Analytic Functions:

Differentiability:

$$w = f(z) \qquad \Delta w = f(z_0 + z) - f(z_0)$$

$$f'(z) = \frac{w(z)}{\Delta z} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

Ex.1: $f(z) = 4x + y + i(4y - x)$

Sol^n:-

$$u = 4x + y$$

$$v = -x + 4y$$

Now,

$$f(z+\delta z) - f(z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y)$$

$$- 4x - y + ix - 4iy$$

$$= 4\delta x + \delta y - i\delta x + 4i\delta y$$

$$f'(z) = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + \delta i y}$$

Ex.2: if $f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ then discuss $\dfrac{df}{dz}$ at $z = 0$

Soln:- if $z \to 0$ along radius vector $y = mx$,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0}\left[\frac{\dfrac{x^3 y(y-ix)}{x^6+y^2}}{x+iy}\right]$$

$$= \lim_{x\to 0}\left[\frac{-ix^3 y(x+iy)}{(x^6+y^2)(x+iy)}\right]$$

$$= \lim_{z\to 0}\left[\frac{-ix^3 y}{x^6+y^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{-ix^4 m}{x^6+m^2x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{-ix^2 m}{x^4+m^2}\right] = 0$$

But along $y = x^3$

$$\lim_{z\to 0}\left[\frac{-ix^3 y}{x^6+y^2}\right]$$

$$= \lim_{z\to 0}\left[\frac{-ix^6}{x^6+x^6}\right]$$

$$= \lim_{x\to 0}\left[\frac{-i}{2}\right]$$

$$= \frac{-i}{2}$$

Since we get different values for different paths, approaching 0. $f(z)$ is not differentiable at $z=0$.

Analytic at a Point;

L'hopital Rule:

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

when $f(z_0)=0$, $g(z_0)=0$ and $g'(z_0)\neq 0$

Ex. 4.

Soln:-

$i^2=-1$ $i^3=-i$ $i^4=1$ $i^5=i$

23. $$\lim_{z \to i} \frac{z^7 + i}{z^{14} + 1}$$

Now

$f(z) = z^7 + i$

$f(i) = -i + i = 0$

$g(z) = z^{14} + 1$

$g(i) = 0$

$f'(z) = 7z^6 \quad f'(i) = -7$

$g'(z) = 14z^{13} \quad g'(i) = 14i$

Thus

$$\lim_{z \to i} \frac{f'(z)}{g'(z)} = \frac{-1}{2i} \quad \text{(Ans)}$$

Cauchy-Riemann Equation:

Theorem: Suppose $f(z) = u(x,y) + iv(x,y)$ is differentiable at point $z = x + iy$.

Then at z, u' and v' exist and satisfy,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

Proof:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

since the limit is assume to exist, Δz can approach from any directions,

However, $f(z) = u(x,y) + iv(x,y)$

$\Delta z = \Delta x + i\Delta y$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - v(x,y)}{\Delta x + i\Delta y} \quad \text{——— (iii)}$$

If from a horizontal direction,

$\Delta y = 0$ and $\Delta z = \Delta x$

then,

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \quad \text{——— (iv)}$$

This implies that each limit exists. Thus both $\delta u/\delta x$ and $\delta v/\delta x$ exist at point z.

$$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \quad \text{——— (v)}$$

We now let $\Delta z \to 0$ along a vertical line. With $\Delta x = 0$ and $\Delta z = i\Delta y$

(iii) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

Thus, we can show

$$f'(z) = -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \text{ ——— (vii)}$$

<u>N.B:</u> Cauchy - Riemann Theorem is used to figure out if a function is differential at a certain point

<u>Ex.5:</u>

Solⁿ:- $f(z) = z^2+z$

$= x^2-y^2+x+i(2xy+y)$

$u(x,y) = x^2-y^2+x$

$v(x,y) = 2xy+iy$

$\frac{\delta u}{\delta x} = 2x+1 = \frac{\delta v}{\delta y}$

$\frac{\delta u}{\delta y} = -2y = -\frac{\delta v}{\delta x}$

(Proved)

Necessary Condition:

(i) $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ (ii) $\frac{\delta u}{\delta y} = \frac{-\delta v}{\delta x}$ provided they exist (4 diff.)

Sufficient Condition:

(i) $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$, $\frac{\delta u}{\delta y} = \frac{-\delta v}{\delta x}$ (ii) they are continuous functions of x and y in Region R

<u>Ex.6:</u> $f(z) = z\bar{z}$

Solⁿ:- $f(z) = z\bar{z} = (x+iy)(x-iy)$

$= x^2-i^2y^2$

$= x^2+y^2 = u+iv$

$u = x^2+y^2$

$v = 0$

At origin,

$$\frac{\delta u}{\delta x} = \lim_{h\to 0} \frac{u(0+h, 0) - u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$\frac{\delta u}{\delta y} = \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$\frac{\delta v}{\delta y} = 0$$

$$\frac{\delta v}{\delta x} = 0$$

Thus, $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ & $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

Hence, C-R is satisfied at the origin

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{(x^2+y^2) - 0}{x+iy}$$

Now, if $y = mx$. and $z \to 0$

$$f'(0) = \lim_{x\to 0} \frac{(x^2+m^2x^2)}{(x+imx)} = \lim_{x\to 0} \frac{(1+m^2)x}{1+im} = 0$$

Thus, $f'(z)$ is differentiable at $z=0$

C-R Equation in Polar form;

$$\frac{\delta u}{\delta r} = \frac{1}{r}\frac{\delta v}{\delta \theta}, \quad \frac{\delta u}{\delta \theta} = -r\frac{\delta v}{\delta r}$$

Ex.8: $\omega = e^z$

Soln:- $\omega = e^z$

$\Rightarrow u+iv = e^{x+iy}$

$= e^x \cdot e^{iy}$

$= e^x(\cos y + i\sin y)$

$u = e^x \cos y$

$v = e^x i\sin y$

$\frac{\delta u}{\delta x} = e^x \cos y \quad \frac{\delta v}{\delta x} = e^x \sin y$

$\frac{\delta u}{\delta y} = -e^x \sin y \quad \frac{\delta v}{\delta y} = e^x \cos y$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$

$\therefore \frac{\delta u}{\delta y} = \frac{-\delta v}{\delta x}$

C-R Equations are satisfied. Also e^x, $\sin y$, $\cos y$ are continuous for all value of x and y.

Thus, e^z is analytic everywhere

$$\frac{dw}{dz} = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$ [this comes from Cauchy-Riemann equation $\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \frac{\delta v}{\delta y} - i\frac{\delta u}{\delta y}$]

$= e^x \cos y + i e^x \sin y$

$= e^x (\cos y + i \sin y)$

$= e^z$ (Ans)

Harmonic Functions:

Ex. 9: $u = x^2 - y^2$, $V = \frac{y}{x^2+y^2}$

Soln:- $\frac{\delta u}{\delta x} = 2x$, $\frac{\delta^2 u}{\delta x^2} = 2$, $\frac{\delta u}{\delta y} = -2y$, $\frac{\delta^2 u}{\delta y^2} = -2$

$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 2-2 = 0$

$u(x,y)$ satisfies Laplace Equation, hence $u(x,y)$ is harmonic

$$v = \frac{y}{x^2+y^2}, \quad \frac{\delta v}{\delta x} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\delta^2 v}{\delta x^2} = \frac{(x^2+y^2)^2(-2y) - (-2xy)\,2(x^2+y^2)\,2x}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y) - (-2xy)4x}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3} \quad \text{---(i)}$$

$$\frac{\delta v}{\delta y} = \frac{(x^2+y^2).1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\delta^2 v}{\delta y^2} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)\,2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(4y)}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3} \quad \text{---(ii)}$$

Now, adding (i) and (ii),

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0$$; thus v is also a harmonic function

But, $\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} \neq \frac{-\delta v}{\delta x}$

Therefore, u and v are not harmonic conjugates.

(Ans)

Milne – Thomson Method:

Definition: $z = x + iy$ $\qquad x = \frac{z+\bar{z}}{2}$

$\bar{z} = x - iy$ $\qquad y = \frac{z-\bar{z}}{2i}$

$f(z) = u(x,y) + iv(x,y)$

$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

Replacing $\bar{z}$ by z, we gets

$$f(z) = u(z,0) + iv(z,0)$$

which can be obtained by replacing x by z and y by 0 in (1)

Construction of Analytic Function:

1. To find Analytic $f(z)$ when $u(x,y)$ is given;

i) Find u_x and u_y

ii) Find u_x and u_y at $(z,0)$

iii) $f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz$

2. When $v(x,y)$ is given (Imaginary part)

i) Find v_x and v_y

ii) Find v_x and v_y at $(z,0)$

iii) $f(z) = \int v_y(z,0)\,dz + i\int v_x(z,0)\,dz$

Ex. 10: $v = e^x(x\cos y - y\sin y)$

Soln:-

$$\frac{\delta v}{\delta x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$= \psi_2(x,y) \text{ ——— (i)}$$

$$\frac{\delta v}{\delta y} = e^x(-x\sin y - y\cos y - \sin y)$$

$$= \psi_1(x,y) \text{ ——— (ii)}$$

$$\frac{\delta^2 v}{\delta x^2} = e^x(x\cos y - y\sin y) + e^x\cos y + e^x\cos y$$

$$= e^x(x\cos y - y\sin y + 2\cos y) \text{ ——— (iii)}$$

$$\frac{\delta^2 v}{\delta y^2} = e^x\left(-x\cos y - (\cos y - \sin y) - \cos y\right)$$

$$= e^x(-x\cos y + y\sin y - 2\cos y) \text{ ——— (iv)}$$

Adding (iii) and (iv),

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0 \Rightarrow v \text{ is a harmonic function}$$

Now putting $x = z$, $y = 0$ in (i) and (ii) we get,

$$\psi_1(z,0) = 0,\ \psi_2(z,0) = ze^z + e^z$$

Hence, by Milne Thomson Method,

$$F(z) = \int[\psi_1(z,0) + i\psi_2(z,0)]\,dz + C$$

$$= \int[0 + i(ze^z + e^z)]\,dz + C = ize^z + C$$; This is the required Analytic Function

[5] Determination of Conjugate Harmonic;

Ex.11:

Soln:-

(a) If $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

$u_x = 6xy + 4x$

$u_{xx} = 6y + 4$

$u_y = 3x^2 - 3y^2 - 4y$

$u_{yy} = -6y - 4$

$\nabla^2 u = u_{xx} + u_{yy}$

$= 6y + 4 + (-6y - 4)$

$= 0$

Hence, u is harmonic.

Now, by Cauchy-Riemann equations;

$u_x = v_y$; $v_x = -u_y$

However, $u_x = v_y = 6xy + 4x$

Thus,

$v = \int (6xy + 4x)\,dy = 3xy^2 + 4xy + g(x)$

$v_x = 3y^2 + 4y + g'(x)$

Since, $v_x = -u_y$

$3y^2 + 4y + g'(x) = -3x^2 + 3y^2 + 4y$

$g'(x) = -3x^2$

$g(x) = -x^3$

Therefore, the harmonic conjugate is $v(x,y) = 3xy^2 + 4xy - x^3$

(b) $v(x,y) = \ln(x^2+y^2)$

$v_x = \dfrac{2x}{x^2+y^2}$

$v_{xx} = \dfrac{(x^2+y^2).2 - 2x.2x}{(x^2+y^2)^2}$

$= \dfrac{2x^2 + 2y^2 - 4x^2}{(x^2+y^2)^2}$

$= \dfrac{2y^2 - x^2}{(x^2+y^2)^2}$

$v_y = \dfrac{2y}{x^2+y^2}$

$v_{yy} = \dfrac{(x^2+y^2).2y - 2y.2y}{(x^2+y^2)^2}$

$= \dfrac{2x^2 - 2y^2}{(x^2+y^2)^2}$

Now,

$$\nabla^2 v = \frac{0}{(x^2+y^2)^2} = 0 \text{ ; Thus } v \text{ is harmonic function}$$

Now, to find the conjugate Harmonic,

$u_x = v_y$, $u_y = -v_x$

Thus, $u_x = \dfrac{2y}{x^2+y^2}$

$$u = \int \left[\frac{2y}{x^2+y^2}\right] dx$$

$$= 2y \int \frac{1}{x^2+y^2}\, dx$$

$$= \frac{2y}{y}\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= 2\tan^{-1}\left(\frac{x}{y}\right) + c$$

$$= 2\tan^{-1}\left(\frac{x}{y}\right) + g(y)$$

Now, $u_y = 2\left[\dfrac{1}{1+(x/y)^2} \cdot \dfrac{\delta}{\delta y}\left(\dfrac{x}{y}\right)\right] + g'(y)$

$$= 2\left[\frac{1}{1+(x/y)^2} \cdot \frac{-x}{y^2}\right] + g'(y)$$

$$= 2\left[\frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{y^2}\right] + g'(y)$$

$$= 2\left[\frac{y^2}{x^2+y^2} \cdot \frac{-x}{y^2}\right] + g'(y)$$

$$= \frac{-2x}{x^2+y^2} + g'(y)$$

However, $u_y = -v_x$

$$\Rightarrow \frac{-2x}{x^2+y^2} + g'(y) = \frac{-2x}{x^2+y^2}$$

$$g'(y) = 0$$

$$g(y) = 0$$

Thus, $u = 2\tan^{-1}\left(\dfrac{x}{y}\right) + c$ (Ans)