13:2-Limits and Continuity

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Objectives

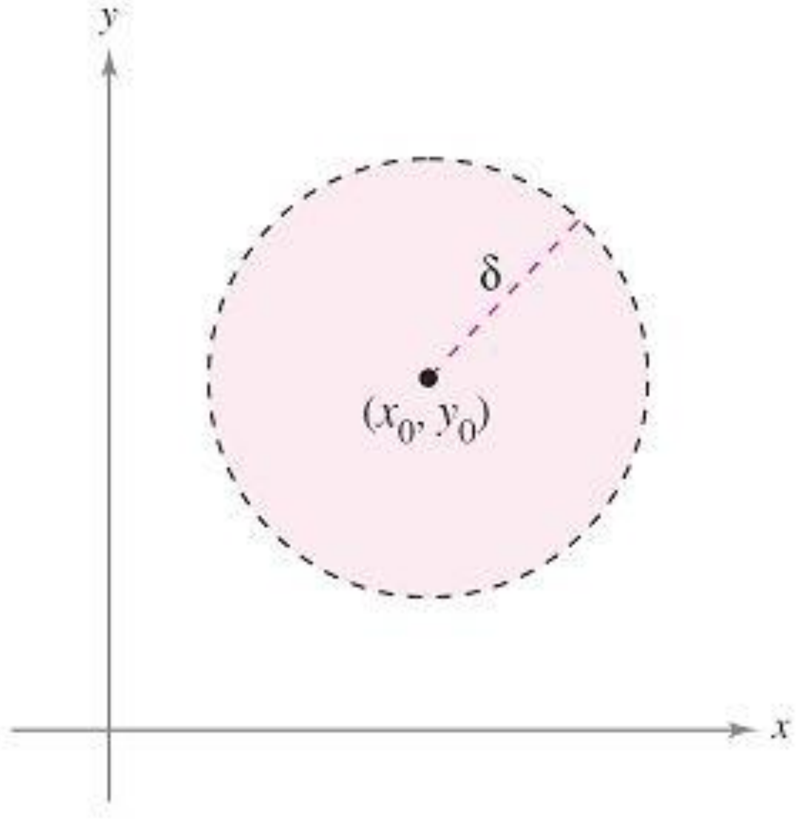
- Understand the definition of a neighborhood in the plane.
- Understand and use the definition of the limit of a function of two variables.
- Extend the concept of continuity to a function of two variables.
- Extend the concept of continuity to a function of three variables.

Neighborhoods in the Plane

- Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the **δ-neighborhood** about (x_0, y_0) to be the **disk** centered at (x_0, y_0) with radius $\delta > 0$

$$\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\} \qquad \text{Open disk}$$

as shown in Figure.

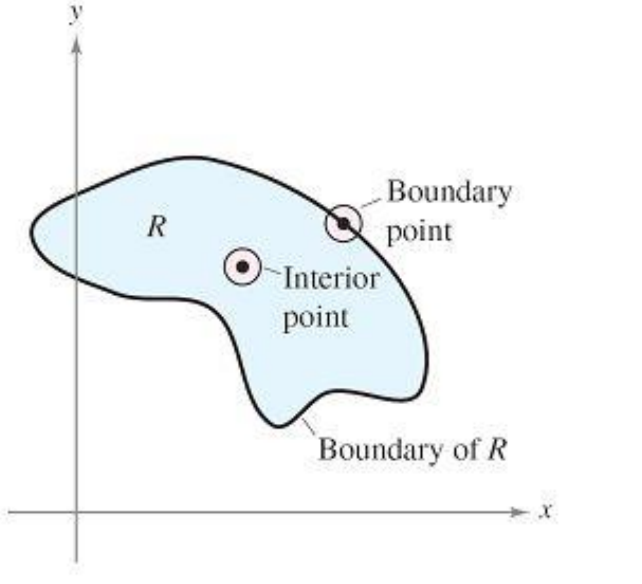


An open disk

Neighborhoods in the Plane

- When this formula contains the *less than* inequality sign, <, the disk is called **open,** and when it contains the *less than or equal to* inequality sign, ≤, the disk is called **closed.** This corresponds to the use of < and ≤ to define open and closed intervals.

- Let the region R be a set of points in the plane. A point (x_0, y_0) in R is an **interior point** of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure.



The boundary and interior points of a region R

Neighborhoods in the Plane

- If every point in R is an interior point, then R is an **open region.** A point (x_0, y_0) is a **boundary point** of R if every open disk centered at (x_0, y_0) contains points inside R *and* points outside R. If R contains all its boundary points, then R is a **closed region**.

Limit of a Function of Two Variables

Definition of the Limit of a Function of Two Variables

Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

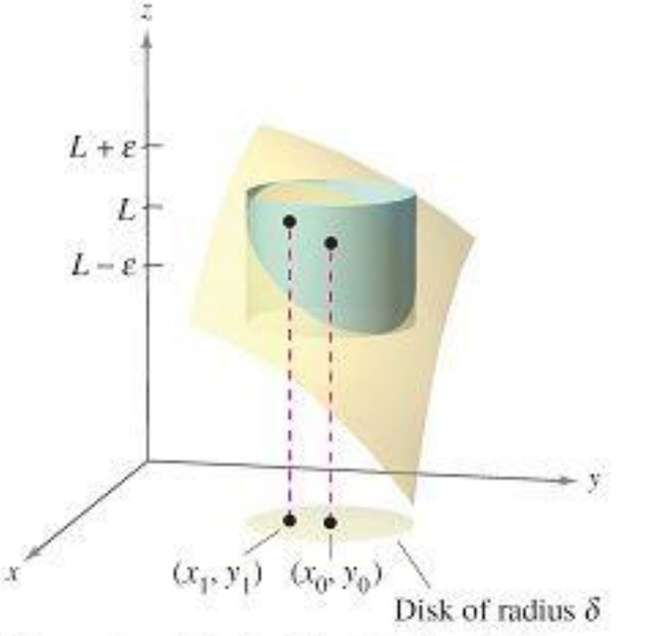
$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Limit of a Function of Two Variables

- Graphically, the definition of the limit of a function of two variables implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure.



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.

Limit of a Function of Two Variables

- The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference.

- To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left.

- When the function approaches the same limit from the right and from the left, you can conclude that the limit exists.

Limit of a Function of Two Variables

- For a function of two variables, however, the statement

  $(x, y) \to (x_0, y_0)$

  means that the point (x, y) is allowed to approach (x_0, y_0) from any direction.

- If the value of $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$

  is not the same for all possible approaches, or **paths,** to (x_0, y_0), then the limit does not exist.

Example 1 – *Verifying a Limit by the Definition*

- Show that $\lim_{(x,\, y)\to(a,\, b)} x = a.$

- Solution:

Let $f(x, y) = x$ and $L = a$.

You need to show that for each $\varepsilon > 0$, there exists a δ-neighborhood about (a, b) such that

$$|f(x, y) - L| = |x - a| < \varepsilon$$

whenever $(x, y) \neq (a, b)$ lies in the neighborhood.

Example 1 – *Solution*

- You can first observe that from

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

it follows that

$$\begin{aligned} |f(x, y) - L| &= |x - a| \\ &= \sqrt{(x-a)^2} \\ &\le \sqrt{(x-a)^2 + (y-b)^2} \\ &< \delta. \end{aligned}$$

- So, you can choose $\delta = \varepsilon$, and the limit is verified.

Limit of a Function of Two Variables

- Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables.

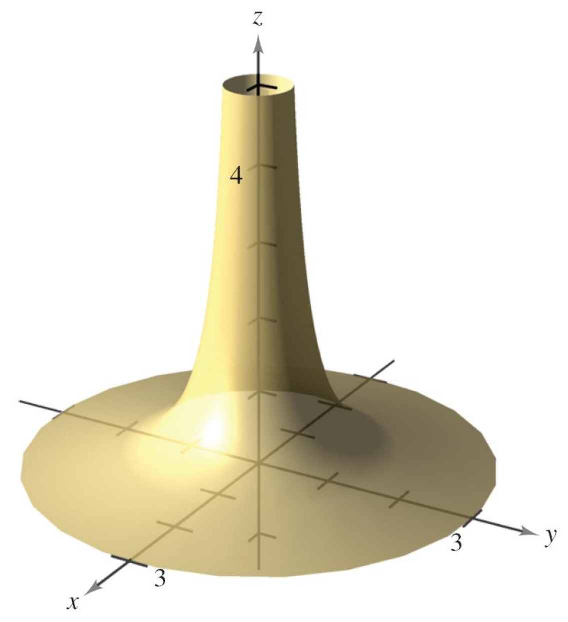
Limit of a Function of Two Variables

- For some functions, it is easy to recognize that a limit does not exist.

- For instance, it is clear that the limit

$$\lim_{(x,\,y)\to(0,\,0)} \frac{1}{x^2 + y^2}$$

does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure).



$\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{1}{x^2 + y^2}$ does not exist.

Continuity of a Function of Two Variables

- The limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution.

- That is, the limit is $f(1, 2) = 2$.

- In such cases, the function f is said to be **continuous** at the point (1, 2).

Continuity of a Function of Two Variables

Definition of Continuity of a Function of Two Variables

A function f of two variables is **continuous at a point (x_0, y_0)** in an open region R if $f(x_0, y_0)$ is defined and is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Continuity of a Function of Two Variables

- The function

$$f(x, y) = \frac{5x^2y}{x^2 + y^2}$$

is not continuous at (0, 0). Because the limit at this point exists, however, you can remove the discontinuity by defining *f* at (0, 0) as being equal to its limit there. Such a discontinuity is called **removable.**

- The function

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

is not continuous at (0, 0), and this discontinuity is **nonremovable.**

Continuity of a Function of Two Variables

THEOREM 13.1 Continuous Functions of Two Variables

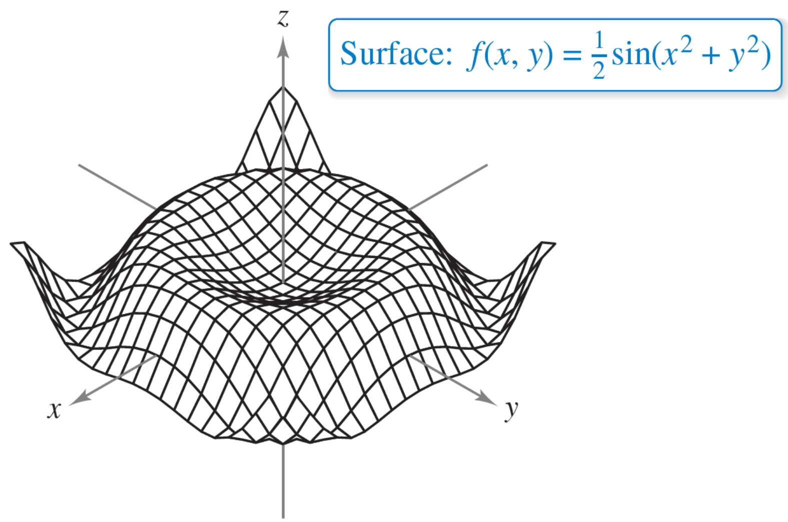
If k is a real number and $f(x, y)$ and $g(x, y)$ are continuous at (x_0, y_0), then the following functions are also continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, $g(x_0, y_0) \neq 0$

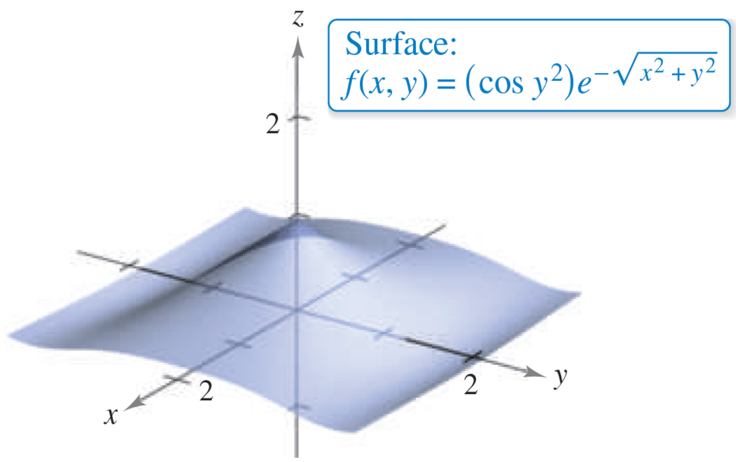
- Theorem 13.1 establishes the continuity of *polynomial* and *rational* functions at every point in their domains.
- Furthermore, the continuity of other types of functions can be extended naturally from one to two variables.

Continuity of a Function of Two Variables

- For instance, the functions whose graphs are shown in Figures are continuous at every point in the plane.



The function f is continuous at every point in the plane.



The function f is continuous at every point in the plane.

Continuity of a Function of Two Variables

THEOREM 13.2 Continuity of a Composite Function

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

- Note in Theorem 13.2 that h is a function of two variables and g is a function of one variable.

Example 5 – *Testing for Continuity*

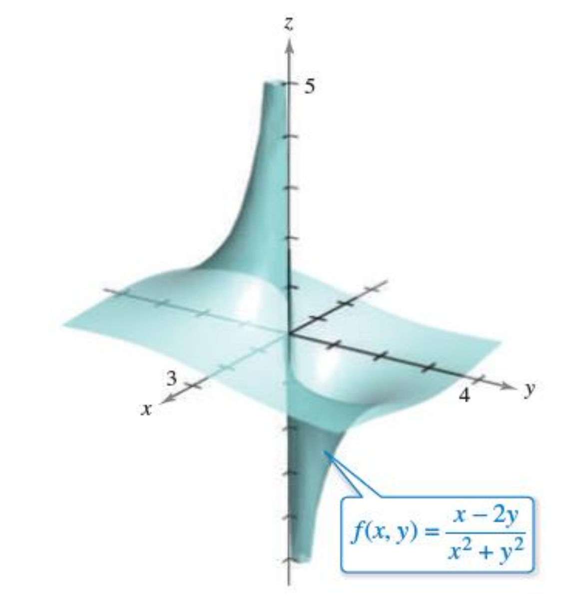
- Discuss the continuity of each function.

a. $f(x, y) = \dfrac{x - 2y}{x^2 + y^2}$

b. $g(x, y) = \dfrac{2}{y - x^2}$

Example 5(a) – *Solution*

- Because a rational function is continuous at every point in its domain, you can conclude that f is continuous at each point in the xy-plane except at (0, 0), as shown in Figure.



The function f is not continuous at (0, 0).

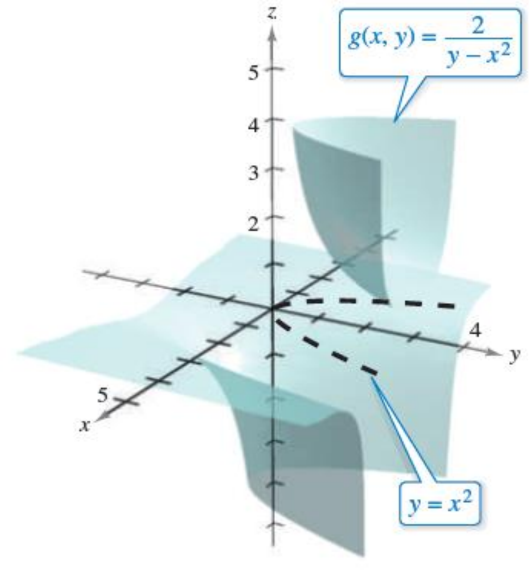
Example 5(b) – *Solution*

- The function $g(x, y) = 2/(y - x^2)$ is continuous except at the points at which the denominator is 0. These points are given by the equation
$y - x^2 = 0.$

- So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$.

Example 5(b) – *Solution*

- Inside this parabola, you have $y > x^2$, and the surface represented by the function lies above the *xy*-plane, as shown in Figure 13.26.



The function g is not continuous on the parabola $y = x^2$.

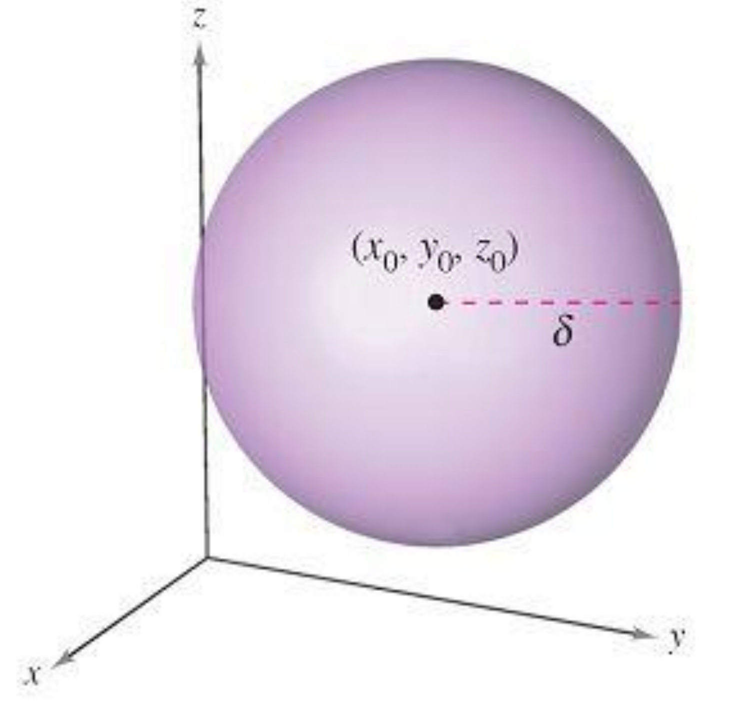
- Outside the parabola, $y < x^2$, and the surface lies below the *xy*-plane.

Continuity of a Function of Three Variables

- The definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the *open sphere*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2. \quad \text{Open sphere}$$

- The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0), as shown in Figure.



Open sphere in space

Continuity of a Function of Three Variables

- A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called **open**.

Definition of Continuity of a Function of Three Variables

A function f of three variables is **continuous at a point (x_0, y_0, z_0)** in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x, y, z)\to(x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Example 6 – *Testing Continuity of a Function of Three Variables*

- Discuss the continuity of $$f(x, y, z) = \frac{1}{x^2 + y^2 - z}.$$

- Solution:

  The function f is continuous except at the points at which the denominator is 0, which are given by the equation

$$x^2 + y^2 - z = 0.$$

- So, f is continuous at each point in space except at the points on the paraboloid

$$z = x^2 + y^2.$$

Suggested Problems

Exercise 13.2:16,20,22,23,30,31,42,45,46

Thanks a lot …