Removable Singularities:

if $w=f(z)$ becomes infinite at the point $z=a$

$g(z)=(z-a)^n f(z)$ where if n makes $g(z)$ analytic

at $z=a$, then the pole has been removed in the form of $g(z)$

Essential Singularities:

Functions that have infinite many terms that becomes unbounded as $z \to a$. Ex. Functions that can be expanded in a descending power series,

$$w = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots + \underbrace{\frac{1}{n!\,z^n} + \dots}_{\text{infinite}}$$

Isolated Singularities:

- Removable Singularities $\downarrow$ $\dfrac{\sin z}{z}$, $\dfrac{1-\cos z}{z^2}$
- Poles of finite Order $\downarrow$ $\dfrac{1}{z}$, $\dfrac{1}{z^2}$, $\dfrac{1}{(z-1)^m}$, $\dfrac{2z+3}{(z-1)^2(z+2)}$
- Essential Singularities (Infinite order) $\downarrow$ $\sin\left(\dfrac{1}{z}\right)$, $e^{1/z}$

Non-Isolated: $f(z) = \dfrac{1}{\sin(1/z)}$

Taylor-series Theorem:

If f is analytic, then around each point z_0 where its analytic, f actually equals a convergent power series

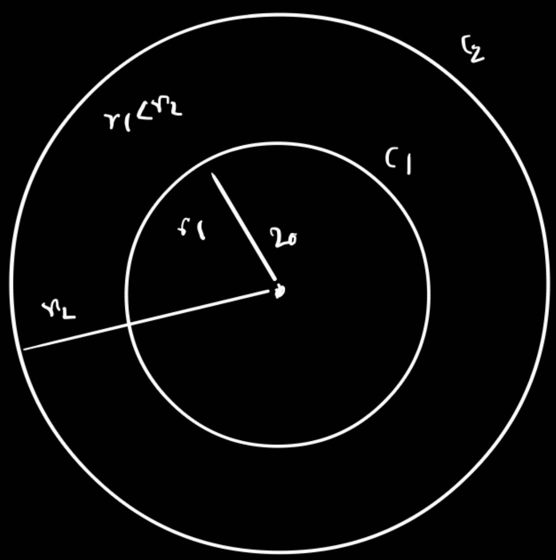
$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n}$$

GPT

formula for co-efficient, $a_n = \dfrac{1}{2\pi i} \oint_c \dfrac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$ (where c is a closed curve around z_0 in A)

$$\boxed{a_n = \frac{f^{(n)}(z_0)}{n!}} = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

Laurent Series:



for any z in the annulus,

$$\boxed{f(z) = \sum_{n=0} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}} \quad \text{—— (i)}$$

$$a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-z_0)^{-n+1}} \, dz$$

Problem-1: Expand $f(z)=\cos z$ as a Taylor series about $z=\pi/4$

Sol^n:-

$f(z)=\cos z$	$f(\pi/4)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$
$f'(z)=-\sin z$	$f'(\pi/4)=-\sin\frac{\pi}{4}=\frac{-1}{\sqrt{2}}$
$f''(z)=-\cos z$	$f''\left(\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=\frac{-1}{\sqrt{2}}$
$f'''(z)=\sin z$	$f'''(\pi/4)=\sin\pi/4=\frac{1}{\sqrt{2}}$

$$\therefore \cos z=\frac{1}{\sqrt{2}}+\left(z-\frac{\pi}{4}\right)\left(\frac{\frac{-1}{\sqrt{2}}}{1!}\right)+\left(z-\frac{\pi}{4}\right)\left(\frac{\frac{-1}{\sqrt{2}}}{2!}\right)+\left(z-\frac{\pi}{4}\right)\left(\frac{\frac{1}{\sqrt{2}}}{3!}\right)+\ldots\ldots$$

Problem-2: $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}=1+\dfrac{3}{z+2}-\dfrac{8}{z+3}$

Sol^n:-

(i) Here singularities at $z=-2$, $z=-3$

from $z_0=0$, the distances are 2 and 3,

regions $|z|<2$, $2<|z|<3$, $|z|>3$

before expansion we take stricter rule, $|z|<2 \therefore \frac{|z|}{2}<1$

$$f(z)=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\left[1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\ldots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\ldots\right]$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$=1+\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right)(z)^n$$

Required Taylor series valid for $|z|<2$

(ii) when $2<|z|<3$ then $\frac{2}{|z|}<1$; $\frac{|z|}{3}<1$

$$f(z) = 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\dots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\dots\right]$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1+\sum_{n=0}^{\infty}(-1)^n\left(\frac{3.2^n}{z^{n+1}}-\frac{8z^n}{3^{n+1}}\right)$$; required Laurent series valid for $2<|z|<3$

(iii) Lauren series valid for $|z|>3$ (left for later)

<u>Problem-3:</u> $f(z)=\frac{1}{z^2-3z+2}$ as Taylor and Laurent series for all possible intervals.

solⁿ:- $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$

(i) when $1<|z|<2$, $\frac{1}{|z|}<1$; $\frac{|z|}{2}<1$;

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{-1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+\dots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right]$$

$$= \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n}$$

(Ans)

(iii) when $|z|>2$ then $\frac{2}{|z|}<1$

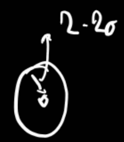
$$f(z)=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}z^{-n}=\sum_{n=0}^{\infty}\frac{2^n-1}{z^{n+1}}$$

(Ans)

Lecture-6:



Residue:

z_0 is isolated singular point of $f(z)$

$f(z)$ has Laurent expansion,

$$\boxed{f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n},\quad 0<|z-z_0|<\delta$$

$$\oint_C f(z)\,dz=\sum_{n=-\infty}^{\infty}c_n\oint_C(z-z_0)^n\,dz$$

$$\left[I_n=\oint_C\frac{1}{(z-z_0)^n}\,dz=\begin{cases}2\pi i, & n=1\\ 0, & n\neq 1\end{cases}\quad (n\in\mathbb{N})\right]$$

Now if $n\neq -1$, then $(z-z_0)^n$ has an anti-derivative locally, so its integral over a closed loop is 0.

If $n=-1$, then we get,

$$\boxed{\oint_C\frac{1}{z-z_0}\,dz=2\pi i}$$

Therefore only $n=-1$ survives,

$$\oint_C f(z)\,dz=c_{-1}\oint_C\frac{1}{z-z_0}\,dz=c_{-1}(2\pi i)\quad\therefore\ \boxed{\frac{1}{2\pi i}\oint_C f(z)\,dz=c_{-1}}$$

Meaning c_{-1} is directly obtained from Laurent integral

c_{-1} is the residue,

Thus, if f has an isolated singularity at z_0, then

$$\boxed{\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \oint_C f(z)\,dz} \qquad \operatorname{Res}(f, z_0) = c_{-1}$$

$\therefore$ Residue = coeff. of $(z-z_0)^{-1}$ in the Laurent series

$\therefore$ Residue = $\frac{1}{2\pi i}$ × the integral of a small loop around z_0.

Notes;

(i) Removable singularity → Res 0

If the singularity is removable, no negative power in Laurent series (Thus just Taylor series)

So, no $(z-z_0)^{-1}$ term.

$$c_{-1} = 0 \Rightarrow \operatorname{Res}(f, z_0) = 0$$

(ii) Essential Singularity → Infinitely many negative powers but only c_{-1} is the residue,

Thus, $\operatorname{Res}(f, z_0) = c_{-1}$

(iii) Pole → Finitely many negative powers (use shortcut)

Laurent's Theorem (LS):

Statement:

Analytical part

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \dots \quad \cdots A$$

Principal part

where,

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz, \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i}\oint_{C_2} (z-a)^{n-1} f(z)\,dz, \quad n = 1, 2, \dots$$

Ex. 1: $f(z) = \dfrac{1}{(z-1)(z-2)}$ for $1 < |z| < 2$ by LS

Soln:- $1 < |z| < 2$

$$\Rightarrow \frac{1}{|z|} < 1 \text{ and } \frac{|z|}{2} < 1$$

To ensure that LS converges, we need to expand by $\frac{1}{|z|}$ or $\frac{|z|}{2}$,

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{(-2)\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$\Rightarrow -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}\right]$$

P.T.O

$$= \frac{-1}{2} - \frac{2}{4} - \frac{2^2}{8} - \frac{2^3}{16} - \cdots - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} \cdots$$

which is required LS where $\frac{1}{|z|} < 1$ and $\frac{|z|}{2} < 1$ QED,

Residue in Complex Numbers

Method:

1. **Rule-1:** Residue at a simple pole is given by,

$$\text{Res}(f, z_0) = \underset{z \to a}{\text{limit}} (z-a) f(z)$$

2. **Rule-2:** Residue at a pole of order n is given by,

$$\text{Res}(f, z_0) = \lim_{z \to a} \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right] \right]_{z=a}$$

Calculation of Residues:

$$a_{-1} = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

$$\boxed{a_{-1} = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]}$$

if $n=1$ (simple pole), then the result

$$a_{-1} = \lim_{z \to a} (z-a) f(z)$$

Ex.1. If $f(z) = \frac{z}{(z-1)(z+1)^2}$ then $z=1$ and $z=-1$ are poles of orders one and two respectively. Res when $z=1$

$$\lim_{z \to 1} (z-1) \left[\frac{z}{(z-1)(z+1)^2} \right] = \frac{1}{4}$$

Res when $z=-1$

$$\lim_{z \to -1} \frac{d}{dz} \left[(z+1)^2 \left(\frac{z}{(z-1)(z+1)^2} \right) \right] = \frac{-1}{4}$$

if $z=a$ is a essential singularity, residue can be found by using known series of expansions.

The residue Theorem;

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\dots)$$

Ex 1: Poles of $\dfrac{4-3z}{z(z-1)(z-2)}$ over C: $|z|=\dfrac{3}{2}$ and residue at each pole.

Solⁿ:-

$$z(z-1)(z-2)=0$$

$z=0,1,2$ where the poles are.

The given circle has radius $r=\dfrac{3}{2}$ and origin at 0. Thus encloses pole $z=1$ as well.

Residue of $f(z)$ at the simple pole, $z=0$

$$\operatorname{Res} f(z) = \lim_{z\to 0}(z-0)f(z)$$

$$= \lim_{z\to 0} z\,\frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{4-0}{(0-1)(0-2)} = 2$$

Now, at simple pole $z=1$

$$\operatorname{Res} f(z) = \lim_{z\to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1}\frac{4-3z}{z(z-2)}$$

$$= \frac{4-3}{-1} = -1$$

(Ans)

Ex.2: evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ over $C: |z| = \frac{1}{2}$

Soln:- From example 1, we got residue for pole $z=0$ and $z=1$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i \times \left(\text{sum of residues inside } |z| = \frac{1}{2}\right)$$

$$= 2\pi i\,(2-1)$$

$$= 2\pi i \quad \text{(Ans)}$$

Ex. 3:

(a) $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$

$\therefore f(z)$ has a double pole at $z=-1$ and simple pole at $z=\pm 2i$

Residue at $z=-1$ is,

$$\lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\left[(z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2(z^2+4)}\right] = \lim_{z\to -1} \frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}$$

$$= \lim_{z\to 1} \frac{5\times(-4)-(1+2)(-2)}{(1+4)^2} = \frac{-14}{25}$$

(Ans)

Residue at $z=2i$

$$\lim_{z\to 2i}\left[(z-2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right] = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

for $z=-2i$ $\operatorname{Res} f(z) = \frac{7-i}{25}$

(b) $f(z) = e^z \operatorname{cosec}^2 z = \dfrac{e^z}{\sin^2 z}$ has double poles at $z = 0, \pm\pi, \pm 2\pi, \ldots$

$z = m\pi$ where $m = 0, \pm1, \pm2, \ldots$

Residue at $z = m\pi$ is

$$\lim_{z \to m\pi} \frac{1}{1!} \frac{d}{dz}\left[(z-m\pi)^2 \frac{e^z}{\sin^2 z}\right]$$

$$F'(z) = e^z \cdot \frac{(z-m\pi)^2}{\sin^2 z} + e^z \cdot \frac{d}{dz}\left(\frac{(z-m\pi)^2}{\sin^2 z}\right)$$

$$\sigma(z) = \frac{(z-m\pi)^2}{\sin^2 z}$$

$$\sigma'(z) = \frac{\sin^2 z \cdot 2(z-m\pi) - (z-m\pi)^2(2\sin z\cos z)}{\sin^4 z}$$

$$F'(z) = e^z \cdot \frac{(z-m\pi)^2 \sin^2 z + 2(z-m\pi)\sin^2 z - 2(z-m\pi)^2 \sin z\cos z}{\sin^4 z}$$

$$= e^z \frac{(z-m\pi)^2 \sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2 \cos z}{\sin^3 z}$$

Now, substituting with u,

$$\lim_{u \to 0} e^{u+m\pi}\left[\frac{u^2 \sin z + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right] = e^{m\pi}\left[\lim_{u\to 0} \frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right]$$

we know,

$$\lim_{u\to 0} \frac{u^3}{\sin^3 u} = \lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3 = 1$$

Then,

$$e^{m\pi} \lim_{u\to 0}\left(\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3} \cdot \frac{u^3}{\sin^3 u}\right) = e^{m\pi} \lim_{u\to 0} \frac{u^2\sin u + 2u\sin u - u^2\cos u}{u^3}$$

[removing $\sin u$] $= e^{m\pi} \times 1$ [using L hopital's several times]

Ex. 4: $f(z) = \dfrac{e^{zt}}{z^2(z^2+2z+2)}$ around $C: |z|=3$

$f(z)$ has double pole at $z=0$, two simple poles at $z=-1\pm i$

residue at $z=0$ is

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left[z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right] = \lim_{z\to 0}\frac{(z^2+2z+2)(te^{zt})-(e^{zt})(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{t-1}{2}$$

residue at $z=-1+i$ is,

$$\lim_{z\to -1+i}\{(z-(-1+i))\}\frac{e^{zt}}{z^2(z^2+2z+2)} = \lim_{z\to -1+i}\left[\frac{e^{zt}}{z^2}\right]\lim_{z\to -1+i}\left[\frac{z+1-i}{z^2+2z+2}\right]$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Similarly, $z=-1-i$

then sum of all,

Ex. 5: $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$

Consider $\oint_C dz/z^6+1$

Now, $z^6+1=0$

$\therefore z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6},$
$e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$

There are simple poles of $1/z^6+1$

Only the poles $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$ lie within

C. Then L'Hopital's rule

$$\text{Res at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}}\left[(z-e^{\pi i/6})\frac{1}{z^6+1}\right]$$

$$= \lim_{z \to e^{\pi i/6}} \left[\cancel{(z - e^{\pi i/6})} \frac{1}{\cancel{(z - e^{\pi i/6})} \cdots} \right] = \lim_{z \to e^{\pi i/6}} \frac{1}{6} \times \frac{1}{z^5} = \frac{1}{6} \times e^{-5\pi i/6}$$

$$\text{Res at } e^{3\pi i/6} = \lim_{z \to e^{3\pi i/6}} \left[\frac{1}{6} \times \frac{1}{z^5} \right] = \frac{1}{6} e^{-15\pi i/6}$$

$$\text{Again at } e^{5\pi i/6} = \frac{1}{6} e^{-25\pi i/6}$$

$$\text{Thus } \oint_C \frac{dz}{z^6+1} = 2\pi i \left[\text{Res sum} \right] = \frac{2\pi}{3}$$

That is,

$$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3}$$

Taking the limit of both sides as $R \to \infty$

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} = 2\int_0^{\infty} \frac{dx}{x^6+1}$$

Ex.7:

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$$

$$\cos\theta + i\sin\theta - \cos\theta + i\sin\theta = 2i\sin\theta$$

Soln:-

$$z = e^{i\theta}, \quad \sin\theta = (e^{i\theta} - e^{-i\theta})/2i = (z - z^{-1})/2i$$

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2i = (z + z^{-1})/2$$

$$\frac{d}{d\theta} z = i e^{i\theta} \Rightarrow dz = iz\, d\theta$$

So that,

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint \frac{dz/iz}{3 - 2(z+z^{-1})/2 + (z - z^{-1})/2i} = \oint_C \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i}$$

where C is the circle of unit radius with center at origin.

Now using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$z = \frac{-6i \pm 4i}{2(1-2i)} = 2-i,\ (2-i)/5$$

only $(2-i)/5$ lies inside C.

Now, for $(2-i)/5$, $= \lim_{z \to (2-i)/5} \left[z-(2-i)/5 \right] \left[\frac{2}{(1-2i)z^2+6iz-1-2i} \right]$

$$= \lim_{z \to (2-i)/5} \frac{2}{2(1-2i)z+6i} = \frac{1}{2i}$$

By L's Hopital.

Then

$$\oint \frac{2dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i \left(\frac{1}{2i} \right) = \pi. \text{ (Ans)}$$

Ex. 8:

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}$$

$z = e^{i\theta}$, then $\cos\theta = (z+z^{-1})/2$

$$\cos 3\theta = (e^{3i\theta} + e^{-3i\theta})/2$$

$$= (z^3 + z^{-3})/2$$

$dz = zi\,d\theta$

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2} \frac{dz}{iz} = \frac{-1}{2i} \oint \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz$$

Res at $z=0$,

$$\lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left[z^3 \frac{z^6+1}{z^3(2z-1)(z-2)} \right] = \frac{21}{8}$$

Res at $z = \frac{1}{2}$