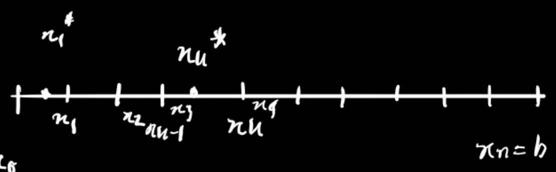
4 Complex Integration:

Definite Integral:

$\|P\|$ = norm of the partition i.e. length of the longest subinterval

The definite integral of f on $[a,b]$ is,

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$



Line Integral:

The line integral of G along C with respect to x,

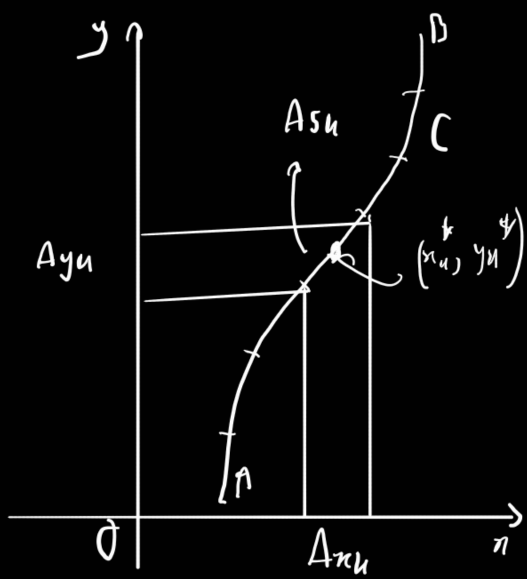
$$\int_C G(x,y)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta x_k$$

w.r.to y,

$$\int_C G(x,y)\,dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta y_k$$

w.r.to arc length s,

$$\int_C G(x,y)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta s_k$$



Ex.1: $x = 4\cos t$, $y = 4\sin t$ where, $0 \le t \le \pi/2$

soln:- $dx = -4\sin t\,dt$

$dy = 4\cos t\,dt$

(a) $$\int_C xy^2\,dx = \int_0^{\pi/2} (4\cos t)(4\sin t)^2(-4\sin t\,dt)$$

$$= -256\int_0^{\pi/2} \sin^3 t\cos t\,dt = -256\left[\frac{1}{4}\sin^4 t\right] = -64$$

[STOOS]

(b) $$\int_C xy^2\,dy = \int_0^{\pi/2} (4\cos t)(4\sin t)^2(4\cos t\,dt)$$

$$= 256\int_0^{\pi/2} \sin^2 t\cos^2 t\,dt$$

$$= 256\int_0^{\pi/2} \frac{1}{4}\sin^2 2t\,dt = 64\int_0 \frac{1}{2}(1-\cos 4t)\,dt = 32\left[t - \frac{1}{4}\sin 4t\right]_0^{\pi/2} = 16\pi$$

(c) $$ds = \sqrt{16(\sin^2 t+\cos^2 t)}\;dt$$

$$\int_C xy^2\,ds = \int_0^{\pi/2} (4\cos t)(4\sin t)^2\,4\,dt$$

$$= 256\int_0^{\pi/2} \sin^2 t\cos t\,dt = 256\left[\frac{1}{3}\sin^3 t\right]_0^{\pi/2} = \frac{256}{3}$$

Contour Integral:

if f is continuous on a smooth curve C given by the parametrization

$z(t) = x(t) + iy(t)$, $a \le t \le b$, then

$$\boxed{\int_C f(z)\,dz = \int f(z(t))\,z'(t)\,dt}$$

Ex. 1:

Here, $f(z) = \bar{z}$

$= \overline{3t + it^2}$

$= 3t - it^2$

However,

$z = 3t + it^2$

thus, $dz = (3t + 2it)\,dt$

$$\int_C \bar{z}\,dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt$$

$$= \int_{-1}^{4} [2t^3 + 9t + 3t^2 i]\,dt$$

$$= \int_C \bar{z}\,dz = \int_{-1}^{4} (2t^3 + 9t)\,dt + i\int_{-1}^{4} 3t^2\,dt$$

$$= \left(\frac{1}{2}t^4 + \frac{9}{2}t^2\right)\Bigg|_{-1}^{4} + it^3\Bigg|_{-1}^{4} = 195 + 65i$$

(Ans)

Ex. 2: $\oint_C \frac{1}{z}\,dz$, $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

Soln:-

$z(t) = \cos t + i\sin t = e^{it}$

$z'(t) = ie^{it}$

$f(z(t)) = \frac{1}{z(t)} = e^{-it}$

$$\oint_C \frac{1}{z}\,dz$$

$$= \int_0^{2\pi} (e^{-it})\,ie^{it}\,dt$$

$$= i\int_0^{2\pi} dt = 2\pi i$$

(Ans)

Properties of Contour Integrals:

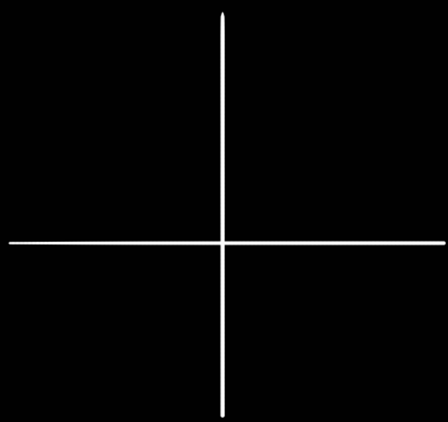
(i) $\int_C k f(z)\,dz = k\int_C f(z)\,dz$, k a complex constant

(ii) $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ where C consists of smooth curve C_1 and C_2 joined end to end,

(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having opposite orientation of C.

Ex. 3: $\int_C (x^2 + iy^2)\,dz$, where C is the contour



Soln:-

$$\int_C (x^2+iy^2)\,dz = \int_{C_1}(x^2+iy^2)\,dz + \int_{C_2}(x^2+iy^2)\,dz$$

since C_1 is defined by $y = x$

$z(x) = x + iy = x + ix$

$z'(x) = 1+i$

$f(z) = x^2 + ix^2$

$$\int_C (x^2+iy^2)\,dz = \int_0^1 (x^2+ix^2)(1+i)\,dx = (1+i)^2\int_0^1 x^2\,dx = \frac{(1+i)^2}{3} = \frac{2}{3}i, \quad \text{——(1)}$$

Again, C_2 is defined by $x = 1$, $1 \le y \le 2$

if we use y as a parameter, then $z(y) = 1+iy$, $z'(y) = i$, $f(z) =$
$f(z(y)) = 1 + iy^2$

$$\int_{C_2} (x^2+iy^2)\,dz = \int_1^2 (1+iy^2)\,i\,dy = -\int_1^2 y^2\,dy + i\int_1^2 dy = \frac{-7}{3} + i. \quad \text{——(ii)}$$

combining (i) and (ii),

$$\frac{2}{3}i + \left(\frac{-7}{3} + i\right) = \frac{5}{3}i - \frac{7}{3} \quad \text{(Ans)}$$

Self-Study Practice to be done later.

4) Cauchy Integral Theorem and Cauchy Integral Formula:

Cauchy's Theorem: suppose f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simply closed contour in D

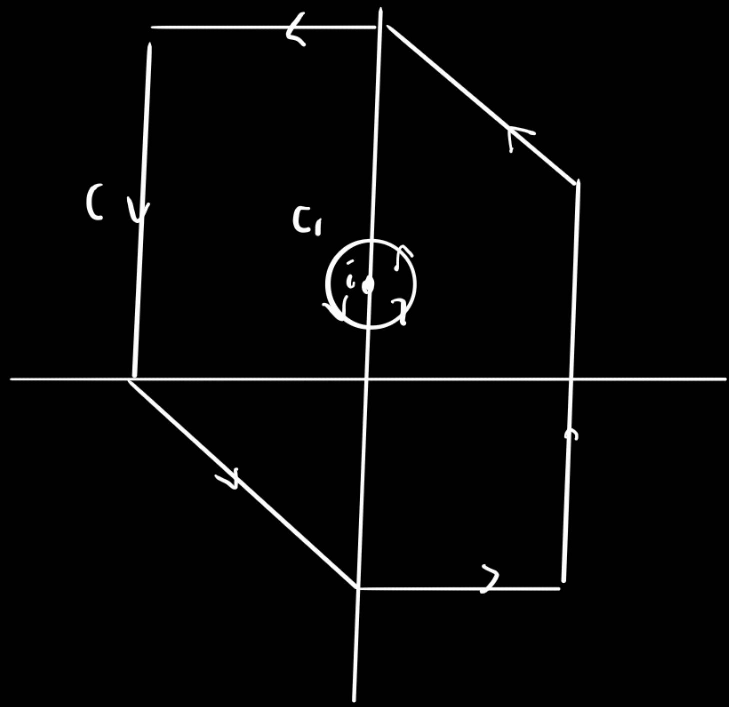
$$\oint_C f(z)\,dz = 0$$

Cauchy–Goursat Theor . Doesn't need to prove f' is continuous

CGT for Multiply Connected Domains:

Ex. 2: $\oint_C \frac{dz}{z-i}$, evaluate

Soln: - We draw C_1, which lies within C



C_1, $|z-i| = 1$

Parametrization: e^{it}, $0 \le t \le 2\pi$

$e^{it} = \cos t + i\sin t$

and $|e^{it}| = 1$

So, $z - i = e^{it}$

$z = i + e^{it}$, $0 \le t \le 2\pi$

Now, $z(t) = i + e^{it}$

$$\frac{dz}{dt} = ie^{it} \Rightarrow dz = ie^{it}\,dt$$

Now, the integral becomes,

$$\oint_{C_1} \frac{dz}{z-i} = \oint_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

N.B: Its easier to compute $|z-i| = 1$ for CGT

Theorem: Suppose there's $C, C_1, C_2, \dots C_n$. If f is analytical on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, 3, \dots, n$, then,

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

$$z^2 + 1 = z^2 - i^2$$

Ex. 3: $\oint_C \frac{dz}{z^2+1}$ where C is the circle $|z| = 4$

soln:- Here, we can say,

$$z^2+1 = (z+i)(z-i)$$

Then consequently the function is not analytical at $z=i$ and $z=-i$

both lie within the contour,

Using partial decomposition once more,

$$\frac{1}{z^2+1} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$\therefore \oint_C \frac{dz}{z^2+1} = \oint_{C_1} \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] + \oint_{C_2} \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

However in C_1,

$\frac{1}{z-i}$ has singularity near i, thus $\oint_{C_1} \frac{1}{z-i} dz = 2\pi i$ [from example 2]

but $\frac{1}{z+i}$ doesn't contain $-i$ near C_1, Thus $\oint_{C_2} \frac{1}{z+i} dz = 0$

Similarly for C_2,

$\frac{1}{z+i}$ has singularity near C_2 $\therefore \oint \frac{1}{z+i} dz = 0$

$$\oint \frac{1}{z-i} dz = 2\pi i$$

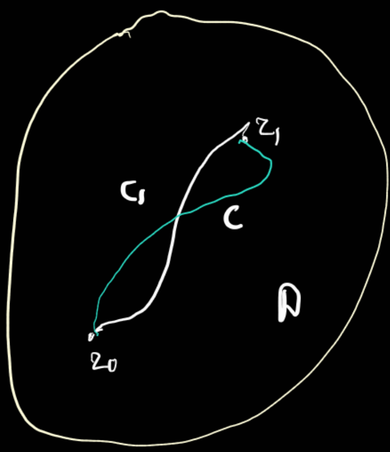
Thus,

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\left[\oint_{C_1} \frac{1}{z-i} dz - \oint_{C_2} \frac{1}{z+i} dz\right]$$

$$= \frac{1}{2i}\left[2\pi i - 2\pi i\right] = 0 \quad \text{(Ans)}$$

Independence of the path;

A contour integral $\oint_C f(z)\,dz$ is said to be independent of the path if its value is same for all contours C in D with initial point z_0 and terminal point z_1



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

Cauchy's Integral Formula:

1st Theorem: Suppose f is analytic in a simple connected domain D and C is any simple closed contour lying entirely in D,

Then for any point z_0 within C,

$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz}$$

- $f(z)$ is analytic inside and on C
- z_0 is inside C

Ex. 4: $\oint_C \dfrac{z^2-4z+4}{z+i}\,dz$, where C is the circle $|z|=2$

soln:- we know, Cauchy Integral Formula

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}\,dz$$

Thus $f(z) = z^2-4z+4$

$z_0 = -i$

$\therefore$ Check $|-i| = 1 < 2$, thus singularity is inside the circle,

and $f(z)$ is a polynomial $\rightarrow$ analytic everywhere.

Now Applying the formulas

$$\oint \frac{f(z)}{z-z_0}\,dz = f(z_0).2\pi i = (-1+4i+4).2\pi i = (3+4i).2\pi i$$

2nd Theorem: Suppose f is analytic in a simply connected domain D and C is any simple closed contour entirely inside D. Then for any point z_0 within C,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

C is

Ex, 5: $\oint_C \frac{z+1}{z^4+2iz^3}\,dz$, where $|z|=1$

Soln: The integrand is not analytic at $z=0$ and $z=-2i$

But only $z=0$ lies in the closed contour,

Rewriting the integral,

$$\frac{z+1}{z^4+2iz^3} = \frac{z+1}{z^3(z+2i)} = \frac{1}{z^3}\cdot\frac{z+1}{z+2i} = \frac{f(z)}{z^3} \qquad f(z) = \frac{z+1}{z+2i}$$

since we have $z=0$ as singularity inside the contour,

$$\oint_C \frac{F(z)}{(z-z_0)^{n+1}}\,dz = \int \frac{f(z)}{(z-0)^3}\,dz = \frac{2\pi i}{2!}f''(0)$$

Now, $f(z) = \frac{z+1}{z+2i}$

$$f'(z) = \frac{(z+2i)\frac{d}{dz}(z+1)-(z+1)\frac{d}{dz}(z+2i)}{(z+2i)^2} \quad \therefore f''(z) = \frac{2(-4i)}{(z+2i)^3}$$

So, at $z=0$

$$f''(0) = \frac{2\cdot(-4)}{8\cdot(-1)} = 1$$

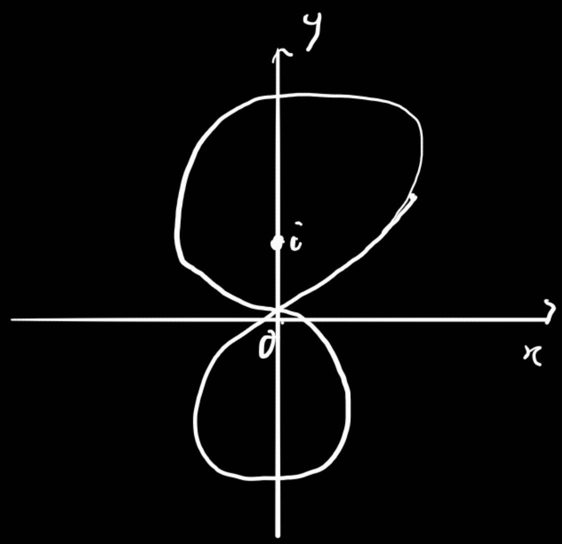
However in slide $f''(0) = \frac{(2i-1)}{4i}$

$$\therefore \oint \frac{z+1}{z^4+2iz^3}\,dz = \frac{2\pi i}{2!} f''(0) = -\frac{\pi}{4} + \frac{\pi i}{2}$$

Ex.7:

We can think of C as C_1 and C_2

C_1 flows clockwise, so we take $-C_1$ for positive orientation.



$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$= -\oint_{-C_1} \frac{(z^3+3)/(z-i)^2}{z}\,dz + \oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2}\,dz = -I_1 + I_2$$

To evaluate I_1, we identify $z_0=0$ $f(z) = \frac{(z^3+3)}{(z-i)^2}$ and $f(0) = -3$

$$I_1 = \oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}\,dz = 2\pi i f(-3) = -6\pi i$$

To evaluate I_2, we now identify $z_0 = i$, $n=1$, $f(z) = \frac{z^3+3}{z}$

$f'(z) = 2z - \frac{3}{z^2}$ and $f'(i) = 3+2i$.

$$I_2 = \int_{C2} \frac{z}{(z-i)^2}\,dz = \frac{2\pi i}{1!} f(i)$$

$$= 2\pi i\,(3+2i) = -4\pi + 6\pi i$$

Finally, we get,

$$\oint_C \frac{z^3+3}{z(z-i)^3} = -I_1 + I_2 = 6\pi + (-4\pi + 6\pi i) = -4\pi + 12\pi i$$

(Ans)