(D|T^+) = \frac{P(T^+|D) \cdot P(D)}{P(T^+)}$$

$$= \frac{0.9 \times 0.008}{0.07664}$$

$$= 0.09395$$

$$\approx 9.4\%$$

$$P(T^+) = P(T^+|D)\,P(D) + P(T^+|D^c)\,P(D^c)$$

$$= (0.9)(0.008) + (0.07)(1-0.008)$$

$$= 0.07664$$

Probability of having a disease who tasted positive.

Problem 2 (Monty Hall Dilemma)

The dilemma is taken from an old American game show called Let's Make a Deal.
The original host of the show, Monty Hall, would select a member of the audience and show that person three large closed doors labeled 1, 2, and 3. Behind one of the doors was a new car. Behind the remaining two doors were joke prizes, such as a live goat.
The contestant was asked to pick a door. Then, Hall would ask that one of the doors the contestant didn't pick be opened, naturally one that didn't have a car behind it. After the audience stopped laughing at whatever joke prize was behind that door, Hall would ask the contestant if they wanted to keep the originally selected door, or if they would rather change their selection to the remaining door. The dilemma is simply that: do they keep their original guess, or do they switch to the remaining door?

Solⁿ: Assuming, I pick door number 1.

The probability of having a car in that door is $\frac{1}{3}$.

Let's say, Monty opens door number 3 and it's a goat. Now,

$$P(\text{Car behind door } 1 \mid \text{pick door } 1) = \frac{1}{3}.$$

$$P(\text{Car behind door } 2 \mid \text{shows goat behind door } 3) = \frac{2}{3}.$$

Here, The act of opening door 3 does not change the initial probabilities, it just gives more information

So, switching is a better choice. It doubles the chance as it inherits the $2/3$ chance of being correct.

Congratulations CSE

Keep making IUT

Blue ♡