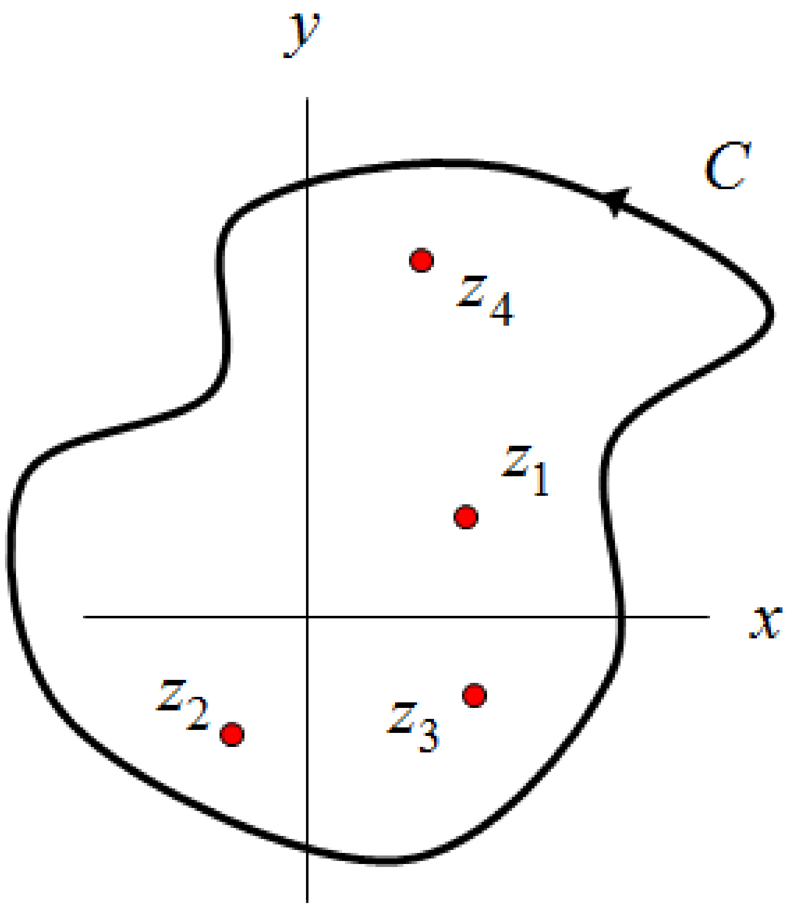


SINGULARITIES, POWER SERIES AND RESIDUE THEOREM

ftp://10.220.20.26/*academic*/mce/maka

Singularity

In Mathematics, a singularity is in general a point at which a given mathematical object is not defined, or a point of exceptional set where it fails to be well- behaved in some particular way, such as differentiability.

For Example: the function $f(x) = \frac{1}{x}$ *on the real line has a singularity at* $x = 0$*, where it is not defined.*

But if $f(x) = |x|$ *also has a singularity at* $x = 0$*, since it is not differentiable at* $x = 0$.

Definition: A point z_s is a singular point of the function $f(z)$ if the function is not analytic at z_s.

(*The function does not necessarily have to be infinite there.*)

Poles or unessential singularities

A pole is a point in the complex plane at which the value of a function becomes infinite.

For example, *$w = 1/z$ is infinite at $z = 0$, and we say that the function $w = 1/z$ has a pole at the origin.*

A pole has an "order":

The pole in $w = 1/z$ is first order.

The pole in $w = 1/Z^2$ is second order.

Removable singularities

If $w = f(z)$ becomes infinite at the point $z = a$,
we define: $g(z) = (z-a)^n f(z)$ where n is an integer.

If it is possible to find a finite value of n which makes $g(z)$ analytic at $z = a$, then, the pole of $f(z)$ has been "removed" in forming $g(z)$ and is called removable singularity.

Essential singularities

Certain functions of complex variables have an infinite number of terms which all approach infinity as the complex variable approaches a specific value. These could be thought of as poles of infinite order, but as the singularity cannot be removed by multiplying the function by a finite factor, they cannot be poles.

Essential singularities

This type of singularity is called an essential singularity and is portrayed by functions which can be expanded in a descending power series of the variable.

Example: e=1/z has an essential singularity at z = 0.

Essential singularities can be distinguished from poles by the fact that they cannot be removed by multiplying by a factor of finite value.

Example:

$$\boxed{w = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n} + \ldots} \longrightarrow \text{infinite at the origin}$$

Try to remove the singularity of the function at the origin by multiplying z_p

$$\boxed{z^p w = z^p e^{\frac{1}{z}} = z^p + z^{p-1} + \frac{z^{p-2}}{2!} + \ldots + \frac{z^{p-n}}{n!} + \ldots}$$

Essential singularities

$$\boxed{w = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n} + \ldots} \longrightarrow$$ infinite at the origin

Try to remove the singularity of the function at the origin by multiplying z_p

$$\boxed{z^p w = z^p e^{\frac{1}{z}} = z^p + z^{p-1} + \frac{z^{p-2}}{2!} + \ldots + \frac{z^{p-n}}{n!} + \ldots}$$

It consists of a finite number of positive powers of z, followed by an infinite number of negative powers of z.

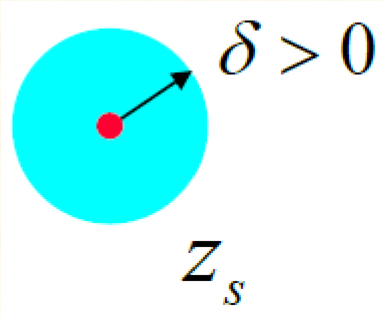
As $\boxed{z \to 0,\ z^p w \to \infty}$

It is impossible to find a finite value of p which will remove the singularity in $e^{1/z}$ at the origin.

The singularity is “essential”.

Isolated Singularity

The function is singular at z_s but is analytic for $0 < |z - z_s| < \delta.$



Examples:

$$\boxed{\frac{\sin z}{z}, \frac{1}{z}, e^{\frac{1}{z}}, \frac{1}{\sin z} \quad at \ z = 0}$$

Classification of Isolated Singularities

Isolated singularities

Removable singularities

$$\boxed{\frac{\sin z}{z}, \frac{1-\cos z}{z^2}}$$

Poles of finite order

$$\boxed{\frac{1}{z}, \frac{1}{z^2}, \frac{1}{(z-1)^m}, \frac{2z+3}{(z-1)^2(z+2)}}$$

Essential singularities (poles of infinite order)

$$\boxed{\sin\left(\frac{1}{z}\right), e^{\frac{1}{z}}}$$

Non-Isolated Singularity

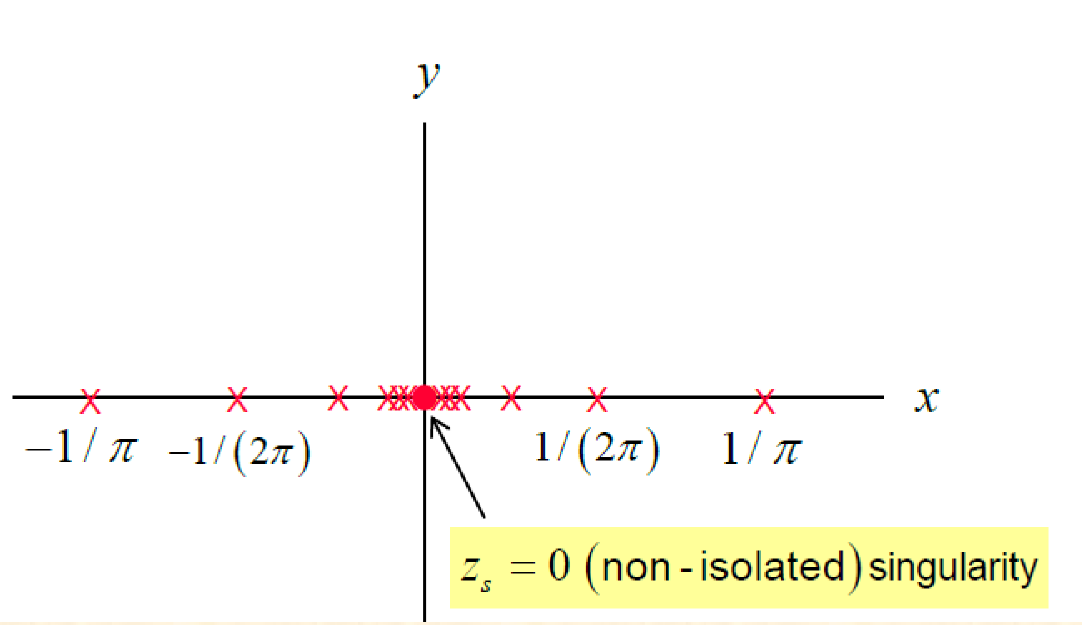
The singularities described above arise from the non-analytic behaviour of single-valued functions.

However, multi-valued functions frequently arise in the solution of engineering problems.

This is a singularity that is not isolated.

Example:

$$\boxed{f(z)=\frac{1}{\sin\left(\frac{1}{z}\right)}}$$



Taylor series

Taylor's theorem completes the story by giving the converse: around each point of analyticity an analytic function equals a convergent power series.

Theorem: Suppose that *f(z)* is an analytic in a region *A*. let $z_0 \epsilon A$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n}$$

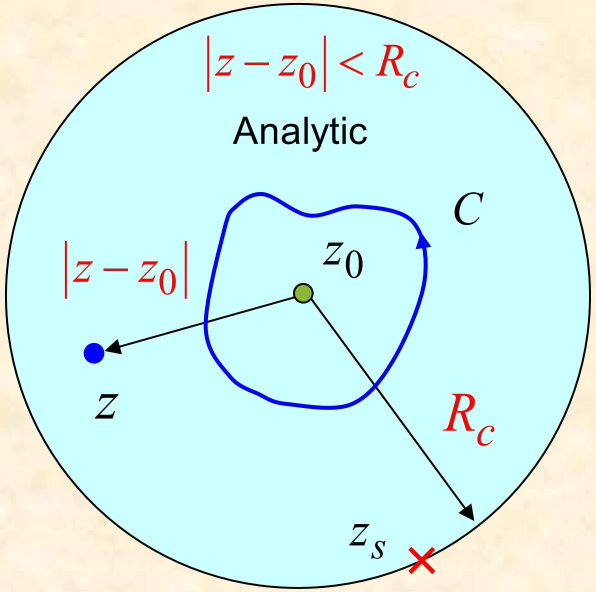
where the series converges on any disk $|z - z_0| < \mathrm{r}$ contained in *A*

$$\boxed{a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z - z_0)^{n+1}} dz}$$

Where c is a closed curve in A around z_0.

Taylor series

The Taylor series will converge <u>within</u> the radius of convergence and diverge outside.



LAURENTS SERIES

If c_1 *and* c_2 are two concentric circles with centre at $z = z_0$ and radii r_1 *and* r_2 $(r1 < r2)$ and if $f(z)$ is analytic inside on the circles and within the annulus between c_1 *and* c_2 then for any z in the annulus, we have

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}} \quad \text{.................}(1)$$

Where,

$$a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z - z_0)^{n+1}} dz, \; b_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

and the integration being taken in positive direction. This series (1) is called Laurent series of $f(z)$ about the point $z = z_0$

Problem 1

Expand $f(z)=cosz$ as a Taylor's series about $z=\pi/4$.

Solution:

Function	Value of function at z = 0
$f(z) = cosz$	$f(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$
$f'(z) = -sinz$	$f'\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f''(z) = -cosz$	$f''\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f'''(z) = sinz$	$f'''\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
…………….	……………………………..

The Taylor series of *f(z)* about $z = \pi/4$ is

$$cosz = \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\left(\frac{-\frac{1}{\sqrt{2}}}{1!}\right) + \left(z-\frac{\pi}{4}\right)\left(\frac{-\frac{1}{\sqrt{2}}}{2!}\right) + \left(z-\frac{\pi}{4}\right)\left(\frac{\frac{1}{\sqrt{2}}}{3!}\right) + \cdots$$

Problem 2

Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Taylor's and Laurent's series.

Solution: $$f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(i) When $|z| < 2$ then $\frac{|z|}{2} < 1$

$$f(z) = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left\{1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right\} - \frac{8}{3}\left\{1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right\}$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right)(z)^n$$

This is the required Taylor's series valid for $|z| < 2$

Problem 2

Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Taylor's and Laurent's series.

Solution: $$f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(ii) When $2 < |z| < 3$ then $\frac{2}{|z|} < 1; \frac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left\{1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right\} - \frac{8}{3}\left\{1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right\}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^n\left(\frac{3.2^n}{z^{n+1}} - \frac{8z^n}{3^{n+1}}\right)$$

This is the required Laurent's series valid for $2 < |z| < 3$

Problem 2

Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Taylor's and Laurent's series.

Solution: $$f(z) = \frac{z^2 - 1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

(iii) When $|z| > 3$ then $\frac{3}{|z|} < 1$;

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left\{1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right\} - \frac{8}{z}\left\{1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right\}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

$$= 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}(3.2^n - 8.3^n)$$

This is the required Laurent's series valid for $|z| > 3$

Problem 3

Expand $f(z) = \frac{1}{z^2-3z+2}$ as a Taylor's and Laurent's series for all possible intervals.

Solution: $f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-2)(z-1)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

(i) When $|z| < 1$;

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = (1-z)^{-1} - \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= \{1 + z + z^2 + z^3 + \cdots\} - \frac{1}{2}\left\{1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right\}$$

$$= \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)(z)^n$$

This is the required Taylor's series valid for $|z| < 1$

Problem 3

Expand $f(z) = \frac{1}{z^2-3z+2}$ as a Taylor's and Laurent's series for all possible intervals.

Solution:

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

(ii) When $1 < |z| < 2$ then $\frac{1}{|z|} < 1; \frac{|z|}{2} < 1;$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left\{1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right\} - \frac{1}{z}\left\{1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right\}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n}$$

This is the required Laurent's series valid for $1 < |z| < 2$

Problem 3

Expand $f(z) = \frac{1}{z^2-3z+2}$ as a Taylor's and Laurent's series for all possible intervals.

Solution:

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

(iii) When $|z| > 2$ then $\frac{2}{|z|} < 1$;

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left\{1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right\} - \frac{1}{z}\left\{1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right\}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty}\frac{2^n - 1}{z^{n+1}}$$

This is the required Laurent's series valid for $|z| > 2$

Practice Problem

Expand as a Taylor's and Laurent's series for all possible intervals.

(i) $f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$

(ii) $f(z) = \frac{z}{(z-1)(z-3)}$

Thanks a lot …