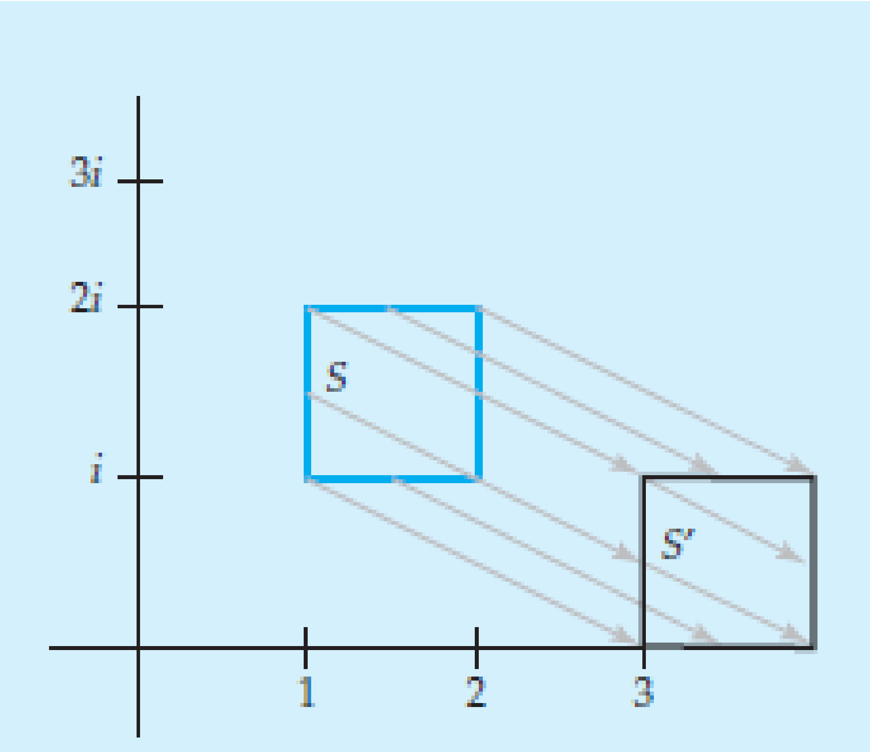

ANALYTIC FUNCTION

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Differentiability

Derivative: Suppose $z = x+iy$ and $z_0 = x_0+iy_0$; then the change in z_0 is the difference $\Delta z = z - z_0$ or $\Delta z = x - x_0 + i(y - y_0) = \Delta x + i\Delta y$. If a complex function $w = f(z)$ is defined at z and z_0, then the corresponding change in the function is the difference $\Delta w = f(z_0 + \Delta z) - f(z_0)$. The **derivative** of the function f is defined in terms of a limit of the difference quotient $\Delta w/\Delta z$ as $\Delta z \to 0$.

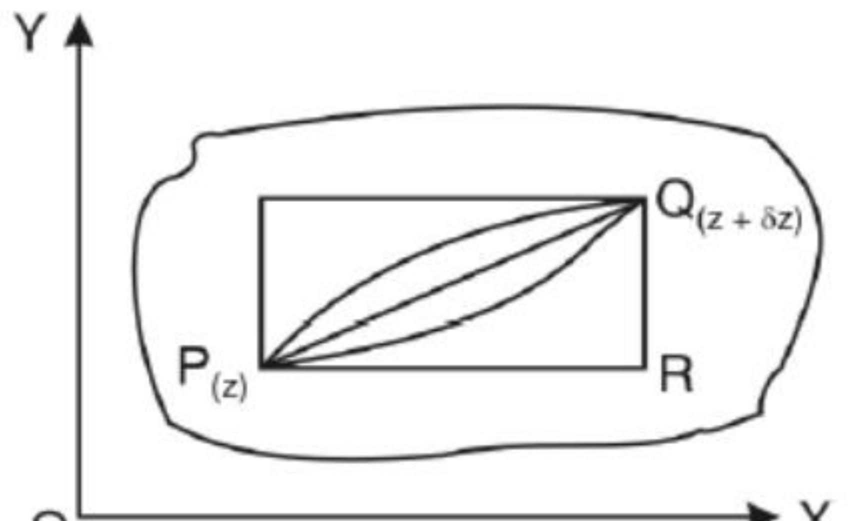
Differentiability

Let $f(z)$ be a single valued function of the variable z, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided that the limit exists and is independent of the path along which $\delta z \to 0$.

Let P be a fixed point and Q be a neighbouring point. The point Q may approach P along any straight line or curved path



Differentiability

Example 1: Consider function $f\ z = 4x + y + i(4y - x)$ and discuss $\frac{df}{dz}$

Solution. Here, $f(z) = 4x + y + i(-x + 4y) = u + iv$

so $u = 4x + y$ and $v = -x + 4y$

$$f(z+\delta z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y)$$

$$f(z+\delta z) - f(z) = 4(x+\delta x) + (y+\delta y) - i(x+\delta x) + 4i(y+\delta y) - 4x - y + ix - 4iy$$

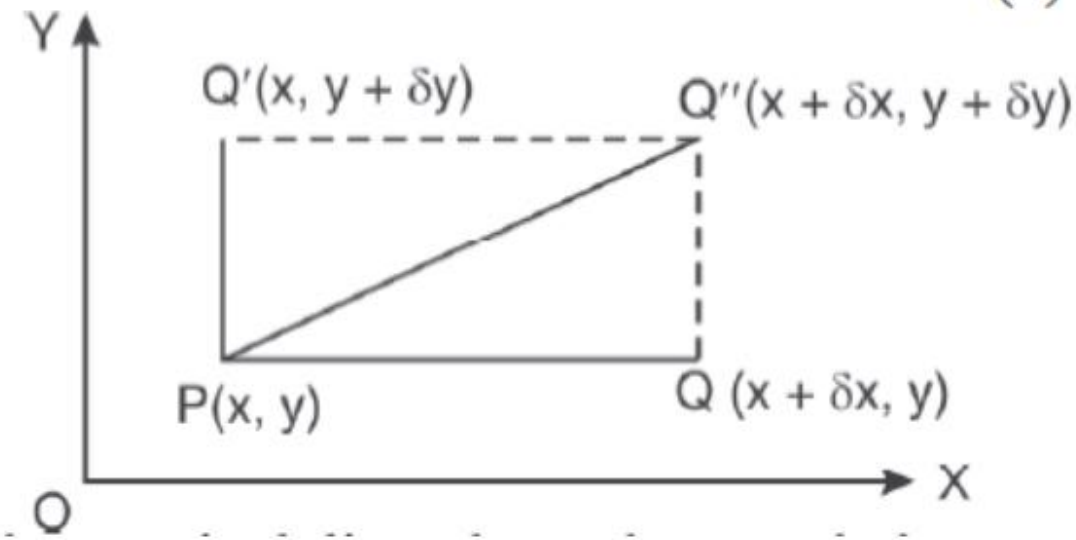
$$= 4\delta x + \delta y - i\,\delta x + 4i\delta y$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{4\delta x + \delta y - i\,\delta x + 4\,i\delta y}{\delta x + i\delta y}$$

Differentiability

Example 1: Consider function $f\ z = 4x + y + i(4y - x)$ and discuss $\frac{df}{\ }$

$$\Rightarrow \qquad \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y} \qquad \ldots (1)$$



(*a*) **Along real axis:** If Q is taken on the horizontal line through $P\ (x, y)$ and Q then approaches P along this line, we shall have $\delta y = 0$ and $\delta z = \delta x$.

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = 4 - i$$

Differentiability

Example 1: Consider function $f\ z = 4x + y + i(4y - x)$ and discuss $\dfrac{df}{dz}$

(*b*) **Along imaginary axis:** If Q is taken on the vertical line through P and then Q approaches P along this line, we have

$$z = x + iy = 0 + iy,\ \delta z = i\delta y,\ \delta x = 0.$$

Putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = \frac{1}{i}(1 + 4i) = 4 - i$$

(*c*) **Along a line $y = x$:** If Q is taken on a line $y = x$.

$$z = x + iy = x + ix = (1 + i)x$$

$$\delta z = (1 + i)\delta x \quad \text{and} \quad \delta y = \delta x$$

On putting these values in (1), we have

$$\frac{\delta f}{\delta z} = \frac{4\delta x + \delta x - i\delta x + 4i\delta x}{\delta x + i\delta x} = \frac{4 + 1 - i + 4i}{1 + i} = \frac{5 + 3i}{1 + i} = \frac{(5 + 3i)(1 - i)}{(1 + i)(1 - i)} = 4 - i$$

Differentiability

Example 1: Consider function $f\ z = 4x + y + i(4y - x)$ and discuss $\frac{df}{dz}$

In all the three different paths approaching Q from P, we get the same values of $\frac{\delta f}{\delta z} = 4 - i$.
In such a case, the function is said to be differentiable at the point z in the given region.

Differentiability

Example 2: If $f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0, \\ 0, & z=0 \end{cases}$ then discuss $\dfrac{df}{dz}$ at $z = 0$.

Solution. If $z \to 0$ along radius vector $y = mx$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left[\frac{\dfrac{x^3 y(y-ix)}{x^6+y^2} - 0}{x+iy}\right] = \lim_{z\to 0}\left[\frac{-ix^3 y\,(x+iy)}{(x^6+y^2)\,(x+iy)}\right]$$

$$= \lim_{z\to 0}\left[\frac{-ix^3 y}{x^6+y^2}\right] = \lim_{x\to 0}\left[\frac{-ix^3 (mx)}{x^6+m^2x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{-imx^2}{x^4+m^2}\right] = 0$$

Differentiability

Example 2:

But along $y = x^3$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left[\frac{-ix^3y}{x^6+y^2}\right] = \lim_{x\to 0}\frac{-ix^3(x^3)}{x^6+(x^3)^2} = -\frac{i}{2}$$

In different paths we get different values of $\frac{df}{dz}$ *i.e.* 0 and $\frac{-i}{2}$. In such a case, the function is not differentiable at $z = 0$.

Differentiability

Example 3: Using the Rules of Differentiation

Differentiate:

(a) $f(z) = 3z^4 - 5z^3 + 2z$ (b) $f(z) = \dfrac{z^2}{4z+1}$ (c) $f(z) = \left(iz^2 + 3z\right)^5$

Solution

(a) Using the power rule (7), the sum rule (3), along with (2), we obtain

$$f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 \cdot 1 = 12z^3 - 15z^2 + 2.$$

(b) From the quotient rule (5),

$$f'(z) = \frac{(4z+1) \cdot 2z - z^2 \cdot 4}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}.$$

(c) In the power rule for functions (8) we identify $n = 5$, $g(z) = iz^2 + 3z$, and $g'(z) = 2iz + 3$, so that

$$f'(z) = 5(iz^2 + 3z)^4(2iz + 3).$$

Analytic at a Point

Definition: A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

A function ***f* is analytic in a domain *D*** if it is analytic at every point in D. The phrase "analytic *on* a domain D" is also used. Although we shall not use these terms.

Holomorphic: a function f that is analytic throughout a domain D is called **holomorphic** or **regular**.

Entire Function: A function that is analytic at every point z in the complex plane is said to be an **entire function**.

Analytic Function

Theorem 3.1 Polynomial and Rational Functions

(*i*) A polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a nonnegative integer, is an entire function.

(*ii*) A rational function $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

Singular Points: Since the rational function $f(z) = 4z/(z^2 - 2z + 2)$ is discontinuous at $1+i$ and $1-i$, f fails to be analytic at these points. Thus by (*ii*) of Theorem 3.1, f is not analytic in any domain containing one or both of these points. In general, a point z at which a complex function $w = f(z)$ fails to be analytic is called a **singular point** of f.

Analytic Function

An Alternative Definition of $f'(z)$ Sometimes it is convenient to define the derivative of a function f using an alternative form of the difference quotient $\Delta w/\Delta z$. Since $\Delta z = z - z_0$, then $z = z_0 + \Delta z$, and so (1) can be written as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{12}$$

Theorem 3.2 Differentiability Implies Continuity

If f is differentiable at a point z_0 in a domain D, then f is continuous at z_0.

Analytic Function

Theorem 3.3 L'Hôpital's Rule

Suppose f and g are functions that are analytic at a point z_0 and $f(z_0) = 0$, $g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}. \tag{13}$$

Example 4: Using L'Hˆopital's Rule

Solution If we identify $f(z) = z^2 - 4z + 5$ and $g(z) = z^3 - z - 10i$, you should verify that $f(2+i) = 0$ and $g(2+i) = 0$. The given limit has the indeterminate form 0/0. Now since f and g are polynomial functions, both functions are necessarily analytic at $z_0 = 2 + i$. Using

Analytic Function

Example 4: Using L'Hˆopital's Rule

$$f'(z) = 2z - 4, \quad g'(z) = 3z^2 - 1, \quad f'(2+i) = 2i, \quad g'(2+i) = 8 + 12i,$$

we see that (13) gives

$$\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i} = \frac{f'(2+i)}{g'(2+i)} = \frac{2i}{8+12i} = \frac{3}{26} + \frac{1}{13}i.$$

Self Study: Using L'Hˆopital's Rule

23. $\displaystyle\lim_{z \to i} \frac{z^7 + i}{z^{14} + 1}$

24. $\displaystyle\lim_{z \to \sqrt{2}+\sqrt{2}i} \frac{z^4 + 16}{z^2 - 2\sqrt{2}z + 4}$

25. $\displaystyle\lim_{z \to 1+i} \frac{z^5 + 4z}{z^2 - 2z + 2}$

26. $\displaystyle\lim_{z \to \sqrt{2}i} z\frac{z^3 + 5z^2 + 2z + 10}{z^5 + 2z^3}$

Analytic Function

Self Study: Determine the points at which the given function is not analytic.

27. $f(z) = \dfrac{iz^2 - 2z}{3z + 1 - i}$

28. $f(z) = -5iz^2 + \dfrac{2+i}{z^2}$

29. $f(z) = \left(z^4 - 2iz^2 + z\right)^{10}$

30. $f(z) = \left(\dfrac{(4+2i)z}{(2-i)z^2 + 9i}\right)^3$

Cauchy-Riemann Equations

Theorem 3.4 Cauchy-Riemann Equations

Suppose $f(z) = u(x,\ y) + iv(x,\ y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1}$$

Proof: Cauchy-Riemann Equations

Proof The derivative of f at z is given by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{2}$$

By writing $f(z) = u(x,\ y) + iv(x,\ y)$ and $\Delta z = \Delta x + i\Delta y$, (2) becomes

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}. \tag{3}$$

Since the limit (2) is assumed to exist, Δz can approach zero from any convenient direction. In particular, if we choose to let $\Delta z \to 0$ along a horizontal line, then $\Delta y = 0$ and $\Delta z = \Delta x$. We can then write (3) as

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y) + i\,[v(x + \Delta x, y) - v(x, y)]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}. \end{aligned} \tag{4}$$

Proof: Cauchy-Riemann Equations

The existence of $f'(z)$ implies that each limit in (4) exists. These limits are the definitions of the first-order partial derivatives with respect to x of u and v, respectively. Hence, we have shown two things: both $\partial u/\partial x$ and $\partial v/\partial x$ exist at the point z, and that the derivative of f is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{5}$$

We now let $\Delta z \to 0$ along a vertical line. With $\Delta x = 0$ and $\Delta z = i\Delta y$, (3) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}. \tag{6}$$

In this case (6) shows us that $\partial u/\partial y$ and $\partial v/\partial y$ exist at z and that

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{7}$$

Verifying : Cauchy-Riemann Equations

Example 5: The polynomial function $f(z) = z^2+z$ is analytic for all z and can be written as $f(z) = x^2-y^2+x+i(2xy+y)$. Thus, $u(x,\ y) = x^2-y^2+x$ and $v(x,\ y) = 2xy+y$. For any point $(x,\ y)$ in the complex plane we see that the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Criterion for Non-analyticity

If the Cauchy-Riemann equations are not satisfied at every point z in a domain D, then the function $f(z) = u(x, y)+iv(x, y)$ cannot be analytic in D.

Cauchy-Riemann Equations

THE NECESSARY CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. *The necessary conditions for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (*ii*) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ provided $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist.

THE SUFFICIENT CONDITION FOR $f(z)$ TO BE ANALYTIC

Theorem. *The sufficient condition for a function $f(z) = u + iv$ to be analytic at all the points in a region R are*

(*i*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(*ii*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in region R.*

Cauchy-Riemann Equations

Remember: **1.** If a function is analytic in a domain D, then u, v satisfy $C - R$ conditions at all points in D.

2. $C - R$ conditions are necessary but not sufficient for analytic function.

3. $C - R$ conditions are sufficient if the partial derivative are continuous.

Example 6: *Discuss the analyticity of the function* $f(z) = z\bar{z}$.

Solution. $f(z) = z\bar{z} = (x+iy)(x-iy) = x^2 - i^2 y^2 = x^2 + y^2$

$$f(z) = x^2 + y^2 = u + iv.$$

$$u = x^2 + y^2,\ v = 0$$

At origin, $$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,\ 0) - u(0,\ 0)}{h} = \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0, 0+k) - u(0, 0)}{k} = \lim_{k\to 0} \frac{k^2}{k} = 0$$

Cauchy-Riemann Equations

Example 6: *Discuss the analyticity of the function* $f(z) = z\bar{z}$.

Also, $$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,0) - v(0,0)}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,0+k) - v(0,0)}{k} = 0$$

Thus, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, C – R equations are satisfied at the origin.

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{(x^2 + y^2) - 0}{x + iy}$$

Let $z \to 0$ along the line $y = mx$, then

$$f'(0) = \lim_{x\to 0} \frac{(x^2 + m^2 x^2)}{(x + imx)} = \lim_{x\to 0} \frac{(1+m^2)x}{1+im} = 0$$

Therefore, $f'(0)$ is unique. Hence the function $f(z)$ is analytic at $z = 0$.

Cauchy-Riemann Equations

Example 7: *Determine whether* $\frac{1}{z}$ *is analytic or not?*

Solution. Let $w = f(z) = u + iv = \frac{1}{z} \quad \Rightarrow \quad u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

Equating real and imaginary parts, we get

$$u = \frac{x}{x^2+y^2}, \qquad v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Cauchy-Riemann Equations

Example 7: *Determine whether $\frac{1}{z}$ is analytic or not?*

Thus, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Thus C – R equations are satisfied. Also partial derivatives are continuous except at (0, 0).

Therefore $\frac{1}{z}$ is analytic everywhere except at $z = 0$.

C-R EQUATION IN POLAR FORM

If $f(z) = u + iv$ is an analytic function where u and v are functions of r, θ and $z = re^{i\theta}$ then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Example 8:

Show that $w = e^z$ is analytic everywhere in the complex plane. Hence, find $\frac{dw}{dz}$.

Example 8:

Show that $w = e^z$ is analytic everywhere in the complex plane. Hence, find $\frac{dw}{dz}$.

Solution

$$w = e^z$$

$$u + iv = e^{x+iy}$$

$$= e^x e^{iy}$$

$$= e^x(\cos y + i\sin y)$$

$$= e^x \cos y + ie^x \sin y$$

Comparing real and imaginary parts,

$$u = e^x \cos y, \qquad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R EQUATION IN POLAR FORM

Example 8:

Show that $w = e^z$ is analytic everywhere in the complex plane. Hence, find $\frac{dw}{dz}$.

C–R equations are satisfied. Also e^x, $\cos y$ and $\sin y$ are continuous for all values of x and y.

Hence, e^z is analytic everywhere in the complex plane.

Since $w = u + iv$ is analytic everywhere,

$$\begin{aligned}
\frac{dw}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= e^x \cos y + ie^x \sin y \\
&= e^x (\cos y + i\sin y) \\
&= e^x e^{iy} \\
&= e^{x+iy} \\
&= e^z
\end{aligned}$$

Harmonic Functions

Definition of Harmonic Function: A real-valued function ϕ of two real variables x and y that has continuous first and second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

Theorem on Harmonic Functions: Suppose the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic in D. i. e.:

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0 \qquad or, \qquad \nabla^2 u = 0$$

$$\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial^2 y} = 0 \qquad or, \qquad \nabla^2 v = 0$$

Harmonic Functions

Definition of Harmonic Conjugate Functions : We have just shown that if a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its real and imaginary parts u and v are necessarily harmonic in D. Now suppose $u(x, y)$ is a given real function that is known to be harmonic in D. If it is possible to find another real harmonic function $v(x, y)$ so that u and v satisfy the Cauchy-Riemann equations throughout the domain D, then the function $v(x, y)$ is called a **harmonic conjugate** of $u(x, y)$. By combining the functions as $u(x, y) + iv(x, y)$ we obtain a function that is analytic in D.

Harmonic Functions

Example 9:

Example : *Prove that* $u = x^2 - y^2$ *and* $v = \dfrac{y}{x^2 + y^2}$ *are harmonic functions of* (x, y), *but are not harmonic conjugates.*

Solution. We have, $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial u}{\partial y} = -2y, \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$u\ (x, y)$ satisfies Laplace equation, hence $u\ (x, y)$ is harmonic

$$v = \frac{y}{x^2 + y^2}, \qquad \frac{\partial v}{\partial x} = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2 (-2y) - (-2xy)\, 2(x^2 + y^2)\, 2x}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)(-2y) - (-2xy)4x}{(x^2 + y^2)^3} = \frac{6x^2 y - 2y^3}{(x^2 + y^2)^3}$$

Harmonic Functions

$$= \frac{(x^2+y^2)(-2y)-(-2xy)4x}{(x^2+y^2)^3} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)\cdot 1 - y\,(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \qquad \text{... (1)}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(-2y)-(x^2-y^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{(x^2+y^2)(-2y)-(x^2-y^2)(4y)}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y-2y^3-4x^2y+4y^3}{(x^2+y^2)^3} = \frac{-6x^2y+2y^3}{(x^2+y^2)^3} \qquad \text{... (2)}$$

On adding (1) and (2), we get $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$v\,(x, y)$ also satisfies Laplace equations, hence $v\,(x, y)$ is also harmonic function.

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

Therefore u and v are not harmonic conjugates. **Proved.**

MILNE-THOMSON METHOD

By this method $f(z)$ is directly constructed without finding v and the method is given below:

Since $z = x + iy$ and $\bar{z} = x - iy$

$$\therefore \quad x = \frac{z+\bar{z}}{2}, \; y = \frac{z-\bar{z}}{2i}$$

$$f(z) \equiv u(x, y) + iv(x, y) \qquad \dots (1)$$

$$f(z) \equiv u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

This relation can be regarded as a formal identity in two independent variables z and $\bar{z}$. Replacing $\bar{z}$ by z, we get

$$f(z) \equiv u(z, 0) + iv(z, 0)$$

Which can be obtained by replacing x by z and y by 0 in (1)

MLINE THMSON METHOD

Case I. When u is given

Step 1. Find $\frac{\partial u}{\partial x}$ and equate it to $\phi_1(x, y)$.

Step 2. Find $\frac{\partial u}{\partial y}$ and equate it to $\phi_2(x, y)$.

Step 3. Replace x by z and y by 0 in $\phi_1(x, y)$ to get $\phi_1(z, 0)$.

Step 4. Replace x by z and y by 0 in $\phi_2(x, y)$ to get $\phi_2(z, 0)$.

Step 5. Find $f(z)$ by the formula $f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\}\, dz + c$

MLINE THMSON METHOD

Case II. When v is given

Step 1. Find $\dfrac{\partial v}{\partial x}$ and equate it to $\psi_2(x, y)$.

Step 2. Find $\dfrac{\partial v}{\partial y}$ and equate it to $\psi_1(x, y)$.

Step 3. Replace x by z and y by 0 in $\psi_1(x, y)$ to get $\psi_1(z, 0)$.

Step 4. Replace x by z and y by 0 in $\psi_2(x, y)$ to get $\psi_2(z, 0)$.

Step 5. Find $f(z)$ by the formula

$$f(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c$$

MLINE THMSON METHOD

Example 10:

Show that e^x (x cos y – y sin y) is a harmonic function. Find the analytic function for which e^x (x cos y – y sin y) is imaginary part.

Solution. Here $v = e^x (x \cos y - y \sin y)$

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial v}{\partial x} = e^x \ (x \cos y - y \sin y) + e^x \cos y = \psi_2 (x, y), \qquad \text{(say)} \qquad \dots (1)$$

$$\frac{\partial v}{\partial y} = e^x \ (-x \sin y - y \cos y - \sin y) = \psi_1 \ (x, \ y) \qquad \text{(say)} \qquad \dots (2)$$

$$\frac{\partial^2 v}{\partial x^2} = e^x \ (x \cos y - y \sin y) + e^x \cos y + e^x \cos y$$

$$= e^x (x \cos y - y \sin y + 2 \cos y) \qquad \dots (3)$$

MLINE THMSON METHOD

Example 10:

and $$\frac{\partial^2 v}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y) \qquad \ldots (4)$$

Adding equations (3) and (4), we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \Rightarrow v \text{ is a harmonic function.}$$

Now putting $x = z$, $y = 0$ in (1) and (2), we get

$$\psi_2(z,0) = ze^z + e^z \qquad \psi_1(z,0) = 0$$

Hence by Milne-Thomson method, we have

$$f(z) = \int [\psi_1(z, 0) + i\,\psi_2(z, 0)]\, dz + C$$

$$= \int [0 + i(ze^z + e^z)]\, dz + C = i(ze^z - e^z + e^z) + C = i\,z\,e^z + C.$$

This is the required analytic function. **Ans.**

Determination of Conjugate Harmonic

Example 11: Verify that the following functions u are harmonic, and in each case give a conjugate harmonic function v.

(a) $u(x;y) = 3x^2y+2x^2 -y^3 -2y^2$,

(b) $v(x;y) = \ln(x^2 +y^2)$.

Solution: (a) If $u(x,y) = 3x^2y+2x^2 -y^3 -2y^2$,

then, $u_x = 6xy+4x$, $u_{xx} = 6y+4$, $u_y = 3x^2 -3x^2 -4y$, $u_{yy} = -6y-4$

Thus, $\nabla^2 u = u_{xx} + u_{yy} = 6y+4+ (-6y-4) = 0.$

Hence, u is harmonic.

The harmonic conjugate of u will satisfy the Cauchy-Riemann equations and have continiuos partials of all orders.

By Cauchy-Riemann equations

$$u_x = v_y ; v_x = -u_y ;$$

Determination of Conjugate Harmonic

Solution (cont.): we have that $u_x = v_y = 6xy+4x$.

Thus, $v = \int(6xy + 4x)dy = 3xy^2 + 4xy + g(x)$

Thus, $v_x = 3y^2 + 4y + g'(x)$

Since, $v_x = -u_y$,

$$3y^2 + 4y + g'(x) = -3x^2 + 3y^2 + 4y$$

$$g'(x) = -3x^2$$

$$g(x) = -x^3$$

Therefore, the harmonic conjugate is $v(x,y) = 3xy^2 + 4xy - x^3$.

(b) Do yourself

Thanks a lot …