

13:6-Directional Derivatives and Gradients

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Objectives

- Find and use directional derivatives of a function of two variables.
- Find the gradient of a function of two variables.
- Use the gradient of a function of two variables in applications.
- Find directional derivatives and gradients of functions of three variables.

Directional Derivative

- You are standing on the hillside represented by $z = f(x, y)$ in Figure 13.42 and want to determine the hill's incline toward the z -axis. You already know how to determine the slopes in two different directions—the slope in the y -direction is given by the partial derivative $f_y(x, y)$, and the slope in the x -direction is given by the partial derivative $f_x(x, y)$.
- In this section, you will see that these two partial derivatives can be used to find the slope in *any* direction.

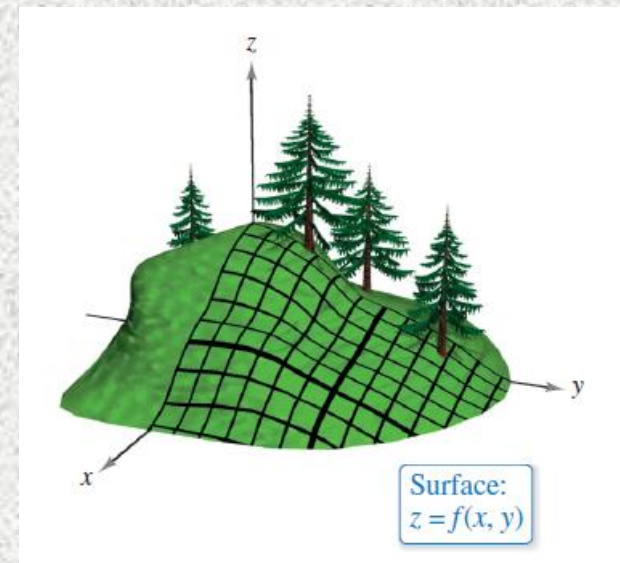


Figure 13.42

Directional Derivative

- To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative**.
- Begin by letting $z = f(x, y)$ be a *surface* and $P(x_0, y_0)$ be a *point* in the domain of f , as shown in Figure 13.43.
- The “direction” of the directional derivative is given by a unit vector $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ where θ is the angle the vector makes with the positive x -axis.

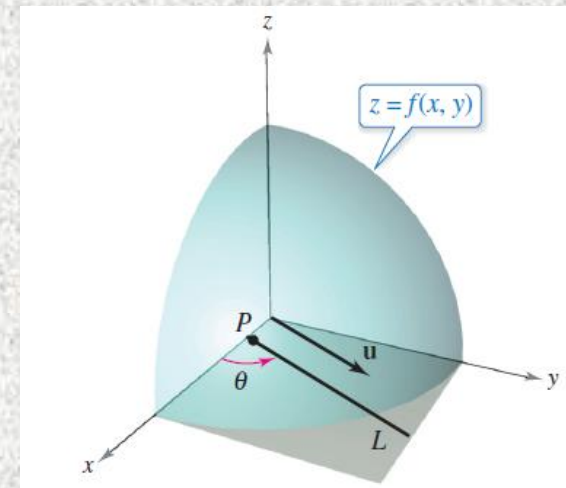


Figure 13.43

Directional Derivative

- To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to \mathbf{u} , as shown in Figure 13.44.
- This vertical plane intersects the surface to form a curve C .
- The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of \mathbf{u} is defined as the slope of the curve C at that point.

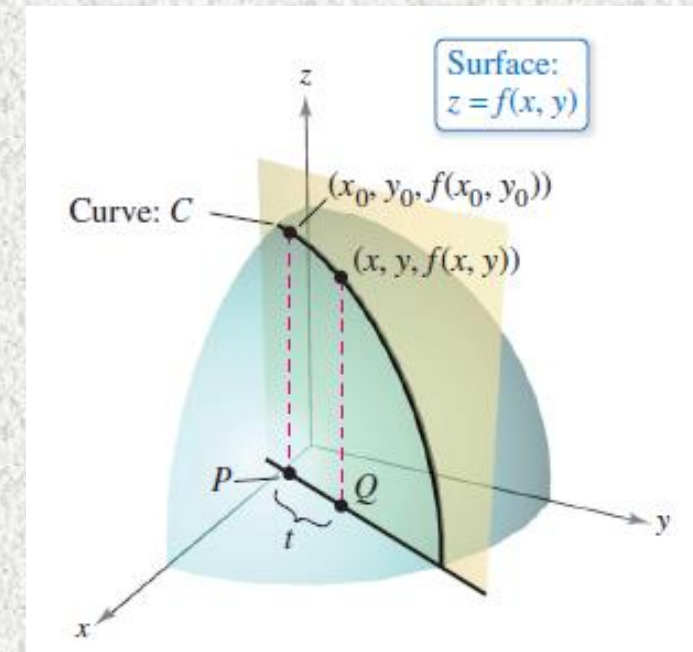


Figure 13.44

Directional Derivative

- Informally, you can write the slope of the curve C as a limit that looks much like those used in single-variable calculus.
- The vertical plane used to form C intersects the xy -plane in a line L , represented by the parametric equations
$$x = x_0 + t \cos \theta \quad \text{and} \quad y = y_0 + t \sin \theta$$
so that for any value of t , the point $Q(x, y)$ lies on the line L .
- For each of the points P and Q , there is a corresponding point on the surface.

$$(x_0, y_0, f(x_0, y_0))$$

$$(x, y, f(x, y))$$

Point above P

Point above Q

Directional Derivative

- ✓ Moreover, because the distance between P and Q is

$$\begin{aligned}\sqrt{(x - x_0)^2 + (y - y_0)^2} &= \sqrt{(t \cos \theta)^2 + (t \sin \theta)^2} \\ &= |t|\end{aligned}$$

- ✓ you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t \cos \theta, y_0 + t \sin \theta) - f(x_0, y_0)}{t}.$$

- ✓ Finally, by letting t approach 0, you arrive at the following definition.

Directional Derivative

Definition of Directional Derivative

Let f be a function of two variables x and y and let $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of \mathbf{u}** , denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

provided this limit exists.

- Calculating directional derivatives by this definition is similar to finding the derivative of a function of one variable by the limit process. A simpler formula for finding directional derivatives involves the partial derivatives f_x and f_y .

Directional Derivative

THEOREM 13.9 Directional Derivative

If f is a differentiable function of x and y , then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

- There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by \mathbf{u} , as shown in Figure 13.45.

PROOF For a fixed point (x_0, y_0) , let $x = x_0 + t \cos \theta$ and let $y = y_0 + t \sin \theta$. Then, let $g(t) = f(x, y)$. Because f is differentiable, you can apply the Chain Rule given in Theorem 13.6 to obtain

$$g'(t) = f_x(x, y)x'(t) + f_y(x, y)y'(t) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0) \cos \theta + f_y(x_0, y_0) \sin \theta.$$

By the definition of $g'(t)$, it is also true that

$$\begin{aligned} g'(0) &= \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(x_0 + t \cos \theta, y_0 + t \sin \theta) - f(x_0, y_0)}{t}. \end{aligned}$$

Consequently, $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0) \cos \theta + f_y(x_0, y_0) \sin \theta$.

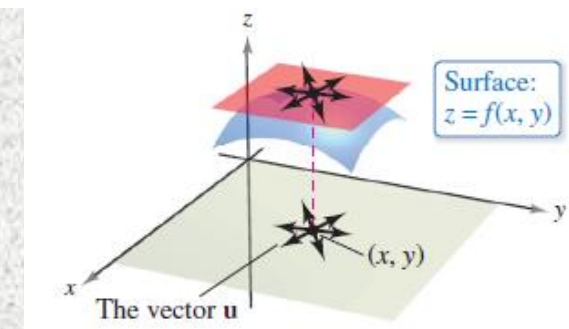


Figure 13.45

Directional Derivative

- Two of these are the partial derivatives f_x and f_y .

1. Direction of positive x -axis ($\theta = 0$): $\mathbf{u} = \cos 0 \mathbf{i} + \sin 0 \mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}}f(x, y) = f_x(x, y) \cos 0 + f_y(x, y) \sin 0 = f_x(x, y)$$

2. Direction of positive y -axis ($\theta = \pi/2$): $\mathbf{u} = \cos \frac{\pi}{2} \mathbf{i} + \sin \frac{\pi}{2} \mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}}f(x, y) = f_x(x, y) \cos \frac{\pi}{2} + f_y(x, y) \sin \frac{\pi}{2} = f_y(x, y)$$

Example 1 – Finding a Directional Derivative

- Find the directional derivative of

$$f(x, y) = 4 - x^2 - \frac{1}{4}y^2$$

Surface

at $(1, 2)$ in the direction of

$$\mathbf{u} = \left(\cos \frac{\pi}{3}\right)\mathbf{i} + \left(\sin \frac{\pi}{3}\right)\mathbf{j}.$$

Direction

- Solution:**

Because $f_x(x, y) = -2x$ and $f_y(x, y) = -y/2$ are continuous, f is differentiable, and you can apply Theorem 13.9.

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

Example 1 – Solution

$$= (-2x) \cos \theta + \left(-\frac{y}{2}\right) \sin \theta$$

Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$D_{\mathbf{u}}f(1, 2) = (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right)$$

$$= -1 - \frac{\sqrt{3}}{2}$$

$$\approx -1.866.$$

See Figure 13.46.

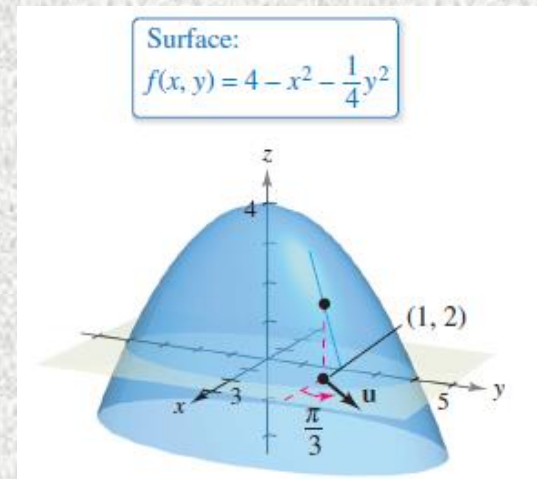


Figure 13.46

The Gradient of a Function of Two Variables

- The **gradient** of a function of two variables is a vector-valued function of two variables.

Definition of Gradient of a Function of Two Variables

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f** , denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

(The symbol ∇f is read as “del f .”) Another notation for the gradient is given by **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y) , the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

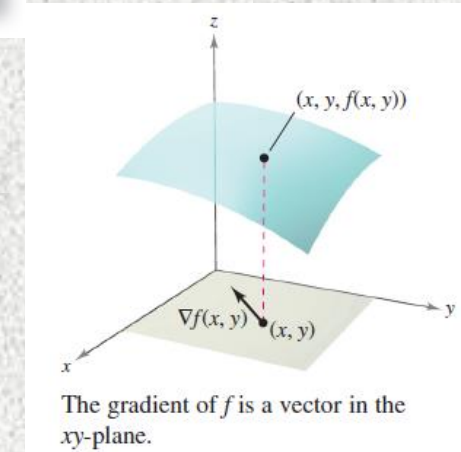


Figure 13.48

Example 3 – Finding the Gradient of a Function

- Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point $(1, 2)$.

- Solution:**

Using $f_x(x, y) = \frac{y}{x} + y^2$ and $f_y(x, y) = \ln x + 2xy$ you have

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

$$= \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}.$$

- At the point $(1, 2)$, the gradient is

$$\nabla f(1, 2) = \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j}$$

$$= 6\mathbf{i} + 4\mathbf{j}.$$

The Gradient of a Function of Two Variables

- Because the gradient of f is a vector, you can write the directional derivative of f in the direction of \mathbf{u} as

$$D_{\mathbf{u}}f(x, y) = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot [\cos \theta\mathbf{i} + \sin \theta\mathbf{j}].$$

- In other words, the directional derivative is the dot product of the gradient and the direction vector.

THEOREM 13.10 Alternative Form of the Directional Derivative

If f is a differentiable function of x and y , then the directional derivative of f in the direction of the unit vector \mathbf{u} is

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Example 4 – Using $\nabla f(x, y)$ to Find a Directional Derivative

- Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $(-\frac{3}{4}, 0)$ in the direction from $P(-\frac{3}{4}, 0)$ to $Q(0, 1)$.

- Solution:**

Because the partials of f are continuous, f is differentiable and you can apply Theorem 13.10.

- ✓ A vector in the specified direction is

$$\overrightarrow{PQ} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j}$$

$$= \frac{3}{4}\mathbf{i} + \mathbf{j}$$

Example 4 – *Solution*

cont'd

✓ And a unit vector in this direction is

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

Unit vector in direction of \overrightarrow{PQ}

✓ Because $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 6x\mathbf{i} - 4y\mathbf{j}$, the gradient at $(-\frac{3}{4}, 0)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\mathbf{i} + 0\mathbf{j}.$$

Gradient at $(-\frac{3}{4}, 0)$

Example 4 – Solution

cont'd

✓ Consequently, at $\left(-\frac{3}{4}, 0\right)$, the directional derivative is

$$D_{\mathbf{u}}f\left(-\frac{3}{4}, 0\right) = \nabla f\left(-\frac{3}{4}, 0\right) \cdot \mathbf{u}$$

$$= \left(-\frac{9}{2}\mathbf{i} + 0\mathbf{j}\right) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$$

$$= -\frac{27}{10}.$$

Directional derivative at $\left(-\frac{3}{4}, 0\right)$

✓ See Figure 13.49.

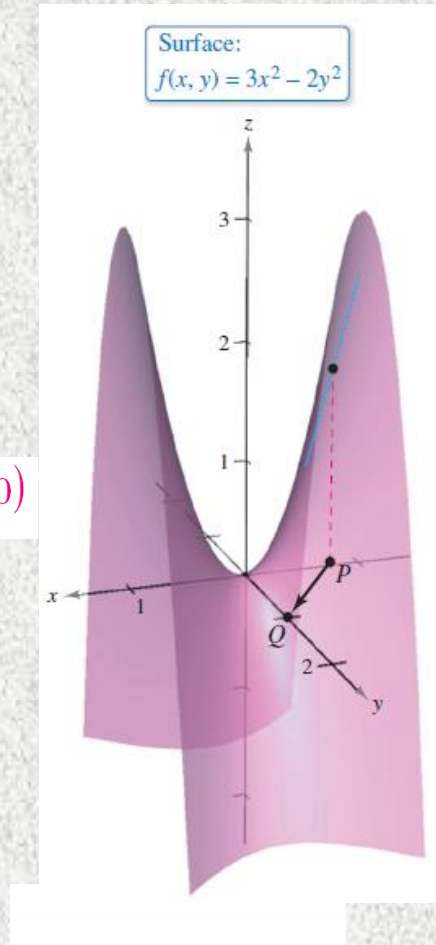


Figure 13.49

Applications of the Gradient

THEOREM 13.11 Properties of the Gradient

Let f be differentiable at the point (x, y) .

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y) = 0$ for all \mathbf{u} .
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}}f(x, y)$ is

$$\|\nabla f(x, y)\|. \quad \text{Maximum value of } D_{\mathbf{u}}f(x, y)$$

3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}}f(x, y)$ is

$$-\|\nabla f(x, y)\|. \quad \text{Minimum value of } D_{\mathbf{u}}f(x, y)$$

Example 5 – *Finding the Direction of Maximum Increase*

- The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

- ✓ where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

- **Solution:**

The gradient is

$$\begin{aligned}\nabla T(x, y) &= T_x(x, y)\mathbf{i} + T_y(x, y)\mathbf{j} \\ &= -8x\mathbf{i} - 2y\mathbf{j}.\end{aligned}$$

Example 5 – Solution

- It follows that the direction of maximum increase is given by $\nabla T(2, -3) = -16\mathbf{i} + 6\mathbf{j}$ as shown in Figure 13.51, and the rate of increase is

$$\begin{aligned}\|\nabla T(2, -3)\| &= \sqrt{256 + 36} \\ &= \sqrt{292} \\ &\approx 17.09^\circ \text{ per centimeter.}\end{aligned}$$

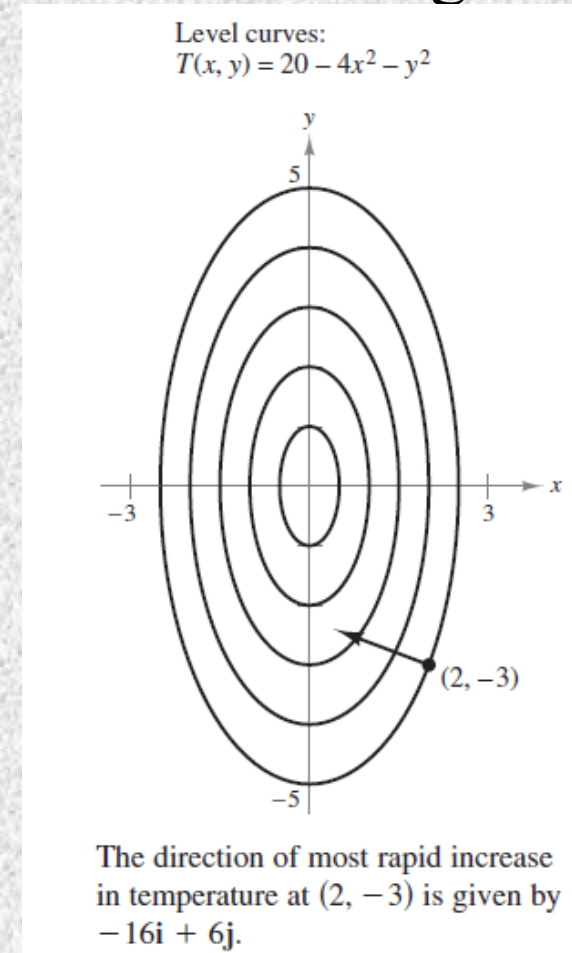


Figure 13.51

Applications of the Gradient

THEOREM 13.12 Gradient Is Normal to Level Curves

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0) .

Example 7 – Finding a Normal Vector to a Level Curve

- Sketch the level curve corresponding to $c = 0$ for the function given by $f(x, y) = y - \sin x$ and find a normal vector at several points on the curve.

- **Solution:**

The level curve for $c = 0$ is given by

$$0 = y - \sin x \Rightarrow y = \sin x$$

as shown in Figure 13.53(a).

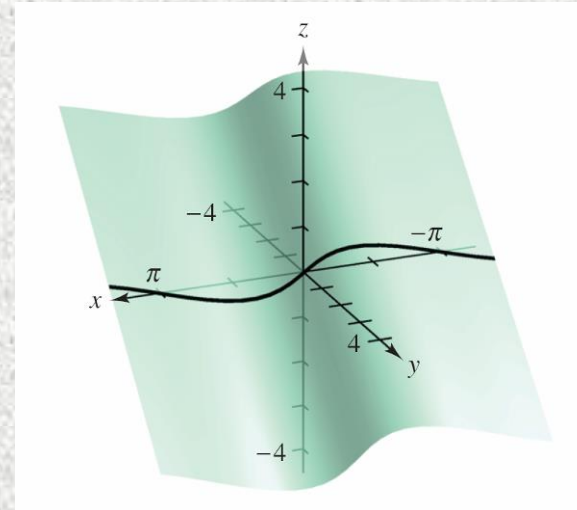


Figure 13.53(a)

Example 7 – Solution

cont'd

- ✓ Because the gradient of f at (x, y) is

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= -\cos x\mathbf{i} + \mathbf{j}\end{aligned}$$

- ✓ you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point (x, y) .
- ✓ Some gradient are

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j}$$

Example 7 – Solution

cont'd

$$\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j}$$

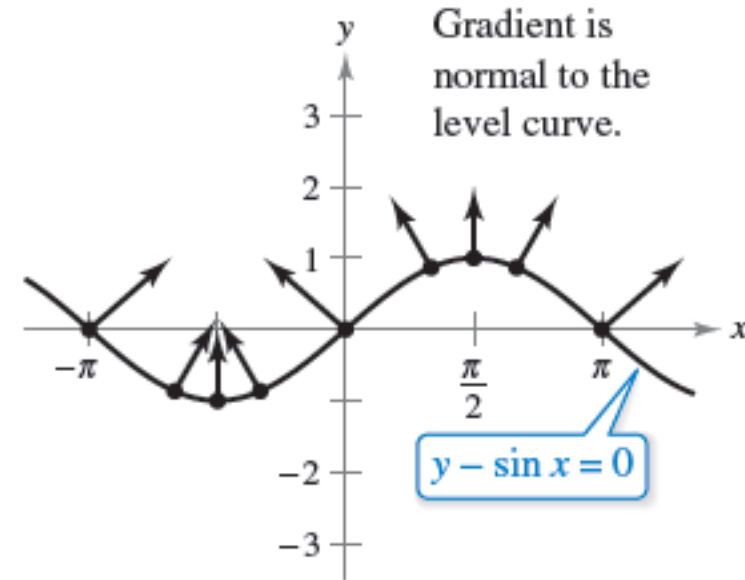
$$\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$

$$\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j}.$$

$$\nabla f\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

and

$$\nabla f(\pi, 0) = \mathbf{i} + \mathbf{j}.$$



The level curve is given by $f(x, y) = 0$.

Figure 13.53(b)

These are shown in Figure 13.53(b).

Functions of Three Variables

Directional Derivative and Gradient for Three Variables

Let f be a function of x , y , and z with continuous first partial derivatives. The **directional derivative** of f in the direction of a unit vector

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

is given by

$$D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient** of f is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y, z) = 0$ for all \mathbf{u} .
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$\|\nabla f(x, y, z)\|.$$

Maximum value of $D_{\mathbf{u}}f(x, y, z)$

4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$-\|\nabla f(x, y, z)\|.$$

Minimum value of $D_{\mathbf{u}}f(x, y, z)$

Example 8 – *Finding the Gradient of a Function*

- Find $\nabla f(x, y, z)$ for the function

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.

- Solution:**

The gradient is

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

$$= 2x\mathbf{i} + 2y\mathbf{j} - 4\mathbf{k}.$$

Example 8 – Solution

cont'd

✓ So, it follows that the direction of maximum increase at $(2, -1, 1)$ is

$$\nabla f(2, -1, 1) = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}.$$

See Figure 13.54.

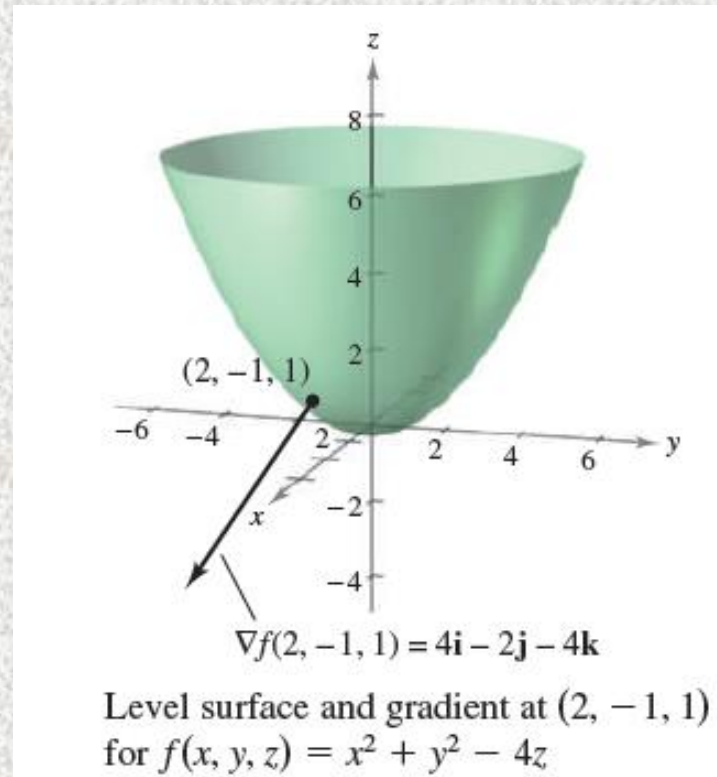


Figure 13.54

Suggested Problems

Exercise 13.6:5,8,12,18,23,28,35.

Thanks a lot ...