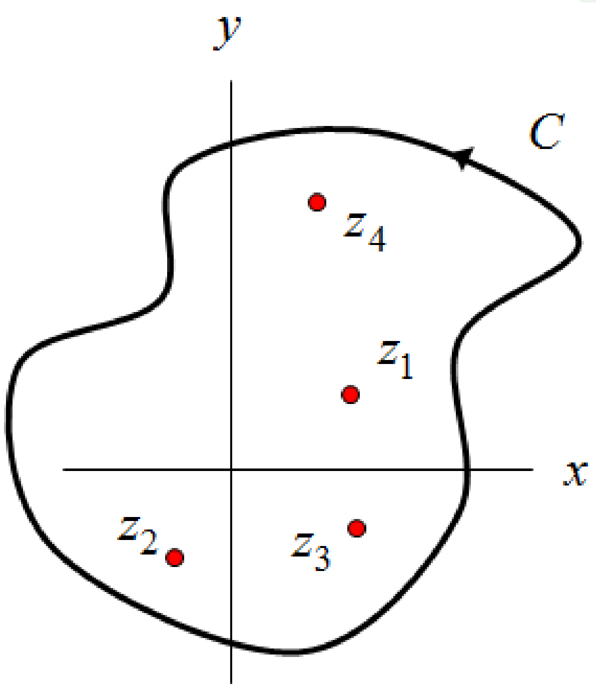

RESIDUE, AND APPLICATION OF RESIDUE THEORY TO INTEGRALS

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Isolated Singularity

$$\begin{cases} z_0 \text{ is the removable singular point } of \ f(z) \Leftrightarrow \lim\limits_{z\to z_0} f(z) \text{ exists and finite;} \\ z_0 \text{ is the pole of } f(z) \Leftrightarrow \lim\limits_{z\to z_0} f(z) = \infty; \\ z_0 \text{ is the essential singular point of } f(z) \Leftrightarrow \lim\limits_{z\to z_0} f(z) \text{ not exists and } \neq \infty. \end{cases}$$

We can use the above different situations to judge the types of isolated singular points.

Residue

z_0 is isolated singular point of $f(z)$,

$\Rightarrow f(z)$ has Laurent expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \ ,0 < |z - z_0| < \delta.$$

$$\oint_C f(z)dz = \sum_{n=-\infty}^{\infty} c_n \oint_C (z - z_0)^n dz$$

$$\left[\text{In} = \oint_C \frac{1}{(z - z_0)^n} dz = \begin{cases} 2\pi i, n = 1, \\ 0, \quad n \neq 1. \end{cases} (n \in N) \right]$$

$$\Rightarrow \oint_C f(z)dz = c_{-1} \cdot 2\pi i, i.e. \frac{1}{2\pi i} \oint_C f(z)dz = c_{-1}.$$

Residue

If $f(z)$ has an isolated singularity at $z_0 (z_0 \neq \infty), C$ is the positive oriented simple closed curve in $0 < |z - z_0| < \delta$,

then $\frac{1}{2\pi i} \oint_C f(z)dz$ is called the residue of $f(z)$ at z_0

and is denoted by $\mathrm{Res}[f(z), z_0]$,

$$i.e. \mathrm{Res}[f(z), z_0] = \frac{1}{2\pi i} \oint_C f(z)dz = c_{-1}.$$

Note:

(1). z_0 is removable singularity $\Rightarrow \mathrm{Res}[f(z), z_0] = 0.$

(2). z_0 is essential singularity $\Rightarrow \mathrm{Res}[f(z), z_0] = c_{-1}.$

(3). z_0 is a pole,then we can get the following rules.

Laurent's Theorem (LS)

If we are required to expand *f(z)* about a point where *f(z)* is not analytic, then it is expanded by Laurent's series.

Statement: If *f(z)* is analytic on C_1 and C_2 and on the annular region R bounded by the two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ having centers at a, then for all z in R,

$$f(z) = \left.\begin{array}{l} a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots\ldots\ldots + \\ + \dfrac{b_1}{(z-a)} + \dfrac{b_2}{(z-a)^2} + \ldots\ldots\ldots\ldots\ldots\ldots + \end{array}\right\} \ldots\ldots\ldots\ldots(A)$$

where

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz, \quad n = 0, 1, 2, \ldots..$$

$$b_n = \frac{1}{2\pi i}\oint_{C_2} (z-a)^{n-1} f(z)\, dz, \quad n = 1, 2, \ldots..$$

Here the part $a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$ is called the analytic part of Laurent's series, while the remaining part, that is $\dfrac{b_1}{(z-a)} + \dfrac{b_2}{(z-a)^2} + \ldots\ldots\ldots\ldots$ is called the principal part. If the principal part is zero, then the LS reduces to Taylor's series.

Example 1: Expand f(z) = $f(z)=\dfrac{1}{(z-1)(z-2)}$ for $1<|z|<2$ by LS.

Solution: Given that

$$1<|z|<2$$

$$\Rightarrow \frac{1}{|z|}<1 \text{ and } \frac{|z|}{2}<1$$

To ensure that the LS converses, we have to expand f(z) in terms of $\dfrac{1}{|z|}$ or $\dfrac{|z|}{2}$. Therefore, we should write f(z) as

$$f(z)=\frac{1}{(z-1)(z-2)}$$

$$=\frac{1}{(z-2)}-\frac{1}{(z-1)}$$

$$=\frac{1}{(-2)\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}......\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}.....\right]$$

$$= -\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}-........-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}....$$

which is required LS where $\frac{1}{|z|}<1$ and $\frac{|z|}{2}<1$ QED.

Exercises : Expand $f(z)=\dfrac{z}{(z-1)(2-z)}$ in a LS vaild for (a) $|z|<1$, (b) $|z|>2$, (c) $1<|z|<2$, (d) $|z-1|>1$, (e) $0<|z-2|<1$

Residue in Complex Number

In complex analysis we can represent a function as a series called the Laurent series. Now if we want to integrate this function we can just integrate the series term by term. Well thanks to some theorems about integrals a lot of these integrals will be zero. What isn't zero is called a **Residue** and represents the value of the integral in question.

The coefficient of $\frac{1}{z-a}$, i.e., b_1 in LS

$$f(z) = \left.\begin{matrix} a_0 + a_1(z-a) + a_2(z-a)^2 + \dots\dots\dots\dots + \\ + \frac{b_1}{(z-a)} + \frac{a_2}{(z-a)^2} + \dots\dots\dots\dots\dots\dots + \end{matrix}\right\} \dots\dots\dots\dots(A)$$

about an isolated singular point z =a is called the **residue** of *f(z)* at z = a.

Method of finding residue:

1. **Rule 1:** Residue at a simple pole is given by

   $$\text{Res } f(z) = \underset{z \to a}{Lt}\,(z-a)f(z)$$

2. **Rule 2:** Residue at a pole of order n is given by

   $$\text{Res } f(z) = \frac{1}{\lfloor (n-1)}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z-a)^n f(z)\right\}\right]_{z=a}$$

Calculation of Residues

To obtain the residue of a function $f(z)$ at $z = a$, it may appear from (7.1) that the Laurent expansion of $f(z)$ about $z = a$ must be obtained. However, in the case where $z = a$ is a pole of order k, there is a simple formula for a_{-1} given by

$$a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \{(z-a)^k f(z)\}$$

If $k = 1$ (simple pole), then the result is especially simple and is given by

$$a_{-1} = \lim_{z \to a} (z-a)\, f(z)$$

which is a special case of (7.5) with $k = 1$ if we define $0! = 1$.

EXAMPLE 1: If $f(z) = z/(z-1)(z+1)^2$, then $z = 1$ and $z = -1$ are poles of orders one and two, respectively.

We have, using (7.6) and (7.5) with $k = 2$,

$$\text{Residue at } z = 1 \text{ is } \lim_{z \to 1} (z-1) \left\{ \frac{z}{(z-1)(z+1)^2} \right\} = \frac{1}{4}$$

$$\text{Residue at } z = -1 \text{ is } \lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} \left\{ (z+1)^2 \left(\frac{z}{(z-1)(z+1)^2} \right) \right\} = -\frac{1}{4}$$

If $z = a$ is an essential singularity, the residue can sometimes be found by using known series expansions.

The Residue Theorem

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \ldots$ inside C, which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ [see Fig. 7-1]. Then, the *residue theorem* states that

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \tag{7.7}$$

i.e., the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C. Note that (7.7) is a generalization of (7.3). Cauchy's theorem and integral formulas are special cases of this theorem

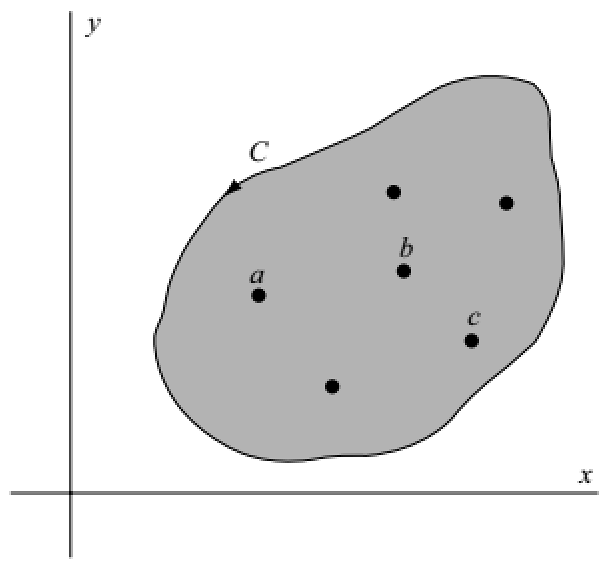

Fig. 7-1

Residue Theorem (RT)

If a function *f(z)* is analytic in a closed curve C except at a finite poles within C, then

$$\int_C f(z)\,dz = 2\pi i\left(\text{sum of resideues at all the poles within C}\right)$$

Example 1: Determine the poles of $\dfrac{4-3z}{z(z-1)(z-2)}$, over $C:|z|=\dfrac{3}{2}$ and residue at each pole.

Solution: The poles of the integrand are given by putting the denominator to zero. That is,

$$z(z-1)(z-2)=0$$
$$\Rightarrow z=0, z=1, z=2$$

The given circle $|z|=\dfrac{3}{2}$ with centre at the origin (z = 0) and radius $r=\dfrac{3}{2}$ encloses two simple poles z = 0, and z = =1 only and the other pole lies out side of the circle (domain). Now using Rule 1 above, we have

Residue of f(z) at the simple pole z = 0 is

$$\text{Res } f(z) = \underset{z\to 0}{Lt}(z-0)f(z)$$

$$= \underset{z\to 0}{Lt}\, z\frac{4-3z}{z(z-1)(z-2)}$$

$$= \underset{z\to 0}{Lt}\frac{4-3z}{(z-1)(z-2)}$$

$$= \frac{4-0}{(0-1)(0-2)} = 2$$

$$\therefore \text{Res } f(0) = 2 ----------------(1)$$

Next, residue of f(z) at the simple pole z = 1 is

$$\text{Res } f(z) = \underset{z\to 1}{Lt}(z-1)f(z)$$

$$= \underset{z\to 1}{Lt}(z-1)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \underset{z \to 1}{Lt} \frac{4-3z}{z(z-2)}$$

$$= \frac{4-3}{(1)(1-2)} = -1$$

$$\therefore \operatorname{Res} f(1) = -1 \text{------------------------} (2)$$

Example 2: Determine the poles of $\dfrac{4-3z}{z(z-1)(z-2)}$, over $C:|z|=\dfrac{3}{2}$ and residue at each pole. Hence evaluate $\displaystyle\int_C \frac{4-3z}{z(z-1)(z-2)}dz$, over $C:|z|=\dfrac{3}{2}$.

Solution: The poles of the integrand and the residues have already been evaluated in example 1. Now by RT we have

$$\int_C \frac{4-3z}{z(z-1)(z-2)}dz, \text{ over } C:|z|=\frac{3}{2}$$

$$=2\pi i\left(\text{sum of resideues at all the poles within } |z|=\frac{3}{2}\right)$$

$$=2\pi i(2-1)\ \ [\text{By eqn (1) and (2)}]$$

$$=2\pi i$$

$$\therefore \int_C \frac{4-3z}{z(z-1)(z-2)}dz=2\pi i \square$$

One can verify the result by comparing the results obtained earlier by CIF.

Example 3:

Find the residues of (a) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z \csc^2 z$ at all its poles in the finite plane.

Solution

(a) $f(z)$ has a double pole at $z = -1$ and simple poles at $z = \pm 2i$.

Residue at $z = -1$ is

$$\lim_{z \to -1} \frac{1}{1!} \frac{d}{dz} \left\{ (z+1)^2 \cdot \frac{z^2 - 2z}{(z+1)^2(z^2+4)} \right\} = \lim_{z \to -1} \frac{(z^2+4)(2z-2) - (z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

Residue at $z = 2i$ is

$$\lim_{z \to 2i} \left\{ (z-2i) \cdot \frac{z^2 - 2z}{(z+1)^2(z-2i)(z+2i)} \right\} = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

Residue at $z = -2i$ is

$$\lim_{z \to -2i} \left\{ (z+2i) \cdot \frac{z^2 - 2z}{(z+1)^2(z-2i)(z+2i)} \right\} = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

(b) $f(z) = e^z \csc^2 z = e^z/\sin^2 z$ has double poles at $z = 0, \pm\pi, \pm 2\pi, \ldots$, i.e., $z = m\pi$ where $m = 0, \pm 1, \pm 2, \ldots$.

Residue at $z = m\pi$ is

$$\lim_{z\to m\pi} \frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2 \frac{e^z}{\sin^2 z}\right\} = \lim_{z\to m\pi} \frac{e^z[(z-m\pi)^2 \sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2 \cos z]}{\sin^3 z}$$

Letting $z - m\pi = u$ or $z = u + m\pi$, this limit can be written

$$\lim_{u\to 0} e^{u+m\pi}\left\{\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u}\right\} = e^{m\pi}\left\{\lim_{u\to 0}\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u}\right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that

$$\lim_{u\to 0}\frac{u^3}{\sin^3 u} = \lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3 = 1$$

and thus write the limit as

$$e^{m\pi}\lim_{u\to 0}\left(\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3}\cdot\frac{u^3}{\sin^3 u}\right) = e^{m\pi}\lim_{u\to 0}\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3} = e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit, we can instead use the series expansions $\sin u = u - u^3/3! + \cdots$, $\cos u = 1 - u^2/2! + \cdots$.

Example 4: Evaluate $\frac{1}{2\pi i}\oint\limits_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$ around the circle C with equation $|z| = 3$.

Solution

The integrand $e^{zt}/\{z^2(z^2+2z+2)\}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$ [roots of $z^2 + 2z + 2 = 0$]. All these poles are inside C.

Residue at $z = 0$ is

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left\{z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to 0}\frac{(z^2+2z+2)(te^{zt}) - (e^{zt})(2z+2)}{(z^2+2z+2)^2} = \frac{t-1}{2}$$

Residue at $z = -1 + i$ is

$$\lim_{z\to -1+i}\left\{[z-(-1+i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\}\lim_{z\to -1+i}\left\{\frac{z+1-i}{z^2+2z+2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1 - i$ is

$$\lim_{z\to -1-i}\left\{[z-(-1-i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \frac{e^{(-1-i)t}}{4}$$

Then, by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i \text{ (sum of residues)} = 2\pi i\left\{\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t\right\}$$

that is,

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t$$

Example 5: Evaluate $\int_0^{\infty} \frac{dx}{x^6+1}$.

Solution

Consider $\oint_C dz/(z^6+1)$, where C is the closed contour of Fig. 7-5 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6+1=0$ when $z=e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles of $1/(z^6+1)$. Only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$, and $e^{5\pi i/6}$ lie within C. Then, using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}} \left\{(z-e^{\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}} \left\{(z-e^{3\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z\to e^{5\pi i/6}} \left\{(z-e^{5\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6}$$

Thus $$\oint_C \frac{dz}{z^6+1} = 2\pi i\left\{\frac{1}{6}e^{-5\pi i/6} + \frac{1}{6}e^{-5\pi i/2} + \frac{1}{6}e^{-25\pi i/6}\right\} = \frac{2\pi}{3}$$

that is, $$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3} \qquad (1)$$

Taking the limit of both sides of (1) as $R \to \infty$ and using Problems 7.7 and 7.8, we have

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \tag{2}$$

Since

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_{0}^{\infty} \frac{dx}{x^6+1}$$

the required integral has the value $\pi/3$.

Example 6: Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}.$

Solution

The poles of $z^2/(z^2+1)^2(z^2+2z+2)$ enclosed by the contour C of Fig. 7-5 are $z = i$ of order 2 and $z = -1+i$ of order 1.

Residue at $z = i$ is

$$\lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$$

Residue at $z = -1+i$ is

$$\lim_{z\to -1+i} (z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$$

Then

$$\oint_C \frac{z^2\, dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or

$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_{\Gamma} \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem **5**. , we obtain the required result.

Example 7: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

Solution

Let $z = e^{i\theta}$. Then $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i = (z - z^{-1})/2i$, $\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + z^{-1})/2$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint_C \frac{dz/iz}{3 - 2(z + z^{-1})/2 + (z - z^{-1})/2i} = \oint_C \frac{2\,dz}{(1-2i)z^2 + 6iz - 1 - 2i}$$

where C is the circle of unit radius with center at the origin (Fig. 7-6).

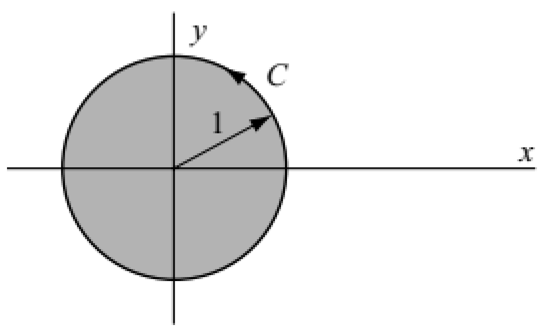

Fig. 7-6

The poles of $2/\{(1-2i)z^2+6iz-1-2i\}$ are the simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)}$$
$$= \frac{-6i \pm 4i}{2(1-2i)} = 2-i,\ (2-i)/5$$

Only $(2-i)/5$ lies inside C.

Residue at

$$(2-i)/5 = \lim_{z\to(2-i)/5} \{z-(2-i)/5\}\left\{\frac{2}{(1-2i)z^2+6iz-1-2i}\right\}$$
$$= \lim_{z\to(2-i)/5} \frac{2}{2(1-2i)z+6i} = \frac{1}{2i}$$

by L'Hospital's rule.

Then

$$\oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right) = \pi,$$

the required value.

Example 8: Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}$.

Solution

Let $z = e^{i\theta}$. Then $\cos\theta = (z+z^{-1})/2$, $\cos 3\theta = (e^{3i\theta}+e^{-3i\theta})/2 = (z^3+z^{-3})/2$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2}\frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}dz$$

where C is the contour of Fig. 7-6.

The integrand has a pole of order 3 at $z=0$ and a simple pole $z=\frac{1}{2}$ inside C.

Residue at $z=0$ is

$$\lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{21}{8}$$

Residue at $z=\frac{1}{2}$ is

$$\lim_{z\to 1/2}\left\{\left(z-\frac{1}{2}\right)\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = -\frac{65}{24}$$

Then

$$-\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8}-\frac{65}{24}\right\} = \frac{\pi}{12} \text{ as required.}$$

SUPPLEMENTARY PROBLEMS

Ex 1: For each of the following functions, determine the poles and the residues at the poles:

$$\text{(a)}\ \frac{2z+1}{z^2-z-2},\quad \text{(b)}\ \left(\frac{z+1}{z-1}\right)^2,\quad \text{(c)}\ \frac{\sin z}{z^2}$$

Ex 2: Find the zeros and poles of $f(z) = \dfrac{z^4+4}{z^3+2z^2+2z}$ and determine the residues at the poles.

Ex 3: Evaluate $\displaystyle\oint_C e^{-1/z}\sin(1/z)\,dz$ where C is the circle $|z| = 1$. essential singularity use w=1/z

Ex 4: Evaluate $\displaystyle\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}dz$ where C is (a) $|z-2i| = 6$, (b) the square with vertices at $1+i$, $2+i$, $2+2i$, $1+2i$.

Ex 5: Evaluate $\displaystyle\oint_C \frac{2+3\sin \pi z}{z(z-1)^2}dz$ where C is a square having vertices at $3+3i$, $3-3i$, $-3+3i$, $-3-3i$.

Ex 6: Evaluate $\displaystyle\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z(z^2+1)}dz$, $t > 0$ around the square with vertices at $2+2i$, $-2+2i$, $-2-2i$, $2-2i$.

Thanks a lot …