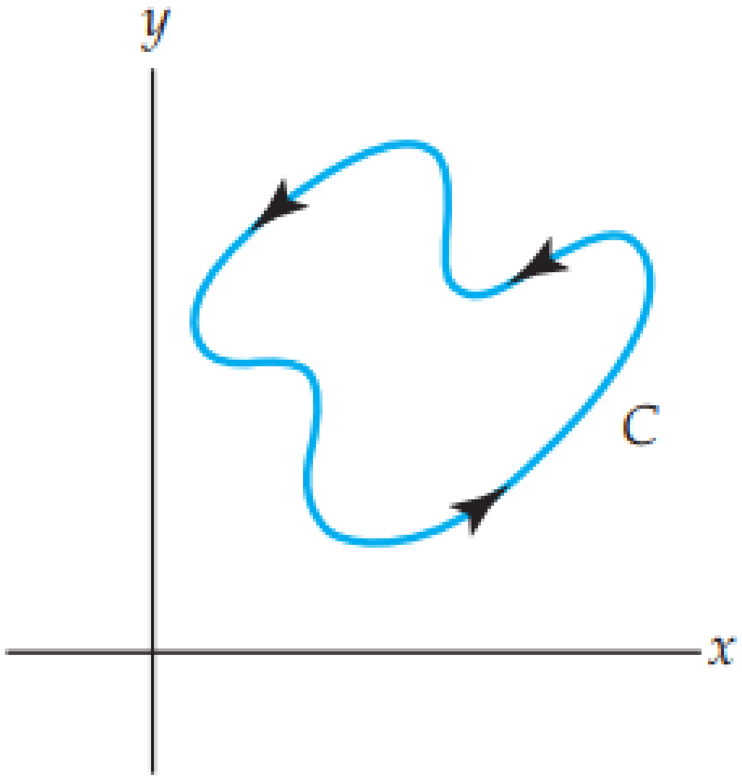

CAUCHY INTEGRAL THEOREM AND CAUCHY INTEGRAL FORMULA

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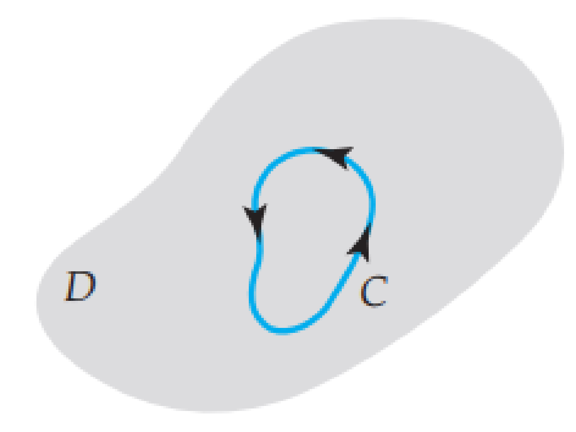
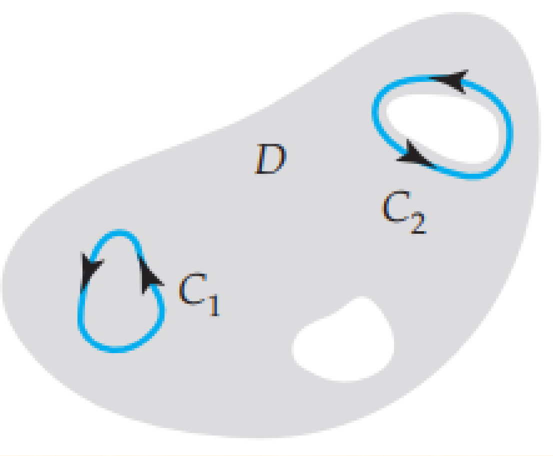
Simply and Multiply Connected Domains

A domain D is **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D. See Figure 5.26. Intuitively, a simply-connected region is one that does not have any have any holes. A domain that is not simply connected is called a **multiply connected domain**; that is, a multiply connected domain has "holes" in it.



Simply connected domain D



Multiply connected domain D

Cauchy's Theorem

In 1825 the French mathematician Louis-Augustin Cauchy proved one the most important theorems in complex analysis.

Cauchy's Theorem: *Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D,*

$$\oint_c f(z)dz = 0.$$

Cauchy-Goursat Theorem

In 1883 the French mathematician Edouard Goursat proved that the assumption of continuity of f' is not necessary to reach the conclusion of Cauchy's theorem. The resulting modified version of Cauchy's theorem is known today as the **Cauchy-Goursat theorem**.

Cauchy-Goursat Theorem: Suppose that a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D,

$$\oint_C f(z)dz = 0$$

Cauchy-Goursat Theorem

EXAMPLE 1: Applying the Cauchy-Goursat Theorem

Evaluate $\oint_C \dfrac{dz}{z^2}$, where the contour C is the ellipse

$$(x - 2)^2 + \frac{(y-5)^2}{4} = 1.$$

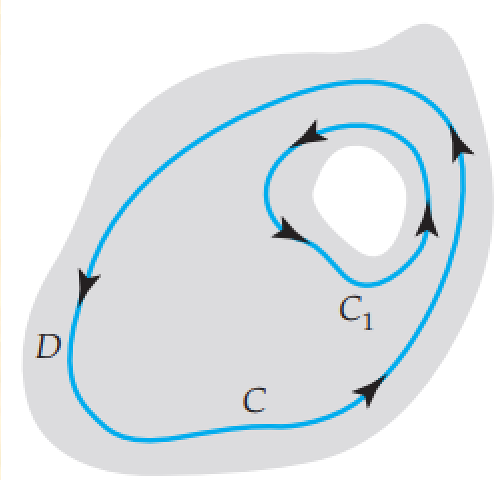
Solution: The rational function $f(z) = \dfrac{1}{z^2}$ is analytic everywhere except at $z = 0$. But $z = 0$ is not a point interior to or on the simple closed elliptical contour C. Thus, according to CGT we have that

$$\oint_C \frac{dz}{z^2} = 0$$

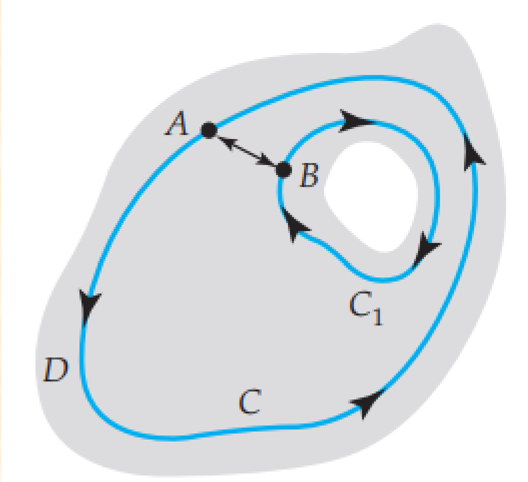
Cauchy-Goursat Theorem for Multiply Connected Domains

If f is analytic in a multiply connected domain D then we cannot conclude that $\oint_C f(z)dz = 0$ for every simple closed contour C in D. To begin, suppose that D is a doubly connected domain and C and C_1 are simple closed contours such that C_1 surrounds the "hole" in the domain and is interior to C. See the following Figure 5.29(a). Suppose, also, that f is analytic on each contour and at each point interior to C but exterior to C_1. By introducing the crosscut AB shown in Figure 5.29(b), the region bounded between the curves is now simply connected. From (*iv*) of Theorem5.2, the integral from A to B has the opposite value of the integral from B to A, and so from(4) we have

Cauchy-Goursat Theorem for Multiply Connected Domains



(a)



(b)

$$\therefore \oint_{c} f(z)dz + \int_{AB} f(z)dz + \int_{BA} f(z)dz + \oint_{c_1} f(z)dz = 0$$

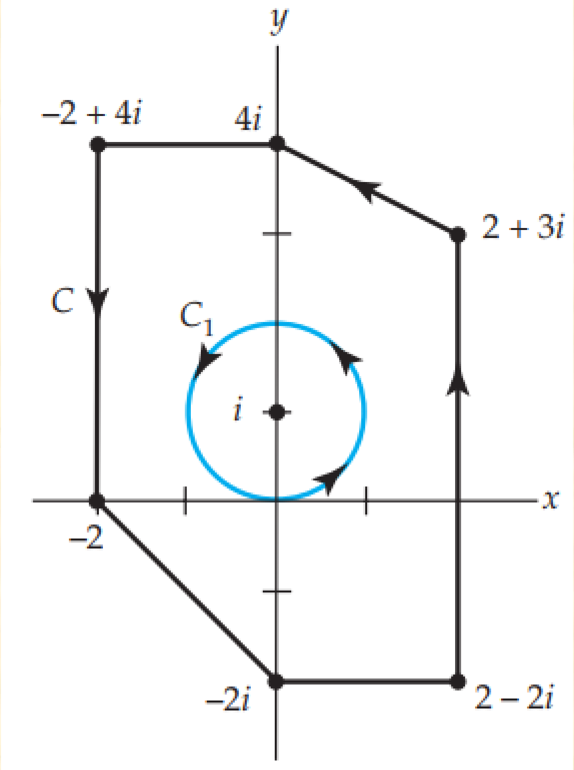
$$or, \oint_{c} f(z)dz + \int_{AB} f(z)dz + \int_{-AB} f(z)dz - \oint_{c_1} f(z)dz = 0$$

$$or, \oint_{c} f(z)dz = \oint_{c_1} f(z)dz$$

Cauchy-Goursat Theorem for Multiply Connected Domains

EXAMPLE 2: Applying Deformation of Contours

Evaluate $\oint_C \frac{dz}{z-i}$, where C is the contour shown in black in Figure 5.30.



Solution: In view of (5), we choose the more convenient circular contour C_1 drawn in color in the figure. By taking the radius of the circle to be $r = 1$, we are guaranteed that C_1 lies within C. In other words, C_1 is the circle $|z - i| = 1$, which from(10) of Section 2.2 can be parametrized by $z = i+e^{it}$, $0 \le t \le 2\pi$. From $z - i = e^{it}$ and $dz = ie^{it}dt$,

we obtain
$$\oint_c \frac{dz}{z-i} = \oint_{c_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = i\int_0^{2\pi} dt = 2\pi i$$

Cauchy-Goursat Theorem for Multiply Connected Domains

Suppose C, C_1, . . ., C_n are simple closed curves with a positive orientation such that C_1, C_2, . . ., C_n are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then

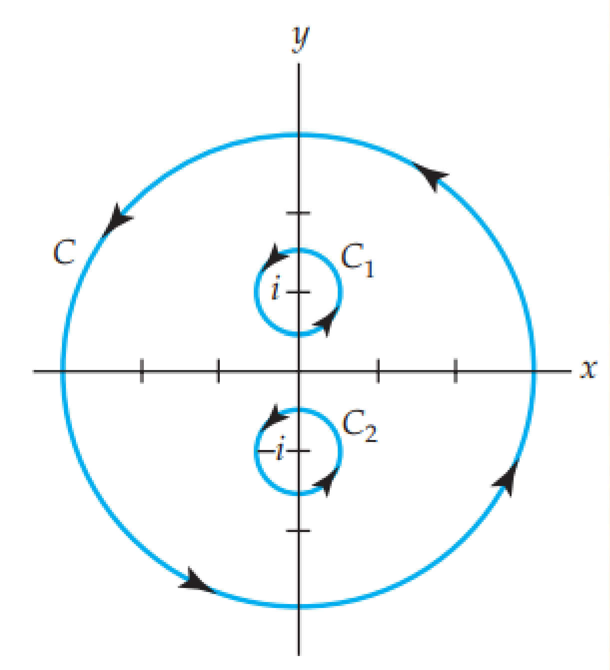
$$\oint_c f(z)\,dz = \sum_{k=1}^{n} \oint_{c_k} f(z)\,dz$$

EXAMPLE 3: Applying Theorem 5.5

Evaluate

$$\oint_c \frac{dz}{z^2+1}$$

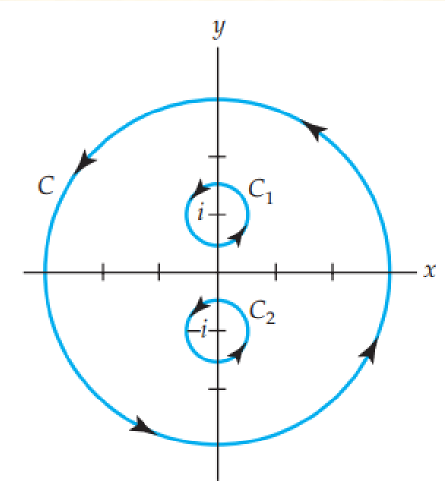
where C is the circle $|z| = 4$.



Solution: In this case the denominator of the integrand factors as $z^2 + 1 = (z - i)(z + i)$. Consequently, the integrand $1/(z^2 + 1)$ is not analytic at $z = i$ and at $z = -i$. Both of these points lie within the contour C. Using partial fraction decomposition once more, we have

Cauchy-Goursat Theorem for Multiply Connected Domains

$$\frac{1}{z^2+1}=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$$

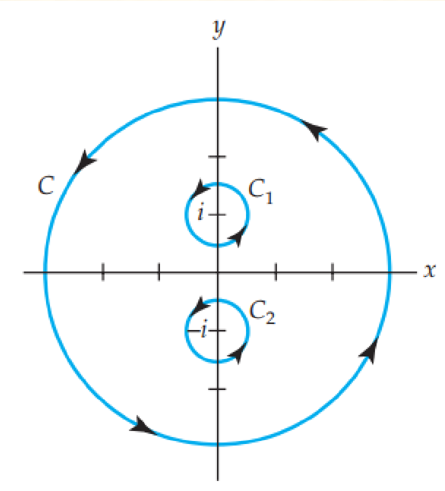


$$\therefore \oint_c \frac{dz}{z^2+1}=\oint_{c1}\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]+\oint_{c2}\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$$

$$=\oint_{c1}\frac{1}{2i}\left[\frac{1}{z-i}\right]-\oint_{c1}\frac{1}{2i}\left[\frac{1}{z+i}\right]+\oint_{c2}\frac{1}{2i}\left[\frac{1}{z-i}\right]-\oint_{c2}\frac{1}{2i}\left[\frac{1}{z+i}\right]$$

Because $1/(z+i)$ is analytic on C_1 and at each point in its interior and because $1/(z - i)$ is analytic on C_2 and at each point in its interior, it follows from(4) that the second and third integrals in (9) are zero. Moreover, it follows from (6), with $n = 1$, that

Cauchy-Goursat Theorem for Multiply Connected Domains



$$\frac{1}{z^2+1}=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$$

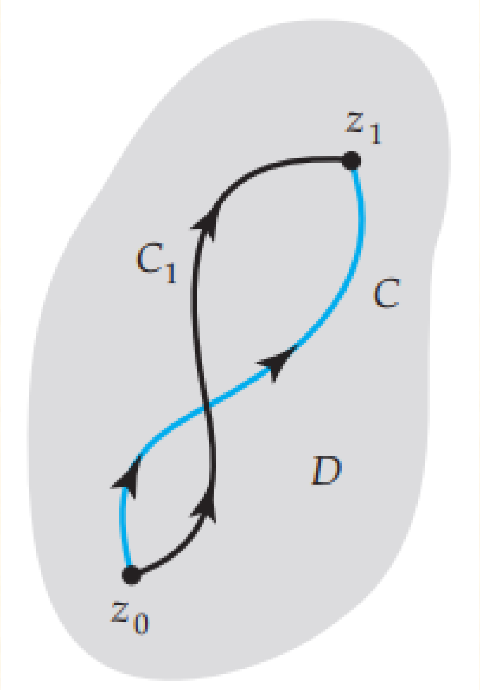
$$\therefore \oint_c \frac{dz}{z^2+1}=\oint_{c1}\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]+\oint_{c2}\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$$

$$=\oint_{c1}\frac{1}{2i}\left[\frac{1}{z-i}\right]-\oint_{c1}\frac{1}{2i}\left[\frac{1}{z+i}\right]+\oint_{c2}\frac{1}{2i}\left[\frac{1}{z-i}\right]-\oint_{c2}\frac{1}{2i}\left[\frac{1}{z+i}\right]$$

$$=2\pi i-0+0-2\pi i=0$$

Independence of the Path

Let z_0 and z_1 be points in a domain D. A contour integral $\oint_c f(z)dz$ is said to be **independent of the path** if its value is the same for all contours C in D with initial point z_0 and terminal point z_1.



$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz$$

If f is analytic in D, integrals on C and C_1 are equal.

Cauchy's Integral Formula

1st Theorem: Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$$

EXAMPLE 4: Using Cauchy's Integral Formula

Evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i} dz$, where C is the circle $|z| = 2$.

Solution: First, we identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula (1) we obtain

Cauchy’s Integral Formula

EXAMPLE 4: Using Cauchy’s Integral Formula

Evaluate $\oint_C \frac{z^2-4z+4}{z+i}\,dz$, where C is the circle $|z| = 2$.

Solution: First, we identify $f(z) = z2 -4z +4$ and $z0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula (1) we obtain

$$\oint_c \frac{z^2-4z+4}{z+i}\,dz = 2\pi i f\left(-i\right) = 2\pi i\left(3+4i\right) = \pi\left(-8+6i\right)$$

Cauchy's Integral Formula

2nd Theorem: Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

EXAMPLE 5: Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}\,dz$, where C is the circle $|z| = 1$.

Solution: Inspection of the integrand shows that it is not analytic at $z = 0$ and $z = -2i$, but only $z = 0$ lies within the closed contour. By writing the integrand as

Cauchy's Integral Formula

EXAMPLE 5: Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}\,dz$, where C is the circle $|z| = 1$.

Solution:

$$\frac{(z+1)}{z^4+2iz^3} = \frac{(z+1)}{z^3(z+2i)} = \frac{\frac{(z+1)}{z+2i}}{z^3}$$

we can identify, $z_0 = 0, n = 2$, and $f(z) = \frac{(z+1)}{(z+2i)}$. The quotient rule gives $f''(z) = \frac{(2-4i)}{(z+2i)^3}$ and so $f''(0) = \frac{(2i-1)}{4i}$. Hence from (6) we find

$$\oint_c \frac{z+1}{z^4+2iz^3}\,dz = \frac{2\pi i}{2!} f''(0)$$

$$= \pi i\left(\frac{2i-1}{4i}\right) = -\frac{\pi}{4} + \frac{\pi i}{2}$$

Cauchy's Integral Formula

EXAMPLE 6: Evaluate $\oint_C \frac{z}{z^2+9}\,dz$ where C is the path:

(a) C_1 : $|z - 3i| = 1$
(b) C_2 : $|z + 3i| = 1$
(c) C_3 : $|z| = 6$

Solution: (a) We have

$$\frac{z}{z^2+9} = \frac{z}{(z+3i)(z-3i)} = \frac{\frac{z}{z+3i}}{z-3i}$$

Z=-3i is outside the circle, $|z - 3i| = 1$. The numerator $\frac{z}{z+3i}$ is analytic inside and on the path C_1 so putting $z_0 = 3i$ in Cauchy's integral formula

Cauchy's Integral Formula

Solution: (a) We have *Z=-3i is outside the circle,* |*z* - 3i| = 1. The numerator $\frac{z}{z+3i}$ is analytic inside and on the path C_1 so putting $z_0 = 3i$ in Cauchy's integral formula

$$\frac{z}{z^2+9} = \frac{z}{(z+3i)(z-3i)} = \frac{\frac{z}{z+3i}}{z-3i}$$

$$\therefore f(z) = \frac{z}{z+3i}$$

$$\oint_c \frac{z}{z^2+9i}\,dz = 2\pi i\left[f(3i)\right]$$

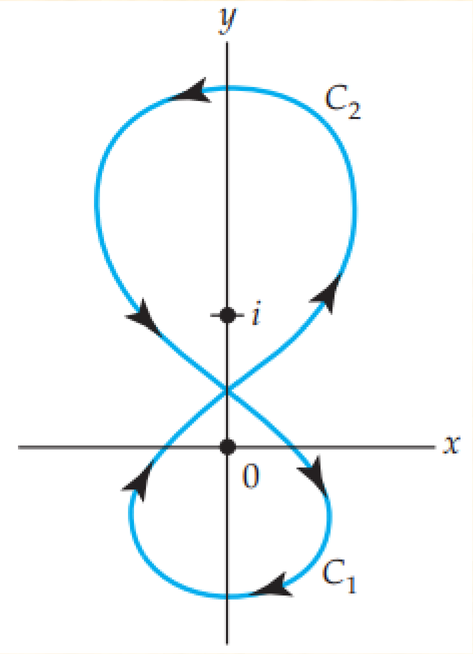
$$= 2\pi i\left[\frac{3i}{3i+3i}\right]$$

$$= 2\pi i\left(\frac{1}{2}\right) = \pi i$$

Solution: (b) and (c) – Do yourself

Cauchy's Integral Formula

EXAMPLE 7: Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2} dz$ where C is the path figure-eight contour shown in Figure 5.45. :



Contour for Example 7

Solution: Although C is not a simple closed contour, we can think of it as the union of two simple closed contours C_1 and C_2 as indicated in Figure 5.45. Since the arrows on $C1$ flow clockwise or in the negative direction, the opposite curve $-C_1$ has positive orientation. Hence, we write

$$\oint_C \frac{z^3+3}{z(z-i)^2} dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2} dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2} dz$$

$$= -\oint_{-C_1} \frac{(z^3+3)/(z-i)^2}{z} dz + \oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2} dz = -I_1 + I_2$$

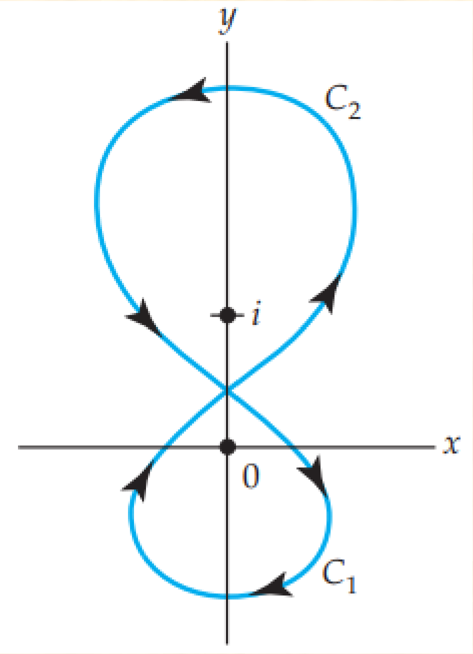
Cauchy's Integral Formula

Solution: $\oint_C \frac{z^3+3}{z(z-i)^2}dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz$

$= -\oint_{-C_1} \frac{(z^3+3)/(z-i)^2}{z}dz + \oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2}dz = -I_1 + I_2$

To evaluate I_1 we identify $z_0 = 0$, $f(z) = \frac{z^3+3}{(z-i)^2}$, and $f(0) = -3$. By 1st theorem, it follows that

$$I_1 = \oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z} = 2\pi i f(0) = 2\pi i(-3) = -6\pi i$$

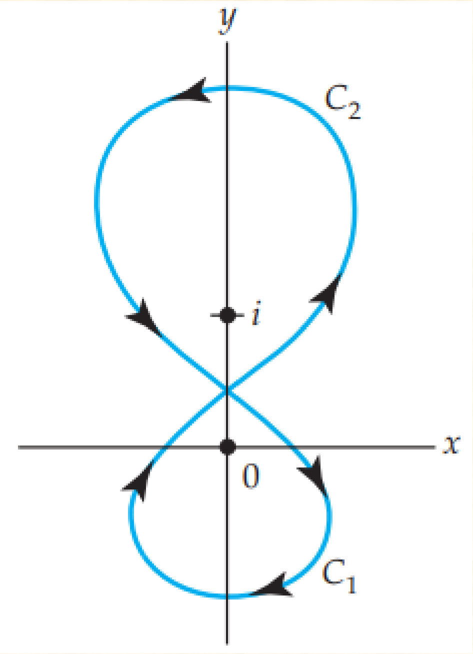


Contour for Example 7

Cauchy's Integral Formula

Solution: $\oint_C \frac{z^3+3}{z(z-i)^2}dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz$

$= -\oint_{-C_1} \frac{(z^3+3)/(z-i)^2}{z}dz + \oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2}dz = -I_1 + I_2$



Contour for Example 7

To evaluate I_2 we now identify $z_0 = i, n = 1, f(z) = \frac{z^3+3}{z}$, $f'(z) = 2z - \frac{3}{z^2}$, and $f'(i) = 3 + 2i$. From 2nd theorem we obtain

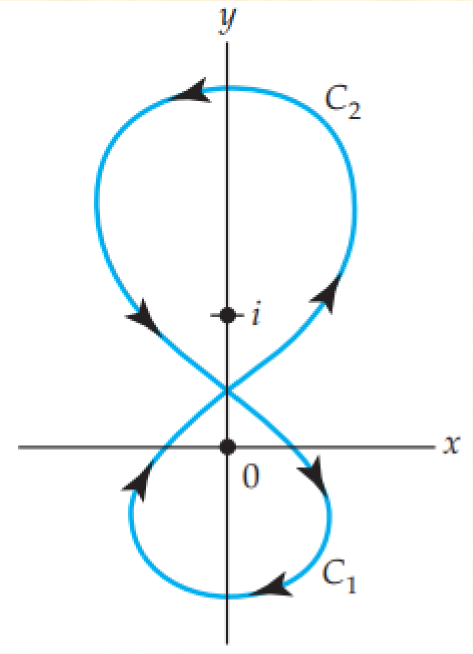
$$I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = \frac{2\pi i}{1!} f(i)$$

$$= 2\pi i(3+2i) = -4\pi + 6\pi i$$

Cauchy's Integral Formula

Solution: $\oint_C \frac{z^3+3}{z(z-i)^2}dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz$

$= -\oint_{-C_1} \frac{(z^3+3)/(z-i)^2}{z}dz + \oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2}dz = -I_1 + I_2$



Contour for Example 7

Finally, we get

$$\oint_C \frac{z^3+3}{z(z-i)^2} = -I_1 + I_2$$

$$= 6\pi i + (-4\pi + 6\pi i)$$

$$= -4\pi + 12\pi i$$

Thanks a lot …