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The asymptotical distribution of the mutual winding angles of several independent planar Wiener processes was studied in the work of M. Yor. A more general question is to study the asymptotical distribution of the full system of Vassiliev invariants for braids formed by independent planar Wiener processes. This problem is considered in the current article.

1. Introduction

A lot of works were devoted to the study of geometrical properties of planar Brownian motion trajectory (see, e.g., [1], [2], [3]). The asymptotical distribution of the winding angle of the Brownian trajectory was studied by F. Spitzer [4]. The more subtle results on Brownian winding angle were obtained in the works of Zhan Shi [5], J. Bertoin and W. Werner [6].

The related geometrical questions arise in the theory of stochastic flows. In isotropic Brownian stochastic flows the individual trajectories are Brownian motions. A question of great importance is to study the multi-particle motion in these flows. This question is also motivated by the study of turbulence, as Brownian stochastic flows are used to model turbulent flows. The characterization of an n-point motion in such flows is the problem of great importance in turbulence theory [7].

A natural question concerning multi-particle motion in Brownian flows is to characterize braids formed by particle trajectories. There is a full system of invariants characterizing braids, that is, the Vassiliev invariants system. This system distinguishes braids up to homotopy preserving beginning and endpoints. Maxim Kontsevich established integral formulas for Vassiliev invariants for smooth knots. These formulas can be generalised for smooth braids [8]. In the present article we study the asymptotical distribution of invariants of braids formed by independent planar Wiener processes. In section 2 we discuss Vassiliev invariants and Kontsevich integral. In section 3 we obtain the representation of Vassiliev invariants for braids formed by continuous semimartingales under common flitration in the form of multiple Stratonovich integrals. In section 4 we discuss some facts about the weak asymptotics of stochastic integrals. In section 5 we use these results to prove the main result about the asymptotical distribution of Vassiliev invariants for braids formed by independent Brownian motions.

2. Kontsevich integral for smooth braids

We give the necessary definitions concerning braids and their invariants.

Definition 1. *[8] A braid with n strands is a continuous curve*

$$Z(t) = (Z_1(t), \dots, Z_n(t)), t \in [0, T], Z_k(t) \in \mathbb{C}, t \in [0, T]$$

in the space $C_{0,n} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j, i \neq j\}$, that is, a continuous mapping from $[0, T]$ to the space $C_{0,n}$. The trajectories $Z_k(t), t \in [0, T]$ are called the strands of a braid.

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As follows from the definition, the braid's strands do not intersect, that is, for any $k \neq l$:

$$\forall t \in [0, T] \quad Z_k(t) \neq Z_l(t).$$

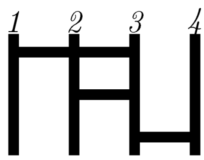
There exists a system of Vasssiliev invariants for braids which is full in a sence that for two braids with the same starting and endpoints all the invariants coinside if and only if the braids are homotopous with the homotopy preserving starting and endpoints [10]. There is an integral representation for Vassiliev invariants for smooth or piecewise-smooth braids given by M. L. Kontsevich [9]. We describe it briefly based on the article [8]. These invariants take values in the space of diagrams. So, we first describe objects called diagrams.

Let $\mathbb{P}_{mn}$ be the set of all possible matrices

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \\ \dots & \dots \\ P_{m1} & P_{m2} \end{pmatrix}$$

of the size $m \times 2$, where for all $i = 1, \dots, m \quad P_{i1}, P_{i2} \in \{1, \dots, n\}, P_{i1} \neq P_{i2}$. It is obvious that $|\mathbb{P}_{mn}| = (n(n-1)/2)^m$. Here n is a number of strands of the braid in concern. Put in correspondence to any matrix $P \in \mathbb{P}_{mn}$ some object $D(P)$ — "the diagram". The diagram $D(P)$ corresponding to the matrix $P \in \mathbb{P}_{mn}$ consists of n vertical segments corresponding to the braid's strands, and of connecting them horizontal segments representing the rows of the matrix P. A horizontal segment representing the ith row $(P_{i1}P_{i2})$ connects vertical segments with numbers P_{i1} and P_{i2}. For $i < j$ the segment which corresponds to the ith row is higher then the segment which corresponds to the jth row.

Example 1. *Let $n = 4$ (the braid for 4 braids), $m = 3$, and the matrix $P \in \mathbb{P}_{34}$ has the form* $P = \begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}$. *Then the diagram $D(P)$ has the form*



We define addition on diagrams which will obey the following relations.

- One-term relation: let matrix P' be obtained from $P \in \mathbb{P}_{mn}$ by interchanging two rows:

$$P = \begin{pmatrix} \dots & \dots \\ \dots & \dots \\ i & j \\ k & l \\ \dots & \dots \\ \dots & \dots \end{pmatrix}, P' = \begin{pmatrix} \dots & \dots \\ \dots & \dots \\ k & l \\ i & j \\ \dots & \dots \\ \dots & \dots \end{pmatrix},$$

where i, j, k, l correspond to four pairwise distinct strands. Then $D(P) = D(P')$.

- Four-term relation. Let matrices $P_1, P_2, P_3, P_4, P_5, P_6 \in \mathbb{P}_{mn}$ be:

$$P_1 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ i & j \\ j & k \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}, P_2 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ j & k \\ i & k \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}, P_3 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ i & k \\ i & j \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix},$$

$$P_4 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ j & k \\ i & j \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}, P_5 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ i & k \\ j & k \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}, P_6 = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ i & j \\ i & k \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix},$$

  where i, j, k correspond to four pairwise distinct strands. Then

$$D(P_1) - D(P_4) = D(P_2) - D(P_5) = D(P_3) - D(P_6).$$

Now introduce the following definition.

Definition 2. *Let V be the complex vvector space of the dimension $(n(n-1)/2)^m$ with the basis $\{D(P), P \in \mathbb{P}_{mn}\}$, that is, the space of formal finite linear combinations of the diagrams $D(P), P \in \mathbb{P}_{mn}$ with coefficients from $\mathbb{C}$. We consider its subspace $U \subset V$, that is the span of vectors of the form $D(P) - D(P'), D(P_1) - D(P_4) - D(P_2) + D(P_5), D(P_1) - D(P_4) - D(P_3) + D(P_6)$ for all possible matrices $P, P', P_1, P_2, P_3, P_4, P_5, P_6$ of the form we mentioned in the definition of one-term and four-term relations. The factorspace of the space V over its subspace U is called the space of the diagrams of the order m.*

Definition 3. *Any linear function on the space of the diagrams of the order m is called the system of weights of the order m.*

We shall also need the definition of the product of the diagrams.

Definition 4. *The product of the diagrams $D(P)$ and $D(P')$, where*

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \\ \cdots & \cdots \\ P_{r1} & P_{r2} \end{pmatrix}, P' = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \\ \cdots & \cdots \\ Q_{s1} & Q_{s2} \end{pmatrix},$$

is the diagram $D(P'')$, where

$$P'' = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \\ \cdots & \cdots \\ P_{r1} & P_{r2} \\ Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \\ \cdots & \cdots \\ Q_{s1} & Q_{s2} \end{pmatrix}.$$

We shall write: $D(P'') = D(P) \times D(P')$.

Now we can give the definition of the Kontsevich integral.

Definition 5. *[8] The Kontsevich integral of the order m for a piecewise-smooth braid $Z(t) = (Z_1(t), \ldots, Z_n(t))$ is the following element of the space of diagrams of the order m:*

$$K_m = \sum_{P\in\mathbb{P}_{mn}} \int\limits_{\Delta_m} \omega_{P_{11}P_{12}}(t_1) \wedge \ldots \wedge \omega_{P_{m1}P_{m2}}(t_m) D(P),$$

where $\Delta_m = \Delta_m(T) = \{(t_1, \ldots, t_m) \mid 0 \le t_1 \le \cdots \le t_m \le T\}$,

$$\omega_{kl}(t) = \omega_{lk}(t) = \frac{1}{2\pi i}\frac{dZ_k(t) - dZ_l(t)}{Z_k(t) - Z_l(t)}.$$

The numerical invariants of the order m (we shall call them, as in [8], the invariants of the order m) are obtained from the integrals K_m and linear functions on the space of diagrams of the order m with the help of exchange of diagrams on the corresponding weights (thus, any system of weights defines its own invariant).

Example 2. *The invariants of the first order are the all possible linear combinations of the variable*

$$\lambda_{kl}(T) = \tfrac{1}{2\pi i} \int\limits_0^T \tfrac{dZ_k(t')-dZ_l(t')}{Z_k(t')-Z_l(t')} = \tfrac{1}{2\pi i}\left(\ln \tfrac{R_{kl}(T)}{R_{kl}(0)} + i(\Phi_{kl}(T) - \Phi_{kl}(0))\right),$$

where $R_{kl}(t) = |Z_k(t) - Z_l(t)|$, $\Phi_{kl}(t)$ is a continuous on t version of the argument of the complex number $Z_k(t) - Z_l(t)$. In other words, $\Phi_{kl}(T) - \Phi_{kl}(0)$ is an angle that the strand Z_k wounds around the strand Z_l up to time T.

Example 3. *[8] An example of the invariant of the first order is the following invariant for the smooth braid of three strands $(Z_1(t), Z_2(t), Z_3(t)), t \in [0, T]$:*

$$\Psi_{123} = \frac{1}{2}\int_0^T (\lambda_{12}(s)d\lambda_{13}(s) - \lambda_{13}(s)d\lambda_{12}(s)) + \frac{1}{2}\int\limits_0^T (\lambda_{13}(s)d\lambda_{23}(s) - \lambda_{23}(s)d\lambda_{13}(s)) +$$
$$+\frac{1}{2}\int\limits_0^T (\lambda_{23}(s)d\lambda_{12}(s) - \lambda_{12}(s)d\lambda_{23}(s)),$$

where $\lambda_{kl}(t) = \frac{1}{2\pi i}\int\limits_0^t \frac{dZ_k(t')-dZ_l(t')}{Z_k(t')-Z_l(t')}$.

3. Vassiliev invariants for random braids

In this section we obtain the expressions for the Vassiliev invariants for braids formed by continuous semimartingales. First we will consider the invariants for non-smooth braids.

Definition 6. *Let Z be the braid formed by continuous curves $Z_k(t), t \in [0, T], k = 1, \ldots, n$. Let $\tau = \{0 = t_0 < t_1 < \cdots < t_p = T\}$ be some partition of the interval $[0, T]$. Consider the polygonal lines Z_k^τ with vertices $Z_k(t_0), \ldots, Z_k(t_p)$ inscribed into the curve Z_k. We shall refer to the braid formed by polygonal lines Z_k^τ as to the braid inscribed into the braid Z on the partition τ and denote it by Z^τ.*

Remark 1. *The polygonal lines Z_k^τ can intersect and thus to form no braid. But it is clear that for smooth enough partitions τ this situation will not occur, and we will not make such reservation further.*

The following statement takes place.

Theorem 1. *Any Vassiliev invariant for the continuous braid Z is the limit of the corresponding invariants for the sequence of the inscribed braids Z^{τ_l}, if the diameters of the partitions τ_l tend to 0.*

Proof. Let $Z(t) = (Z_1(t), \ldots, Z_n(t)), t \in [0, T]$ be the braid. Let us show that for fine enough partitions of the time interval the braids formed by the polygonal lines built over this partition are homotopous to the braid Z.

The following well-known lemma is used in the proof.

Lemma 1. *(the Lebesgue number lemma, [12, p. 179]). For any covering of the compact metric space X by the open sets there exists $\varepsilon > 0$ such that for any $x \in X$ the ball $B_\varepsilon(x)$ is contained in one of the sets of the covering.*

Let U_α be the covering $\mathbb{C}^n \setminus \{\exists i \neq j : z_i = z_j\}$ by the open balls. Then the sets $Z^{-1}(U_\alpha)$ form an open covering of $X = [0, T]$. For this covering we will choose $\varepsilon > 0$ from lemma. Beginning from some fineses of the partition, for all i we have $|t_{i+1} - t_i| < \varepsilon$, and any interval of the partition $[t_i, t_{i+1}]$ is mapped into the fixed set U_α. Then the change of the trajectory $Z(t)$ on any interval $[t_i, t_{i+1}]$ for the segment with the ends $Z(t_i), Z(t_{i+1})$ changes the trajectory for the homotopous to it. To say more precisely, the homotopy is realized by the family of the curves $Z_\mu(t), \mu \in [0, 1], t \in [0, T]$, where

$$Z_\mu(t) = (1 - \mu)Z(t) + \mu \frac{(t - t_i)Z(t_{i+1}) + (t_{i+1} - t)Z(t_i)}{t_{i+1} - t_i}, t \in [t_i, t_{i+1}].$$

When $\mu = 0$ we have the initial curve, when $\mu = 1$ we have the polygonal line with the ends in points $Z(t_i)$. $\square$

Remark 2. *It follows from the proof that not only all the braid and the corresponding polygonal line will be homotopous, but also all the braids that correspond to time intervals $[0, t_i]$. That is, the braids*

$$Z(t)|_{t \in [0, t_i]} = (Z_1(t), \ldots, Z_n(t)), t \in [0, t_i],$$

will be homotopous to the corresponding polygonal lines. This fact will be used in the future considerations.

For any two nonintersecting trajectories $Z_1(t), Z_2(t)$, $t \in [0, T]$ the function

$$\lambda_{12}(t) = \frac{1}{2\pi}\Phi_{12}(t) - \frac{i}{2\pi} \ln \frac{R_{12}(t)}{R_{12}(0)}$$

is defined, where $R_{12}(t) = |Z_1(t) - Z_2(t)|$, a $\Phi_{12}(t)$ is the winding angle of the trajectory Z_2 around Z_1 up to time t, i.e. the winding angle of the trajectory $Z(t) = Z_2(t) - Z_1(t)$ around zero.

We give the proof of the following fact as it was not found in the literature in this formulation.

Statement 1. *Any numerical invariant L_m of the order m for the piecewise-smooth braid $Z(t)$s can be expressed as the sum of the form*

$$\sum_i \int_0^T L^i_{m-1}(t)d\lambda_i(t),$$

where L^i_{m-1} are some invariants of the order $m-1$ for the same braid, $\lambda_i = \lambda_{kl}$ for some $k \neq l$, where the functions λ_{kl} were introduces previously.

Proof. An invariant L_m is obtained from some Kontsevich integral K_m of the order m by replacing diagrams for the corresponding weights. We have

$$K_m = \sum_{P\in\mathbb{P}_{mn}} \int_{\Delta_m} \omega_{P_{11}P_{12}}(t_1)\dots\omega_{P_{m1}P_{m2}}(t_m)D(P) =$$

$$= \sum_{i\neq j}\sum_{P\in\mathbb{P}_{m-1,n}} \int_0^T \left\{ \int_{\Delta_{m-1}(t_m)} \omega_{P_{11}P_{12}}(t_1)\dots\omega_{P_{m-1,1}P_{m-1,2}}(t_{m-1}) \right\} \omega_{ij}(t_m)\times$$

$$\times (D(P)\times D(P^{ij})),$$

where $P^{ij} = \begin{pmatrix} i & j \end{pmatrix}$.

For the fixed weights system, putting in correspondence to any diagram D the number $w(D)$, we get for the corresponding invariant $L_m = w(K_m)$:

$$L_m = \sum_{i\neq j}\sum_{P\in\mathbb{P}_{m-1,n}} \int_0^T \left\{ \int_{\Delta_{m-1}(t)} \omega_{P_{11}P_{12}}(t_1)\dots\omega_{P_{m-1,1}P_{m-1,2}}(t_{m-1}) \right\} \omega_{ij}(t)\times$$

$$\times\, w(D(P)\times D(P^{ij})).$$

The system of weights $w'(D(P)) = w(D(P)\times D(P'))$ on the diagrams $P \in \mathbb{P}_{m-1,n}$ is correctly defined, as the one-term and four-term relations for it are implied by the same relations for the weights $w(D(P)\times D(P'))$. Thus, for fixed i, j every sum

$$\sum_{P\in\mathbb{P}_{m-1,n}} \int_{\Delta_{m-1}(t_m)} \omega_{P_{11}P_{12}}(t_1)\dots\omega_{P_{m-1,1}P_{m-1,2}}(t_{m-1})w(D(P)\times D(P^{ij}))$$

is itself an invariant (of the order $m-1$). The proof is complete. □

Let us consider the braid Z, formed by continuous curves

$$Z_k(t), t\in[0,T], k = 1,\dots,n,$$

and the sequence of partitions $\tau = \tau_p = \{0 = t_0 < t_1 < \dots < t_p = T\}$ of the interval $[0,T]$ with $|\tau_p| \to 0, p\to\infty$. Let $L(t)$ be some invariant of the order m for the braid $Z(s), 0\le s\le t$ formed by the curves $Z_k(s), 0\le s\le t$, $\lambda = \lambda_{kl}$ for some $k\neq l$. Let us consider the polygonal lines Z^τ_k inscribed into the curves Z_k with vertices $Z_k(t_0),\dots,Z_k(t_p)$, and let Z^τ be the braid formed by these polygonal lines. Denote by $L_\tau(t)$ the value of the invariant in question on the braid $Z^\tau(s), 0\le s\le t$, and let $\lambda_\tau(t)$ be the corresponding to $\lambda(t)$ function for this polygonal line. In this situation the following takes place.

Theorem 2. *If for any k the sums $\sum\limits_{i=0}^{p-1}|Z_k(t_{i+1})-Z_k(t_i)|^2$ are bounded uniformly over partitions τ_p, then the following convergence takes place:*

$$\sum_{i=0}^{p-1}\frac{L(t_i)+L(t_{i+1})}{2}(\lambda(t_{i+1})-\lambda(t_i))-\int\limits_0^T L_{\tau_p}(t)d\lambda_{\tau_p}(t)\to 0, p\to\infty. \tag{1}$$

Remark 3. *The functions L_τ, λ_τ are piecewise smooth. More precisely, they are differentiable everywhere except the points t_i, where they have right-side and left-side derivatives. This differentiability is implied by the explicit expression of the Vassiliev invariants in the form of the integrals, which holds for the piecewise-smooth curves, exactly as the braid Z_τ is.*

Remark 4. *The condition of the theorem holds for the braid whose strands $Z_1,\ldots,Z_n$ are Hölder continuous with exponent $1/2$.*

Let us prove the theorem 2.

Proof. We first mention that due to homotopical invariance of $L(t),\lambda(t)$ and the remark 2, for all fine enough partitions $\tau=\tau_p$ from the sequence in question the equalities

$$L(t_i)=L_\tau(t_i), \lambda(t_i)=\lambda_\tau(t_i)$$

are satisfied for any i. All the subsequent estimations will be conducted just for these small enough partitions. Replace any L_τ for its piecewise-linear version $\tilde{L}$: for any i $L_\tau(t_i)=\tilde{L}(t_i)$, $\tilde{L}(t)$ is linear on$[t_i,t_{i+1}]$.

We estimate the difference:

$$\left|\int_0^T (L_\tau(t)-\tilde{L}(t))d\lambda_\tau(t)\right| \le \sum_{i=0}^{p-1}\max_{t\in[t_i,t_{i+1}]}|L_\tau(t)-\tilde{L}(t)||\lambda(t_{i+1})-\lambda(t_i)| \le$$

$$\le \sum_{i=0}^{p-1}\max_{t\in[t_i,t_{i+1}]}|L''_\tau(t)|(t_{i+1}-t_i)^2|\lambda(t_{i+1})-\lambda(t_i)|. \tag{2}$$

Indeed, we have $L_\tau(t_i)=\tilde{L}(t_i), L_\tau(t_{i+1})=\tilde{L}(t_{i+1})$. According to Lagrange's theorem applied to the function L_τ differentiable on the line segment $[t_i,t_{i+1}]$ (see reference 3), there exists $\tilde{t}\in[t_i,t_{i+1}]$ such that

$$L'_\tau(\tilde{t})=\frac{L_\tau(t_{i+1})-L_\tau(t_i)}{t_{i+1}-t_i}=\tilde{L}'(\tilde{t}),$$

and thus for all $t\in[t_i,t_{i+1}]$:

$$|L'_\tau(t)-\tilde{L}'(t)|=|L'_\tau(t)-\tilde{L}'(\tilde{t})|=|L'_\tau(t)-L'_\tau(\tilde{t})|\le$$

$$\le \max_{[t_i,t_{i+1}]}|L''_\tau(t)||t-\tilde{t}|\le \max_{[t_i,t_{i+1}]}|L''(t)||t_{i+1}-t_i|.$$

From here we get for $t\in[t_i,t_{i+1}]$:

$$|L_\tau(t)-\tilde{L}(t)|=\left|\int\limits_{t_i}^{t}(L'_\tau(s)-\tilde{L}'(s))ds\right|\le$$

$$\le \max_{s\in[t_i,t_{i+1}]}|L'_\tau(s)-\tilde{L}'(s)||t_{i+1}-t_i|\le \max_{s\in[t_i,t_{i+1}]}|L''_\tau(s)|(t_{i+1}-t_i)^2.$$

Thus, the inequality (2) is established.

The following estimation on $L''_\tau(t), t \in [t_i, t_{i+1}]$ takes place:

$$|L''_\tau(t)| \le C \max_{k=1,\dots,n} \frac{\Delta X_k(t_i)^2 + \Delta Y_k(t_i)^2}{\Delta t_i{}^2}, \tag{3}$$

where X_k, Y_k are the coordinates of the kth strand of the braid Z, i.e. $Z_k(t) = X_k(t) + iY_k(t)$, $\Delta X_k(t_i) = X_k(t_{i+1}) - X_k(t_i)$, $\Delta Y_k(t_i) = Y_k(t_{i+1}) - Y_k(t_i)$, $C > 0$ is some constant that depends only on the initial braid (but not on the partition). Indeed, $L_\tau(t)$ is expressed in the form of the sum of the integrals of the form $\int\limits_0^T L_\tau^{m-1}(t)d\lambda_\tau(t)$, where $L_\tau^{m-1}(t)$ are some invariants of the order $m-1$ (for the polygonal line approximating our braid), $\lambda_\tau(t) = \lambda_{kl}^\tau(t)$ for some $k \ne l, 1 \le k, l \le n$ are the invariants of the first order (also for the polygonal braids). We have for $t \in [t_i, t_{i+1}]$

$$\left(\int\limits_0^t L_\tau^{m-1}(t)d\lambda_\tau(t)\right)'' = (L_\tau^{m-1}(t)\lambda'_\tau(t))' = (L_\tau^{m-1}(t))'\lambda'_\tau(t) + L_\tau^{m-1}(t)\lambda''_\tau(t)). \tag{4}$$

Denote $X_{kl}(t) = X_l(t) - X_k(t), Y_{kl}(t) = Y_l(t) - Y_k(t)$. We have

$$\lambda'_\tau(t) = (\lambda_\tau^{kl})'(t) = \frac{1}{2\pi}\left(\frac{X_{kl}Y'_{kl} - Y_{kl}X'_{kl}}{X_{kl}^2 + Y_{kl}^2} - i\frac{X_{kl}X'_{kl} + Y_{kl}Y'_{kl}}{X_{kl}^2 + Y_{kl}^2}\right).$$

Taking into account that $X''_l = Y''_l = X''_k = Y''_k = 0$, we get

$$\begin{aligned}2\pi\lambda''_\tau(t) = {} & \frac{X_{kl}Y''_{kl} - Y_{kl}X''_{kl}}{X_{kl}^2 + Y_{kl}^2} - 2\frac{(X_{kl}Y'_{kl} - Y_{kl}X'_{kl})(X_{kl}X'_{kl} + Y_{kl}Y'_{kl})}{(X_{kl}^2 + Y_{kl}^2)^2} - \\ & - i\frac{X_{kl}X''_{kl} + Y_{kl}Y''_{kl} + 2X'_{kl}Y'_{kl}}{X_{kl}^2 + Y_{kl}^2} + 2i\frac{(X_{kl}X'_{kl} + Y_{kl}Y'_{kl})(X_{kl}X'_{kl} + Y_{kl}Y'_{kl})}{(X_{kl}^2 + Y_{kl}^2)^2} = \\ & = -2\frac{(X_{kl}Y'_{kl} - Y_{kl}X'_{kl})(X_{kl}X'_{kl} + Y_{kl}Y'_{kl})}{(X_{kl}^2 + Y_{kl}^2)^2} - 2i\frac{X'_{kl}Y'_{kl}}{X_{kl}^2 + Y_{kl}^2} + 2i\frac{(X_{kl}X'_{kl} + Y_{kl}Y'_{kl})^2}{(X_{kl}^2 + Y_{kl}^2)^2},\end{aligned}$$

and we get for some $C_1, C_2 > 0$ for all fine enough partitions when $t \in [t_i, t_{i+1}]$

$$\begin{aligned}|\lambda''_\tau(t)| \le {} & C_1(X'^2_l + Y'^2_l + X'^2_k + Y'^2_k) = \\ & = C_1\frac{\Delta X_k(t_i)^2 + \Delta Y_k(t_i)^2 + \Delta X_l(t_i)^2 + \Delta Y_l(t_i)^2}{\Delta t_i^2} \le \\ & \le C_2 \max_{k=1,\dots,n} \frac{\Delta X_k(t_i)^2 + \Delta Y_k(t_i)^2}{\Delta t_i^2}.\end{aligned}$$

The estimation of $(L_\tau^{m-1}(t))'$ is made in the similar way. From these estimates and from (4) we get (3). From it and from (2), taking into account the piecewise linearity of X, Y, we get

$$\begin{aligned}\left|\int\limits_0^T (L_\tau(t) - \tilde{L}(t))d\lambda_\tau(t)\right| \le {} & C\sum_{i=0}^{p-1} \max_{[t_i,t_{i+1}]} |L''_\tau(t)|(t_{i+1} - t_i)^2|\lambda(t_{i+1}) - \lambda(t_i)| \le \\ & \le C' \max_{i=0,\dots p-1} |\lambda(t_{i+1}) - \lambda(t_i)| \sum_{k=1}^n \sum_{i=0}^{p-1}(\Delta X_k(t_i)^2 + \Delta Y_k(t_i)^2) \xrightarrow[p\to\infty]{} 0\end{aligned}$$

due to the sum $\sum_{i=0}^{p-1}(\Delta X_k(t_i)^2+\Delta Y_k(t_i)^2)$ being uniformly bounded. Further, replace $\lambda_\tau(\cdot)$ by piecewise-smooth version $\tilde{\lambda}(\cdot)$. We have

$$\int\limits_0^T \tilde{L}(t)d\lambda_\tau(t)=\tilde{L}(T)\lambda_\tau(T)-\int\limits_0^T \lambda_\tau(t)d\tilde{L}(t).$$

The difference

$$\left|\int\limits_0^T \lambda_\tau(t)d\tilde{L}(t)-\int_0^T \tilde{\lambda}(t)d\tilde{L}(t)\right|$$

is estimated similarly to the difference

$$\left|\int\limits_0^T (L_\tau(t)-\tilde{L}(t))d\lambda_\tau(t)\right|.$$

Thus, we get:

$$\int\limits_0^T \tilde{L}(t)d\tilde{\lambda}(t)-\int\limits_0^T L_\tau(t)d\lambda_\tau(t)\to 0, p\to\infty.$$

But,

$$\int\limits_0^T \tilde{L}(t)d\tilde{\lambda}(t)=\sum_{i=0}^{p-1}\frac{L(t_i)+L(t_{i+1})}{2}(\lambda(t_{i+1})-\lambda(t_i)).$$

This finishes the proof of the theorem 2. □

Now we proceed to the braids formed by continuous semimartingales. First consider the invariants of the first order.

Lemma 2. *Let X_t, Y_t be such continuous semimartingales with respect to the common filtration $(\mathfrak{F}_t), t\in[0,T]$, that with probability* 1

$$\forall t\in[0,T]\quad Z(t)=X(t)+iY(t)\neq 0.$$

Then the processes $\ln R(t)=\ln|Z(t)|$ and the winding angle $\Phi(t)$ of the process Z around the origin are all semimartingales with respect to $(\mathfrak{F}_t), t\in[0,T]$.

Proof. Using Ito's formula, we get for the process

$$R_t^2=X_t^2+Y_t^2$$

the equality

$$R_t^2=R_0^2+2\int\limits_0^t X_s dX_s+2\int\limits_0^t Y_s dY_s+\langle X\rangle_t+\langle Y\rangle_t,$$

and the process R^2 is a continuous semimrtingale with the characteristics

$$\langle R^2\rangle_t=4\int\limits_0^t X_s^2 d\langle X\rangle_s+4\int\limits_0^t Y_s^2 d\langle Y\rangle_s+8\int\limits_0^t X_sY_s d\langle X,Y\rangle_s.$$

For the process $\ln R(t) = \frac{1}{2}\ln R(t)^2$ with the help of Ito's formula we now obtain

$$\ln R(t) = \ln R(0) + \int_0^t \frac{1}{R_s^2}(X_s dX_s + Y_s dY_s) -$$

$$- \int_0^t \frac{X_s^2 - Y_s^2}{2R_s^4}(d\langle X\rangle_s - d\langle Y\rangle_s) - \int_0^t \frac{4}{R_s^4} X_s Y_s d\langle X, Y\rangle_s .$$

As Ito's integrals of continuous semimartingales with respect to continuous semimartingales are continuous semimartingales ([13], p. 58), then $\ln R(t)$ is a continuous semimartingale.

Now we show that $\Phi(t)$ is also a continuous semimartingale. We consider a sequence of partitions $\tau_l : 0 = t_0 < \ldots < t_l = T$ of the segment $[0, T]$ with finenes $|\tau_l|$ that converges to zero. For large enough $l \geq l_0 = l_0(\omega)$ all the line segments of the trajectories Z on the intervals $[t_i, t_{i+1}]$ will be contained in one of the sets $\{(x, y)\colon x > 0\}$, $\{(x, y)\colon x < 0\}$, $\{(x, y)\colon y > 0\}$, $\{(x, y)\colon y < 0\}$. Thus, for large enough $l \geq l_0 = l_0(\omega)$ the increments $\Phi(t_{i+1}) - \Phi(t_i)$ will coincide with the increments of one of the functions $\operatorname{arctg}\frac{Y_s}{X_s}$, $\operatorname{arcctg}\frac{X_s}{Y_s}$. Thus, we conclude that for large enough $l \geq l_0(\omega)$ $\Phi(t) - \Phi(0)$ for all $t \in [0, T]$ coincides with one of the 2^l sums

$$S_J = \sum_{i=0}^{l-1}(f_{j_i}(X(t_{i+1}\wedge t), Y(t_{i+1}\wedge t)) - f_{j_i}(X(t_i \wedge t), Y(t_i \wedge t))),$$

where $f_1(x, y) = \operatorname{arctg}\frac{y}{x}$, $f_2(x, y) = \operatorname{arcctg}\frac{x}{y}$, and the set $J = (j_1, \ldots, j_l)$ runs over all 2^l possible values from $\{1, 2\}^l$. applying the Ito's formula, we get

$$d\frac{Y_s}{X_s} = -\frac{Y_s}{X_s^2}dX_s + \frac{1}{X_s}dY_s + \frac{Y_s}{X_s^3}d\langle X\rangle_s - \frac{1}{X_s^2}d\langle X, Y\rangle_s ,$$

and therefore

$$d\left\langle \frac{Y}{X}\right\rangle_s = \frac{Y_s^2}{X_s^4}d\langle X\rangle_s + \frac{1}{X_s^2}d\langle Y\rangle_s - 2\frac{Y_s}{X_s^3}d\langle X, Y\rangle_s .$$

From the Ito's formula and the expressions obtained we get

$$d\operatorname{arctg}\frac{Y_s}{X_s} = \frac{X_s^2}{X_s^2 + Y_s^2}d\frac{Y_s}{X_s} - \frac{X_s^3 Y_s}{(X_s^2 + Y_s^2)^2}d\left\langle \frac{Y}{X}\right\rangle_s =$$

$$= \frac{X_s dY_s - Y_s dX_s}{X_s^2 + Y_s^2} + \frac{X_s Y_s}{(X_s^2 + Y_s^2)^2}(d\langle X\rangle_s - d\langle Y\rangle_s) - \frac{X_s^2 - Y_s^2}{(X_s^2 + Y_s^2)^2}d\langle X, Y\rangle_s .$$

The analagous computations allow to obtain the formula

$$d\operatorname{arcctg}\frac{X_s}{Y_s} =$$

$$= \frac{X_s dY_s - Y_s dX_s}{X_s^2 + Y_s^2} + \frac{X_s Y_s}{(X_s^2 + Y_s^2)^2}(d\langle X\rangle_s - d\langle Y\rangle_s) - \frac{X_s^2 - Y_s^2}{(X_s^2 + Y_s^2)^2}d\langle X, Y\rangle_s .$$

Thus, for all $l_0 \geq l_0(\omega)$ we have

$$\Phi(t) - \Phi(0) =$$
$$= \int\limits_0^t \frac{X_s dY_s - Y_s dX_s}{X_s^2 + Y_s^2} + \int\limits_0^t \frac{X_s Y_s}{(X_s^2 + Y_s^2)^2} (d\left\langle X \right\rangle_s - d\left\langle Y \right\rangle_s) - \int\limits_0^t \frac{X_s^2 - Y_s^2}{(X_s^2 + Y_s^2)^2} d\left\langle X, Y \right\rangle_s .$$

So, $\Phi(t)$ is also a continuous semimartingale. □

Theorem 3. *For continuous semimartingales $Z_k(t), t \in [0,T], k = 1, \ldots, n$ with respect to the common filtration $(\mathfrak{F}_t), t \in [0,T]$, such that with probability* 1

$$\forall t \in [0,T] \quad \forall k \neq l \quad Z_k(t) \neq Z_l(t),$$

the Kontsevich integrals are computed as the corresponding multiple Stratonovich integrals.

Proof. The theorem 3 is obviously implied by the theorem 2, and the statement 1, by taking into account the fact that for finite number of continuous semimartingales $U_k(t)$ with respect to the common filtration $\mathfrak{F}_t$ there exists a deterministic sequence of partitions

$$\tau_p : 0 = t_0^{(p)} < t_1^{(p)} \ldots < t_{j_p-1}^{(p)} < t_{j_p}^{(p)} = T$$

with the finenes tending to zero (i.e. $\max\limits_{0 \leq j \leq j_p - 1} (t_{j+1}^{(p)} - t_j^{(p)}) \xrightarrow[p \to \infty]{} 0$), such that all the sequences of the sums

$$\sum_{j=0}^{j_p-1} |U_k(t_{j+1}^{(p)}) - U_k(t_j^{(p)})|^2, k = 1, \ldots, n,$$

are bounded with the probability 1. Indeed, choosing an arbitrary sequence of partitions

$$\Lambda_q : 0 = s_0^{(q)} < s_1^{(q)} \ldots < s_{l_q-1}^{(q)} < s_{l_q}^{(q)} = T$$

with the finenes converging to zero, we get that the corrresponding sums

$$\sum_{j=0}^{l_q-1} |U_k(s_{j+1}^{(q)}) - U_k(s_j^{(q)})|^2$$

converge in probability when $q \to \infty$ [13, p. 51], and as there is a finite number of these sums, then we can choose a subsequence of the sequence Λ_q, for which the corresponing sums will converge with the probability 1. □

Example 4. *The invariant from the example 3 is now written in the form*

$$\Psi_{123} = \frac{1}{2} \int_0^T (\lambda_{12}(s) \circ d\lambda_{13}(s) - \lambda_{13}(s) \circ d\lambda_{12}(s)) +$$
$$+ \frac{1}{2} \int\limits_0^T (\lambda_{13}(s) \circ d\lambda_{23}(s) - \lambda_{23}(s) \circ \lambda_{13}(s)) + \frac{1}{2} \int\limits_0^T (\lambda_{23}(s) \circ d\lambda_{12}(s) - \lambda_{12}(s) \circ d\lambda_{23}(s)). \quad (5)$$

Note that in this case instead of Stratonovich integrals we can write Ito integrals, i.e.

$$\Psi_{123} = \frac{1}{2}\int_0^T (\lambda_{12}(s)d\lambda_{13}(s) - \lambda_{13}(s)d\lambda_{12}(s)) +$$
$$+ \frac{1}{2}\int_0^T (\lambda_{13}(s)d\lambda_{23}(s) - \lambda_{23}(s)\lambda_{13}(s)) + \frac{1}{2}\int_0^T (\lambda_{23}(s)d\lambda_{12}(s) - \lambda_{12}(s)d\lambda_{23}(s)). \quad (6)$$

4. Some facts about weak convergence of stochastic integrals

In order to study the asymptotical behaviour of Vassiliev invariants for braids formed by independent Wiener processes we need some lemmas about weak convergence of stochastic integrals.

Lemma 3. *Let the set of random variables $\xi(T), T > 0$ be such that there are random variables $\eta_n(T), \gamma_n(T) \geq 0, \gamma_n \geq 0, \eta_n$ such that*

- $\forall T\, |\xi(T) - \eta_n(T)| \leq \gamma_n(T)$;
- $\forall n\, \eta_n(T) \xrightarrow[T\to\infty]{d} \eta_n$;
- $\forall \varepsilon > 0\ \exists n_0\ \forall n \geq n_0\ \exists T_0(n)\ \forall T \geq T_0(n)\colon P(\gamma_n(T) > \varepsilon) < \varepsilon$.

Then $\xi(T)$ has a limit in distribution when $T \to \infty$:

$$\xi(T) \xrightarrow[n\to\infty]{d} \xi_0.$$

Moreover, we have $\eta_n(T) \xrightarrow[T\to\infty]{d} \xi_0$.

Proof. Fix $\varepsilon > 0$. Choose n_0 and $T_0(n)$ in such a way that when $n \geq n_0, T \geq T_0(n)$:

$$P(\gamma_n(T) > \varepsilon) < \varepsilon.$$

Then for $n \geq n_0, T \geq T_0(n)$:

$$\mathbb{E}\frac{\gamma_n(T)}{1 + \gamma_n(T)} \leq 2\varepsilon.$$

Let μ_ξ be the distribution of ξ, and let ρ be the Wasserstein distance of the order zero, i.e. for random variables ξ, η

$$\rho(\mu_\xi, \mu_\eta) = \inf_{\xi' \overset{d}{=} \xi, \eta' \overset{d}{=} \eta} \mathbb{E}\frac{|\xi' - \eta'|}{1 + |\xi' - \eta'|}.$$

For any $n \geq n_0$ choose $T_1(n) > T_0(n)$ in such a way that for $T \geq T_1(n)$ the following condition holds:

$$\rho(\mu_{\eta_n(T)}, \mu_{\eta_n}) \leq \varepsilon.$$

As $T_1(n) > T_0(n)$, for $n \geq n_0, T \geq T_1(n)$ we have

$$\mathbb{E}\frac{\gamma_n(T)}{1 + \gamma_n(T)} \leq 2\varepsilon.$$

Then for $n \geq n_0$, $T \geq T_1(n)$, with the help of the inequality $|\xi(T) - \eta_n(T)| \leq \gamma_n(T)$:

$$\rho(\mu_{\xi(T)}, \mu_{\eta_n(T)}) \leq \mathbb{E}\frac{|\xi(T) - \eta_n(T)|}{1 + |\xi(T) - \eta_n(T)|} \leq \mathbb{E}\frac{|\gamma_n(T)|}{1 + |\gamma_n(T)|} \leq 2\varepsilon.$$

Further, for $n \geq n_0$, $T \geq T_1(n)$:

$$\rho(\mu_{\xi(T)}, \mu_{\eta_n}) \leq \rho(\mu_{\xi(T)}, \mu_{\eta_n(T)}) + \rho(\mu_{\eta_n(T)}, \mu_{\eta_n}) \leq 2\varepsilon + \varepsilon = 3\varepsilon.$$

Thus, for any $t_1, t_2 \geq T_1(n_0)$ we have

$$\rho(\mu_{\xi(t_1)}, \mu_{\xi(t_2)}) \leq \rho(\mu_{\xi(t_1)}, \mu_{\eta_N}) + \rho(\mu_{\xi(t_2)}, \mu_{\eta_{n_0}}) \leq 6\varepsilon.$$

Therefore, due to the choice of ε being arbitrary, the sequence $\{\mu_{\xi(T)}\}$ is fundamental in the metrics ρ and has the weak limit:

$$\xi(T) \xrightarrow[T\to\infty]{d} \xi_0.$$

It remains to show that $\eta_n \xrightarrow[n\to\infty]{d} \xi_0$. Choose T_2 in such a way that for $T \geq T_2$:

$$\rho(\mu_{\xi(T)}, \mu_{\xi_0}) \leq \varepsilon.$$

Then for $n \geq N, T \geq \max\{T_1(n), T_2\}$ we have:

$$\rho(\mu_{\xi_0}, \mu_{\eta_n}) \leq \rho(\mu_{\xi_0}, \mu_{\xi(T)}) + \rho(\mu_{\xi(T)}, \mu_{\eta_n}) \leq \varepsilon + 3\varepsilon = 4\varepsilon.$$

Thus, we get the convergence $\rho(\mu_{\eta_n}, \mu_{\xi_0}) \xrightarrow[n\to\infty]{} 0$, and the proof is finished. □

Denote by $\mathbb{D}[0,1]$ the Skorokhod space of right-continuous with left-hand limits at every point functions.

Lemma 4. *Let $\varepsilon > 0$, $f \in \mathbb{D}[0,1]$. Suppose that all jumps of f are in absolute value smaller than ε. Then*

$$\exists n_0 \forall n \geq n_0 \forall k, 0 \leq k \leq 2^n - 1\colon \sup_{s,t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |f(s) - f(t)| < \varepsilon.$$

Proof. Suppose the converse. Then for any n there exist segments of the partition with $\sup\limits_{s,t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |f(s) - f(t)| \geq \varepsilon$. Thus, we can choose points s_n, t_n with $|s_n - t_n| \leq \frac{1}{2^n}$ and $|f(s_n) - f(t_n)| \geq \varepsilon(1 - \frac{1}{2^n})$. There exists a subsequence n_k and a point $t_0 \in [0,1]$ with $s_{n_k} \to t_0, t_{n_k} \to t_0$. Thus, for any $\varepsilon' < \varepsilon$ in any neighbourhood of t_0 we can find points s, t with $|f(s) - f(t)| \geq \varepsilon'$. This contradicts the fact that the jump in t_0 is less than ε. □

Lemma 5. *Let $f \in \mathbb{D}[0,1]$, $\varepsilon > 0$ be specified. Then there exists $K > 0$ such that for all large enough n ($n \geq n_0$) we have*

$$\sum_{k=0}^{2^n-1} \mathbb{1}_{\sup\limits_{s,t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |f(s)-f(t)| > \varepsilon} \leq K,$$

and for $n \geq n_0$ all the segments $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, for which $\sup\limits_{s,t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |f(s) - f(t)| > \varepsilon$, are among the segments containing the discontinuities of the value not less then ε.

Proof. Remove from f all the jumps of the magnitude not less then ε. Then the lemma is reduced to the lemma 4. $\square$

Lemma 6. *Let $f, g \in \mathbb{D}[0,1]$ be without simultaneous jumps. Then*

$$\forall \varepsilon > 0 \ \exists n_0 \ \forall n \geq n_0 \sum_{k=0}^{2^n-1} \mathbb{1}_{\sup\limits_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |f(s)-f(t)|>\varepsilon} \mathbb{1}_{\sup\limits_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |g(s)-g(t)|>\varepsilon} = 0.$$

Proof. By lemma 5, for all large enough n ($n \geq n_0$) the inequalities

$$\sup_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |f(s)-f(t)| > \varepsilon,$$

$$\sup_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |g(s)-g(t)| > \varepsilon$$

can be satisfied only for the $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, that contain the jump points of f, g respectively of the magnitudes not less than ε. As there is only finite number of jumps of f, g of the magnitude not less than ε, and they are all in different points, then for large enough n they will be divided by segments of partition, and for large enough n no two such jumps will be on the same segment of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$. $\square$

Lemma 7. *Let $f, g \in \mathbb{D}[0,1]$ have no simultaneous jumps, $\varepsilon > 0$ be fixed. Then*

$$\sum_{k=0}^{2^n-1} \mathbb{1}_{\sup\limits_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |f(s)-f(t)|>\varepsilon} \left| g\left(\frac{k+1}{2^n}\right) - g\left(\frac{k}{2^n}\right) \right| \xrightarrow[n\to\infty]{} 0. = 0.$$

Proof. By lemma 5, for some $K > 0$ for all large enough n there are no more than K summands in the sum we are interested in. From the other side, by lemma 6, for all large enough n on all the segments $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ that contribute to the sum the next inequality takes place: $\left|g\left(\frac{k+1}{2^n}\right) - g\left(\frac{k}{2^n}\right)\right| \leq \varepsilon$. Thus, our sum for large enough n does not surmount $K\varepsilon$. But this is not enough to prove lemma.

So, we need a more sophisticated argument. Choose any $\delta, 0 < \delta < \varepsilon$. For large enough n all the nonzero summands do not surmount δ by lemma 6, and the number of nonzero terms is not more than K (the constant K is the same as in the beginning of the proof). Thus, for large enough n our sum does not surmount $K\delta$. As δ is arbitrary, and K is fixed, the lemma is proved. $\square$

Lemma 8. *Let $f, g \in \mathbb{D}[0,1]$ be nondecreasing functions without simultaneous jumps, $h \in C[0,\infty)$ be a continuous function. Then*

$$S_n = \sum_{k=0}^{2^n-1} \sup_{s,t\in[f(\frac{k}{2^n}),f(\frac{k+1}{2^n})]} (h(s)-h(t))^2 \left| g\left(\frac{k+1}{2^n}\right) - g\left(\frac{k}{2^n}\right) \right| \xrightarrow[n\to\infty]{} 0.$$

Proof. Fix $\varepsilon > 0$. Choose $\delta > 0$ in such a way that for any $s, t \in [f(0), f(1)]$, for which $|s-t| \leq \delta$, the following inequality takes place: $|h(s)-h(t)| \leq \varepsilon$. This choice is possible due to h being continuous. Divide the sum S_n into two parts:

$$S_n = G_n + H_n.$$

To G_n we attribute those summands in which $f(\frac{k+1}{2^n}) - f(\frac{k}{2^n}) > \delta$. To H_n we attribute those summands where $f(\frac{k+1}{2^n}) - f(\frac{k}{2^n}) \leq \delta$. By lemma 7, considering the boundedness of h on segment $[f(0), f(1)]$, we get $G_n \to 0 (n \to \infty)$. From the other side, it is clear that for H_n the following estimation takes place: $H_n \leq \varepsilon^2(g(1) - g(0))$. Thus,

$$\overline{\lim_{n\to\infty}} S_n \leq \varepsilon^2(g(1) - g(0)).$$

Due to ε being arbitrary, we get $\overline{\lim\limits_{n\to\infty}} S_n = 0$, which finishes the proof of lemma. □

Definition 7. *We say that the family of random variables $\gamma_{n,T}$ is stochastically small at infinity, if*

$$\forall \varepsilon > 0 \ \exists n_0 \ \forall n \geq n_0 \ \exists T_0(n) \ \forall T \geq T_0(n) \colon P(\gamma_{n,T} > \varepsilon) < \varepsilon.$$

Definition 8. *We say that the family of random variables $\gamma_{n,T}$ is stochastically bounded at infinity, if*

$$\forall \varepsilon > 0 \ \exists C > 0 \ \exists n_0 \ \forall n \geq n_0 \ \exists T_0(n) \ \forall T \geq T_0(n) \colon P(\gamma_{n,T} > C) < \varepsilon.$$

Lemma 9. *Let $\xi_{n,T}$ be stochastically small at infinity, $\eta_{n,T}$ be stochastically bounded at infinity. Then $\zeta_{n,T} = \eta_{n,T}\xi_{n,T}$ is stochastically small at infinity.*

Proof. Fix $\varepsilon > 0$. Choose C, n_1, n_2 in such a way that

$$\forall n \geq n_1 \ \exists T_1(n) \ \forall T \geq T_1(n) \colon P(\xi_{n,T} > \varepsilon) < \varepsilon.$$

$$\forall n \geq n_2 \ \exists T_2(n) \ \forall T \geq T_2(n) \colon P(\eta_{n,T} > C) < \varepsilon.$$

Then for all $n \geq n_0 = \max\{n_1, n_2\}, T \geq T_0(n) = \max\{T_1(n), T_2(n)\}$ we have

$$P(\zeta_{n,T} > C\varepsilon) < 2\varepsilon.$$

□

Lemma 10. *If $\overline{\lim\limits_{T\to\infty}} \xi_T \leq \xi$ п.н., then for any A*

$$\overline{\lim_{T\to\infty}} P(\xi_T \geq A) \leq P(\xi \geq A).$$

Proof. If $\overline{\lim\limits_{T\to\infty}} a_T \leq a$, then

$$\overline{\lim_{T\to\infty}} \mathbb{1}_{a_T \geq A} \leq \mathbb{1}_{a \geq A}.$$

Thus,

$$\overline{\lim_{T\to\infty}} \mathbb{1}_{\xi_T \geq A} \leq \mathbb{1}_{\xi \geq A}.$$

By Fatou's lemma,

$$P(\xi \geq A) = \mathbb{E}\mathbb{1}_{\xi \geq A} \geq \mathbb{E} \overline{\lim_{T\to\infty}} \mathbb{1}_{\xi_T \geq A} \geq \overline{\lim_{T\to\infty}} \mathbb{E}\mathbb{1}_{\xi_T \geq A} = \overline{\lim_{T\to\infty}} P(\xi_T \geq A).$$

□

In what follows we denote by $\xrightarrow{fd}$ convergence of random processes in a sense of finite-dimentional distributions.

Lemma 11. *Let* $(\xi^{(T)}, \eta^{(T)}) \xrightarrow[t\to\infty]{fd} (\xi, \eta)$, *where* ξ, η *take values in the space* $\mathbb{D}[0,1]$ *and with probability* 1 *have no common jumps. Let* $\varepsilon_0 > 0$ *be fixed,*

$$\gamma_{n,T} = \sum_{k=0}^{2^n-1} \mathbb{1}_{\sup\limits_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} |\xi^{(T)}(t)-\xi^{(T)}(s)|>\varepsilon_0} \left| \eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right|.$$

Then $\gamma_{n,T}$ *is stochastically small at infinity.*

Proof. Fix $\varepsilon > 0$. Let

$$\gamma_n = \sum_{k=0}^{2^n-1} \mathbb{1}_{|\xi(\frac{k+1}{2^n})-\xi(\frac{k}{2^n})|\geq\varepsilon_0/2} \left| \eta\left(\frac{k+1}{2^n}\right) - \eta\left(\frac{k}{2^n}\right)\right|.$$

Then $\gamma_n \to 0$ a.s. by lemma 7. Choose n_0 in such a way that for any $n \geq n_0$ $P(\gamma_n \geq \varepsilon) < \varepsilon$. Then for $n \geq n_0$ we argue in such a way: for $(\xi^{(T)}(\frac{k}{2^n}), \eta^{(T)}(\frac{k}{2^n}), 0 \leq k \leq 2^n-1)$ and $(\xi(\frac{k}{2^n}), \eta(\frac{k}{2^n}), 0 \leq k \leq 2^n-1)$ there exists Skorohod's representation that realizes the almost sure convergence:

$$\left(\xi^{(T)}\left(\frac{k}{2^n}\right), \eta^{(T)}\left(\frac{k}{2^n}\right), 0 \leq k \leq 2^n-1\right) \stackrel{d}{=} \left(\tilde{\xi}^{(T)}\left(\frac{k}{2^n}\right), \tilde{\eta}^{(T)}\left(\frac{k}{2^n}\right), 0 \leq k \leq 2^n-1\right),$$

$$\left(\xi\left(\frac{k}{2^n}\right), \eta\left(\frac{k}{2^n}\right), 0 \leq k \leq 2^n-1\right) \stackrel{d}{=} \left(\tilde{\xi}\left(\frac{k}{2^n}\right), \tilde{\eta}\left(\frac{k}{2^n}\right), 0 \leq k \leq 2^n-1\right),$$

and for fixed n

$$\left(\tilde{\xi}^{(T)}\left(\frac{k}{2^n}\right), \tilde{\eta}^{(T)}\left(\frac{k}{2^n}\right), 0 \leq k \leq 2^n-1\right) \xrightarrow[T\to\infty]{} \left(\tilde{\xi}\left(\frac{k}{2^n}\right), \tilde{\eta}\left(\frac{k}{2^n}\right), 0 \leq k \leq 2^n-1\right) \text{ п.н.}$$

Thus, for fixed $n \geq n_0$

$$\varlimsup_{T\to\infty} \tilde{\gamma}_{n,T} \leq \tilde{\gamma}_n,$$

and by lemma 10

$$\varlimsup_{T\to\infty} P(\gamma_{n,T} \geq \varepsilon) \leq P(\gamma_n \geq \varepsilon).$$

Thus, for given fixed $n \geq n_0$ we have for large enough T:

$$P(\gamma_{n,T} > 2\varepsilon) \leq P(\gamma_n \geq \varepsilon) < \varepsilon.$$

This proves the lemma. □

Lemma 12. *Let* $\xi^{(T)}, \eta^{(T)}$ *be nondecreasing nonnegative processes,* $(\xi^{(T)}, \eta^{(T)}) \xrightarrow[T\to\infty]{fd} (\xi, \eta)$, *where* ξ, η *take values in the space* $\mathbb{D}[0,1]$ *and have no simultaneous jumps with probability* 1. *Let also* $\beta_{n,T}$ *be Wiener processes. Let*

$$\gamma_{n,T} = \sum_{k=0}^{2^n-1} \sup_{s,t\in[\xi^{(T)}(\frac{k}{2^n}),\xi^{(T)}(\frac{k+1}{2^n})]} |\beta_{n,T}(t) - \beta_{n,T}(s)|^2 \left(\eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right).$$

Then $\gamma_{n,T}$ *is stochastically small at infinity.*

Proof. Fix $\varepsilon > 0$. Choose $C > 0$ in such a way that

$$P\left(\xi^{(T)}(1) > C\right) < \varepsilon, T > T_0.$$

Choose $\delta > 0$ in such a way that for Wiener process β

$$P\left(\sup_{s,t\in[0,C],|s-t|\le\delta} |\beta(t) - \beta(s)| > \varepsilon\right) < \varepsilon.$$

Denote

$$G_{n,T} = \sum_{k=0}^{2^n-1} \mathbb{1}_{\xi^{(T)}(\frac{k+1}{2^n})-\xi^{(T)}(\frac{k}{2^n})>\delta} \sup_{s,t\in[\xi^{(T)}(\frac{k}{2^n}),\xi^{(T)}(\frac{k+1}{2^n})]} |\beta_{n,T}(t) - \beta_{n,T}(s)|^2 \left(\eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right),$$

$$H_{n,T} = \sum_{k=0}^{2^n-1} \mathbb{1}_{\xi^{(T)}(\frac{k+1}{2^n})-\xi^{(T)}(\frac{k}{2^n})\le\delta} \sup_{s,t\in[\xi^{(T)}(\frac{k}{2^n}),\xi^{(T)}(\frac{k+1}{2^n})]} |\beta_{n,T}(t) - \beta_{n,T}(s)|^2 \left(\eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right).$$

Notice that

$$G_{n,T} \le 4 \sup_{t\in[0,\xi^{(T)}(1)]} |\beta_{n,T}(t)|^2 \sum_{k=0}^{2^n-1} \mathbb{1}_{\xi^{(T)}(\frac{k+1}{2^n})-\xi^{(T)}(\frac{k}{2^n})>\delta} \left(\eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right).$$

Then $G_{n,T}$ is stochastically small at infinity by lemmas 9, 11 and due to $\sup_{t\in[0,\xi^{(T)}(1)]} |\beta_{n,T}(t)|^2$ being stochastically bounded. For $H_{n,T}$ we have forи $T > T_0$:

$$P\left(H_{n,T} > \varepsilon^2\left(\eta^{(T)}(1) - \eta^{(T)}(0)\right)\right) \le P\left(\xi^{(T)}(1) > C\right) + \\ + P\left(\sup_{s,t\in[0,C],|s-t|\le\delta} |\beta_{n,T}(t) - \beta_{n,T}(s)| > \varepsilon\right) < 2\varepsilon.$$

Thus, $\gamma_{n,T} = G_{n,T} + H_{n,T}$ is stochastically small at infinity due to ε being arbitrary. □

Lemma 13. *Let $\eta^{(T)}$ be nondecreasing nonnegative processes, $(\xi^{(T)}, \eta^{(T)}) \xrightarrow[t\to\infty]{fd} (\xi, \eta)$, where ξ, η take values in $\mathbb{D}[0,1]$ and with probability 1 do not have simultaneous jumps,*

$$\gamma_{n,T} = \sum_{k=0}^{2^n-1} \sup_{s,t\in[\frac{k}{2^n}),\frac{k+1}{2^n}]} \left|\xi^{(T)}(t) - \xi^{(T)}(s)\right| \left(\eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right).$$

Then $\gamma_{n,T}$ is stochastically small at infinity.

Proof. Fix $\varepsilon > 0$. Denote

$$G_{n,T} = \sum_{k=0}^{2^n-1} \mathbb{1}_{\exists s,t\in[\frac{k}{2^n}),\frac{k+1}{2^n}]:\,|\xi(t)-\xi(s)|\ge\varepsilon} \sup_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} \left|\xi^{(T)}(t) - \xi^{T}(s)\right|^2 \left(\eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right),$$

$$H_{n,T} = \sum_{k=0}^{2^n-1} \mathbb{1}_{\forall s,t\in[\frac{k}{2^n}),\frac{k+1}{2^n}]:\, |\xi(t)-\xi(s)|\le\varepsilon} \sup_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} \left|\xi^{(T)}(t)-\xi^{(T)}(s)\right|^2 \left(\eta^{(T)}\left(\frac{k+1}{2^n}\right)-\eta^{(T)}\left(\frac{k}{2^n}\right)\right).$$

Then $G_{n,T}$ is stochastically small at infinity by lemmas 9, 11 and $\sup\limits_{s\in[0,1]} |\xi^{(T)}(s)|$ being stochasticall bounded.

For $H_{n,T}$ we have for $T > T_0$

$$H_{n,T} \le \varepsilon^2 \left(\eta^{(T)}(1) - \eta^{(T)}(0)\right).$$

Thus, $\gamma_{n,T} = G_{n,T} + H_{n,T}$ is stochastically amall at infinity due to ε being arbitrary. □

Lemma 14. *Let ξ be a random process with values in $\mathbb{D}[0,1]$ without fixed jumps, η be a semimartingale with trajectories in $\mathbb{D}[0,1]$. Then the sum*

$$\sum_{k=0}^{n-1} \xi\left(\frac{k}{n}\right)\left(\eta\left(\frac{k+1}{n}\right)-\eta\left(\frac{k}{n}\right)\right)$$

converges in probability to $\int\limits_0^1 \xi(s-)d\eta(s)$.

Proof. First note that the integral $\int\limits_0^1 \xi(s-)d\eta(s)$ exists because $\xi(s-)$ is left-continuous and bounded on $[0,1]$, hence locally bounded, and the integral is defined for any locally bounded predictable integral (see [11], theorem 26.4). This holds by dominated convergence theorem for stochastic integrals (see [14], chapter 8). We will explain this now in more details. By [11], theorem 26.4, the following relation holds: if X is a semimartingale, $V_n \to 0$ and $|V_n| \le V$, where all the processes V_n, V are predictable and locally bounded, then

$$\sup_{t\in[0,1]} \int_0^t V_n(s)dX(s) \xrightarrow[n\to\infty]{P} 0.$$

Set

$$X(s) = \eta(s), s \ge 0,$$
$$H_n(s) = \xi\left(\frac{[ns]}{n}\right),$$
$$G_n(s) = \xi(s-),$$

$$V_n(s) = G_n(s) - H_n(s).$$

Due to the process $\xi(s-)$ being left-continuous, we have

$$\xi\left(\frac{[ns]}{n}\right) \xrightarrow[n\to\infty]{} \xi(s).$$

(Notice that with probability 1 the process ξ has no discontinuities in rational points at all, and thus with probability 1 the convergence takes place for all s). Thus, we conclude that with probability 1

$$V_n \to 0, n \to \infty.$$

As a dominating process we take

$$V(t) = 2 \sup_{s \in [0,t]} |\xi(s-)|.$$

This process is left-continuous, and thus, predictable and locally bounded. □

Theorem 4. *Let $\xi^{(T)}, \eta^{(T)}$ be continuous semimartingales on $[0,1]$ with respect to the common filtrations, and*

$$\left(\xi^{(T)}, \eta^{(T)}, \left\langle \xi^{(T)} \right\rangle, \left\langle \eta^{(T)} \right\rangle\right) \xrightarrow[T\to\infty]{fd} (\xi, \eta, \zeta_1, \zeta_2),$$

where ξ, η take values in $\mathbb{D}[0,1]$; ζ_1, ζ_2 take values in space $\mathbb{D}[0,1]$ and with probability 1 have no simultaneous jumps.

$$\int_0^1 \xi^{(T)}(s) d\eta^{(T)}(s) \xrightarrow[T\to\infty]{d} \int_0^1 \xi(s-) d\eta(s).$$

Proof. Let

$$\gamma_{n,T} = \int_0^1 \xi^{(T)}(s) d\eta^{(T)}(s) - \sum_{k=0}^{2^n-1} \xi^{(T)}\left(\frac{k}{2^n}\right)\left(\eta^{(T)}\left(\frac{k+1}{2^n}\right) - \eta^{(T)}\left(\frac{k}{2^n}\right)\right).$$

Then

$$\gamma_{n,T} = \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \left(\xi^{(T)}(s) - \xi^{(T)}\left(\frac{k}{2^n}\right)\right) d\eta^{(T)}(s) = w_{n,T}\left(\sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \left(\xi^{(T)}(s) - \xi^{(T)}\left(\frac{k}{2^n}\right)\right)^2 d\left\langle \eta^{(T)} \right\rangle_s\right),$$

where $w_{n,T}$ are some Wiener processes [11], theorem 18.4. Let $\beta_{n,T}$ be associated with $\xi^{(T)}$ Wiener processes. We have

$$\mu_{n,T} = \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \left(\xi^{(T)}(s) - \xi^{(T)}\left(\frac{k}{2^n}\right)\right)^2 d\left\langle \eta^{(T)} \right\rangle_s \leq$$

$$\leq \sum_{k=0}^{2^n-1} \sup_{s,t \in \left[\left\langle \xi^{(T)} \right\rangle\left(\frac{k}{2^n}\right), \left\langle \xi^{(T)} \right\rangle\left(\frac{k+1}{2^n}\right)\right]} |\beta_{n,T}(t) - \beta_{n,T}(s)|^2 \left(\left\langle \eta^{(T)} \right\rangle\left(\frac{k+1}{2^n}\right) - \left\langle \eta^{(T)} \right\rangle\left(\frac{k}{2^n}\right)\right).$$

By lemma 12 $\mu_{n,T}$ is stochastically small at infinity. Then $\gamma_{n,T} = w_{n,T}(\mu_{n,T})$ is stochastically small at infinity as well. By lemmas 3, 14 we get the required statement. □

Theorem 5. *Let* $\left(\xi^{(T)}, \eta^{(T)}, \left\langle \xi^{(T)} \right\rangle, \left\langle \eta^{(T)} \right\rangle\right) \xrightarrow[T\to\infty]{fd} (\xi, \eta, \zeta_1, \zeta_2)$, *where* ξ, η *are* $\mathbb{D}[0,1]$*-valued;* ζ_1, ζ_2 *are* $\mathbb{D}[0,1]$*-valued and have no simultaneous jumps with probability* 1. *Then*

$$\int\limits_0^1 \xi^{(T)}(s)^2 d\left\langle \eta^{(T)} \right\rangle(s) \xrightarrow[T\to\infty]{d} \int\limits_0^1 \xi(s-)^2 d\left\langle \zeta_2 \right\rangle(s).$$

Proof. Let

$$\gamma_{n,T} = \int\limits_0^1 \xi^{(T)}(s)^2 d\left\langle \eta^{(T)} \right\rangle(s) - \sum_{k=0}^{2^n-1} \xi^{(T)}\left(\frac{k}{2^n}\right)^2 \left(\left\langle \eta^{(T)} \right\rangle \left(\frac{k+1}{2^n}\right) - \left\langle \eta^{(T)} \right\rangle \left(\frac{k}{2^n}\right)\right) =$$

$$= \sum_{k=0}^{2^n-1} \int\limits_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \left(\xi^{(T)}(s)^2 - \xi^{(T)}\left(\frac{k}{2^n}\right)^2\right) d\left\langle \eta^{(T)} \right\rangle(s).$$

We have

$$|\gamma_{n,T}| \leq \sum_{k=0}^{2^n-1} \sup_{s,t\in[\frac{k}{2^n},\frac{k+1}{2^n}]} \left|\xi^{(T)}(t)^2 - \xi^{(T)}(s)^2\right| \left(\left\langle \eta^{(T)} \right\rangle \left(\frac{k+1}{2^n}\right) - \left\langle \eta^{(T)} \right\rangle \left(\frac{k}{2^n}\right)\right).$$

By lemma 13 $\gamma_{n,T}$ is stochastically small at infinity. By lemma 3 we get the needed statement. □

Definition 9. *Let* β *be a Wiener process. The family of processes* $\mathcal{U}$ *generated by* β *is defined as the family of all the processes of the form*

$$\int\limits_0^t \int\limits_0^{t_{n-1}} \dots \int\limits_0^{t_2} d\eta_1(t_{n-1}) \dots d\eta_n(t_0),$$

where $d\eta_k(s) = d\beta(s)$ *or* $d\eta_k(s) = ds$ *for any* k. *In particular, the processes* $\xi_1(t) = t$, $\xi_2(t) = \beta(t), \xi_3(t) = \int\limits_0^t \beta(s)d\beta(s)$, $\xi_4(t) = \int\limits_0^t \xi_2(s)ds$ *are included into the family* $\mathcal{U}$. *It is clear that all the processes generated by* β *are continuous semimartingales with respect to filtration generated by* β.

Lemma 15. *If* L *is a process generated by* β, *then for any* $n \geq 0$

$$X_t = \int\limits_0^t \beta^n(s)dL_s$$

can be represented in the form $X = F(\beta)$, *where* F *is a continuous mapping*

$$F\colon C[0,\infty) \to C[0,\infty).$$

In particular, for $n = 0$ *we get that the process* L *itself can be represented in such form.*

Proof. We will conduct the proof by induction on minimal length of the chain of processes generating L. We will check that X can be represented in the form $F(\beta)$. We have two cases:

- $X_t = \int_0^t Y_s ds$, and Y is the process generated by β, for which the statement of lemma is proved. Then
$$\int_0^t \beta^n(s) dX_s = \int_0^t \beta^n(s) Y_s ds,$$
and it is clear that this is a continuous functional of β.

- $X_t = \int_0^t Y_s d\beta(s)$, and for Y the statement is proved. Then
$$\int_0^t \beta^n(s) dX_s = \int_0^t \beta^n(s) Y_s d\beta(s) = \frac{1}{n} \int_0^t Y_s d\beta^n(s) - \frac{n-1}{2} \int_0^t \beta(s)^{n-2} Y_s ds =$$
$$= \frac{1}{n} Y_t \beta^n(t) - \frac{1}{n} \int_0^t \beta^n(s) dY_s - \frac{1}{n} \langle Y, \beta^n \rangle_t - \frac{n-1}{2} \int_0^t \beta(s)^{n-2} Y_s ds$$
for all the summands except $\frac{1}{n} \langle Y, \beta^n \rangle_t$ it is clear that they are continuous functionals of β. There are two possible cases for Y:
  - $Y_t = \int_0^t Q_s ds$, and in this case $\langle Y, \beta^n \rangle_t = 0$;
  - $Y_t = \int_0^t Q_s d\beta(s)$, and in this case $\frac{1}{n} \langle Y, \beta^n \rangle_t = \int_0^t \beta(s)^{n-1} Q_s ds$, and it is clear that this is a continuous functional of β.

  The lemma is proved.

$\square$

5. Weak convergence of Vassiliev invariants for Wiener braids

Definition 10. *Let $Y_1^{(T)}, \ldots, Y_n^{(T)}$ be the families of continuous semimartingales with respect to the common (for each fixed T) filtration. The collection of families of processes generated by $Y_1, \ldots, Y_n$ is the smallest collection S such that*

- *for any $k = 1, \ldots, n$: $Y_k^{(T)} \in S$, $\left\langle Y_k^{(T)} \right\rangle \in S$;*
- *if $U^{(T)} \in S$, then for any k the families $V^{(T)}, H^{(T)}$, defined by equalities*
$$V_t^{(T)} = \int_0^t U^{(T)}(s) dY_k^{(T)}(s), t \geq 0$$

$$H_t^{(T)} = \int\limits_0^t U^{(T)}(s) d \left\langle Y_k^{(T)} \right\rangle (s), t \geq 0,$$

are in S: $V^{(T)} \in S, H^{(T)} \in S$.

Theorem 6. *Let* $Y^{(T)}$ *be the family of continuous local martingales such that*

$$Y^{(T)}(t) = \beta^{(T)} \left(\langle Y^{(T)} \rangle_t \right),$$

amd for any $t_1, \ldots, t_k$ *the random elements*

$$(\beta^{(T)}, \left\langle Y^{(T)} \right\rangle (t_1), \ldots, \left\langle Y^{(T)} \right\rangle (t_k)) \in C[0, \infty) \times \mathbb{R}^k$$

converge in distribution when $T \to \infty$. *Let* S *be a collection of families generated by* $Y^{(T)}$. *Then for any family* $U^{(T)} \in S$:

$$(Y^{(T)}, \left\langle Y^{(T)} \right\rangle, U_1^{(T)}, \ldots, U_k^{(T)})$$

has a limit in a sense of convergence of finite-dimensional distributions:

$$\left(Y^{(T)}, \left\langle Y^{(T)} \right\rangle, U_1^{(T)}, \ldots, U_k^{(T)} \right) \xrightarrow[T \to \infty]{fd} (Y, Q, U_1, \ldots, U_k), \tag{7}$$

and all the jump points of processes $Y, U_1, \ldots, U_k$ *are with probability* 1 *among the points of discontinuity of the process* Q.

Proof. By lemma 15 we get that all the processes from the specified collection can be expressen in the form

$$U^{(T)} = F(\beta^{(T)})(\langle Y^{(T)} \rangle),$$

where $F \colon C[0, \infty) \to C[0, \infty)$ is a continuous functional of Wiener process. So, for $U_i^{(T)} = F_i(\beta^{(T)})(\langle Y^{(T)} \rangle)$ we get the existence of limit in (7) due to the convergence of random vector

$$(\beta^{(T)}, \left\langle Y^{(T)} \right\rangle (t_1), \ldots, \left\langle Y^{(T)} \right\rangle (t_k)) \xrightarrow[T \to \infty]{d} (\beta_0, Q(t_1), \ldots, Q(t_k))$$

for some Wiener process β_0 and for any $t_1, \ldots, t_k$. Next, the convergence

$$F_i(\beta^{(T)})(\langle Y \rangle) \xrightarrow[T \to \infty]{fd} F_i(\beta_0)(Q)$$

yields that all the jumps of the limiting process $F_i(\beta_0)(Q)$ will be among the discontinuities of Q. $\square$

Theorem 7. *Let* $X^{(T)}, Y^{(T)}$ *be continuous semimartingales with respect to the common (for any fixed* T*) filtration, and*

$$(X^{(T)}, Y^{(T)}) \xrightarrow[T \to \infty]{fd} (X, Y),$$

and processes X, Y *with probability* 1 *have no simultaneous jumps. Let the family* $Y^{(T)}$ *also satisfy the conditions of the theorem 6. Denote*

$$L_0^{(T)}(t) = Y^{(T)}(t),$$

$$L_{k+1}^{(T)}(t) = \int_0^t L_k^{(T)}(s)dY^{(T)}(s).$$

Then for any k for any families $U^{(T)}$ from the collection generated by $Y^{(T)}$ (see theorem 6), the finite-dimensional distributions of the processes

$$\int_0^t L_k^{(T)}(s)U^{(T)}(s)dY^{(T)}(s), \int_0^t L_k^{(T)}(s)U^{(T)}(s)d\left\langle Y^{(T)}\right\rangle(s)$$

converge when $T \to \infty$ jointly with $Y^{(T)}$, and all the jumps of the limiting processes coincide with the jumps of the processes Y.

Proof. We make the proof by induction on k. The basis is checked by theorems 4, 5. Induction step: let the statement be checked for k, check for $k+1$. Denote $V^{(T)}(t) = \int_0^t U_k^{(T)}(s)dY_k^{(T)}(s)$, $Q_t = \int_0^t U_k^{(T)}(s)d\left\langle Y_k^{(T)}\right\rangle(s)$. It is clear that the processes $V^{(T)}, Q^{(T)}$ have limits in the sense of convergence of finite-dimensional distributions. We have

$$L_{k+1}^{(T)}(t) = \int_0^t L_k^{(T)}(s)dY^{(T)}(s),$$

$$\int_0^t L_{k+1}^{(T)}(s)U^{(T)}(s)dY^{(T)}(s) = \int_0^t L_{k+1}^{(T)}(s)dV^{(T)}(s) = L_{k+1}^{(T)}(t)V^{(T)}(t) -$$
$$- \int_0^t L_k^{(T)}(s)V^{(T)}(s)dY^{(T)}(s) - \left\langle L_{k+1}^{(T)}, V^{(T)}\right\rangle_t = \int_0^t L_{k+1}^{(T)}(s)dV^{(T)}(s) =$$
$$= L_{k+1}^{(T)}(t)V^{(T)}(t) - \int_0^t L_k^{(T)}(s)V^{(T)}(s)dY^{(T)}(s) - \int_0^t L_k^{(T)}(s)U^{(T)}(s)d\left\langle Y^{(T)}\right\rangle_s,$$

and by assumption we have the convergence of finite-dimensional distributions for

$$\int_0^t L_{k+1}^{(T)}(s)U^{(T)}(s)dY^{(T)}(s).$$

Further,

$$\int_0^t L_{k+1}^{(T)}(s)U^{(T)}(s)d\left\langle Y^{(T)}\right\rangle(s) = \int_0^t L_{k+1}^{(T)}(s)dQ^{(T)}(s) = L_{k+1}^{(T)}(t)Q^{(T)}(t)-$$

$$-\int_0^t L_k^{(T)}(s)Q^{(T)}(s)dY^{(T)}(s) - \left\langle L_{k+1}^{(T)}, V^{(T)}\right\rangle_t = \int_0^t L_{k+1}^{(T)}(s)dV^{(T)}(s) =$$

$$= L_{k+1}^{(T)}(t)V^{(T)}(t) - \int_0^t L_k^{(T)}(s)V^{(T)}(s)dY^{(T)}(s) - \int_0^t L_k^{(T)}(s)U^{(T)}(s)d\left\langle Y^{(T)}\right\rangle_s,$$

and by assumption we have convergence of finite-dimensional distributions for

$$\int_0^t L_{k+1}^{(T)}(s)U^{(T)}(s)d\left\langle Y^{(T)}\right\rangle(s).$$

□

Theorem 8. *Let $Y_1^{(T)}, \dots, Y_n^{(T)}$ be a family of continuous local martingales with respect to the common (for any fixed T) filtration such that*

$$(Y_1^{(T)}, \dots, Y_n^{(T)}) \xrightarrow[T\to\infty]{fd} (Y_1, \dots, Y_n),$$

and with probability 1 any two of the processes $Y_1, \dots, Y_n$ have no simultaneous jump points, and for any k at least one of two conditions holds:

- *the family $Y_k^{(T)}$ satisfies the conditions of the theorem 6;*
- *the limiting process Y_k has no discontinuities with probability 1, characteristics $\left\langle Y_k^{(T)}\right\rangle$ coincide with the caracteristics $\left\langle Y_l^{(T)}\right\rangle$ for some $l \neq k$, and the family $Y_l^{(T)}$ satisfies the conditions of the theorem 6.*

Let S be a collection of the families of processes generated by $Y_1, \dots, Y_n$. Then for any $X_1, \dots, X_k \in S$ the random vector

$$(X_1^{(T)}, \dots, X_k^{(T)})$$

converges when $T \to \infty$ in a sense of convergence of finite-dimensional distributions.

Proof. Follows from the theorem 7. □

Let $Z_k(t), t \geq 1$ be 2-dimensional Brownian motions starting from pairwise distinct points of the plane. Let $Z_{kl}(t) = \|Z_k(t) - Z_l(t)\|, R_{kl}(t) = |Z_{kl}(t)|$, θ_{kl} ne the winding angle of the process Z_{kl} around the originup to time t, $\theta_{kl}(1) = 0$. Introduce, as in [6], the processes X_{kl} with the help of exponential time change for Z_{kl}:

$$X_{kl}(t) = e^{-t/2}Z_{kl}(e^t), t \geq 0.$$

Then $X_{kl}(t)$ is a 2-dimensional Ornstein-Uhlenbeck process. Let $\alpha_{kl}(t) = \theta_{kl}(e^t)$ be the winding angle of the process X_{kl} around zero up to time t. Let

$$\phi_{kl}^{(T)}(t) = \frac{\alpha_{kl}(tT)}{T/2}, r_{kl}^{(T)} = \frac{\ln R_{kl}(e^{tT})}{T/2}, T > 0.$$

The following convergence in the sense of finite-deimensional distribution takes place:

$$\phi_{kl}^{(T)} \xrightarrow[T\to\infty]{fd} \xi, r_{kl}^{(T)} \xrightarrow[T\to\infty]{fd} \eta,$$

where ξ is a Cauchy process, $\eta(t) = t$. Apply theorem 8 to the families of continuous local martingales $\phi_{kl}^{(T)}, r_{kl}^{(T)}$.

Theorem 9. *Let $\mathfrak{S}$ be a collection of the families of processes generated by*

$$\phi_{kl}^{(T)}, r_{kl}^{(T)}, 1 \le k < l \le n,$$

Then for any $X_1, \ldots, X_k \in \mathfrak{S}$ the random vector

$$(X_1^{(T)}, \ldots, X_k^{(T)})$$

converges when $T \to \infty$ in a sense of convergence of finite-dimensional distributions.

For the processes $\phi_{kl}^{(T)}$ itself we get the well-known result of M. Yor [15] about the convergence of mutual winding angles of independent planar Brownian motions. However, from our theorem we obtain much more. Indeed, it follows that all the Vassiliev invariants for the braid formed by independent planar Brownian motion converge in a sense of finite-dimentional distributions. More precisely, for the numerical invariant L of the order m we got the weak convergence of $\frac{L}{T^m}$, and all these convergences hold jointly, that is, for invariants $L_1, \ldots, L_k$ of the orders $m_1, \ldots, m_k$ we get the convergence of random vector

$$\left(\frac{L_1}{T^{m_1}}, \ldots, \frac{L_k}{T^{m_k}}\right).$$

If we return to the initial processes Z_{kl} instead of X_{kl} then we get the convergence for the numerical invariants $L'_1, \ldots, L'_m$ of the orders $m_1, \ldots, m_k$ for the braid formed by the independent Brownian motions Z_k: the random vector

$$\left(\frac{L'_1}{(\ln T)^{m_1}}, \ldots, \frac{L'_k}{(\ln T)^{m_k}}\right)$$

has a limit in a sense of convergence in finite-dimensional distributions.

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