

# 13:5-Chain Rules for Functions of Several Variables

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## Objectives

- Use the Chain Rules for functions of several variables.
- Find partial derivatives implicitly.

# Chain Rules for Functions of Several Variables

## THEOREM 13.6 Chain Rule: One Independent Variable

Let  $w = f(x, y)$ , where  $f$  is a differentiable function of  $x$  and  $y$ . If  $x = g(t)$  and  $y = h(t)$ , where  $g$  and  $h$  are differentiable functions of  $t$ , then  $w$  is a differentiable function of  $t$ , and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}.$$

The Chain Rule is shown schematically in Figure 13.39.

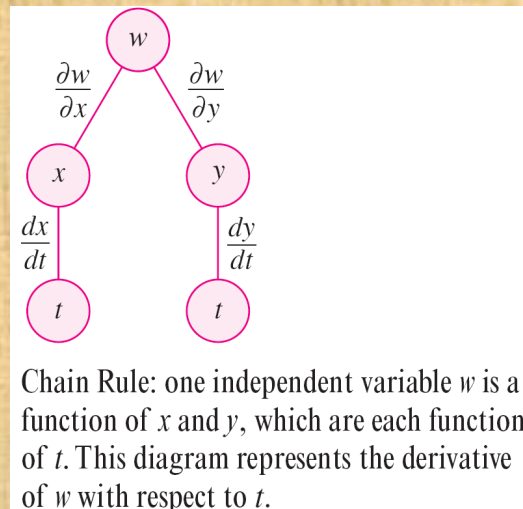


Figure 13.39

## Example 1 – Chain Rule: One Independent Variable

- Let  $w = x^2y - y^2$ , where  $x = \sin t$  and  $y = e^t$ . Find  $dw/dt$  when  $t = 0$ .

- **Solution:**

By the Chain Rule for one independent variable, you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$= 2xy(\cos t) + (x^2 - 2y)e^t$$

## Example 1 – *Solution*

$$= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t$$

$$= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.$$

When  $t = 0$ , it follows that

$$\frac{dw}{dt} = -2.$$

## Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.6 can provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 1, you could have used single-variable techniques to find  $dw/dt$  by first writing  $w$  as a function of  $t$ ,

$$\begin{aligned}w &= x^2y - y^2 \\&= (\sin t)^2(e^t) - (e^t)^2 \\&= e^t \sin^2 t - e^{2t}\end{aligned}$$

and then  $\frac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$

## Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each  $x_i$  is a differentiable function of a single variable  $t$ , then for

$$w = f(x_1, x_2, \dots, x_n)$$

- you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial w}{\partial x_2} \frac{dx_2}{dt} + \dots + \frac{\partial w}{\partial x_n} \frac{dx_n}{dt}.$$

# Chain Rules for Functions of Several Variables

## THEOREM 13.7 Chain Rule: Two Independent Variables

Let  $w = f(x, y)$ , where  $f$  is a differentiable function of  $x$  and  $y$ . If  $x = g(s, t)$  and  $y = h(s, t)$  such that the first partials  $\partial x/\partial s$ ,  $\partial x/\partial t$ ,  $\partial y/\partial s$ , and  $\partial y/\partial t$  all exist, then  $\partial w/\partial s$  and  $\partial w/\partial t$  exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}.$$

The Chain Rule is shown schematically in Figure 13.41.

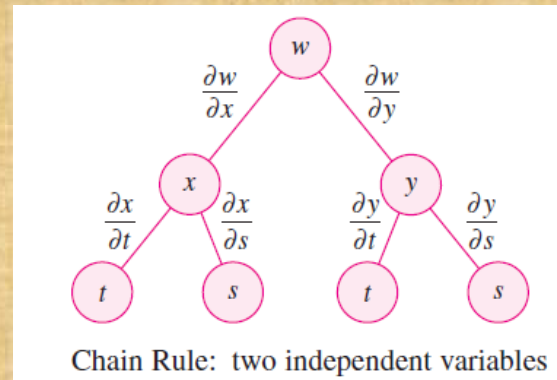


Figure 13.41

### Example 4 – The Chain Rule with Two Independent Variables

- Use the Chain Rule to find  $\partial w/\partial s$  and  $\partial w/\partial t$  for  $w = 2xy$  where  $x = s^2 + t^2$  and  $y = s/t$ .

- **Solution:**

Using Theorem 13.7, you can hold  $t$  constant and differentiate with respect to  $s$  to obtain

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

$$= 2y(2s) + 2x\left(\frac{1}{t}\right)$$

## Example 4 – *Solution*

cont'd

$$= 2\left(\frac{s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right)$$

Substitute  $(s/t)$  for  $y$  and  $s^2 + t^2$  for  $x$ .

$$= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t}$$

$$= \frac{6s^2 + 2t^2}{t}.$$

Similarly, holding  $s$  constant gives

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

## Example 4 – *Solution*

cont'd

$$= 2y(2t) + 2x\left(\frac{-s}{t^2}\right)$$

$$= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right)$$

Substitute  $(s/t)$  for  $y$  and  $s^2 + t^2$  for  $x$ .

$$= 4s - \frac{2s^3 + 2st^2}{t^2}$$

$$= \frac{4st^2 - 2s^3 - 2st^2}{t^2}$$

$$= \frac{2st^2 - 2s^3}{t^2}.$$

## Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.7 can also be extended to any number of variables. For example, if  $w$  is a differentiable function of the  $n$  variables  $x_1, x_2, \dots, x_n$  where each  $x_i$  is a differentiable function of the  $m$  variables  $t_1, t_2, \dots, t_m$ , then for  $w = f(x_1, x_2, \dots, x_n)$  you obtain the following.

$$\begin{aligned}\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \dots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_1} \\ \frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_2} + \dots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_2} \\ &\vdots \\ \frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1} \frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2} \frac{\partial x_2}{\partial t_m} + \dots + \frac{\partial w}{\partial x_n} \frac{\partial x_n}{\partial t_m}\end{aligned}$$

## Implicit Partial Differentiation

- This section concludes with an application of the Chain Rule to determine the derivative of a function defined *implicitly*.
- ✓ Let  $x$  and  $y$  be related by the equation  $F(x, y) = 0$ , where  $y = f(x)$  is a differentiable function of  $x$ . To find  $dy/dx$ , you could use the techniques discussed in Section 2.5. You will see, however, that the Chain Rule provides a convenient alternative. Consider the function  $w = F(x, y) = F(x, f(x))$ .
- You can apply Theorem 13.6 to obtain

$$\frac{dw}{dx} = F_x(x, y) \frac{dx}{dx} + F_y(x, y) \frac{dy}{dx} \cdot$$

# Implicit Partial Differentiation

## THEOREM 13.8 Chain Rule: Implicit Differentiation

If the equation  $F(x, y) = 0$  defines  $y$  implicitly as a differentiable function of  $x$ , then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation  $F(x, y, z) = 0$  defines  $z$  implicitly as a differentiable function of  $x$  and  $y$ , then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

## Example 6 – Finding a Derivative Implicitly

- Find  $dy/dx$  for  $y^3 + y^2 - 5y - x^2 + 4 = 0$ .

- Solution:**

Begin by letting

$$F(x, y) = y^3 + y^2 - 5y - x^2 + 4.$$

Then

$$F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5.$$

- Using Theorem 13.8, you have

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

## Suggested Problems

**Exercise 13.5:** **11,17,22,25,29.**

# Thanks a lot ...