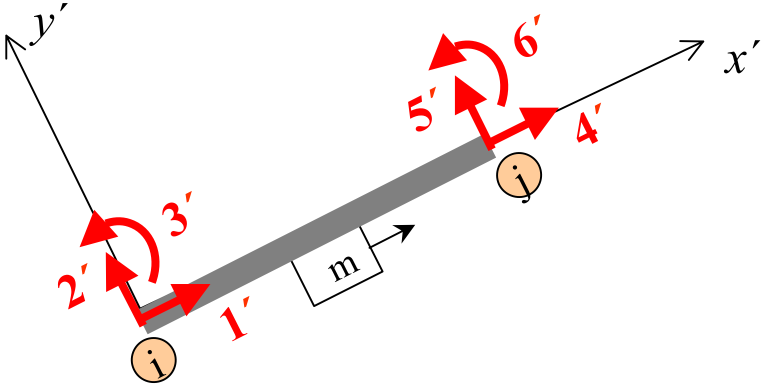
FRAME ANALYSIS USING THE STIFFNESS METHOD

- Simple Frames
  - Frame-Member Stiffness Matrix
  - Displacement and Force Transformation Matrices
  - Frame-Member Global Stiffness Matrix

- Special Frames
  - Frame-Member Global Stiffness Matrix

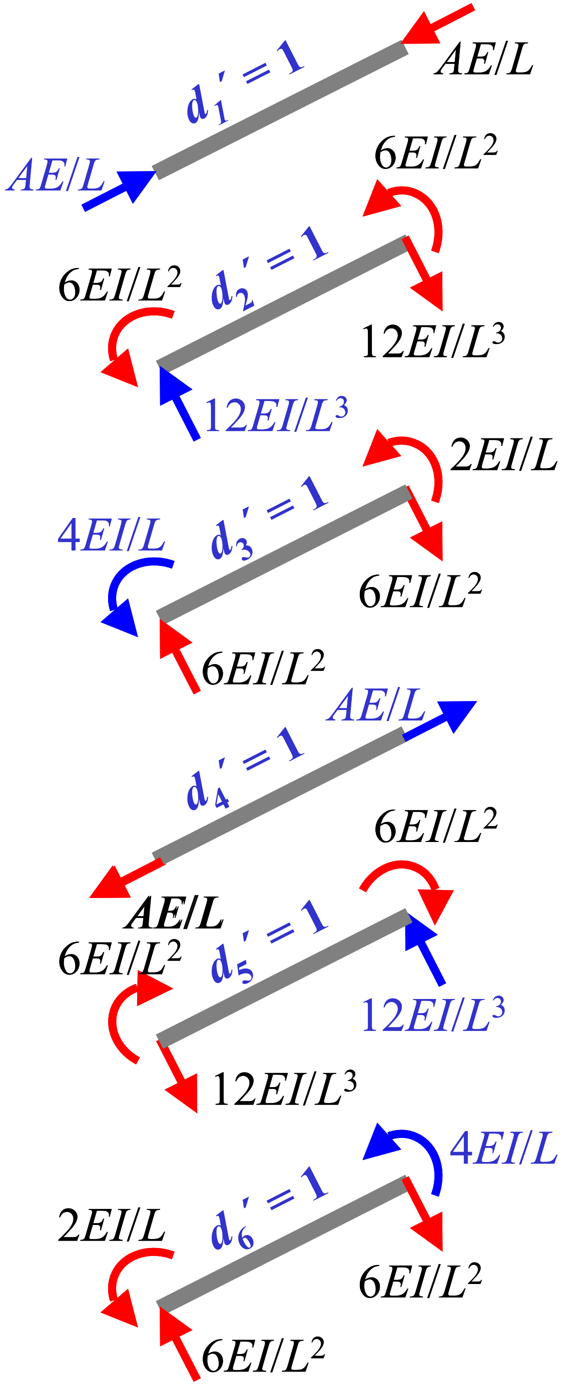
Simple Frames

Frame-Member Stiffness Matrix

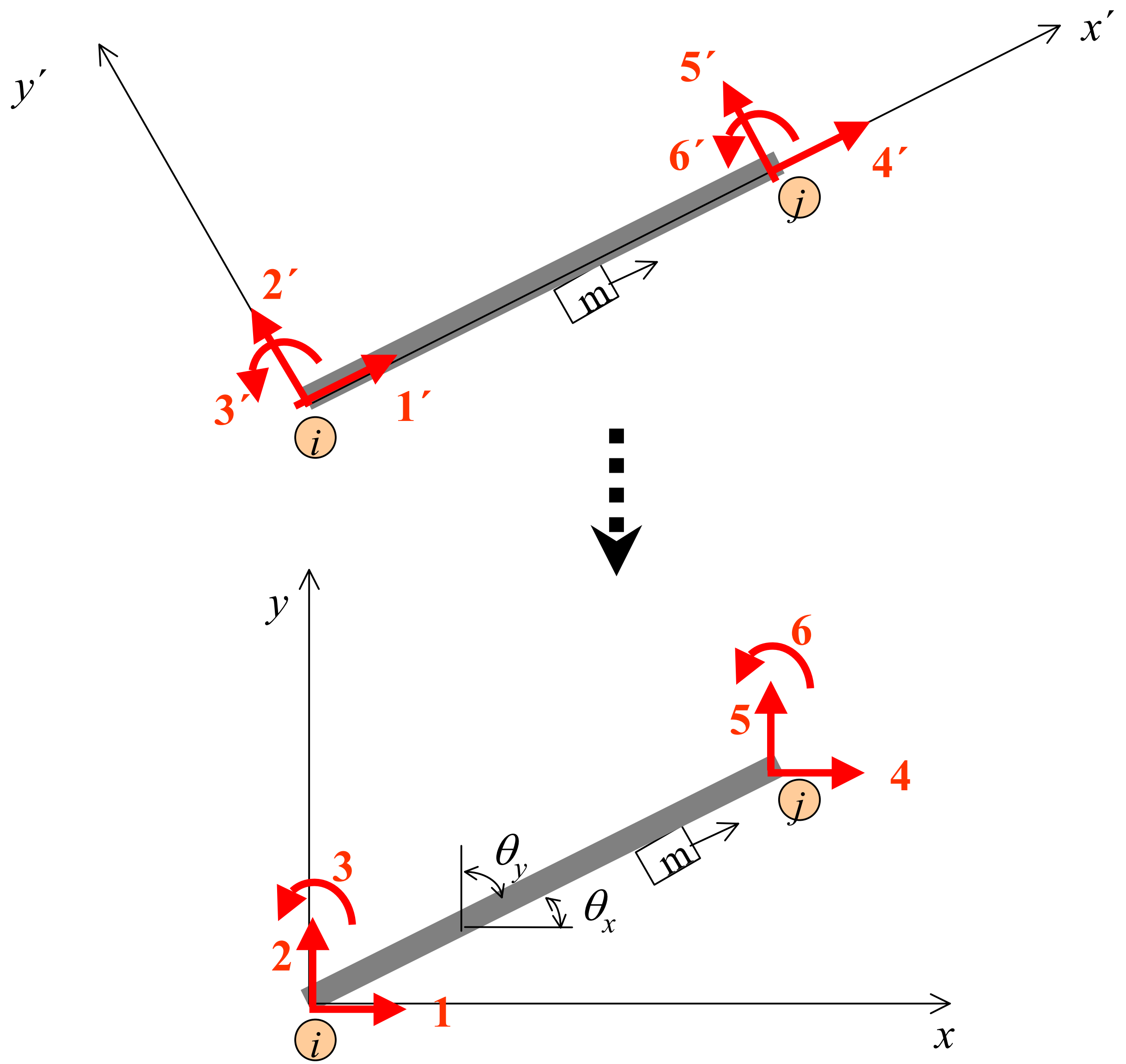


[k´]

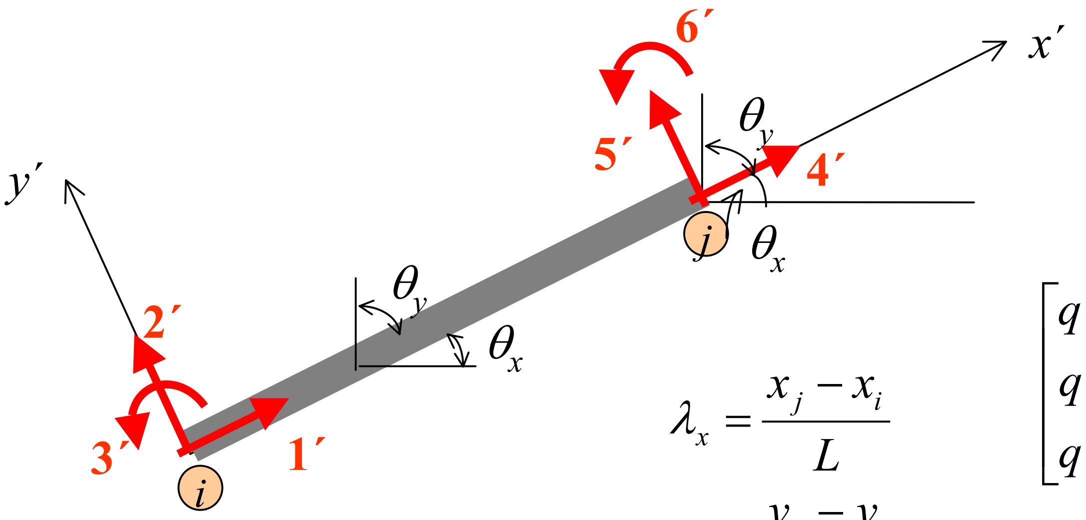
$$
\begin{array}{c|cccccc}
 & 1' & 2' & 3' & 4' & 5' & 6' \\
\hline
1' & AE/L & 0 & 0 & -AE/L & 0 & 0 \\
2' & 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
3' & 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
4' & -AE/L & 0 & 0 & AE/L & 0 & 0 \\
5' & 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
6' & 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{array}
$$



Displacement and Force Transformation Matrices



Force Transformation



$$q_4 = q_{4'}\cos\theta_x - q_{5'}\cos\theta_y$$

$$q_5 = q_{4'}\cos\theta_y + q_{5'}\cos\theta_x$$

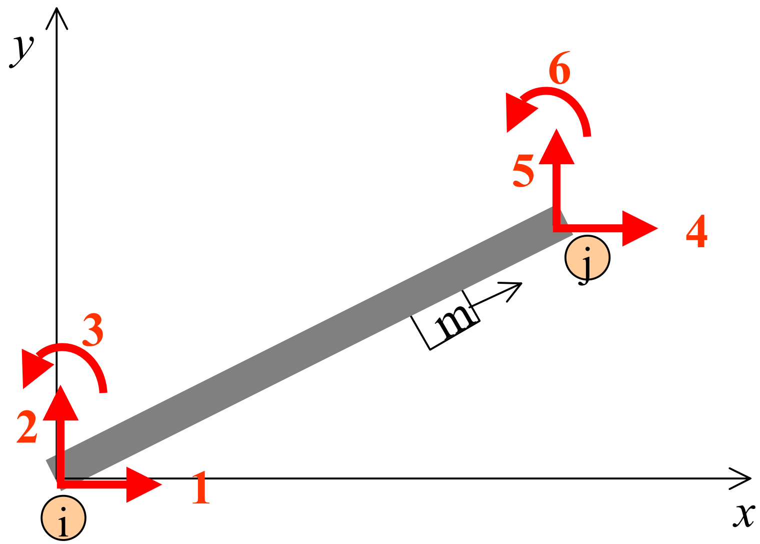
$$q_6 = q_{6'}$$

λ_x λ_y

$$\lambda_x = \frac{x_j - x_i}{L}$$

$$\lambda_y = \frac{y_j - y_i}{L}$$

$$\begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} \lambda_x & -\lambda_y & 0 \\ \lambda_y & \lambda_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$



$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} \lambda_x & -\lambda_y & 0 & 0 & 0 & 0 \\ \lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & -\lambda_y & 0 \\ 0 & 0 & 0 & \lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

$$[q] = [T]^T [q']$$

$$[q] = [T]^T[q']$$

$$= [T]^T \left([k'][d'] + [q'^F] \right)$$

$$= [T]^T [k'][d'] + [T]^T [q'^F]$$

$$[q] = [T]^T [k'][T][d] + [T]^T [q'^F] = [k][d] + [q^F]$$

Therefore, $[k] = [T]^T [k'][T]$

$$[q^F] = [T]^T [q'^F]$$

$$\boxed{[q] = [T]^T[q']}$$

$$\boxed{[d'] = [T][d]}$$

$$\boxed{[k] = [T]^T [k'][T]}$$

Frame Member Global Stiffness Matrix

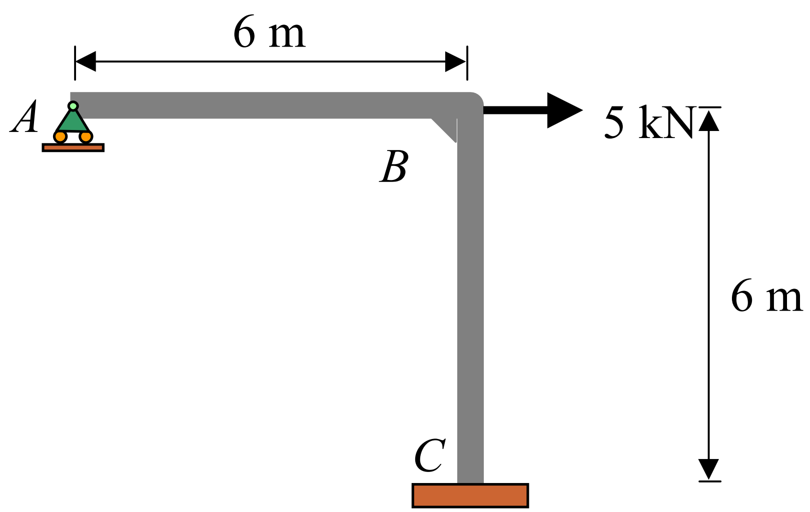
$$[q] = [T]^{\mathrm{T}}[q'] = [T]^{\mathrm{T}}\left([k'][d'] + [q'^{\mathrm{F}}] \right) = [T]^{\mathrm{T}}[k'][d'] + [T]^{\mathrm{T}}[q'^{\mathrm{F}}] = \underbrace{[T]^{\mathrm{T}}[k'][T]}_{[k]}[d] + \underbrace{[T]^{\mathrm{T}}[q'^{\mathrm{F}}]}_{[q^{\mathrm{F}}]}$$

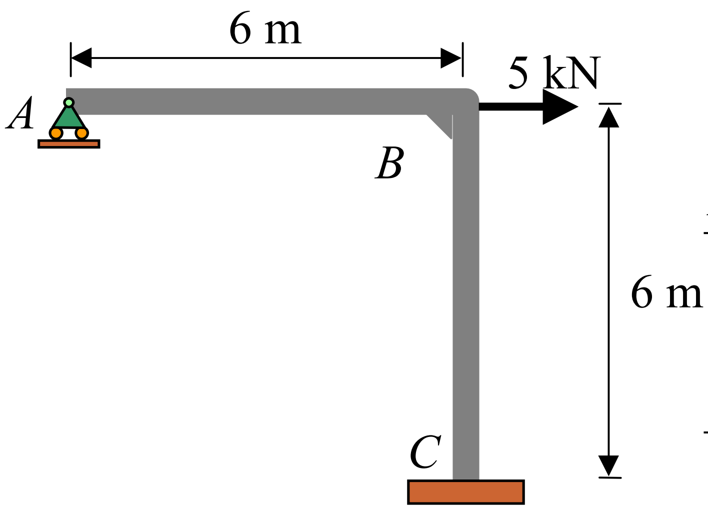
$$[k] = [T]^{\mathrm{T}}[k'][T] =$$

$$\begin{array}{c} \\ U_i \\ V_i \\ M_i \\ U_j \\ V_j \\ M_j \end{array}
\begin{array}{c}
\begin{array}{cccccc} U_i & V_i & M_i & U_j & V_j & M_j \end{array} \\
\begin{pmatrix}
\left(\frac{AE}{L}\lambda_{ix}^2 + \frac{12EI}{L^3}\lambda_{iy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & -\frac{6EI}{L^2}\lambda_{iy} & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{iy} \\
\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy} & \left(\frac{AE}{L}\lambda_{iy}^2 + \frac{12EI}{L^3}\lambda_{ix}^2\right) & \frac{6EI}{L^2}\lambda_{ix} & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & \frac{6EI}{L^2}\lambda_{ix} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{2EI}{L} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right) & \frac{6EI}{L^2}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jx}^2 + \frac{12EI}{L^3}\lambda_{jy}^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \frac{6EI}{L^2}\lambda_{jy} \\
-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right) & -\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right) & -\frac{6EI}{L^2}\lambda_{jx} & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy} & \left(\frac{AE}{L}\lambda_{jy}^2 + \frac{12EI}{L^3}\lambda_{jx}^2\right) & -\frac{6EI}{L^2}\lambda_{jx} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{4EI}{L}
\end{pmatrix}
\end{array}$$

Example 1

For the frame shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.
(c) Draw the **quantitative shear** and **bending moment diagrams.**
$E = 200$ GPa, $I = 60(10^6)$ mm^4, $A = 600$ mm^2

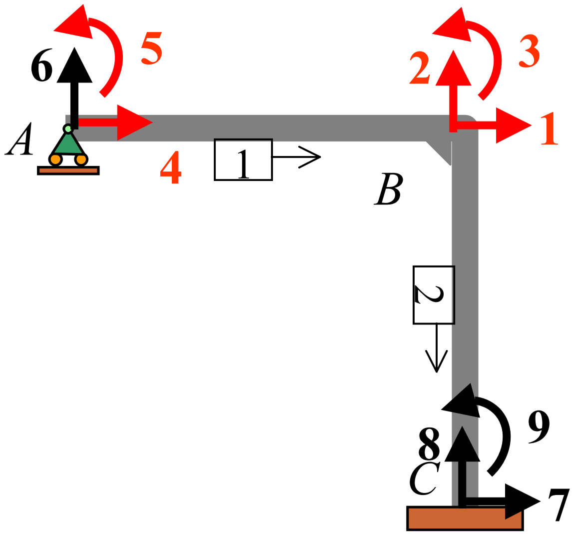




$$\frac{AE}{L} = \frac{(600\times10^{-6}\,\text{m}^2)(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})}{6\text{m}} = 20000\ \text{kN/m}$$

$$\frac{12EI}{L^3} = \frac{12(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{(6\text{m})^3} = 666.667\ \text{kN/m}$$

Global :



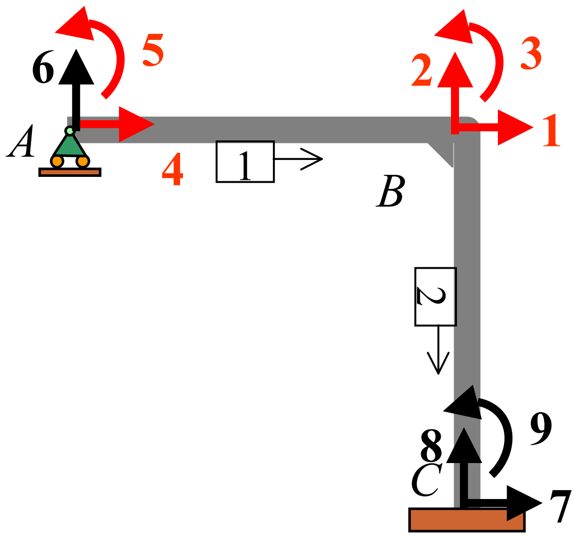
$$\frac{6EI}{L^2} = \frac{6(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{(6\text{m})^2} = 2000\ \text{kN}$$

$$\frac{4EI}{L} = \frac{4(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{6\text{m}} = 8000\ \text{kN}\bullet\text{m}$$

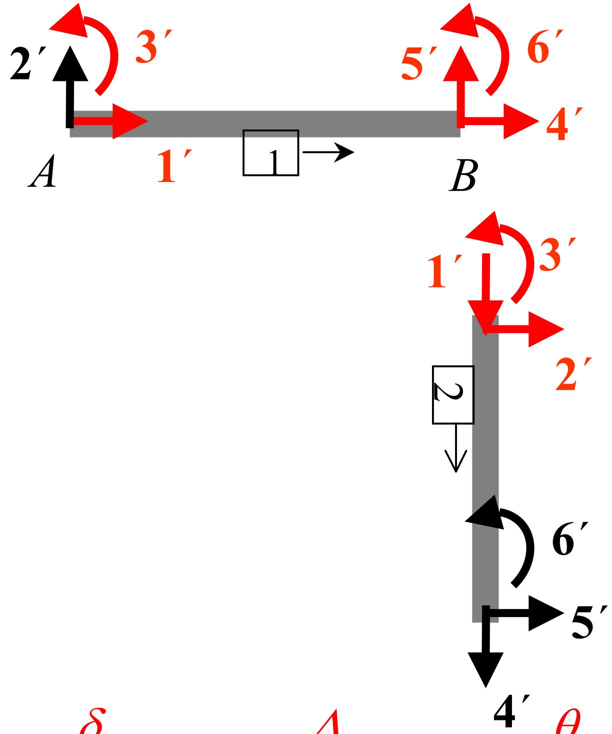
$$\frac{2EI}{L} = \frac{2(200\times10^{6}\,\frac{\text{kN}}{\text{m}^2})(60\times10^{-6}\,\text{m}^4)}{6\text{m}} = 4000\ \text{kN}\bullet\text{m}$$

Using Transformation Matrix:

Global :



Local :

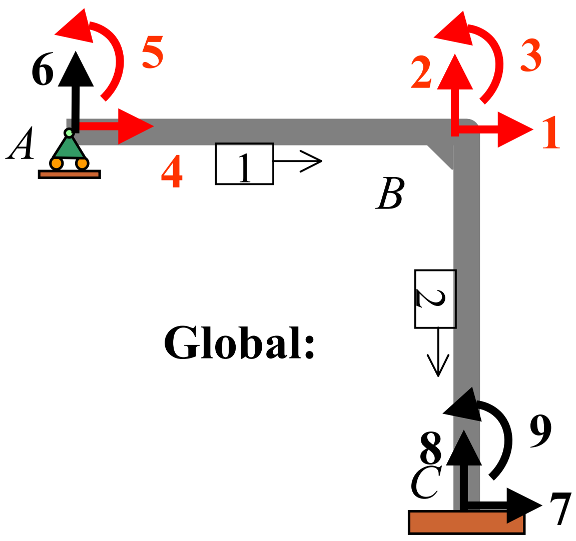


- **Member Stiffness Matrix**

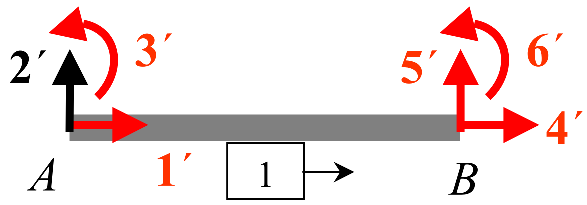
$$
[\mathrm{k'}] = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array}
\begin{array}{c}
\begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

Stiffness Matrix: Member 1

Local :



Global:

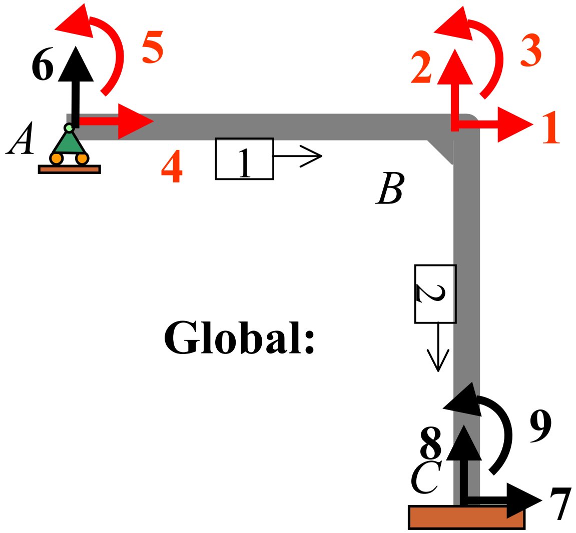


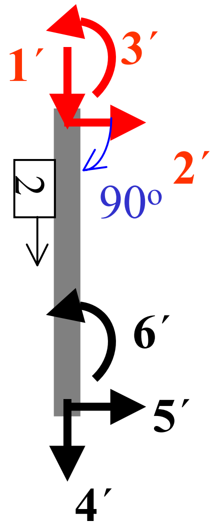
$[q] = [q']$

$\rightarrow [k]_1 = [k']_1$

$$
[k]_1 = \begin{array}{c} \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix}
20000 & 0 & 0 & -20000 & 0 & 0 \\
0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\
0 & 2000 & 8000 & 0 & -2000 & 4000 \\
-20000 & 0 & 0 & 20000 & 0 & 0 \\
0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\
0 & 2000 & 4000 & 0 & -2000 & 8000
\end{pmatrix}
\end{array}
$$

Stiffness Matrix: Member 2





$\lambda_{ix} = \cos(-90^{o}) = 0$
$\lambda_{iy} = \sin(-90^{o}) = -1$

$\lambda_{jx} = \cos(-90^{o}) = 0$
$\lambda_{jy} = \sin(-90^{o}) = -1$

$[q]_2 = [\,T\,]^T [\,q'\,]_2$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

$[T]^T$

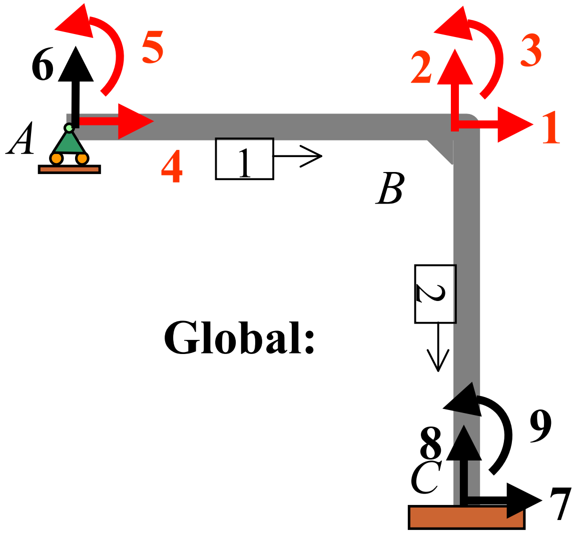
$$[k']_2 = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array}$$

$[k]_2 = [\,T\,]^T[\,k'\,]_2[\,T\,]$

$$[k]_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 666.667 & 0 & 2000 & -666.667 & 0 & 2000 \\ 0 & 20000 & 0 & 0 & -20000 & 0 \\ 2000 & 0 & 8000 & -2000 & 0 & 4000 \\ -666.667 & 0 & -2000 & 666.667 & 0 & -2000 \\ 0 & -20000 & 0 & 0 & 20000 & 0 \\ 2000 & 0 & 4000 & -2000 & 0 & 8000 \end{pmatrix} \end{array}$$

Global Stiffness Matrix:

Global:



$$[k]_1$$

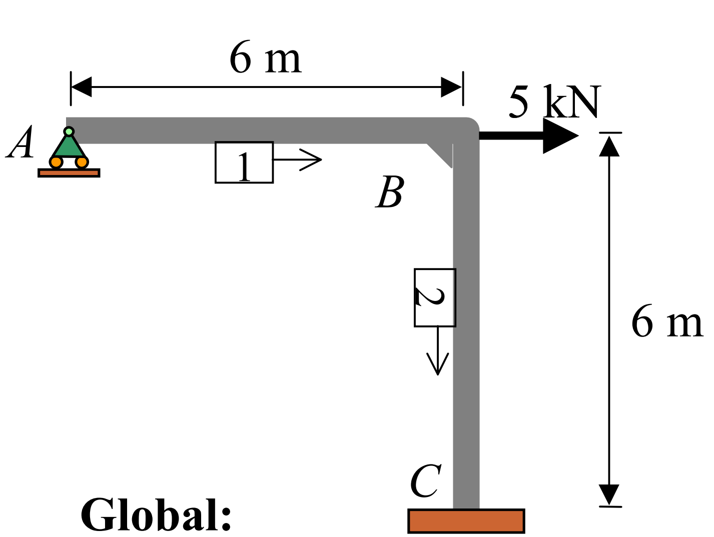
$$\begin{array}{c|c}
 & \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{pmatrix}
20000 & 0 & 0 & -20000 & 0 & 0 \\
0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\
0 & 2000 & 8000 & 0 & -2000 & 4000 \\
-20000 & 0 & 0 & 20000 & 0 & 0 \\
0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\
0 & 2000 & 4000 & 0 & -2000 & 8000
\end{pmatrix}
\end{array}$$

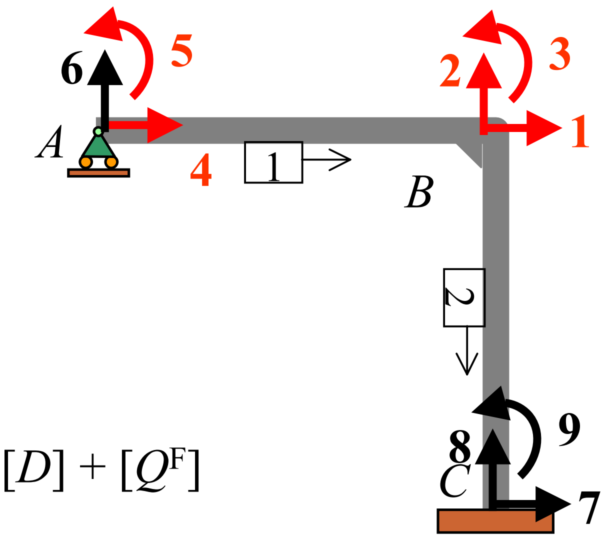
$$[k]_2$$

$$\begin{array}{c|c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} &
\begin{pmatrix}
666.667 & 0 & 2000 & 666.667 & 0 & 2000 \\
0 & 20000 & 0 & 0 & -20000 & 0 \\
2000 & 0 & 8000 & -2000 & 0 & 4000 \\
-666.667 & 0 & -2000 & 666.667 & 0 & -2000 \\
0 & -20000 & 0 & 0 & 20000 & 0 \\
2000 & 0 & 4000 & 2000 & 0 & 8000
\end{pmatrix}
\end{array}$$

$$[K]$$

$$\begin{array}{c|c}
 & \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{pmatrix}
20000 & 0 & -20000 & 0 & 0 \\
0 & 8000 & 0 & -2000 & 4000 \\
-20000 & 0 & 20666.667 & 0 & 2000 \\
0 & -2000 & 0 & 20666.667 & -2000 \\
0 & 4000 & 2000 & -2000 & 16000
\end{pmatrix}
\end{array}$$



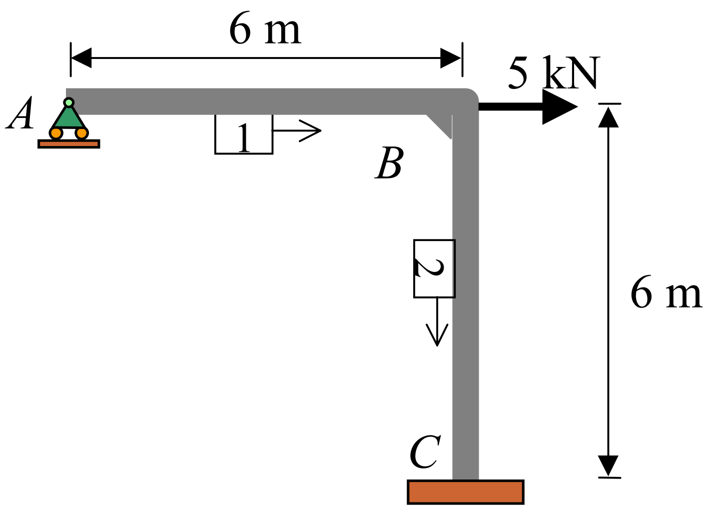


$$[Q] = [K][D] + [Q^F]$$

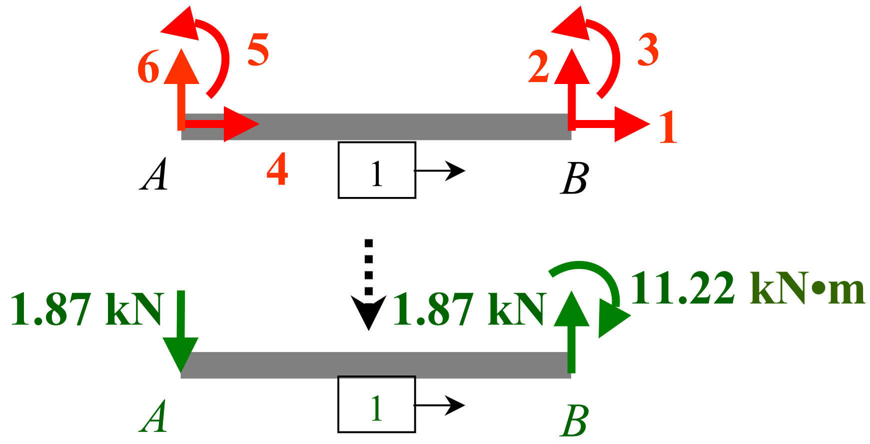
Global:

$$\begin{pmatrix} Q_4 = 0 \\ Q_5 = 0 \\ Q_1 = 5 \\ Q_2 = 0 \\ Q_3 = 0 \end{pmatrix} = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 5 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 20000 & 0 & -20000 & 0 & 0 \\ 0 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 20666.667 & 0 & 2000 \\ 0 & -2000 & 0 & 20666.667 & -2000 \\ 0 & 4000 & 2000 & -2000 & 16000 \end{pmatrix} \end{array} \begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0.01316 \text{ m} \\ 9.199(10^{-4}) \text{ rad} \\ 0.01316 \text{ m} \\ -9.355(10^{-5}) \text{ m} \\ -1.887(10^{-3}) \text{ rad} \end{pmatrix}$$

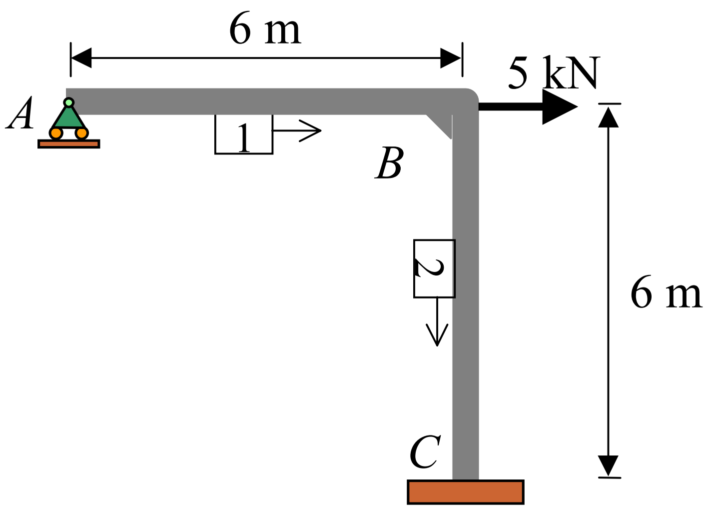


Member 1

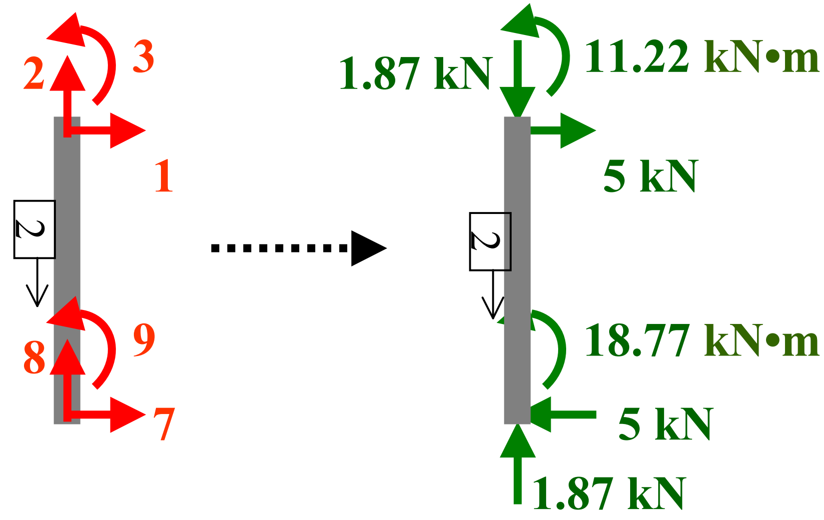


$$[q]_1 = [k]_1[d]_1 + [q^F]_1$$

$$
\begin{pmatrix} q_4 \\ q_6 \\ q_5 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}
=
\begin{array}{c}
 \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix}
20000 & 0 & 0 & -20000 & 0 & 0 \\
0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\
0 & 2000 & 8000 & 0 & -2000 & 4000 \\
-20000 & 0 & 0 & 20000 & 0 & 0 \\
0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\
0 & 2000 & 4000 & 0 & -2000 & 8000
\end{pmatrix}
\end{array}
\begin{pmatrix}
D_4 = 0.01316 \\
D_6 = 0 \\
D_5 = 9.199(10^{-4}) \\
D_1 = 0.01316 \\
D_2 = -9.355(10^{-5}) \\
D_3 = -1.887(10^{-3})
\end{pmatrix}
=
\begin{pmatrix} 0 \\ -1.87 \\ 0 \\ 0 \\ 1.87 \\ -11.22 \end{pmatrix}
$$

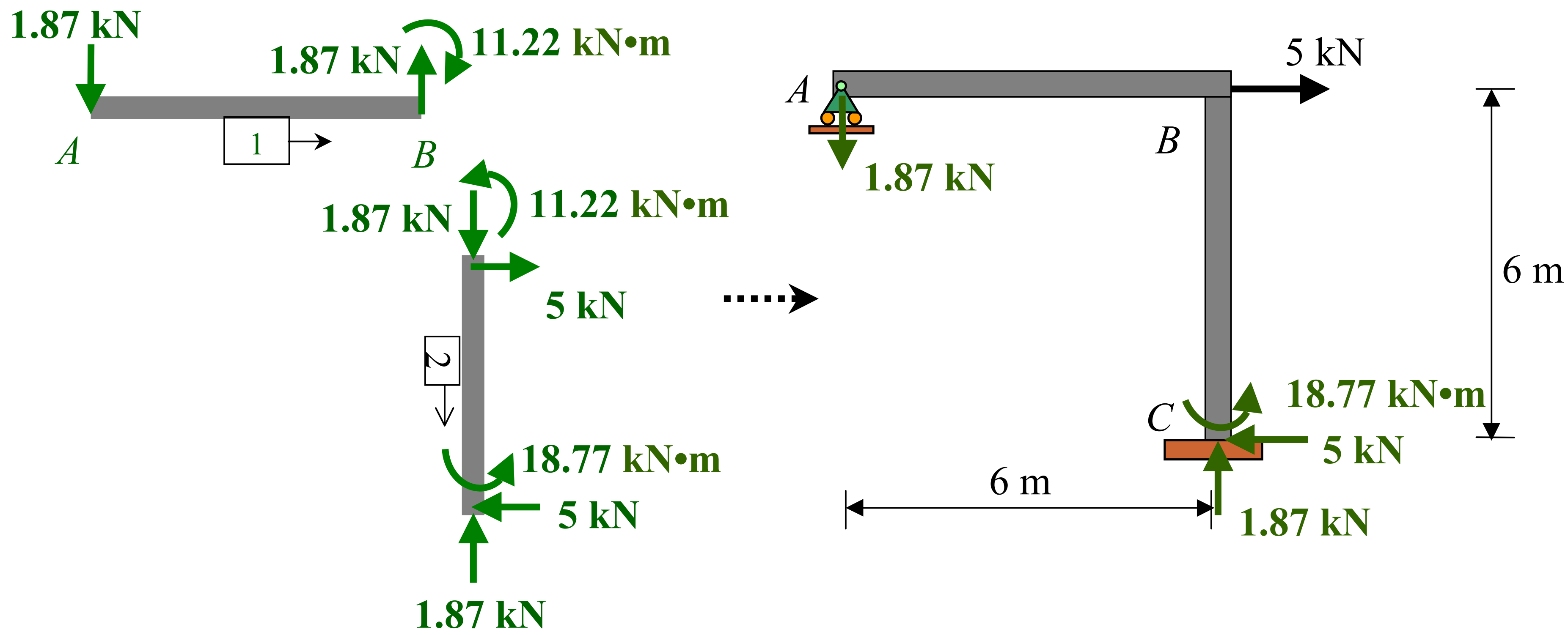


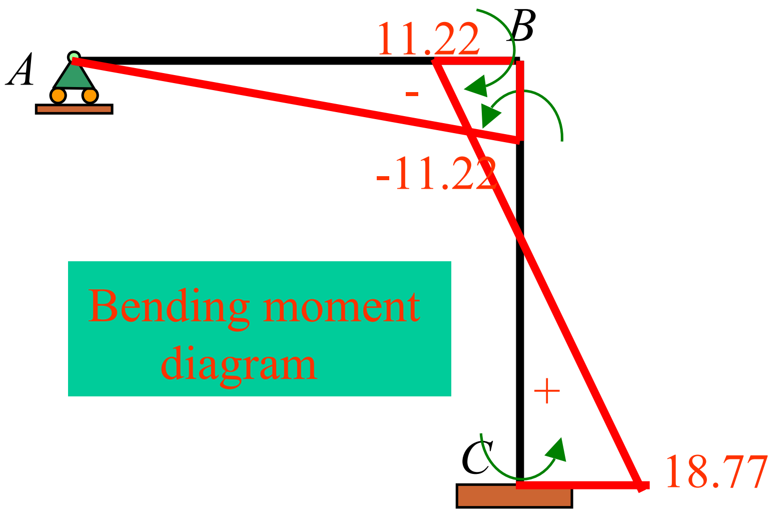
Member 2

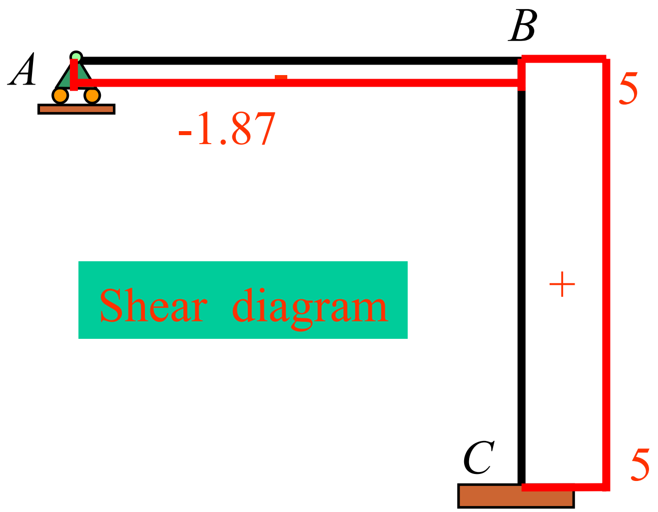


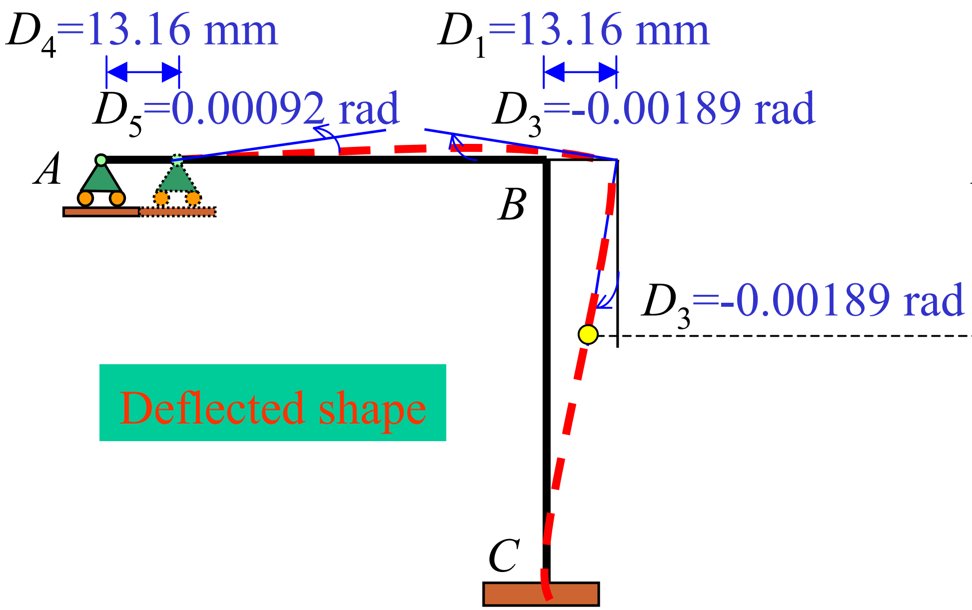
$$[q]_2 = [k]_2[d]_2 + [q^F]_2$$

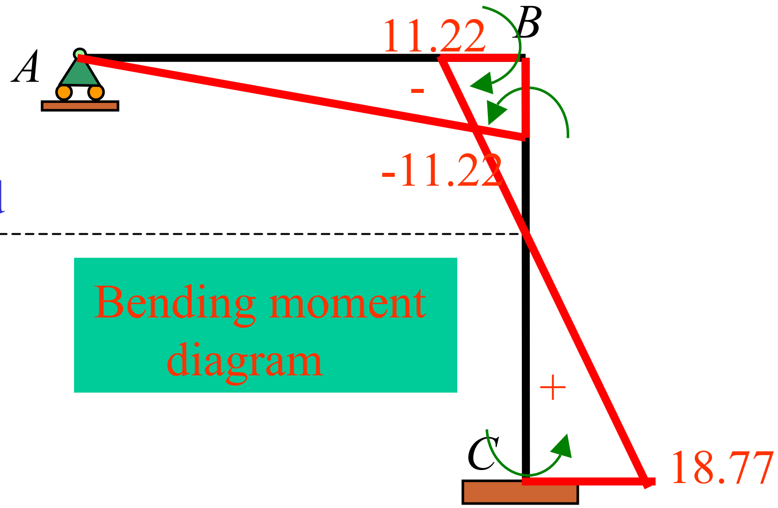
$$
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix}
=
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\
\begin{pmatrix}
666.667 & 0 & 2000 & -666.667 & 0 & 2000 \\
0 & 20000 & 0 & 0 & -20000 & 0 \\
2000 & 0 & 8000 & -2000 & 0 & 4000 \\
-666.667 & 0 & -2000 & 666.667 & 0 & -2000 \\
0 & -20000 & 0 & 0 & 20000 & 0 \\
2000 & 0 & 4000 & -2000 & 0 & 8000
\end{pmatrix}
\end{array}
\begin{pmatrix} D_1 = 0.01316 \\ D_2 = -9.355(10^{-5}) \\ D_3 = -1.887(10^{-3}) \\ D_7 = 0 \\ D_8 = 0 \\ D_9 = 0 \end{pmatrix}
=
\begin{pmatrix} 5 \\ -1.87 \\ 11.22 \\ -5 \\ 1.87 \\ 18.77 \end{pmatrix}
$$


1.87 kN
1.87 kN
11.22 kN•m
A
1
B
1.87 kN
11.22 kN•m
5 kN
2
18.77 kN•m
5 kN
1.87 kN
A
5 kN
B
1.87 kN
6 m
C
18.77 kN•m
5 kN
6 m
1.87 kN

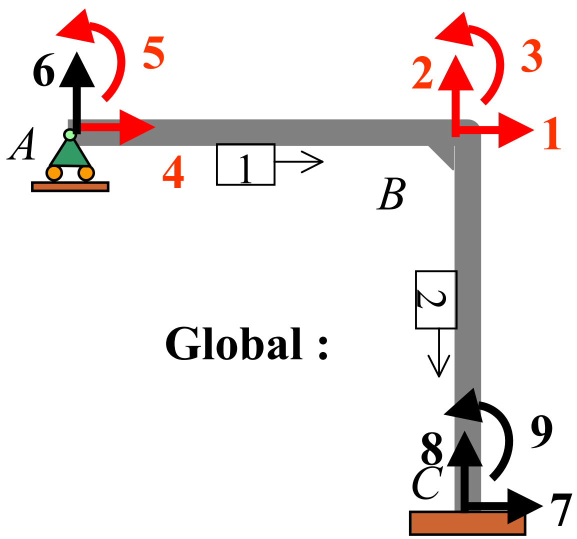

A
11.22
B
-
-11.22
Bending moment diagram
+
C
18.77


A
B
-
-1.87
5
Shear diagram
+
C
5



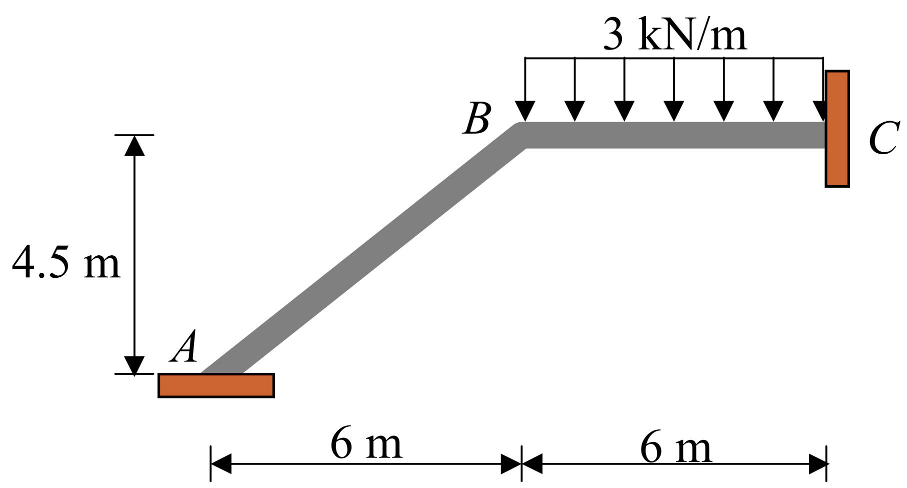


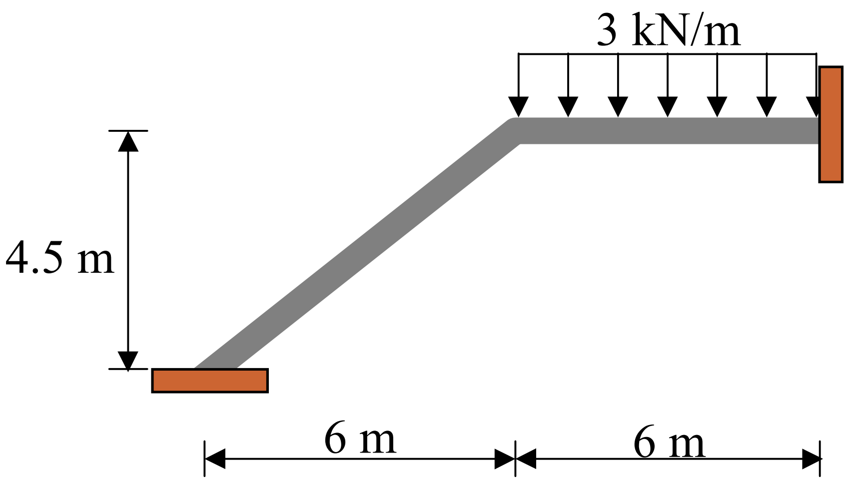
$$
\begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0.01316 \ \text{m} \\ 9.199(10^{-4}) \ \text{rad} \\ 0.01316 \ \text{m} \\ -9.355(10^{-5}) \ \text{m} \\ -1.887(10^{-3}) \ \text{rad} \end{pmatrix}
$$



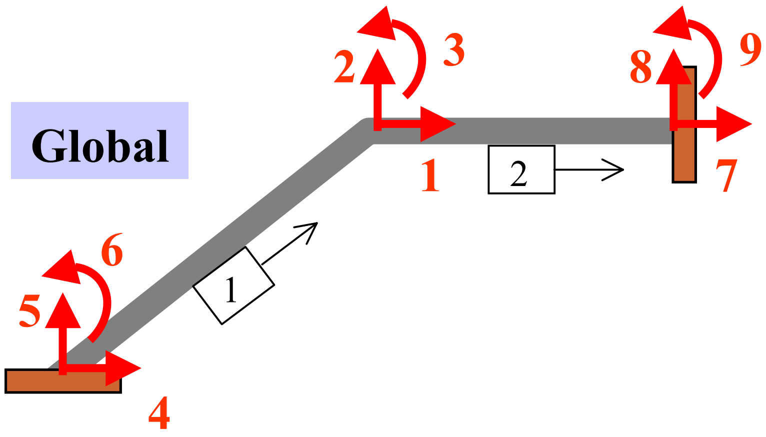
Example 2

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***
(b) Determine all the reactions at supports
(c) Draw the **quantitative shear** and **bending moment diagrams.**
$E = 200$ GPa, $I = 60(10^6)$ mm^4, $A = 600$ mm^2 for each member.

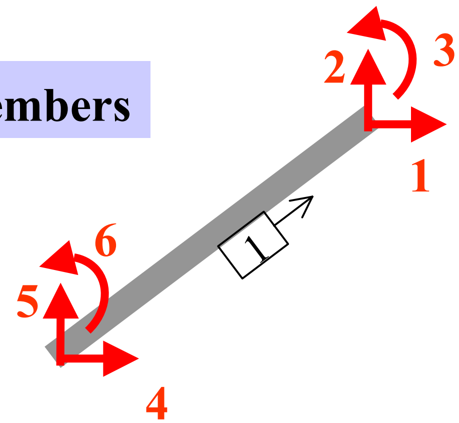


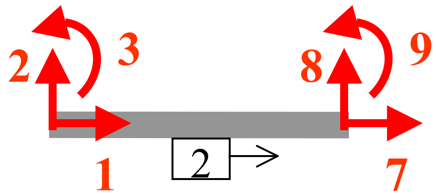


Global

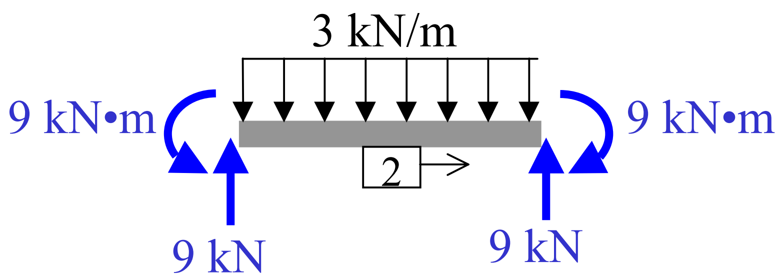


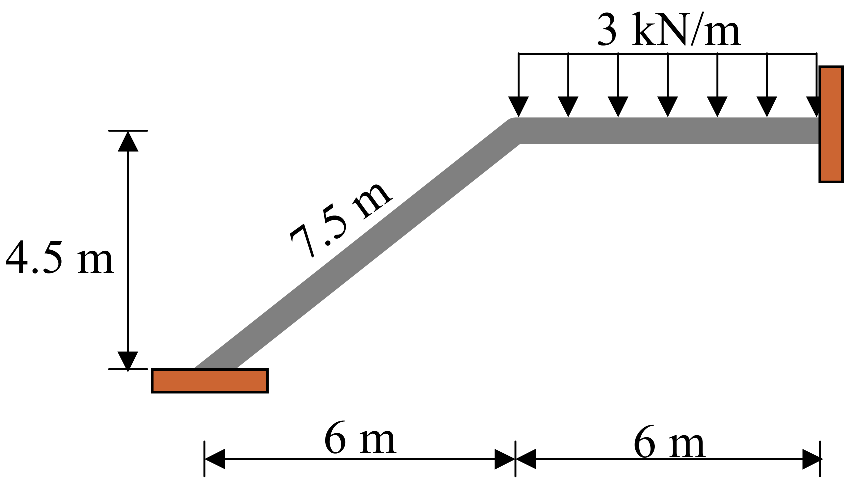
Members



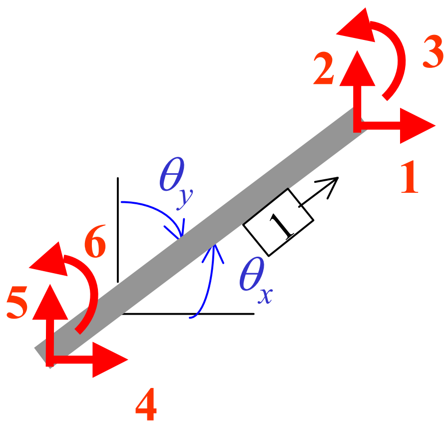


[FEM]





Member 1:



$$\lambda_x = \cos\ \theta_x = 6/7.5 = \ 0.8$$

$$\lambda_y = \cos\ \theta_y = 4.5/7.5 = \ 0.6$$

$$\frac{AE}{L} = \frac{(600\times10^{-6}\ m^2)(200\times10^6\ kN/m^2)}{7.5\ m}$$

$$= 16000\ kN/m$$

$$\frac{12EI}{L^3} = \frac{12(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(7.5\ m)^3}$$

$$= 341.33\ kN/m$$

$$\frac{6EI}{L^2} = \frac{6(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(7.5\ m)^2}$$

$$= 1280\ kN$$

$$\frac{4EI}{L} = \frac{4(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{7.5\ m}$$

$$= 6400\ kN\bullet m$$

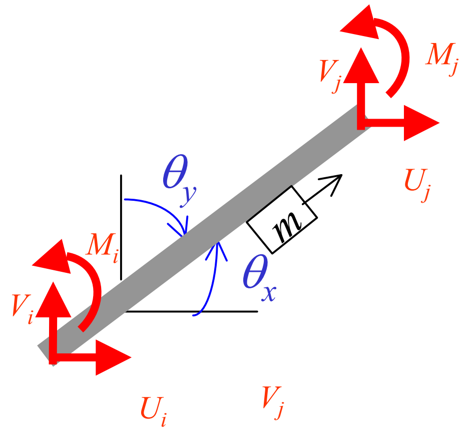
$$\frac{2EI}{L} = \frac{2(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{7.5\ m}$$

$$= 3200\ kN\bullet m$$

Member *m*:

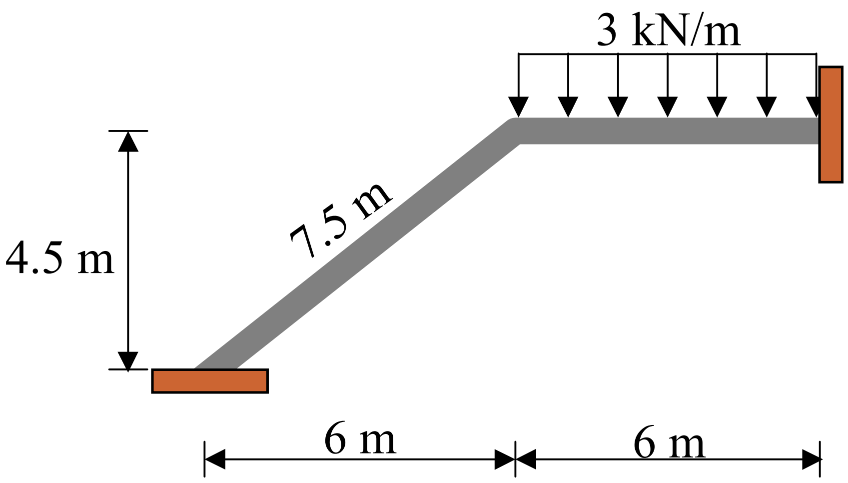
$$\lambda_x = \cos\,\theta_x$$

$$\lambda_y = \cos\,\theta_y$$

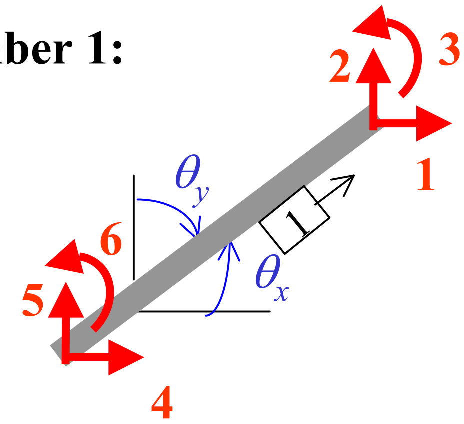


$$[\,\boldsymbol{k_m}\,] = [\,\boldsymbol{T}\,]^{\mathrm{T}}[\,\boldsymbol{k'}\,][\boldsymbol{T}\,] =$$

$$
\begin{array}{c}
 \\ U_i \\ \\ V_i \\ \\ M_i \\ \\ U_j \\ \\ V_j \\ \\ M_j
\end{array}
\begin{array}{c}
\begin{array}{cccccc} U_i & V_i & M_i & U_j & V_j & M_j \end{array} \\
\begin{pmatrix}
(\frac{AE}{L}\lambda_{ix}^2 + \frac{12EI}{L^3}\lambda_{iy}^2) & (\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{ix}\lambda_{iy} & -\frac{6EI}{L^2}\lambda_{iy} & -(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}) & -(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}) & -\frac{6EI}{L^2}\lambda_{iy} \\
(\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{ix}\lambda_{iy} & (\frac{AE}{L}\lambda_{iy}^2 + \frac{12EI}{L^3}\lambda_{ix}^2) & \frac{6EI}{L^2}\lambda_{ix} & -(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}) & -(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}) & \frac{6EI}{L^2}\lambda_{ix} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{2EI}{L} \\
-(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}) & -(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}) & \frac{6EI}{L^2}\lambda_{jy} & (\frac{AE}{L}\lambda_{jx}^2 + \frac{12EI}{L^3}\lambda_{jy}^2) & (\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{jx}\lambda_{jy} & \frac{6EI}{L^2}\lambda_{jy} \\
-(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}) & -(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}) & -\frac{6EI}{L^2}\lambda_{jx} & (\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{jx}\lambda_{jy} & (\frac{AE}{L}\lambda_{jy}^2 + \frac{12EI}{L^3}\lambda_{jx}^2) & -\frac{6EI}{L^2}\lambda_{jx} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{4EI}{L}
\end{pmatrix}
\end{array}
$$



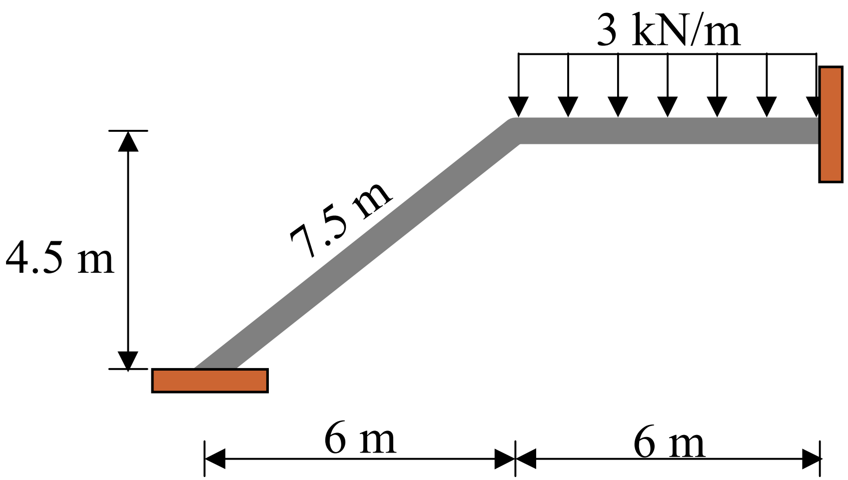
Member 1:



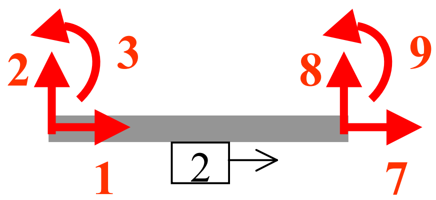
$$\lambda_x = \cos\theta_x = 6/7.5 = 0.8$$

$$\lambda_y = \cos\theta_y = 4.5/7.5 = 0.6$$

$$[k_1] = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 10362.879 & 7516.162 & -768 & -10362.879 & -7516.162 & -768 \\ 7516.162 & 5978.451 & 1024 & -7516.162 & -5978.451 & 1024 \\ -768 & 1024 & 6400 & 768 & -1024 & 3200 \\ -10362.879 & -7516.162 & 768 & 10362.879 & 7516.162 & 768 \\ -7516.162 & -5978.451 & -1024 & 7516.162 & 5978.451 & -1024 \\ -768 & 1024 & 3200 & 768 & -1024 & 6400 \end{pmatrix} \end{array}$$



Member 2:



$\lambda_x = \cos 0^o = 1.0,\ \lambda_y = \cos 90^o = 0$

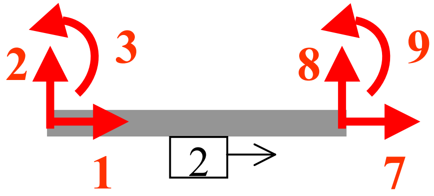
$$\frac{AE}{L} = \frac{(600\times10^{-6}\ m^2)(200\times10^6\ kN/m^2)}{6\ m} = 20000\ kN/m$$

$$\frac{12EI}{L^3} = \frac{12(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(6\ m)^3} = 666.667\ kN/m$$

$$\frac{6EI}{L^2} = \frac{6(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{(6\ m)^2} = 2000\ kN$$

$$\frac{4EI}{L} = \frac{4(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{6\ m} = 8000\ kN\bullet m$$

$$\frac{2EI}{L} = \frac{2(200\times10^6\ kN/m^2)(60\times10^{-6}\ m^4)}{6\ m} = 4000\ kN\bullet m$$

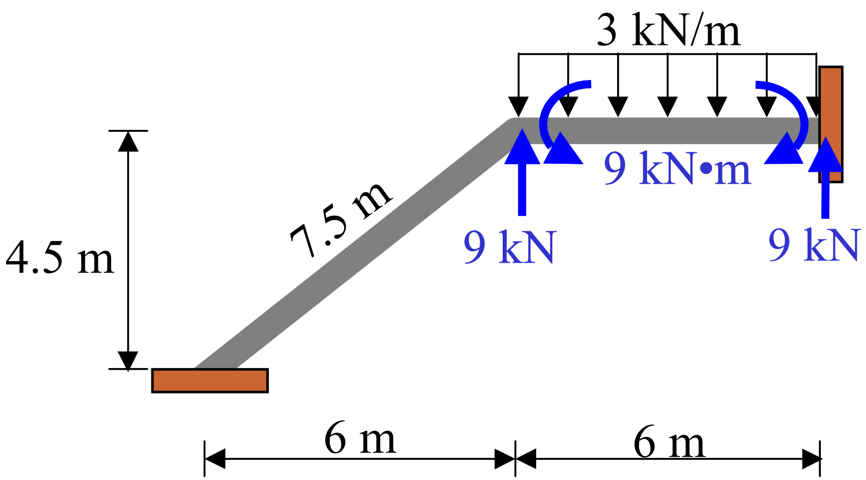


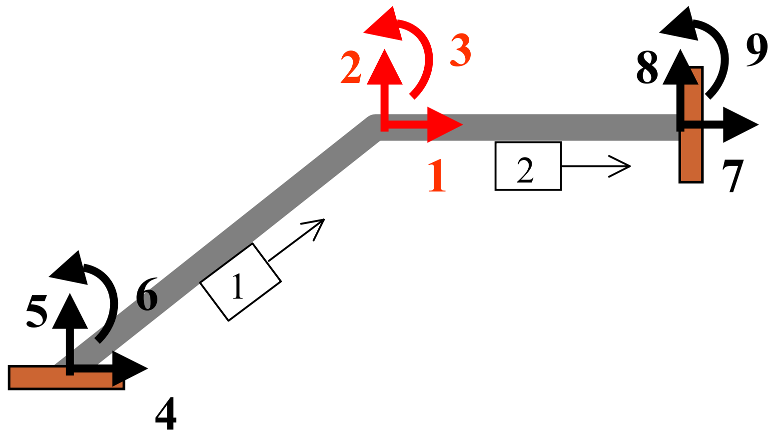
$$[k_2] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{pmatrix} \end{array}$$

$$[k_2] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array}$$

$$[k_1] = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 10362.879 & 7516.162 & -768 & -10362.879 & -7516.162 & -768 \\ 7516.162 & 5978.451 & 1024 & -7516.162 & -5978.451 & 1024 \\ -768 & 1024 & 6400 & 768 & -1024 & 3200 \\ -10362.879 & -7516.162 & 768 & 10362.879 & 7516.162 & 768 \\ -7516.162 & -5978.451 & -1024 & 7516.162 & 5978.451 & -1024 \\ -768 & 1024 & 3200 & 768 & -1024 & 6400 \end{pmatrix} \end{array}$$

$$[k_2] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array}$$



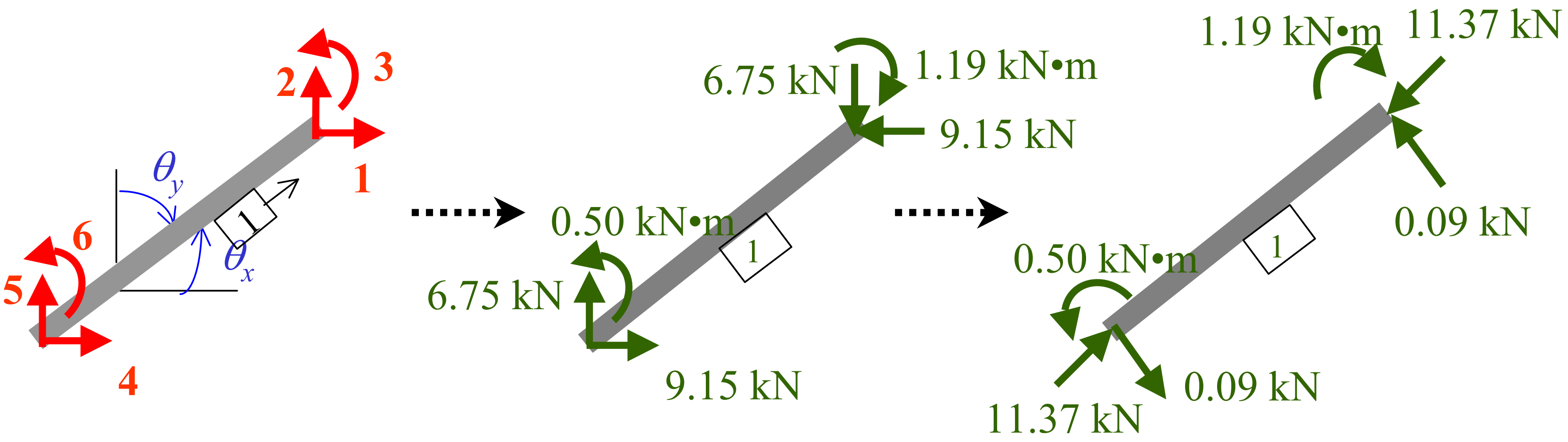


Global:

$$
\begin{pmatrix} \cancel{Q_1}^{0} \\ \cancel{Q_2}^{0} \\ \cancel{Q_3}^{0} \end{pmatrix}
=
\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 30362.9 & 7516.16 & 768 \\ 7516.16 & 6645.12 & 976 \\ 768 & 976 & 14400 \end{pmatrix}
\end{array}
\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}
+
\begin{pmatrix} 0 \\ 9 \\ 9 \end{pmatrix}
$$

$$
\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}
=
\begin{pmatrix} 4.575(10^{-4}) \text{ m} \\ -1.794(10^{-3}) \text{ m} \\ -5.278(10^{-4}) \text{ rad} \end{pmatrix}
$$

Member 1:

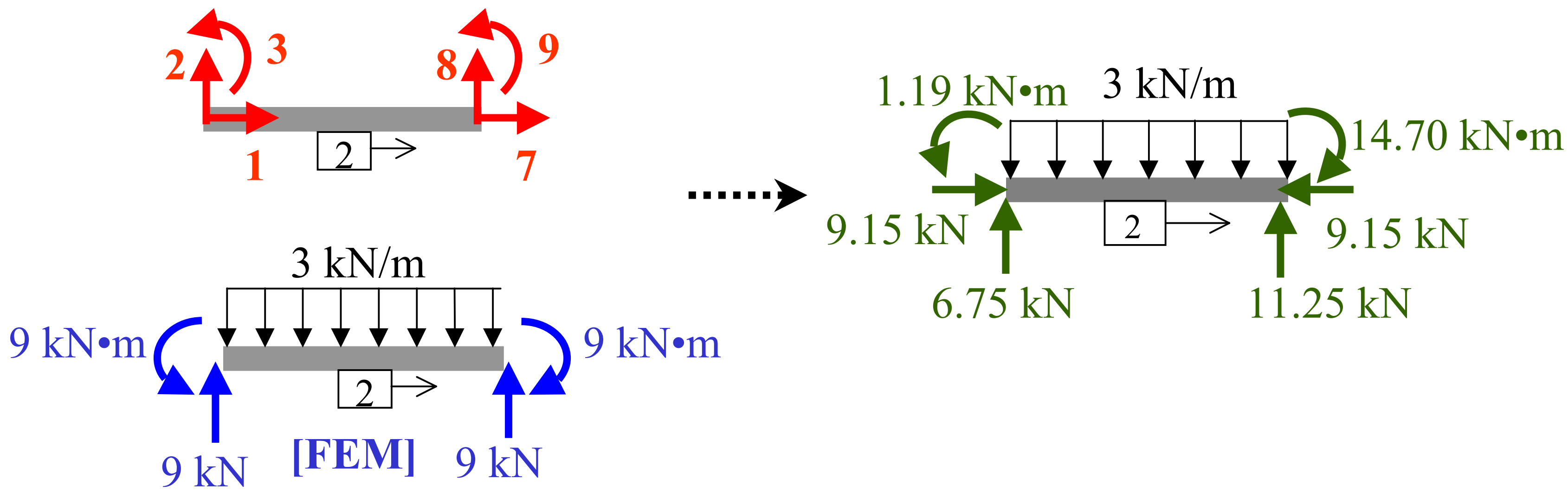


$$\lambda_x = \cos\,\theta_x = 6/7.5 = \ 0.8$$

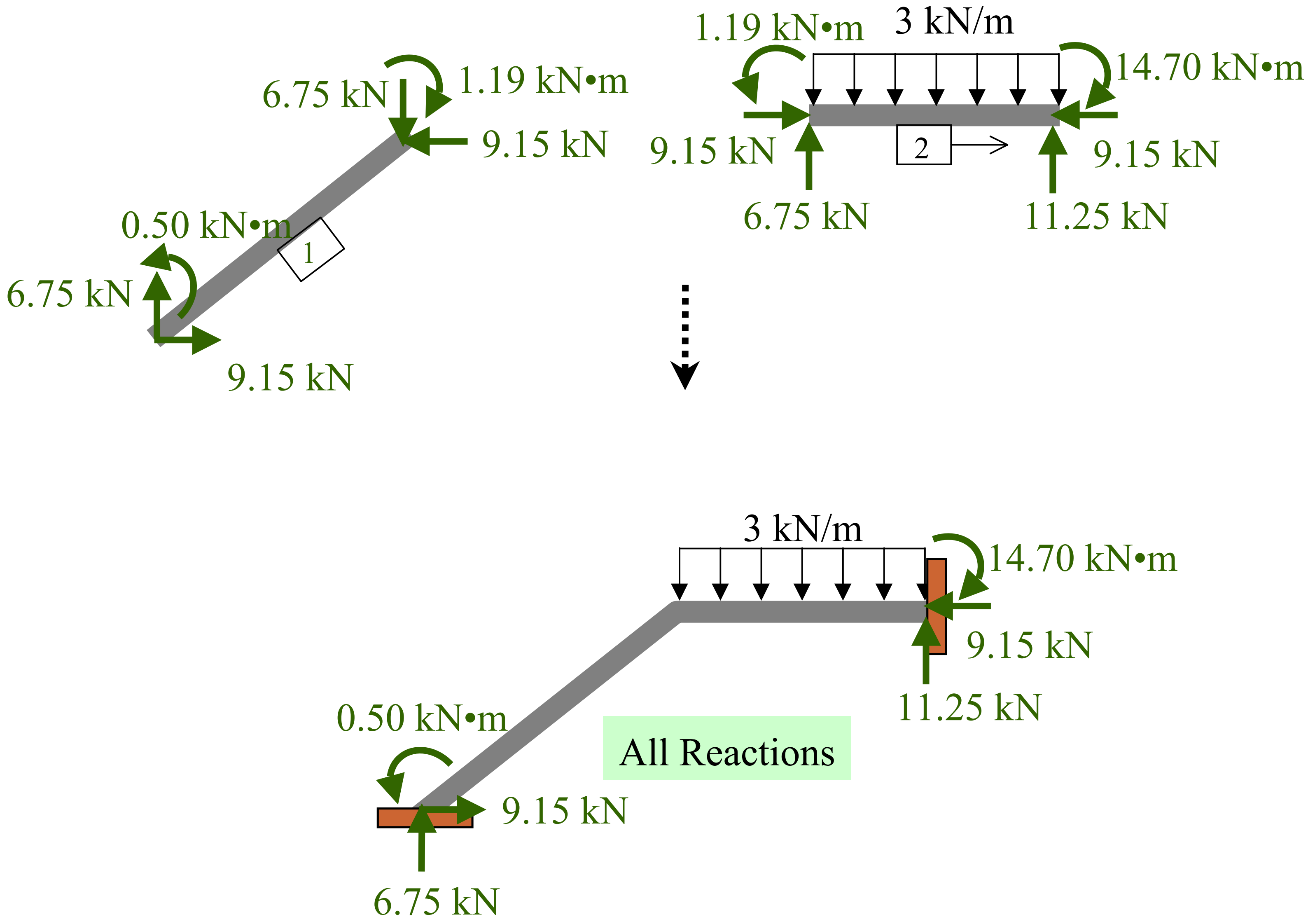
$$\lambda_y = \cos\,\theta_y = 4.5/7.5 = \ 0.6$$

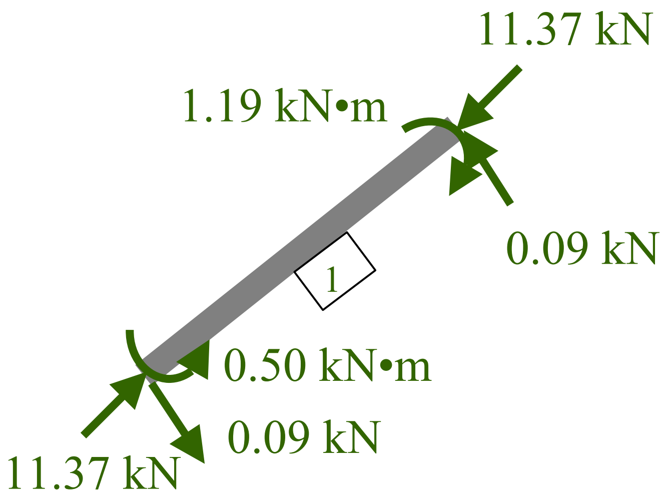
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{c} \\ \\ \\ \boldsymbol{k_1} \\ \\ \\ \end{array} \right) \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ D_1 = 4.575(10^{-4}) \\ D_2 = -1.794(10^{-3}) \\ D_3 = -5.278(10^{-4}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9.15 \\ 6.75 \\ 0.50 \\ -9.15 \\ -6.75 \\ -1.19 \end{pmatrix}$$

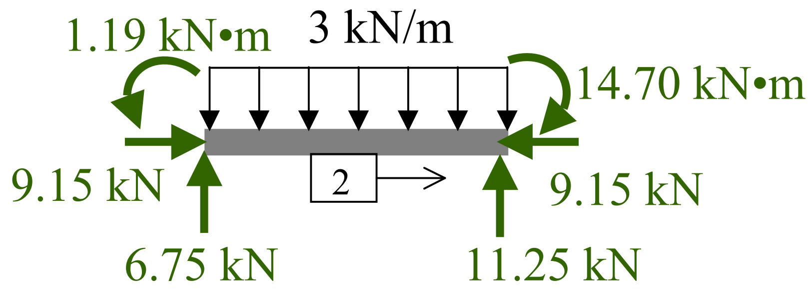
Member 2:



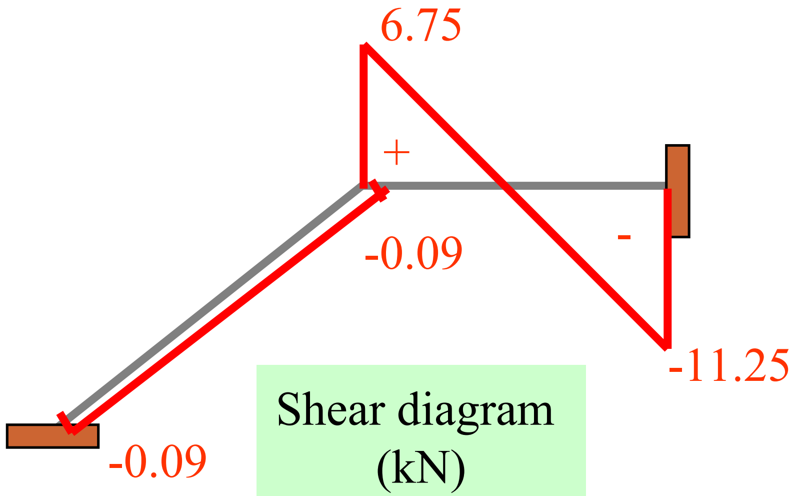
$$
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix}
=
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c} 1 \quad 2 \quad 3 \quad 7 \quad 8 \quad 9 \\ \left(\begin{array}{c} \\ \\ \\ \boldsymbol{k_2} \\ \\ \\ \end{array} \right) \end{array}
\begin{pmatrix} D_1 = 4.575(10^{-4}) \\ D_2 = -1.794(10^{-3}) \\ D_3 = -5.278(10^{-4}) \\ 0 \\ 0 \\ 0 \end{pmatrix}
+
\begin{pmatrix} 0 \\ 9 \\ 9 \\ 0 \\ 9 \\ -9 \end{pmatrix}
=
\begin{pmatrix} 9.15 \\ 6.75 \\ 1.19 \\ -9.15 \\ 11.25 \\ -14.70 \end{pmatrix}
$$


6.75 kN
1.19 kN•m
9.15 kN
0.50 kN•m
1
6.75 kN
9.15 kN
1.19 kN•m
3 kN/m
14.70 kN•m
9.15 kN
2
9.15 kN
6.75 kN
11.25 kN
3 kN/m
14.70 kN•m
9.15 kN
11.25 kN
0.50 kN•m
All Reactions
9.15 kN
6.75 kN

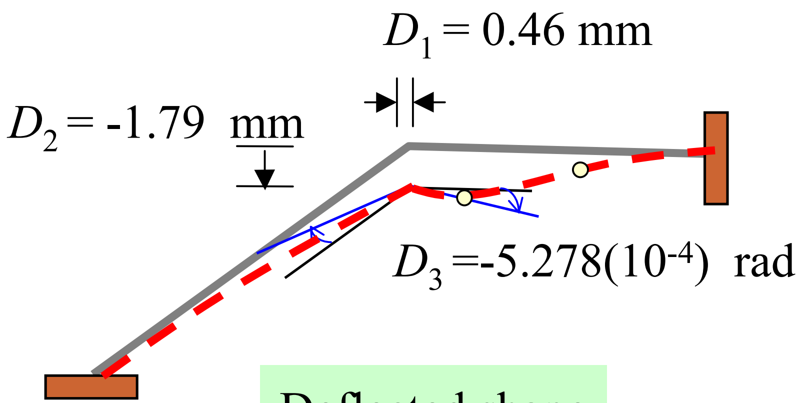




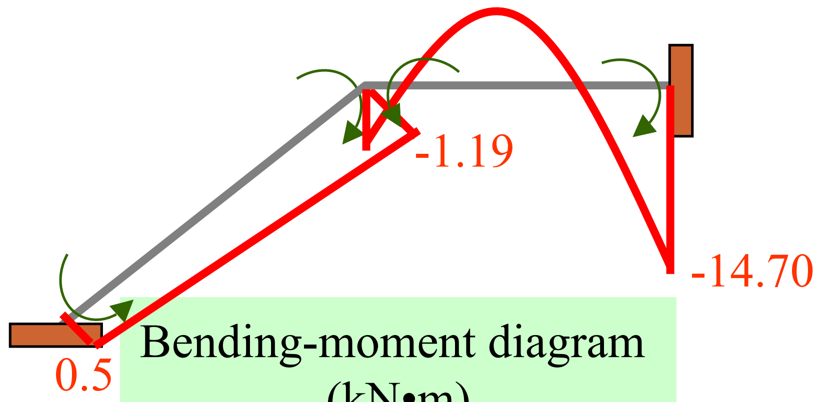
$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 4.575(10^{-4})\ \text{m} \\ -1.794(10^{-3})\ \text{m} \\ -5.278(10^{-4})\ \text{rad} \end{pmatrix}$$



Shear diagram (kN)



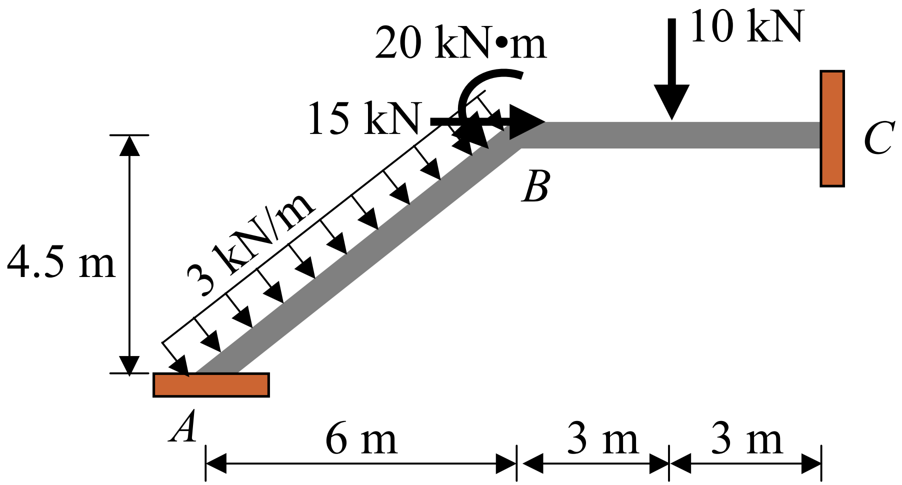
Deflected shape

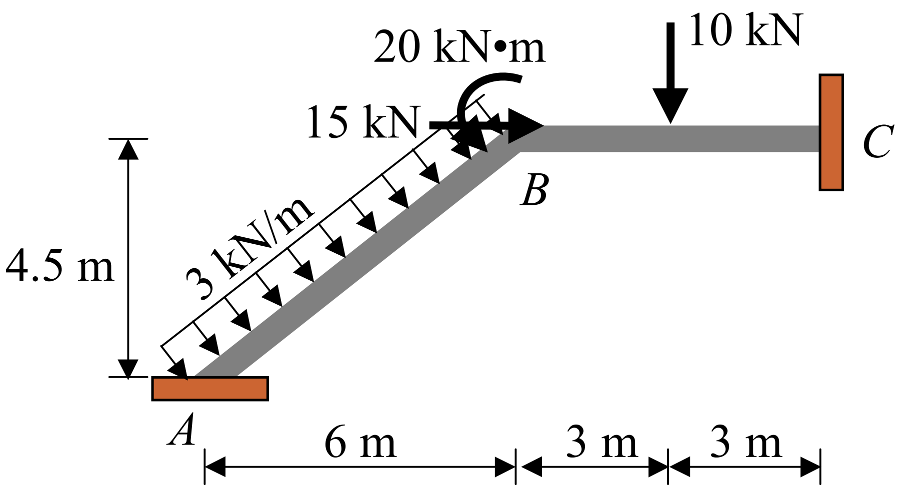


Bending-moment diagram (kN•m)

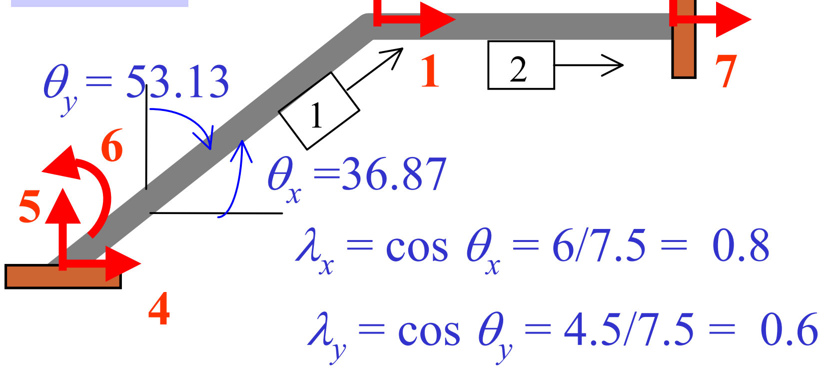
Example 3

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.
(c) Draw the **quantitative shear** and **bending moment diagrams.**
E = 200 GPa, $I = 60(10^6)$ mm^4, A = 600 mm^2 for each member.

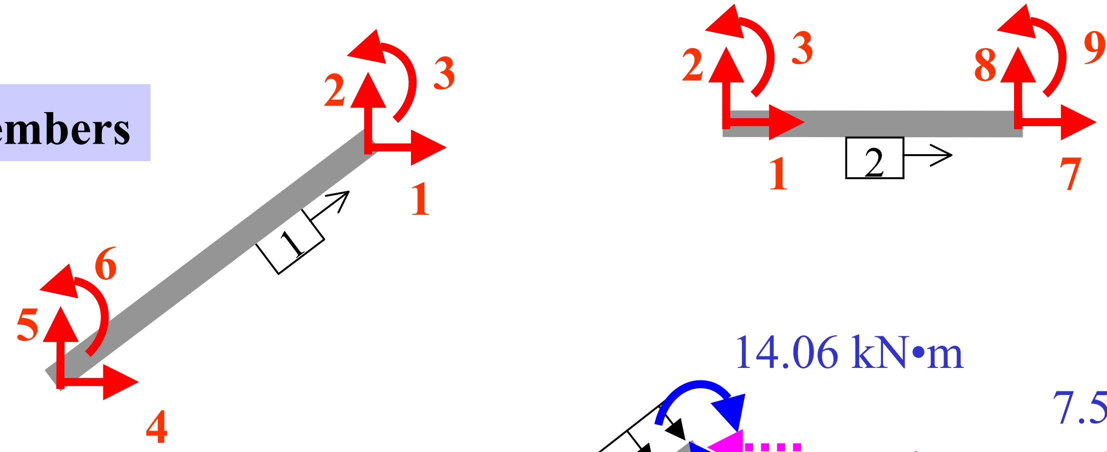




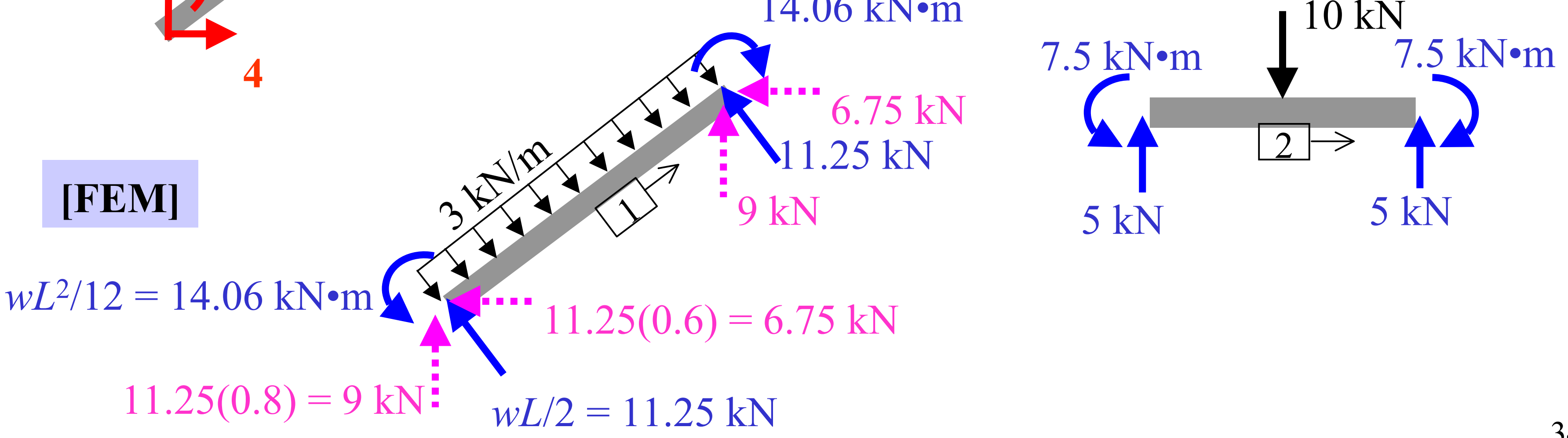
Global

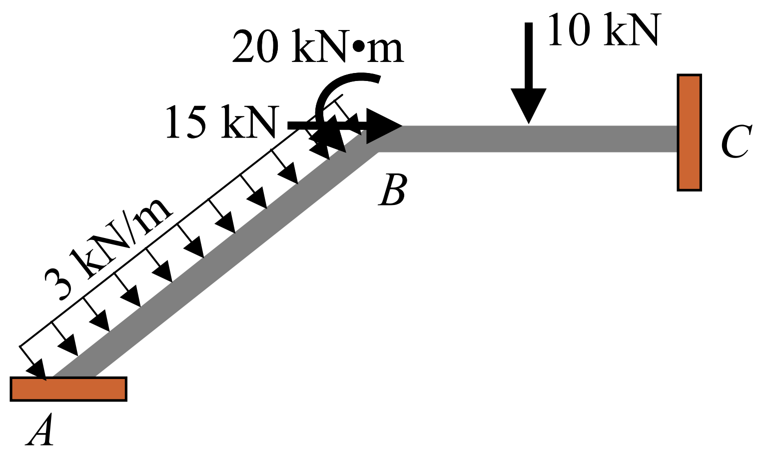


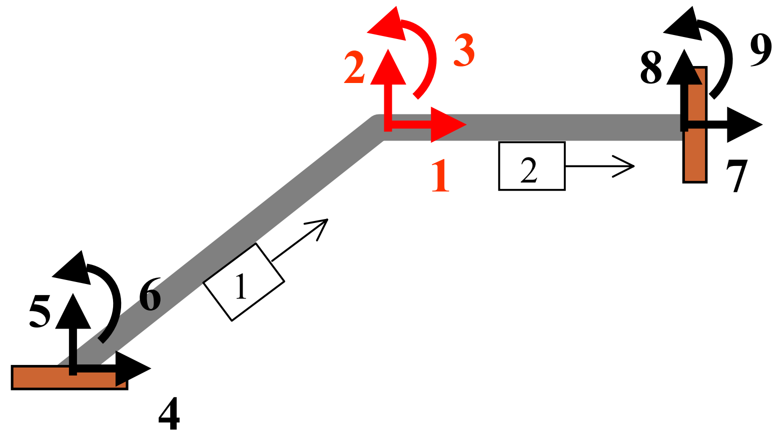
Members



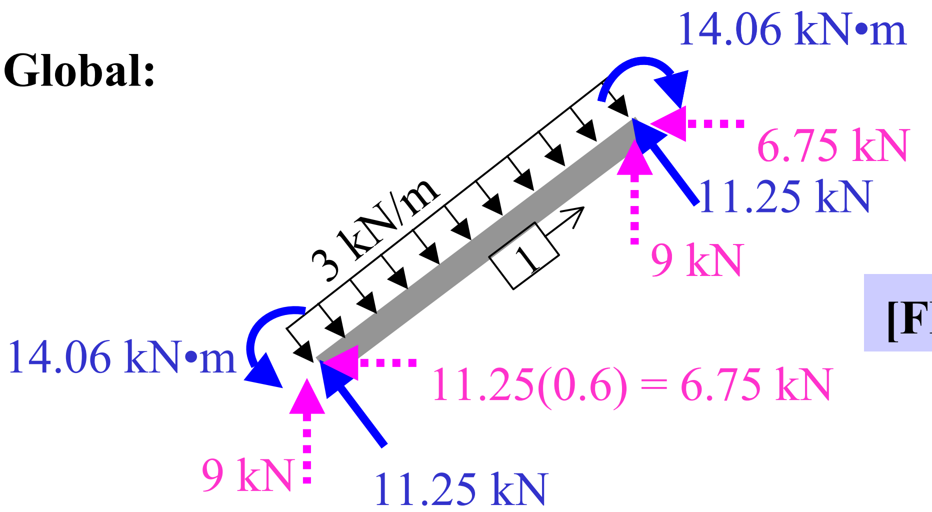
[FEM]

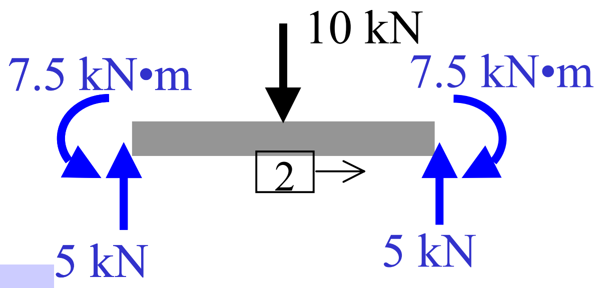






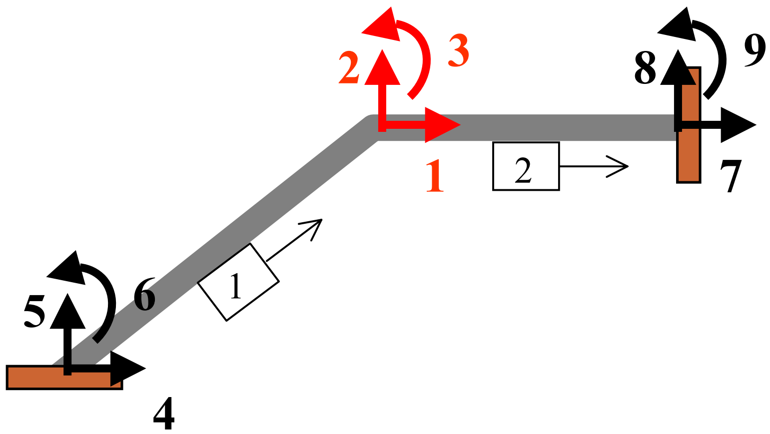
Global:





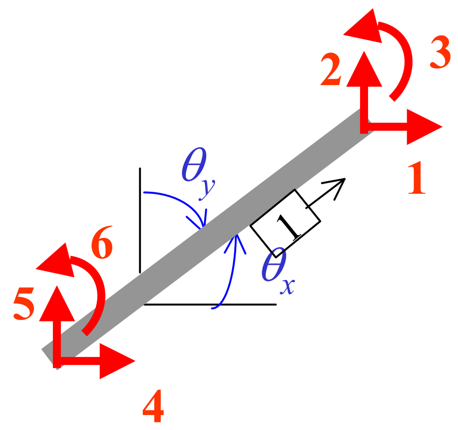
[FEM]

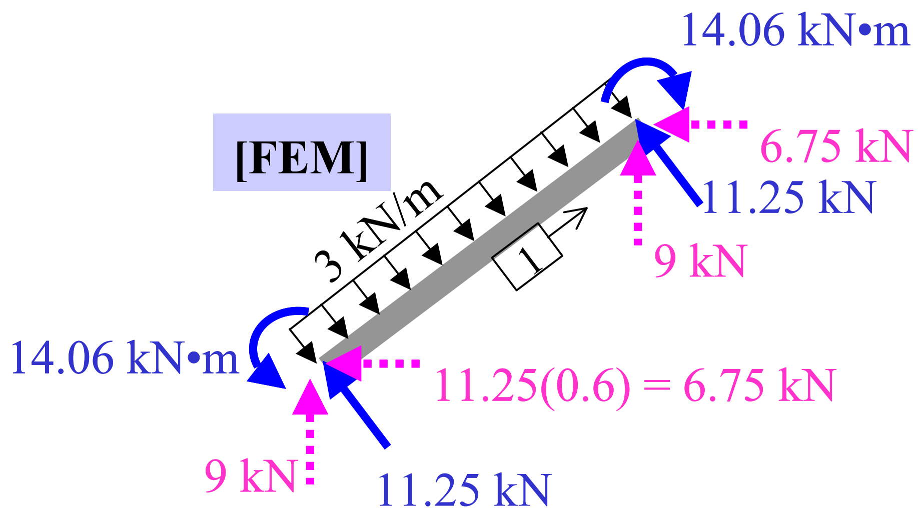
$$
\begin{pmatrix} Q_1 = 15 \\ Q_2 = 0 \\ Q_3 = 20 \end{pmatrix}
=
\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 30362.9 & 7516.16 & 768 \\ 7516.16 & 6645.2 & 976 \\ 768 & 976 & 14400 \end{pmatrix} \end{array}
\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}
+
\begin{pmatrix} -6.75 \\ 9 + 5 \\ -14.06 + 7.5 \end{pmatrix}
$$



$$
\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 1.751(10^{-3})\ \text{m} \\ -4.388(10^{-3})\ \text{m} \\ 2.049(10^{-3})\ \text{rad} \end{pmatrix}
$$

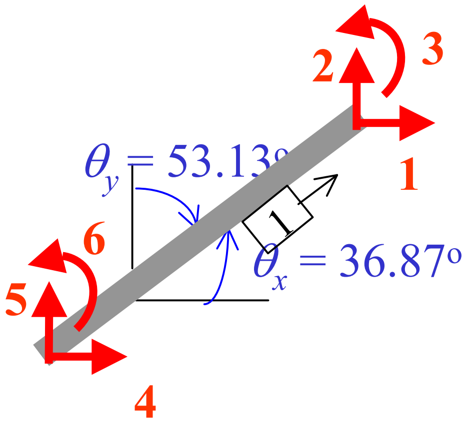
Member 1:



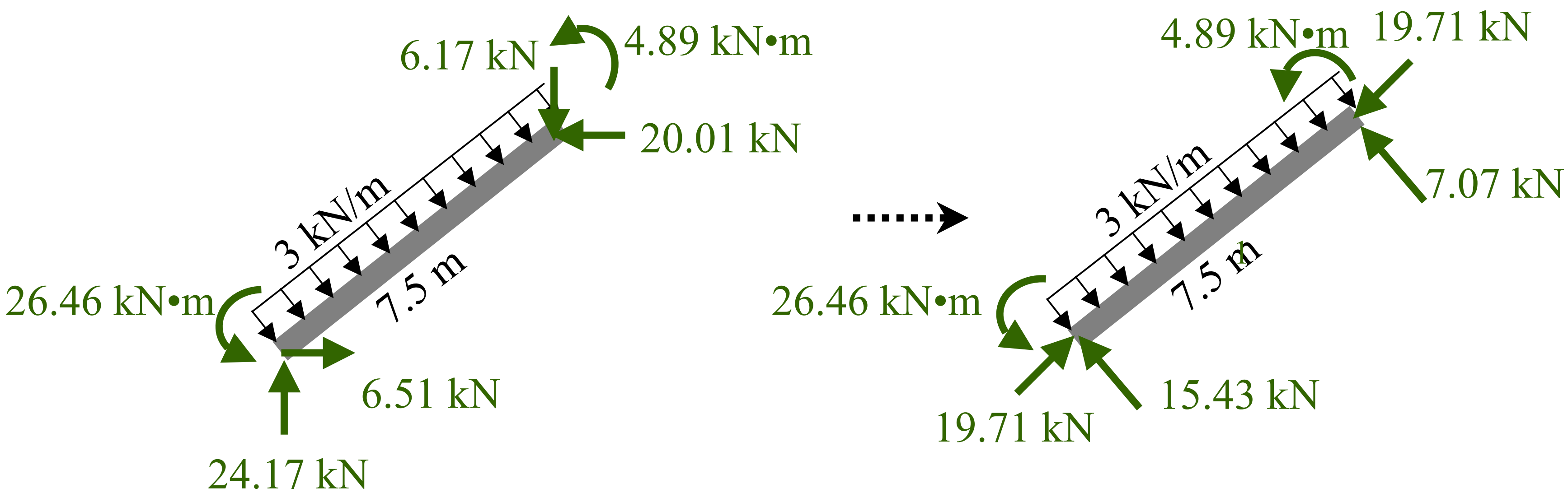


$\lambda_x = \cos 36.87^\circ = 0.8,$
$\lambda_y = \cos 53.13^\circ = 0.6$

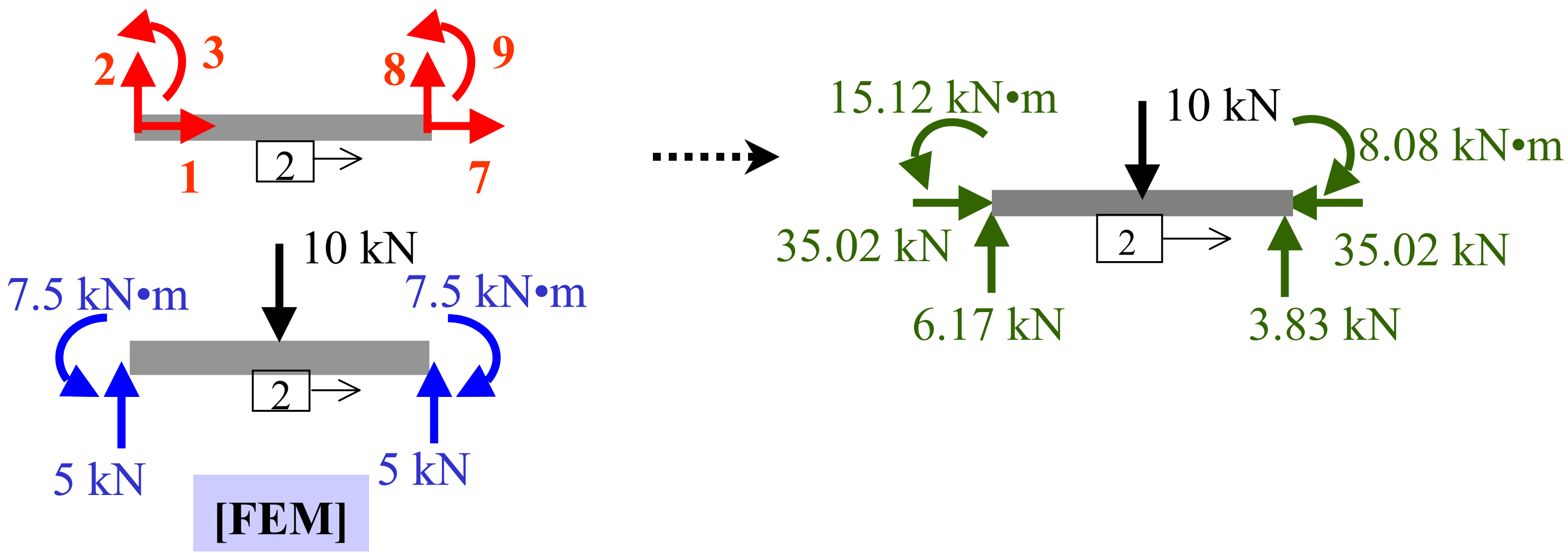
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} & & & & & \\ & & & & & \\ & & \boldsymbol{k_1} & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ D_1 = 1.751(10^{-3}) \\ D_2 = -4.388(10^{-3}) \\ D_3 = 2.049(10^{-3}) \end{pmatrix} + \begin{pmatrix} -6.75 \\ 9 \\ 14.06 \\ -6.75 \\ 9 \\ -14.06 \end{pmatrix} = \begin{pmatrix} 6.51 \\ 24.17 \\ 26.46 \\ -20.01 \\ -6.17 \\ 4.89 \end{pmatrix}$$



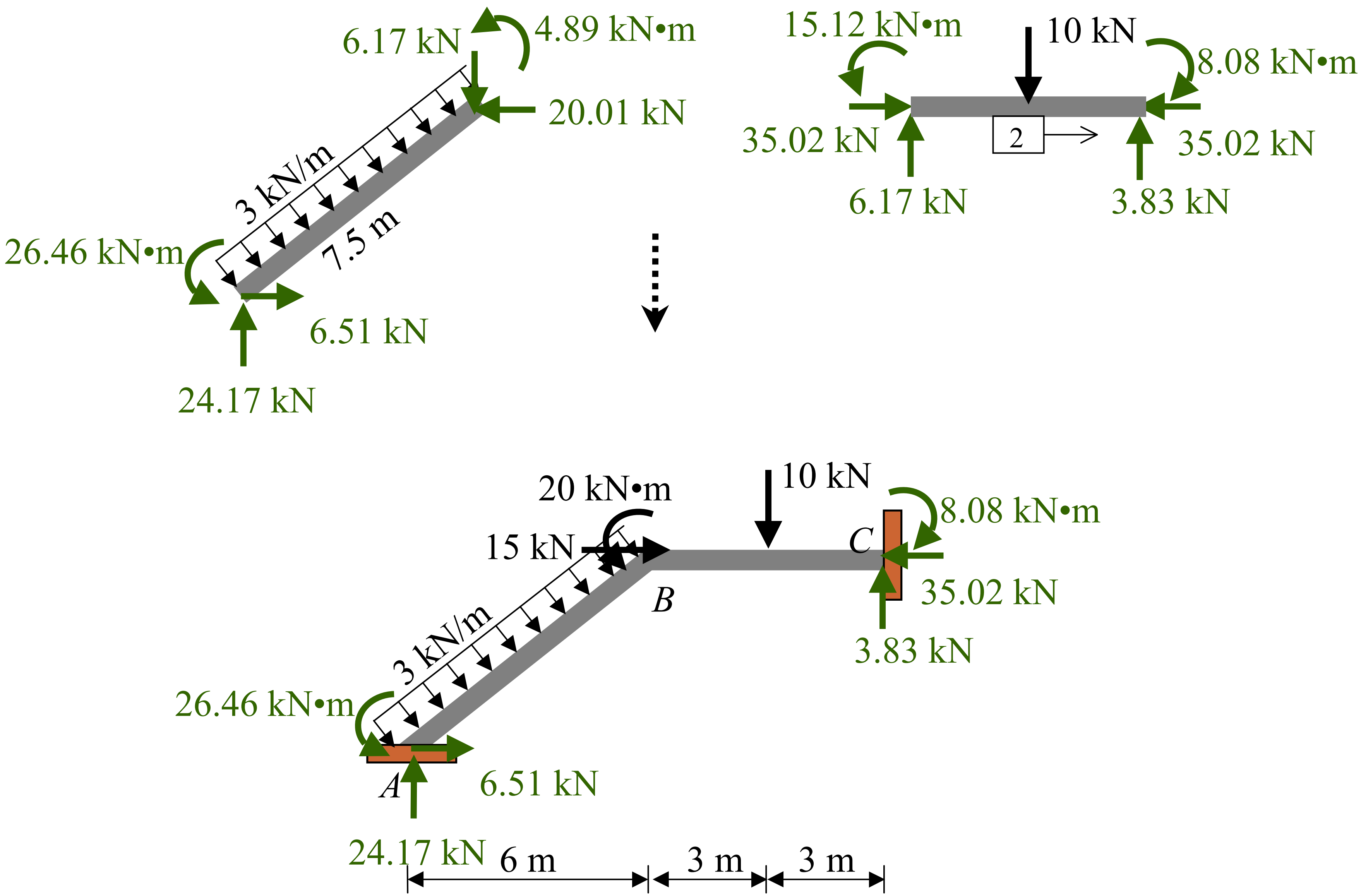
$$
\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 6.51 \\ 24.17 \\ 26.46 \\ -20.01 \\ -6.17 \\ 4.89 \end{pmatrix}
$$

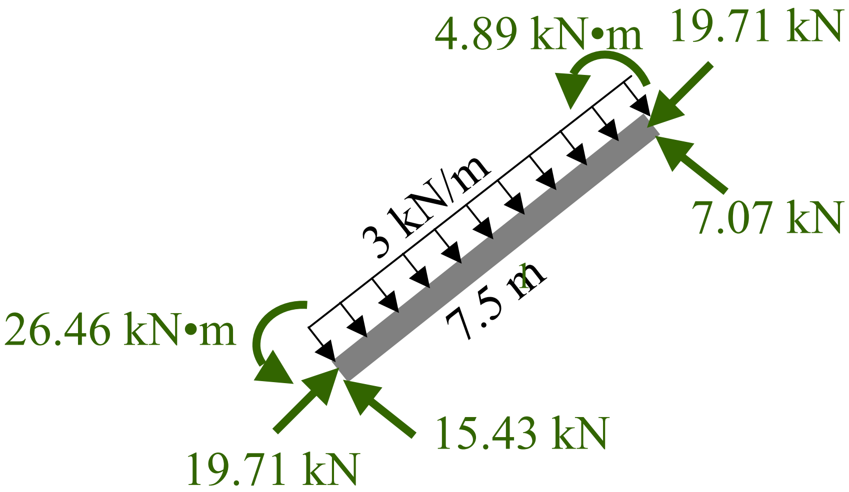


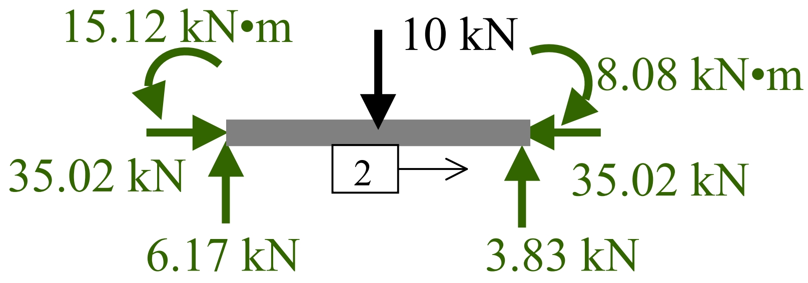
Member 2:



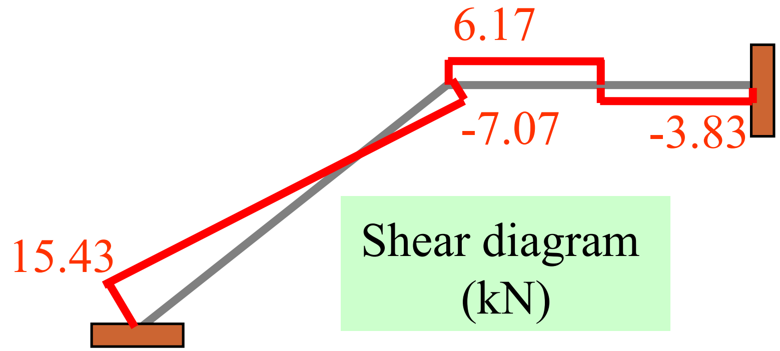
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} & \begin{pmatrix} & & & & & \\ & & & & & \\ & & & \boldsymbol{k_2} & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array} \begin{pmatrix} D_1 = 1.751(10^{-3}) \\ D_2 = -4.388(10^{-3}) \\ D_3 = 2.049(10^{-3}) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ 7.5 \\ 0 \\ 5 \\ -7.5 \end{pmatrix} = \begin{pmatrix} 35.02 \\ 6.17 \\ 15.12 \\ -35.02 \\ 3.83 \\ -8.08 \end{pmatrix}$$


6.17 kN
4.89 kN•m
20.01 kN
3 kN/m
7.5 m
26.46 kN•m
6.51 kN
24.17 kN
15.12 kN•m
10 kN
8.08 kN•m
35.02 kN
2
35.02 kN
6.17 kN
3.83 kN
20 kN•m
15 kN
10 kN
8.08 kN•m
C
35.02 kN
B
3.83 kN
3 kN/m
26.46 kN•m
6.51 kN
A
24.17 kN
6 m
3 m
3 m

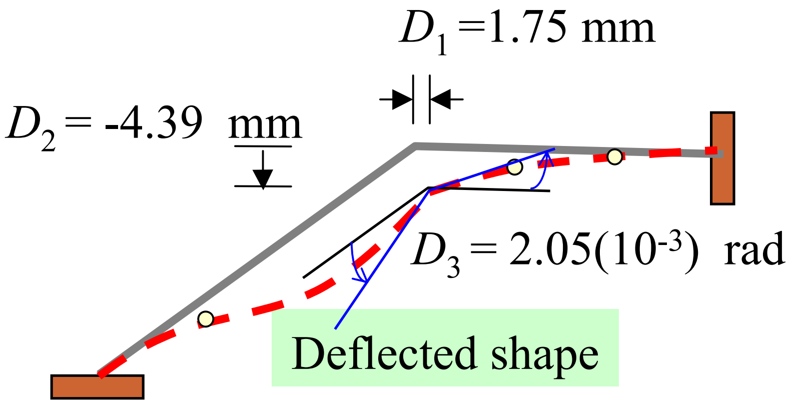




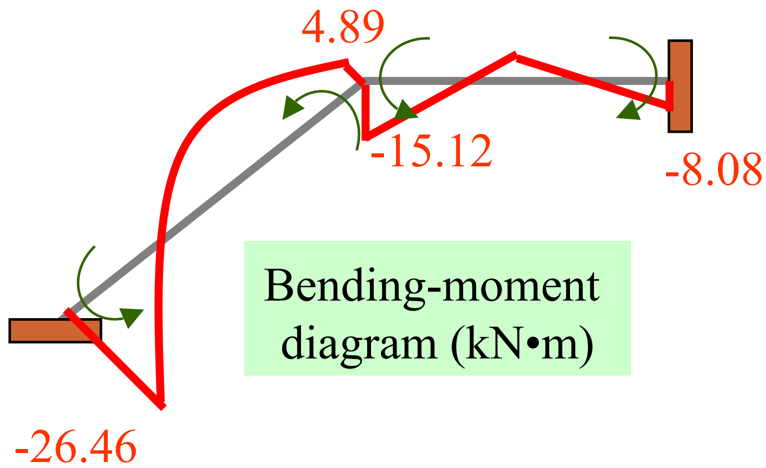
$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 1.751(10^{-3})\ \text{m} \\ -4.388(10^{-3})\ \text{m} \\ 2.049(10^{-3})\ \text{rad} \end{pmatrix}$$



Shear diagram (kN)



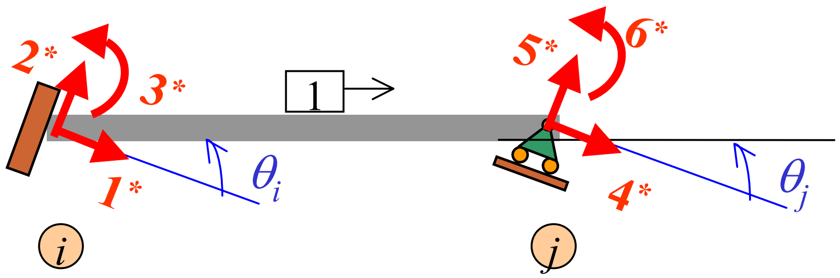
Deflected shape



Bending-moment diagram (kN•m)

Special Frames

Stiffness matrix



$\lambda_{ix} = \cos\ \theta_i$ $\qquad \lambda_{jx} = \cos\ \theta_j$

$\lambda_{iy} = \sin\ \theta_i$ $\qquad \lambda_{jy} = \sin\ \theta_j$

$$[\,\boldsymbol{q}^*\,] = [\,\boldsymbol{T}\,]^{\mathrm{T}}[\,\boldsymbol{q}'\,]$$

$$\begin{bmatrix} q_{1*} \\ q_{2*} \\ q_{3*} \\ q_{4*} \\ q_{5*} \\ q_{6*} \end{bmatrix} = \begin{array}{c} \\ 1* \\ 2* \\ 3* \\ 4* \\ 5* \\ 6* \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{bmatrix} \lambda_{ix} & -\lambda_{iy} & 0 & 0 & 0 & 0 \\ \lambda_{iy} & \lambda_{ix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{jx} & -\lambda_{jy} & 0 \\ 0 & 0 & 0 & \lambda_{jy} & \lambda_{jx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

$[\,\boldsymbol{T}\,]^{\mathrm{T}}$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \mathbf{1^*} & \mathbf{2^*} & \mathbf{3^*} & \mathbf{4^*} & \mathbf{5^*} & \mathbf{6^*} \end{array} \\
[T] = \begin{array}{c} \mathbf{1'} \\ \mathbf{2'} \\ \mathbf{3'} \\ \mathbf{4'} \\ \mathbf{5'} \\ \mathbf{6'} \end{array} &
\begin{bmatrix}
\lambda_{ix} & \lambda_{iy} & 0 & 0 & 0 & 0 \\
-\lambda_{iy} & \lambda_{ix} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_{jx} & \lambda_{jy} & 0 \\
0 & 0 & 0 & -\lambda_{jy} & \lambda_{jx} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

- **Member Stiffness Matrix**

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \mathbf{1'} & \mathbf{2'} & \mathbf{3'} & \mathbf{4'} & \mathbf{5'} & \mathbf{6'} \end{array} \\
[\mathrm{k'}] = \begin{array}{c} \mathbf{1'} \\ \mathbf{2'} \\ \mathbf{3'} \\ \mathbf{4'} \\ \mathbf{5'} \\ \mathbf{6'} \end{array} &
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

$$[\boldsymbol{k}] = [\boldsymbol{T}]^{\mathrm{T}}[\boldsymbol{k}'][\boldsymbol{T}] =$$

$$
\begin{array}{c}
\\ U_i \\ \\ V_i \\ \\ M_i \\ \\ U_j \\ \\ V_j \\ \\ M_j
\end{array}
\begin{array}{c}
\begin{array}{cccccc} U_i & V_i & M_i & U_j & V_j & M_j \end{array} \\
\begin{pmatrix}
(\frac{AE}{L}\lambda_{ix}^2 + \frac{12EI}{L^3}\lambda_{iy}^2) & (\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{ix}\lambda_{iy} & -\frac{6EI}{L^2}\lambda_{iy} & -(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}) & -(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}) & -\frac{6EI}{L^2}\lambda_{iy} \\
(\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{ix}\lambda_{iy} & (\frac{AE}{L}\lambda_{iy}^2 + \frac{12EI}{L^3}\lambda_{ix}^2) & \frac{6EI}{L^2}\lambda_{ix} & -(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}) & -(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}) & \frac{6EI}{L^2}\lambda_{ix} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{2EI}{L} \\
-(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}) & -(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}) & \frac{6EI}{L^2}\lambda_{jy} & (\frac{AE}{L}\lambda_{jx}^2 + \frac{12EI}{L^3}\lambda_{jy}^2) & (\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{jx}\lambda_{jy} & \frac{6EI}{L^2}\lambda_{jy} \\
-(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}) & -(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}) & -\frac{6EI}{L^2}\lambda_{jx} & (\frac{AE}{L} - \frac{12EI}{L^3})\lambda_{jx}\lambda_{jy} & (\frac{AE}{L}\lambda_{jy}^2 + \frac{12EI}{L^3}\lambda_{jx}^2) & -\frac{6EI}{L^2}\lambda_{jx} \\
-\frac{6EI}{L^2}\lambda_{iy} & \frac{6EI}{L^2}\lambda_{ix} & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_{jy} & -\frac{6EI}{L^2}\lambda_{jx} & \frac{4EI}{L}
\end{pmatrix}
\end{array}
$$

Example 4

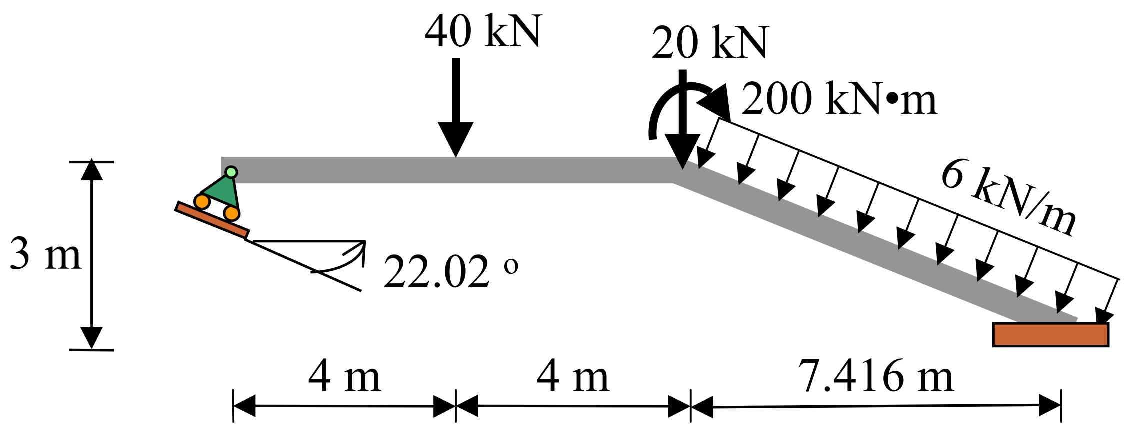
For the beam shown:

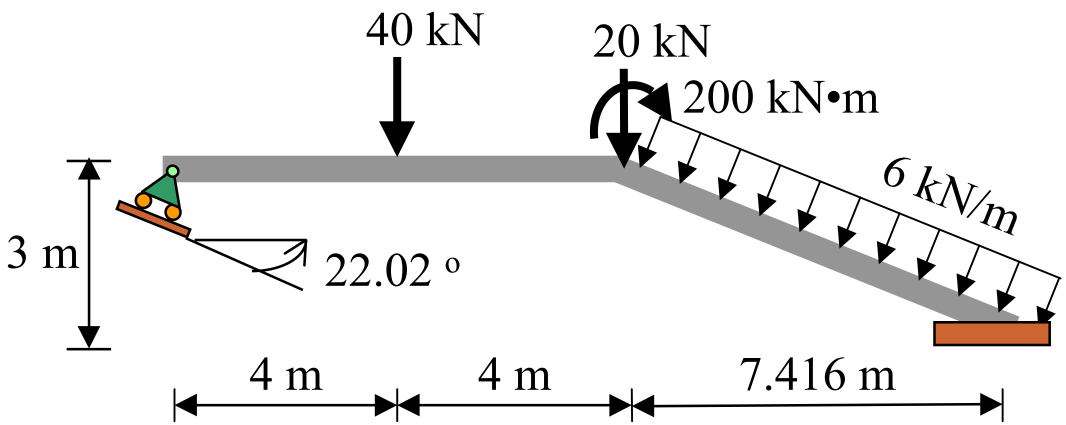
(a) Use the stiffness method to determine all the **reactions** at supports.

(b) Draw the **quantitative free-body diagram** of member.

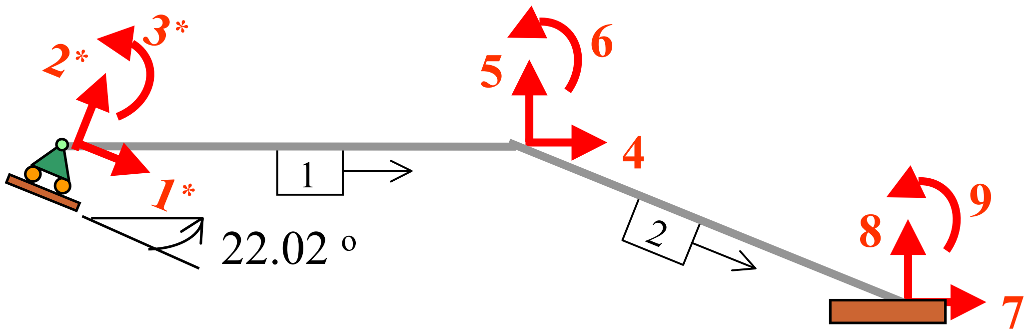
(c) Draw the **quantitative bending moment diagrams** and **qualitative deflected shape.**

Take $I = 200(10^6)$ mm^4 , $A = 6(10^3)$ mm^2, and $E = 200$ GPa for all members. Include axial deformation in the stiffness matrix.

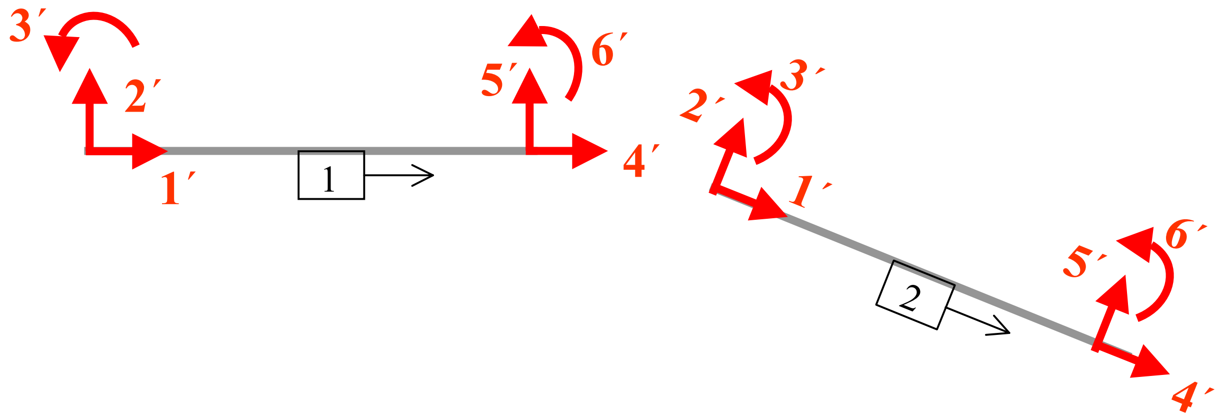




Global



Local



$$
[\mathrm{k}'] = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array}
\begin{array}{c}
\begin{array}{cccccc} \delta_i & \mathit{\Delta}_i & \theta_i & \delta_j & \mathit{\Delta}_j & \theta_j \end{array} \\
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

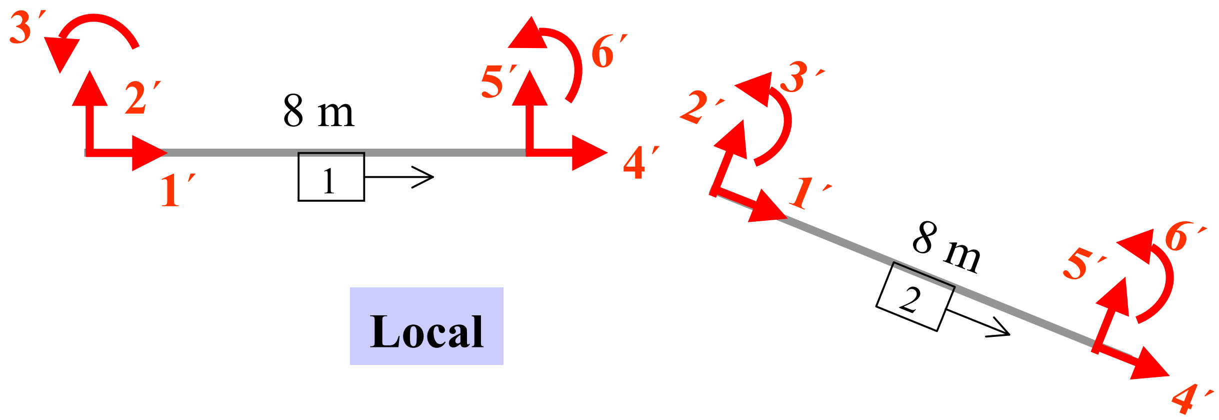
$$\frac{AE}{L} = \frac{(0.006\,m^2)(200\times10^6\,kN/m^2)}{8\,m} = 150\times10^3\ kN/m$$

$$\frac{4EI}{L} = \frac{4(200\times10^6\,kN/m^2)(0.0002\,m^4)}{8\,m} = 20\times10^3\ kN \bullet m$$

$$\frac{2EI}{L} = 10\times10^3\ kN \bullet m$$

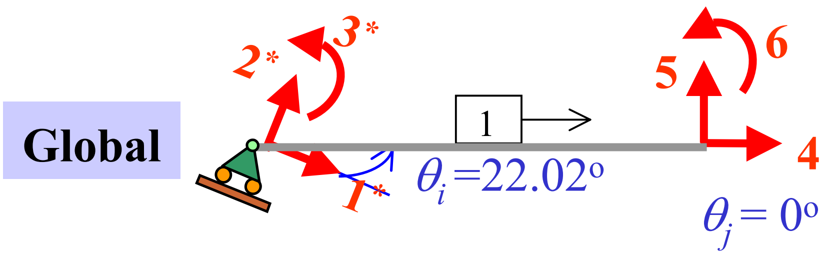
$$\frac{6EI}{L^2} = \frac{6(200\times10^6\,kN/m^2)(0.0002\,m^4)}{(8\,m)^2} = 3.75\times10^3\ kN$$

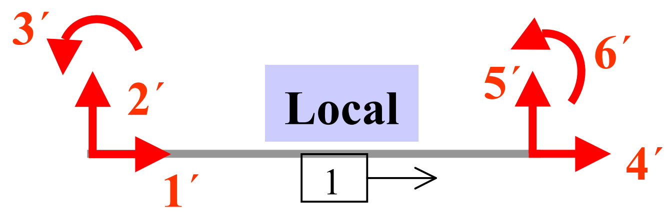
$$\frac{12EI}{L^3} = \frac{12(200\times10^6\,kN/m^2)(0.0002\,m^4)}{(8\,m)^3} = 0.9375\times10^3\ kN/m$$



$$
[\,k'\,]_1 = [\,k'\,]_2 = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array}
\begin{array}{c}
\begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\
\begin{pmatrix}
150000 & 0 & 0 & -150000 & 0 & 0 \\
0 & 937.5 & 3750 & 0 & -937.5 & 3750 \\
0 & 3750 & 20000 & 0 & -3750 & 10000 \\
-150000 & 0 & 0 & 150000 & 0 & 0 \\
0 & -937.5 & -3750 & 0 & 937.5 & -3750 \\
0 & 3750 & 10000 & 0 & -3750 & 20000
\end{pmatrix}
\end{array}
$$

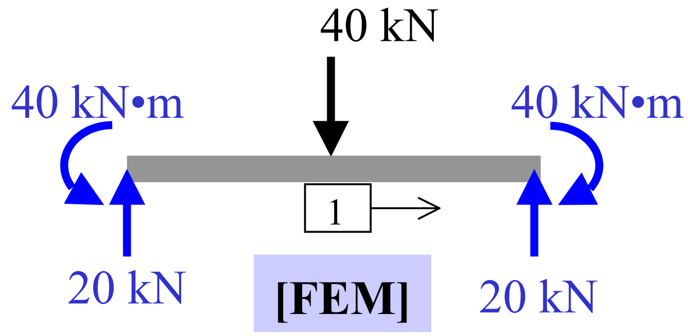
Member 1:





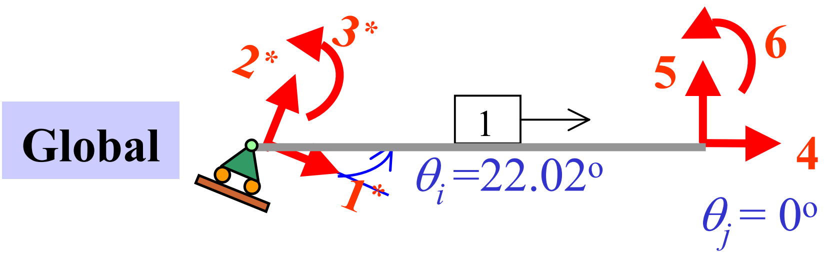
$\lambda_{ix} = \cos(22.02^o) = 0.927$, $\lambda_{jx} = \cos(0^o) = 1$,

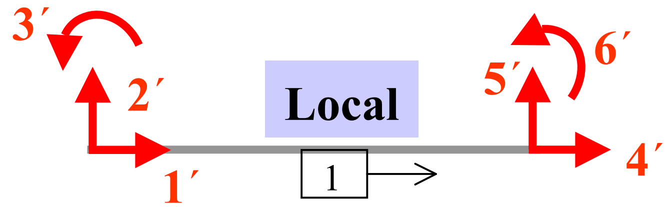
$\lambda_{iy} = \sin(22.02^o) = 0.375$ $\lambda_{jy} = \sin(0^o) = 0$



$$[\, q^* \,] = [\, T \,]^T [\, q' \,]$$

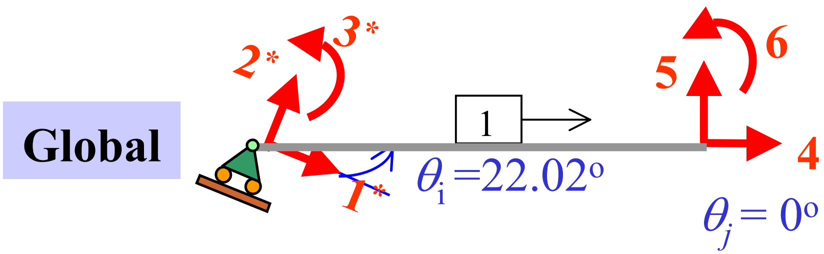
$$\begin{pmatrix} q^*_1 \\ q^*_2 \\ q^*_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4^* \\ 5^* \\ 6^* \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \underbrace{\begin{pmatrix} 0.927 & -0.375 & 0 & 0 & 0 & 0 \\ 0.375 & 0.927 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{[\, T \,]_1^T} \end{array} \begin{pmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{pmatrix}$$

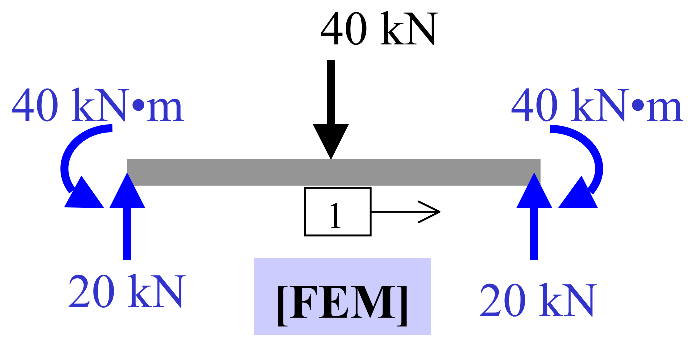




$$[\,\boldsymbol{k}^*\,]_1 = [\,\boldsymbol{T}\,]_1^{\mathrm{T}}[\,\boldsymbol{k}'\,]_1[\,\boldsymbol{T}\,]_1$$

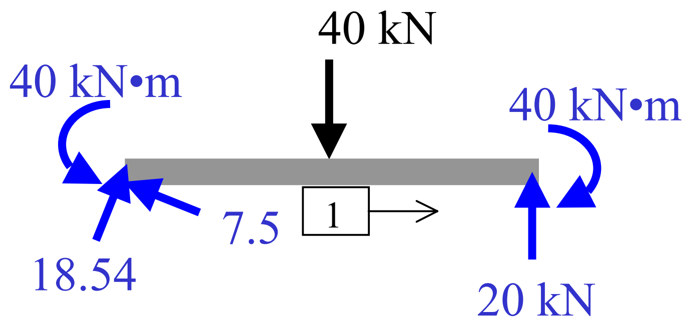
$$[\,k^*\,]_1 = 10^3 \begin{array}{c} \\ \mathbf{1^*} \\ \mathbf{2^*} \\ \mathbf{3^*} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{1^*} & \mathbf{2^*} & \mathbf{3^*} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{pmatrix} 129.046 & 51.811 & -1.406 & -139.058 & 0.351 & -1.406 \\ 51.811 & 21.892 & 3.476 & -56.240 & -0.869 & 3.476 \\ -1.406 & 3.476 & 20.00 & 0 & -3.75 & 10.00 \\ -139.058 & -56.240 & 0.00 & 150 & 0 & 0 \\ 0.351 & -0.869 & -3.75 & 0 & 0.938 & -3.75 \\ -1.406 & 3.476 & 10.00 & 0 & -3.75 & 20 \end{pmatrix} \end{array}$$





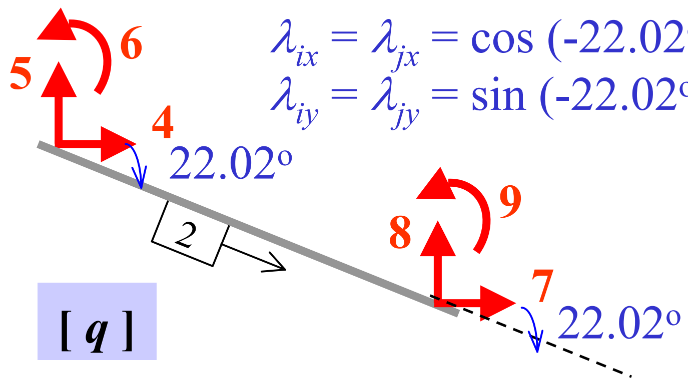
$$[\, q^{F*} \,] = [\, T \,]^T [\, q^{F\prime}]$$

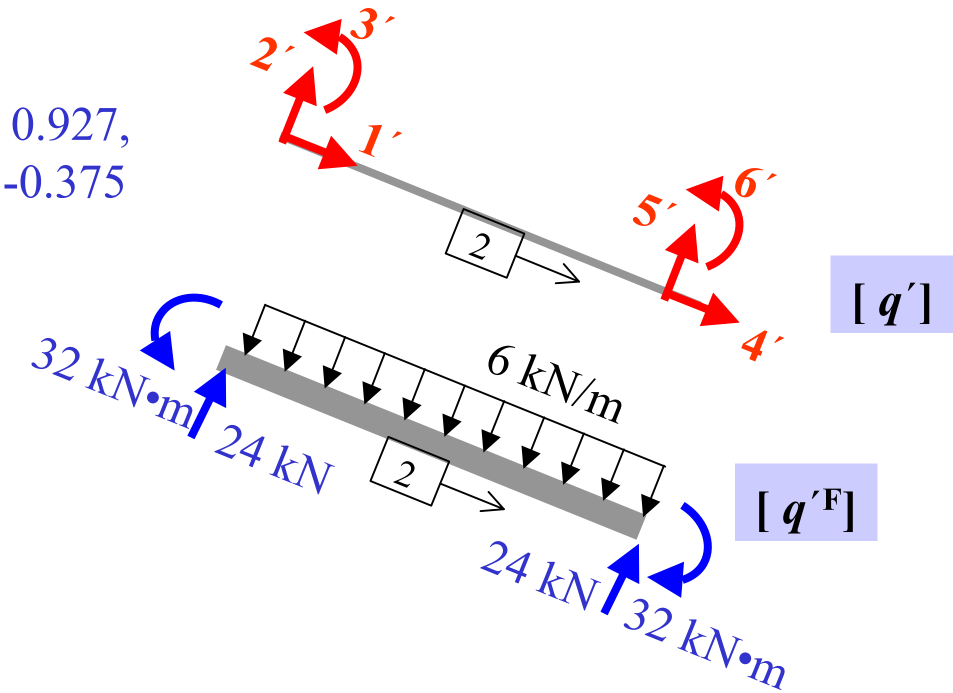
$$[\, q^{F*}] = [\, T \,]_1^{\,T} \begin{pmatrix} 0 \\ 20 \\ 40 \\ 0 \\ 20 \\ -40 \end{pmatrix} = \begin{array}{l} \begin{pmatrix} -7.50 \\ 18.54 \\ 40 \\ 0 \\ 20 \\ -40 \end{pmatrix} \end{array} \begin{array}{l} 1^* \\ 2^* \\ 3^* \\ 4 \\ 5 \\ 6 \end{array}$$



Member 2

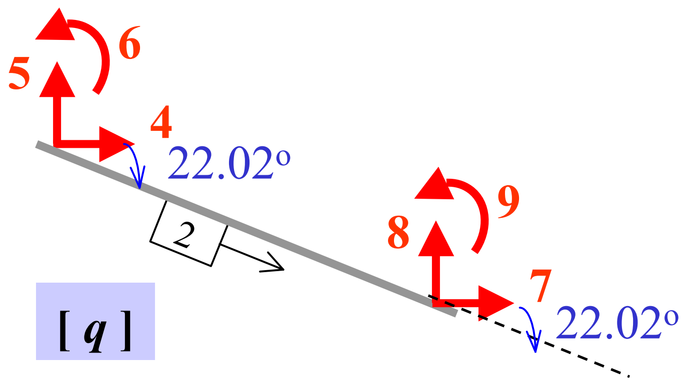
$$\lambda_{ix} = \lambda_{jx} = \cos(-22.02^{o}) = 0.927,$$
$$\lambda_{iy} = \lambda_{jy} = \sin(-22.02^{o}) = -0.375$$

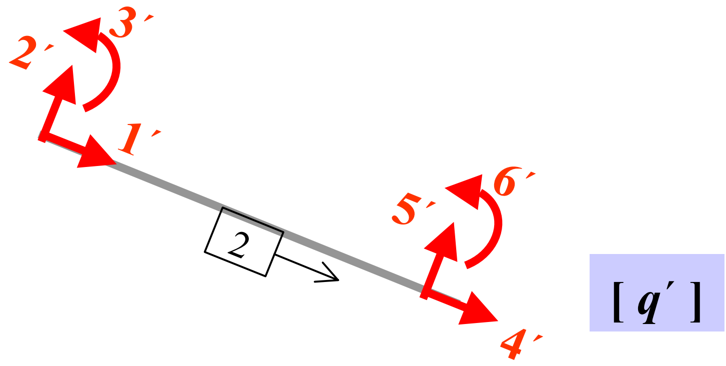




$$[\, q \,] = [\, T \,]^{\mathrm{T}}[\, q' \,]$$

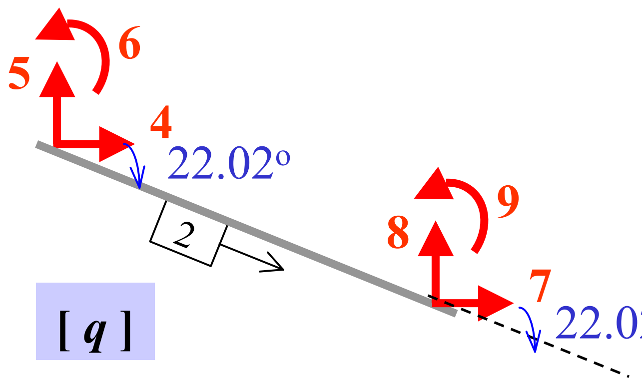
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \underbrace{\begin{pmatrix} 0.927 & 0.375 & 0 & 0 & 0 & 0 \\ -0.375 & 0.927 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.927 & 0.375 & 0 \\ 0 & 0 & 0 & -0.375 & 0.927 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{[\, T \,]_2^{\mathrm{T}}} \end{array} \begin{pmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{pmatrix}$$

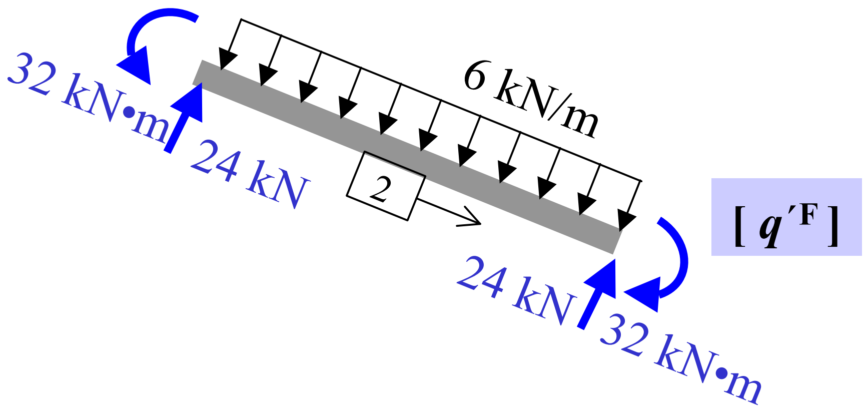




$$[\,\boldsymbol{k}\,]_2 = [\,\boldsymbol{T}\,]_2^{\mathrm{T}}[\,\boldsymbol{k}'\,]_2[\,\boldsymbol{T}\,]_2$$

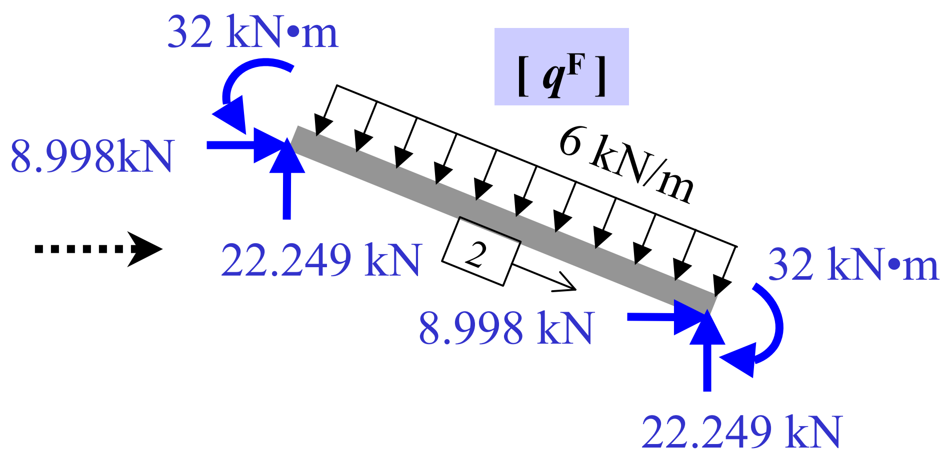
$$[\,k\,]_2 = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 129.046 & -51.811 & 1.406 & -129.046 & 51.811 & 1.406 \\ -51.811 & 21.892 & 3.476 & 51.811 & -21.892 & 3.476 \\ 1.406 & 3.476 & 20 & -1.406 & -3.476 & 10 \\ -129.046 & 51.811 & -1.406 & 129.046 & -51.811 & -1.406 \\ 51.811 & -21.892 & -3.476 & -51.811 & 21.892 & -3.476 \\ 1.4056 & 3.476 & 10 & -1.406 & -3.476 & 20 \end{pmatrix} \end{array}$$



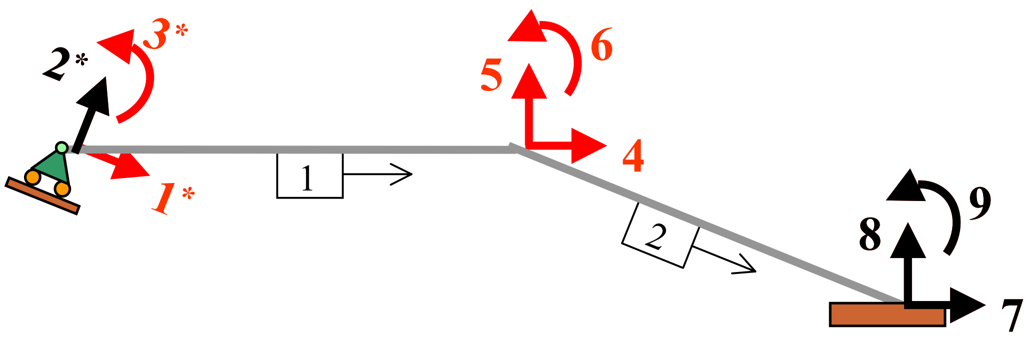


$$[\,q^{F*}\,] = [\,T\,]^{T}[\,q^{F\prime}\,]$$

$$[\,q^{F}\,] = [\,T\,]_2^{T}\begin{pmatrix} 0 \\ 24 \\ 32 \\ 0 \\ 24 \\ -32 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 8.998 \\ 22.249 \\ 32 \\ 8.998 \\ 22.249 \\ -32 \end{pmatrix} \end{array}\begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

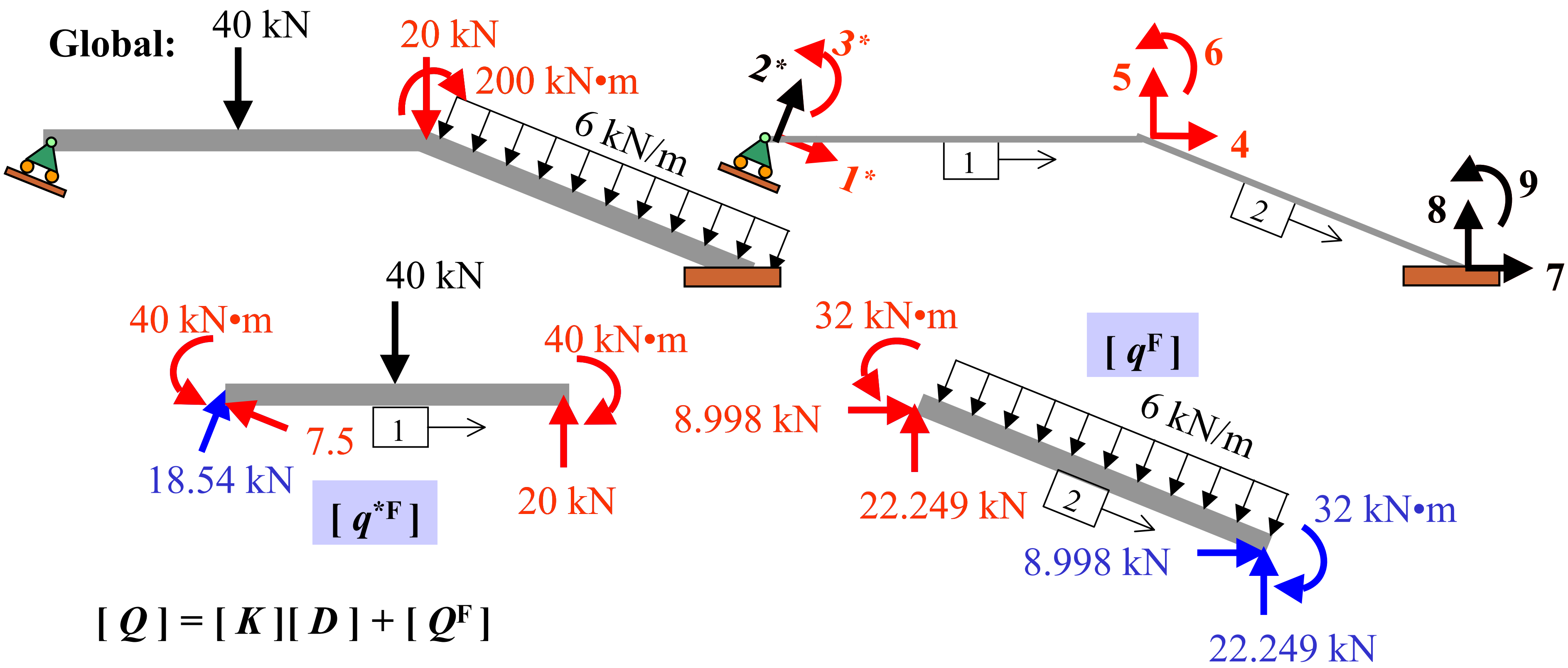


Global Stiffness:



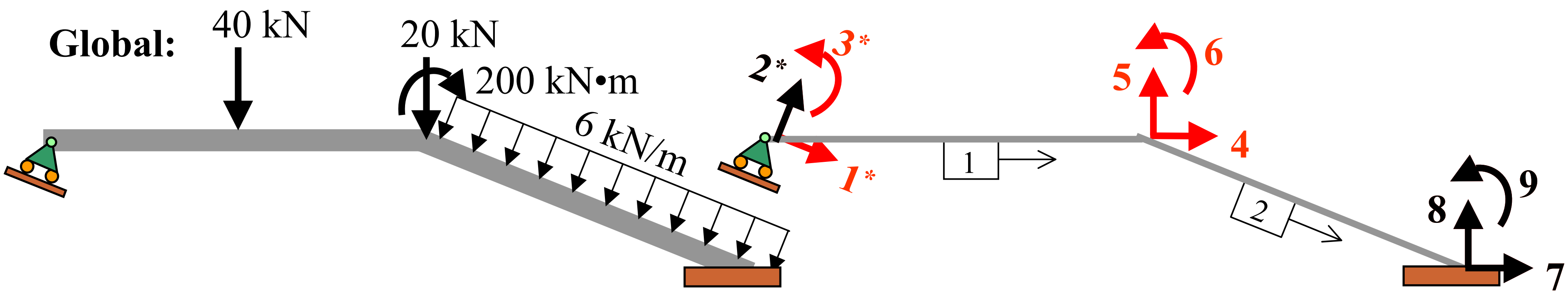
$$[\,k^*\,]_1 = 10^3 \begin{array}{c} \\ 1^* \\ 2^* \\ 3^* \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1^* & 2^* & 3^* & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 129.046 & 51.811 & -1.406 & -139.058 & 0.351 & -1.406 \\ 51.811 & 21.892 & 3.476 & -56.240 & -0.869 & 3.476 \\ -1.406 & 3.476 & 20.00 & 0 & -3.75 & 10.00 \\ -139.058 & -56.240 & 0.00 & 150 & 0 & 0 \\ 0.351 & -0.869 & -3.75 & 0 & 0.938 & -3.75 \\ -1.406 & 3.476 & 10.00 & 0 & -3.75 & 20 \end{array}\right) \end{array}$$

$$[\,k\,]_2 = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{cccccc} 129.046 & -51.811 & 1.406 & -129.046 & 51.811 & 1.406 \\ -51.811 & 21.892 & 3.476 & 51.811 & -21.892 & 3.476 \\ 1.406 & 3.476 & 20 & -1.406 & -3.476 & 10 \\ -129.046 & 51.811 & -1.406 & 129.046 & -51.811 & -1.406 \\ 51.811 & -21.892 & -3.476 & -51.811 & 21.892 & -3.476 \\ 1.4056 & 3.476 & 10 & -1.406 & -3.476 & 20 \end{array}\right) \end{array}$$

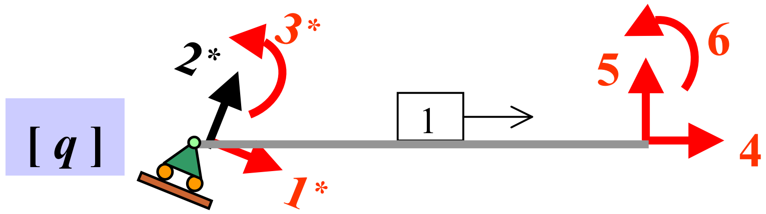


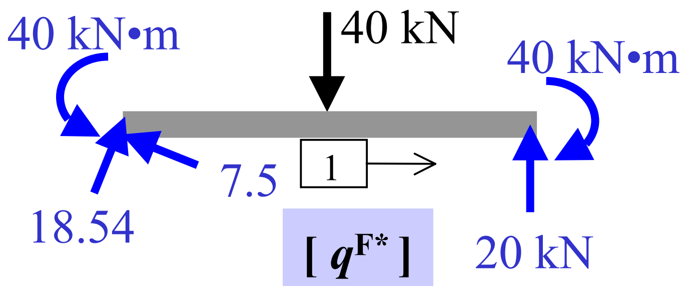
$$[\,\boldsymbol{Q}\,] = [\,\boldsymbol{K}\,][\,\boldsymbol{D}\,] + [\,\boldsymbol{Q}^{F}\,]$$

$$\begin{pmatrix} Q_{1*} = 0.0 \\ Q_{3*} = 0.0 \\ Q_{4} \;= 0.0 \\ Q_{5} \;= -20 \\ Q_{6} \;= -200 \end{pmatrix} = 10^3 \begin{array}{c} \\ \mathbf{1^*} \\ \mathbf{3^*} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{array} \!\! \begin{array}{c} \begin{array}{ccccc} \mathbf{1^*} & \mathbf{3^*} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{pmatrix} 129.046 & -1.406 & -139.058 & 0.351 & -1.406 \\ -1.406 & 20 & 0 & -3.75 & 10 \\ -139.058 & 0 & 279.046 & -51.811 & 1.406 \\ 0.352 & -3.75 & -51.811 & 22.829 & -0.274 \\ -1.406 & 10 & 1.406 & -0.274 & 40 \end{pmatrix} \end{array} \begin{pmatrix} D_{1*} \\ D_{3*} \\ D_{4} \\ D_{5} \\ D_{6} \end{pmatrix} + \begin{pmatrix} -7.5 \\ 40 \\ 8.998 \\ 20 + 22.249 \\ -40 + 32 \end{pmatrix}$$



$$
\begin{pmatrix} D_{1*} \\ D_{3*} \\ D_4 \\ D_5 \\ D_6 \end{pmatrix} = \begin{pmatrix} -0.0205 & \text{m} \\ -0.0112 & \text{rad} \\ -0.0191 & \text{m} \\ -0.0476 & \text{m} \\ -0.0024 & \text{rad} \end{pmatrix}
$$

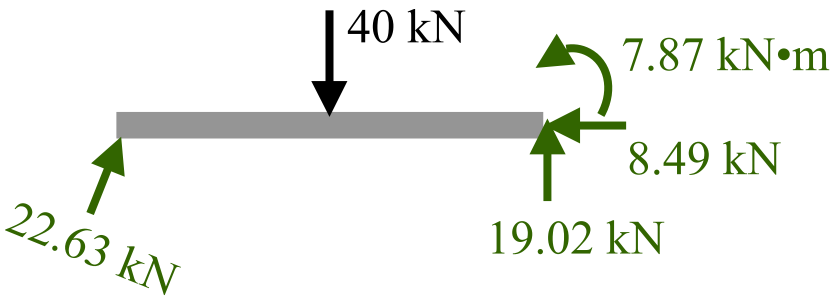


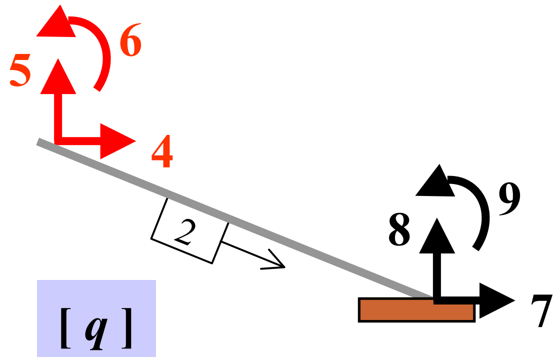


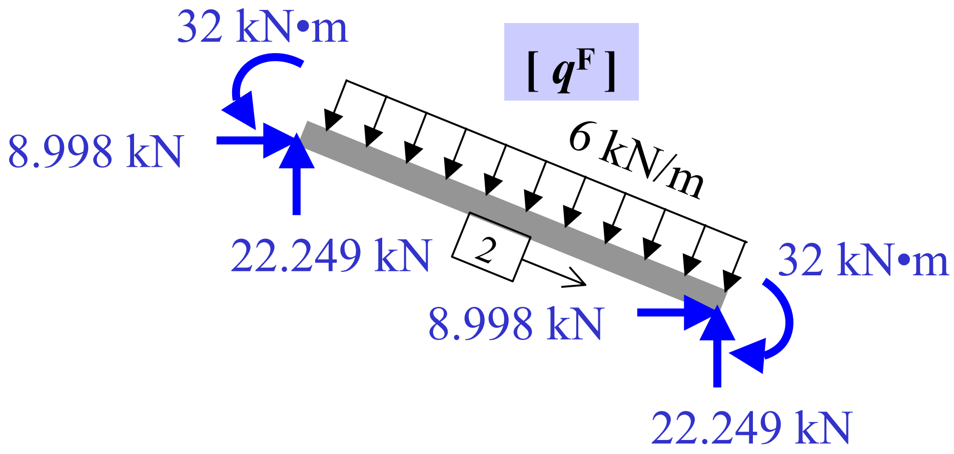
Member 1: [q^*] = [k^*][d^*] + [q^{F*}]

$$
\begin{pmatrix} q_{1*} \\ q_{2*} \\ q_{3*} \\ q_4 \\ q_5 \\ q_6 \end{pmatrix}
= 10^3
\begin{array}{c}
 \\ 1^* \\ 2^* \\ 3^* \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1^* & 2^* & 3^* & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
129.046 & 51.811 & -1.406 & -139.058 & 0.351 & -1.406 \\
51.811 & 21.892 & 3.476 & -56.240 & -0.869 & 3.476 \\
-1.406 & 3.476 & 20.00 & 0 & -3.75 & 10.00 \\
-139.058 & -56.240 & 0.00 & 150 & 0 & 0 \\
0.351 & -0.869 & -3.75 & 0 & 0.938 & -3.75 \\
-1.406 & 3.476 & 10.00 & 0 & -3.75 & 20
\end{pmatrix}
\end{array}
\begin{pmatrix} D_{1*}=-0.0205 \\ D_{2*}= 0.0 \\ D_{3*}=-0.0112 \\ D_4= -0.0191 \\ D_5= -0.0476 \\ D_6=-0.0024 \end{pmatrix}
+
\begin{pmatrix} -7.50 \\ 18.54 \\ 40 \\ 0 \\ 20 \\ -40 \end{pmatrix}
$$

$$
\begin{pmatrix} q_{1*} \\ q_{2*} \\ q_{3*} \\ q_4 \\ q_5 \\ q_6 \end{pmatrix}
=
\begin{pmatrix} 0 & \\ 22.63 & \text{kN} \\ 0 & \\ -8.49 & \text{kN} \\ 19.02 & \text{kN} \\ 7.87 & \text{kN•m} \end{pmatrix}
$$



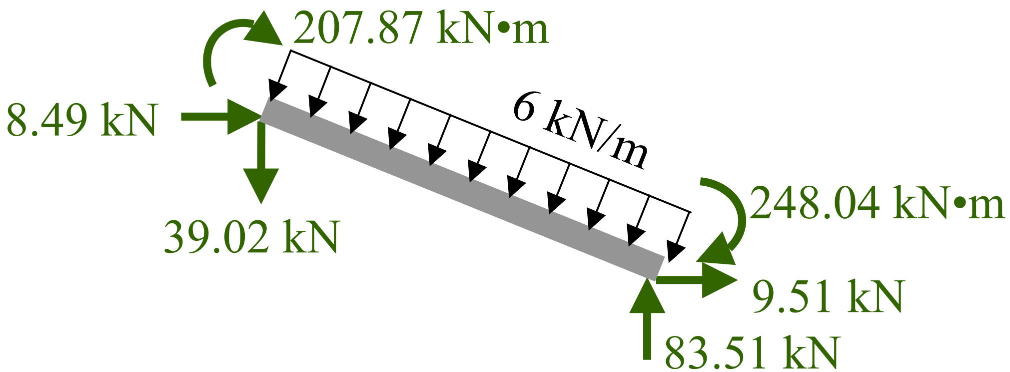


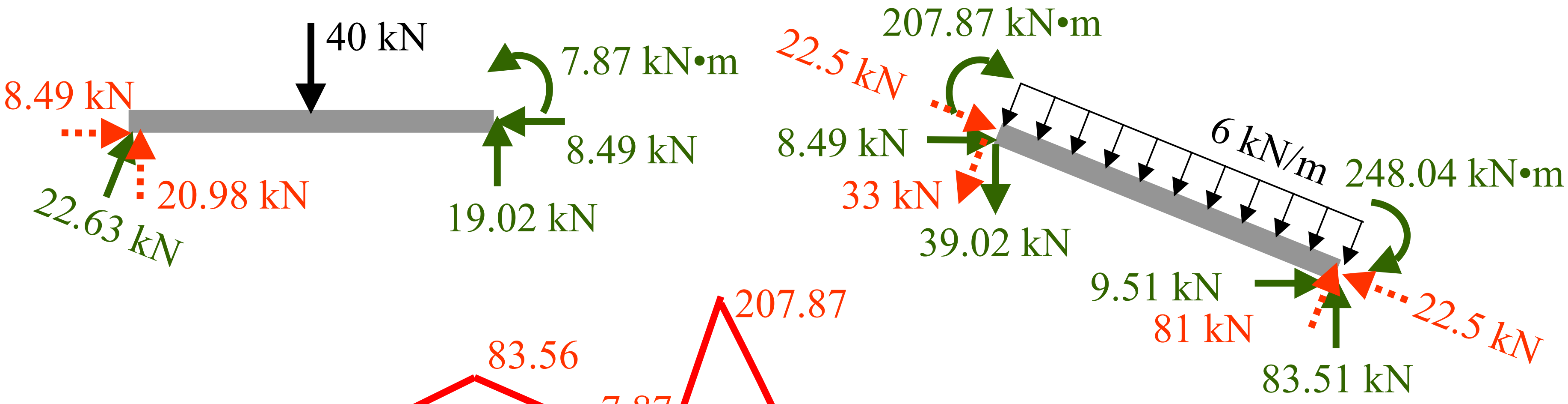


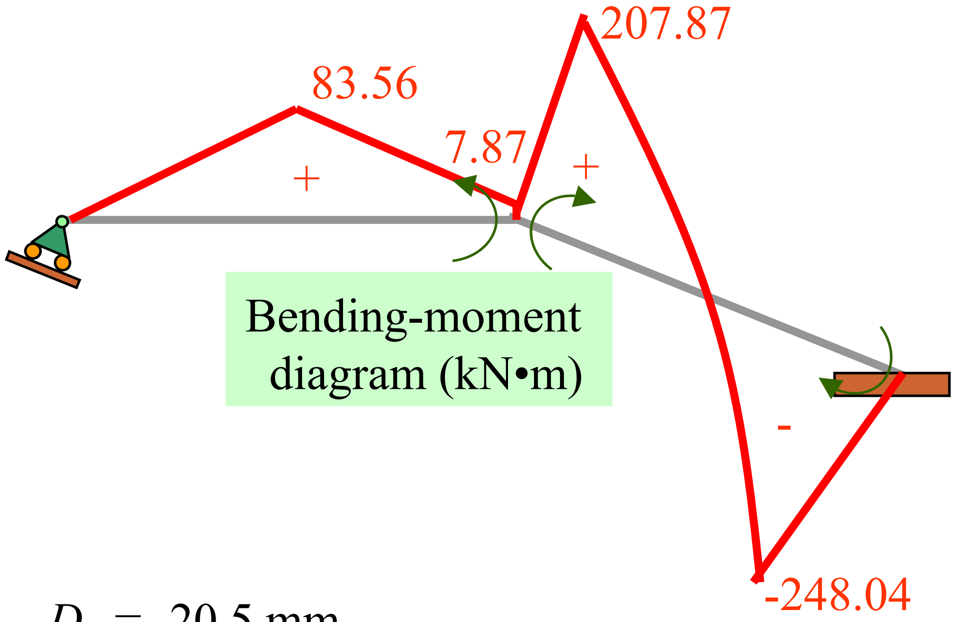
Member 2 : [q] = [k][d] + [q^F]

$$\begin{pmatrix} q_4\\ q_5\\ q_6\\ q_7\\ q_8\\ q_9\end{pmatrix} = 10^3 \begin{array}{c|cccccc} & 4 & 5 & 6 & 7 & 8 & 9\\ \hline 4 & 129.046 & -51.811 & 1.406 & -129.046 & 51.811 & 1.406\\ 5 & -51.811 & 21.892 & 3.476 & 51.811 & -21.892 & 3.476\\ 6 & 1.406 & 3.476 & 20 & -1.406 & -3.476 & 10\\ 7 & -129.046 & 51.811 & -1.406 & 129.046 & -51.811 & -1.406\\ 8 & 51.811 & -21.892 & -3.476 & -51.811 & 21.892 & -3.476\\ 9 & 1.4056 & 3.476 & 10 & -1.406 & -3.476 & 20 \end{array} \begin{pmatrix} D_4 = -0.0191\\ D_5 = -0.0476\\ D_6 = -0.0024\\ D_7 = 0\\ D_8 = 0\\ D_9 = 0\end{pmatrix} + \begin{pmatrix} 8.998\\ 22.249\\ 32\\ 8.998\\ 22.249\\ -32\end{pmatrix}$$

$$\begin{pmatrix} q_4\\ q_5\\ q_6\\ q_7\\ q_8\\ q_9\end{pmatrix} = \begin{pmatrix} 8.49 & \text{kN}\\ -39.02 & \text{kN}\\ -207.87 & \text{kN•m}\\ 9.51 & \text{kN}\\ 83.51 & \text{kN}\\ -248.04 & \text{kN•m}\end{pmatrix}$$

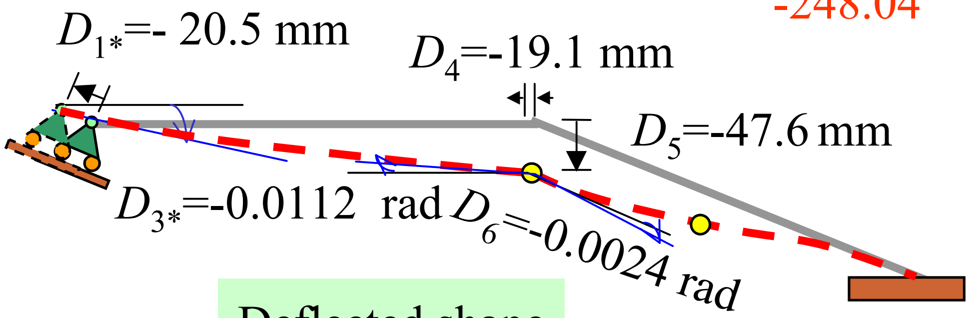






Bending-moment diagram (kN•m)

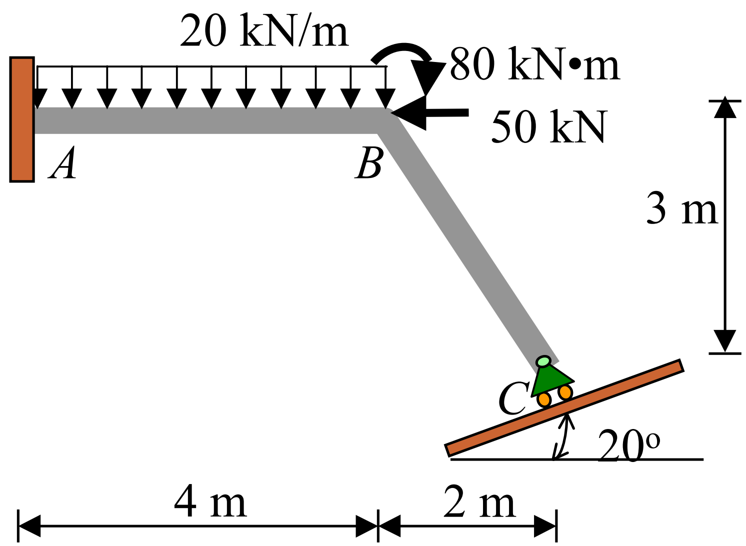
$$\begin{pmatrix} D_{1*} \\ D_{3*} \\ D_4 \\ D_5 \\ D_6 \end{pmatrix} = \begin{pmatrix} -0.0205 & \text{m} \\ -0.0112 & \text{rad} \\ -0.0191 & \text{m} \\ -0.0476 & \text{m} \\ -0.0024 & \text{rad} \end{pmatrix}$$

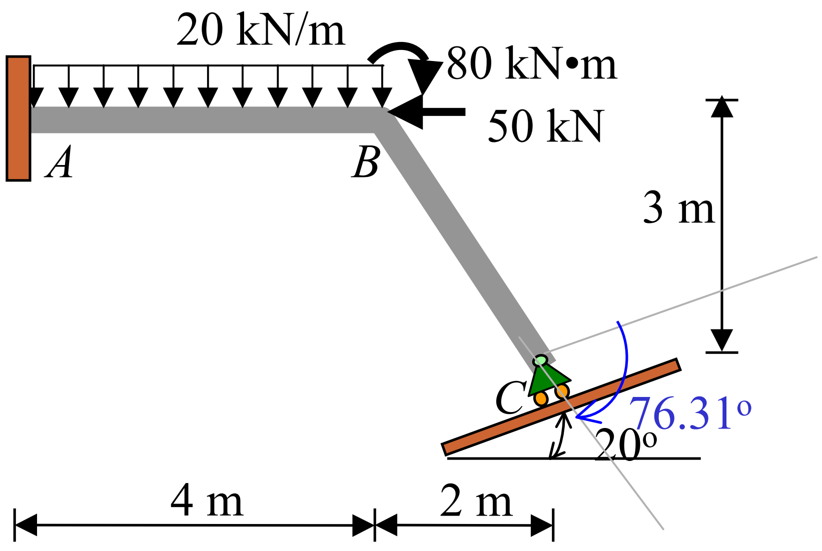


Deflected shape

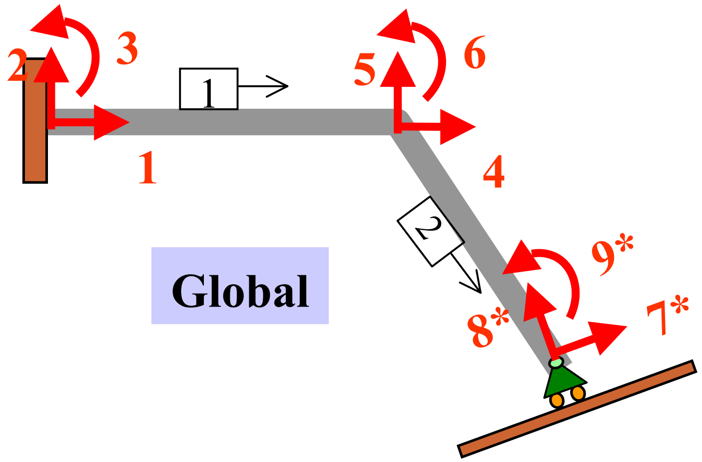
Example 5

For the beam shown:
(a) Use the stiffness method to determine all the **reactions** at supports.
(b) Draw the **quantitative free-body diagram** of member.
(c) Draw the **quantitative bending moment diagrams** and **qualitative deflected shape.**
Take $I = 400(10^6)$ mm^4 , $A = 60(10^3)$ mm^2, and $E = 200$ GPa for all members.

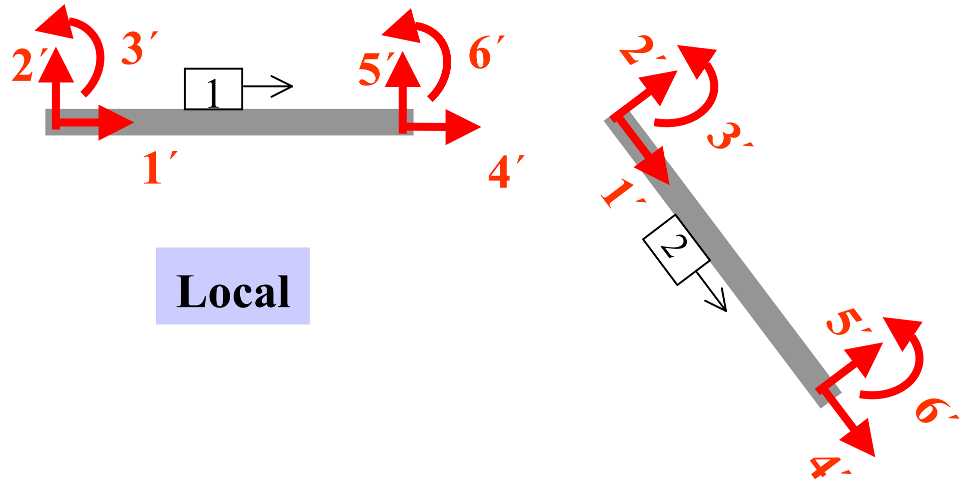




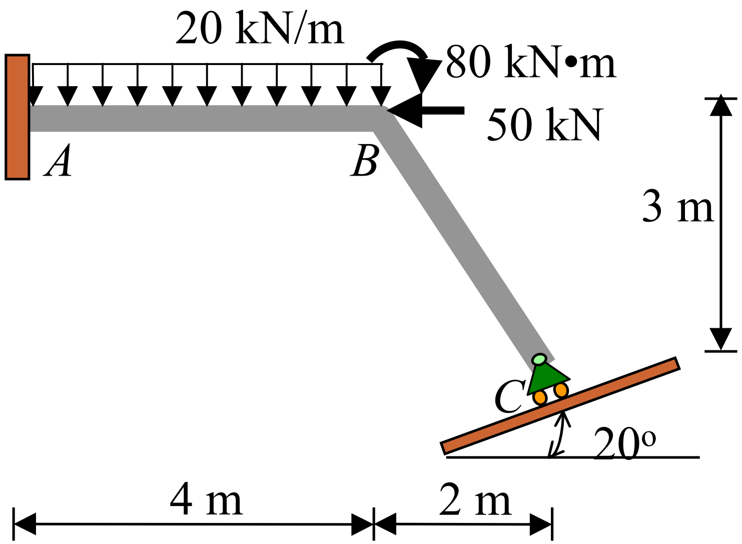
20 kN/m
80 kN•m
50 kN
A
B
C
3 m
76.31°
20°
4 m
2 m

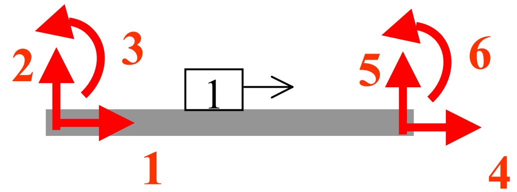


1
2
3
4
5
6
7*
8*
9*
Global



1′
2′
3′
4′
5′
6′
Local





Member 1:

$$\frac{AE}{L} = \frac{(60\times10^{-3}\,\text{m}^2)(200\times10^{6}\,\text{kN/m}^2)}{4\text{ m}}$$

$$= 3000\times10^{3}\ \text{kN/m}$$

$$\frac{4EI}{L} = \frac{4(200\times10^{6}\,\text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{4\text{ m}}$$

$$= 80\times10^{3}\ \text{kN}\bullet\text{m}$$

$$\frac{2EI}{L} = \frac{2(200\times10^{6}\,\text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{4\text{ m}}$$

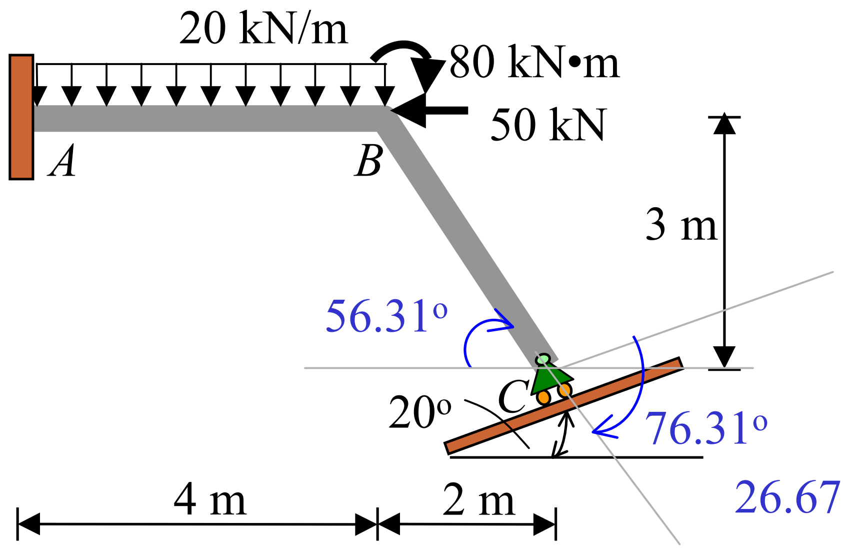
$$= 40\times10^{3}\ \text{kN}\bullet\text{m}$$

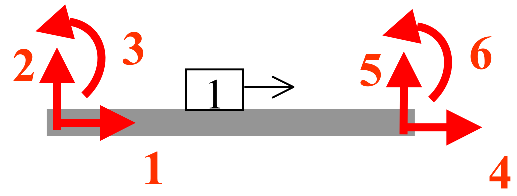
$$\frac{6EI}{L^2} = \frac{6(200\times10^{6}\,\text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{(4\text{ m})^2}$$

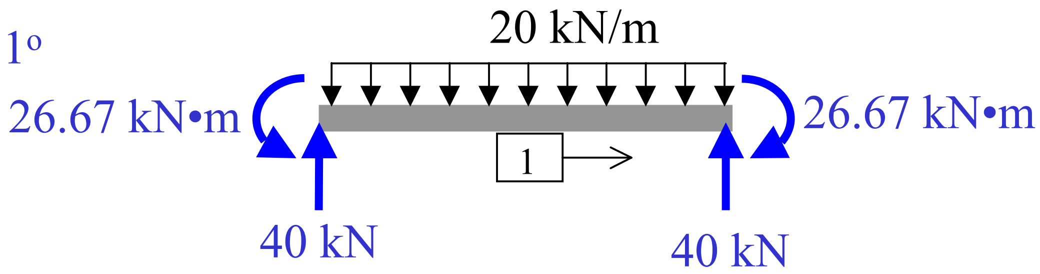
$$= 30\times10^{3}\ \text{kN}$$

$$\frac{12EI}{L^3} = \frac{12(200\times10^{6}\,\text{kN/m}^2)(400\times10^{-6}\,\text{m}^4)}{(4\text{ m})^3}$$

$$= 15\times10^{3}\ \text{kN/m}$$

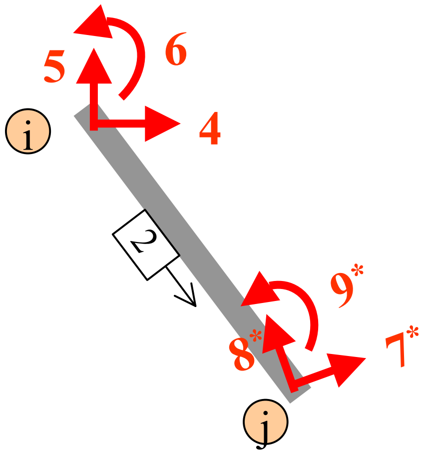


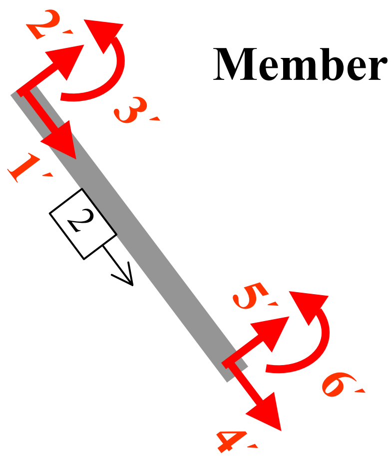




Member 1: [*q*] = [*k*][*d*] + [*q*^F]

$$
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = 10^3
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
3000 & 0 & 0 & -3000 & 0 & 0 \\
0 & 15 & 30 & 0 & -15 & 30 \\
0 & 30 & 80 & 0 & -30 & 40 \\
-3000 & 0 & 0 & 3000 & 0 & 0 \\
0 & -15 & -30 & 0 & 15 & -30 \\
0 & 30 & 40 & 0 & -30 & 80
\end{pmatrix}
\end{array}
\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} +
\begin{pmatrix} 0 \\ 40 \\ 26.67 \\ 0 \\ 40 \\ -26.67 \end{pmatrix}
$$





Member 2:

$$\frac{AE}{L} = \frac{(60\times10^{-3}\ \mathrm{m}^2)(200\times10^{6}\ \mathrm{kN/m}^2)}{3.61\ \mathrm{m}}$$

$$= 3324\times10^{3}\ \mathrm{kN/m}$$

$$\frac{4EI}{L} = \frac{4(200\times10^{6}\ \mathrm{kN/m}^2)(400\times10^{-6}\mathrm{m}^4)}{3.61\ \mathrm{m}}$$

$$= 88.64\times10^{3}\ \mathrm{kN\bullet m}$$

$$\frac{6EI}{L^2} = \frac{6(200\times10^{6}\ \mathrm{kN/m}^2)(400\times10^{-6}\mathrm{m}^4)}{(3.61\ \mathrm{m})^2}$$

$$= 36.83\times10^{3}\ \mathrm{kN}$$

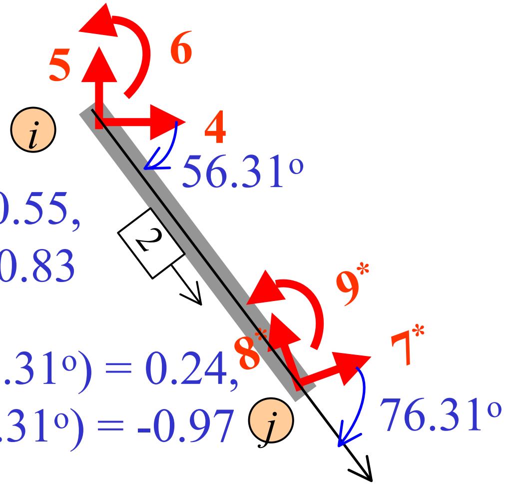
$$\frac{12EI}{L^3} = \frac{12(200\times10^{6}\ \mathrm{kN/m}^2)(400\times10^{-6}\mathrm{m}^4)}{(3.61\ \mathrm{m})^3}$$

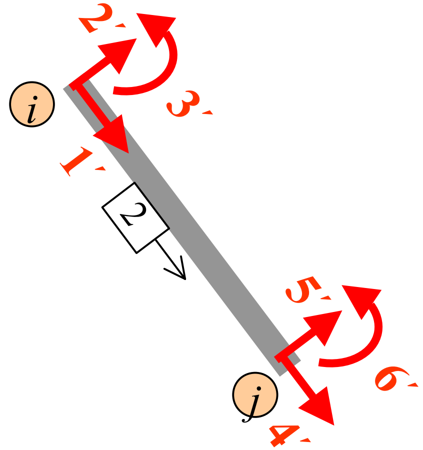
$$= 20.41\times10^{3}\ \mathrm{kN/m}$$

$$\frac{2EI}{L} = \frac{2(200\times10^{6}\ \mathrm{kN/m}^2)(400\times10^{-6}\mathrm{m}^4)}{3.61\ \mathrm{m}}$$

$$= 44.32\times10^{3}\ \mathrm{kN\bullet m}$$

$$[\,k'\,]_2 = 10^3 \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{pmatrix} 3324 & 0 & 0 & -3324 & 0 & 0 \\ 0 & 20.41 & 36.83 & 0 & -20.41 & 36.83 \\ 0 & 36.83 & 88.64 & 0 & -36.83 & 44.32 \\ -3324 & 0 & 0 & 3324 & 0 & 0 \\ 0 & -20.41 & -36.83 & 0 & 20.41 & -36.83 \\ 0 & 36.83 & 44.32 & 0 & -36.83 & 88.64 \end{pmatrix} \end{array}$$





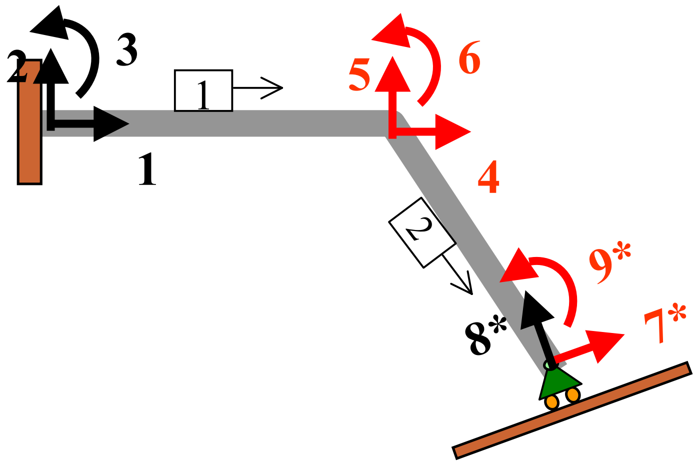
$\lambda_{ix} = \cos(-56.31^{\circ}) = 0.55,$
$\lambda_{iy} = \sin(-56.31^{\circ}) = -0.83$

$\lambda_{jx} = \cos(-76.31^{\circ}) = 0.24,$
$\lambda_{jy} = \sin(-76.31^{\circ}) = -0.97$

$$[\,\boldsymbol{q}^{*}\,] = [\,\boldsymbol{T}\,]^{\mathrm{T}}[\,\boldsymbol{q}'\,]$$

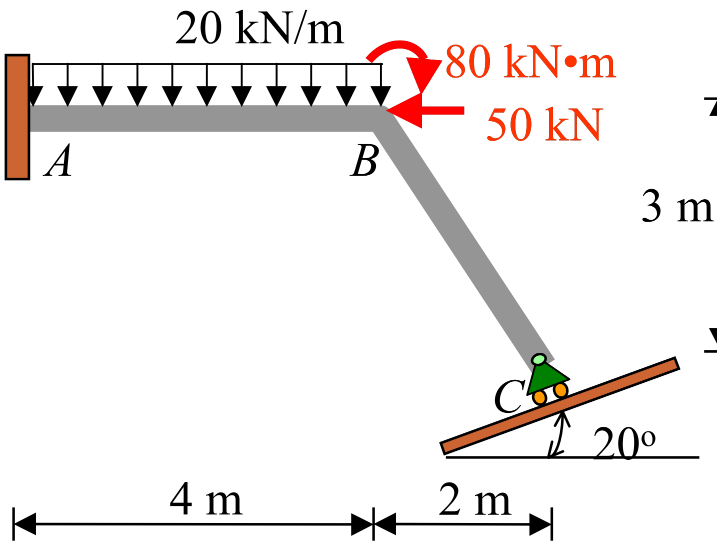
$$\begin{Bmatrix} q_4 \\ q_5 \\ q_6 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{Bmatrix} = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7^* \\ 8^* \\ 9^* \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{bmatrix} 0.55 & 0.83 & 0 & 0 & 0 & 0 \\ -0.83 & 0.55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.24 & 0.97 & 0 \\ 0 & 0 & 0 & -0.97 & 0.24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{Bmatrix}$$

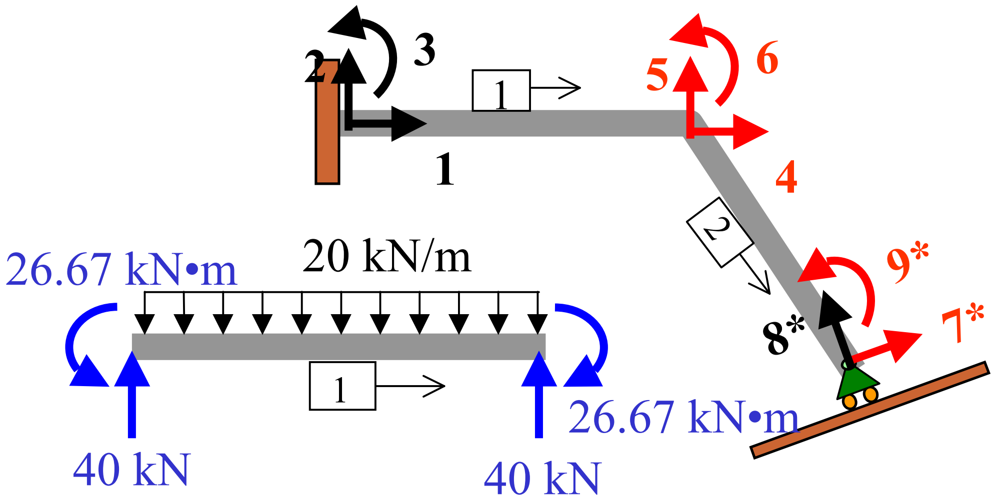
$$[\,k^{*}\,]_2 = [\,T\,]^{\mathrm{T}}[\,k'\,]_2[\,T\,] = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7^* \\ 8^* \\ 9^* \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7^* & 8^* & 9^* \end{array} \\ \begin{bmatrix} 1036.923 & -1524.780 & 30.646 & -452.884 & 1787.474 & 30.646 \\ -1524.780 & 2307.582 & 20.431 & 643.585 & -2689.923 & 20.431 \\ 30.646 & 20.431 & 88.643 & -35.786 & -8.717 & 44.321 \\ -452.884 & 643.585 & -35.786 & 205.452 & -759.668 & -35.786 \\ 1787.474 & -2689.923 & -8.717 & -759.668 & 3139.053 & -8.717 \\ 30.646 & 20.431 & 44.321 & -35.786 & -8.717 & 88.643 \end{bmatrix} \end{array}$$



$$[k]_1 = 10^3 \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 3000 & 0 & 0 & -3000 & 0 & 0 \\ 0 & 15 & 30 & 0 & -15 & 30 \\ 0 & 30 & 80 & 0 & -30 & 40 \\ -3000 & 0 & 0 & 3000 & 0 & 0 \\ 0 & -15 & -30 & 0 & 15 & -30 \\ 0 & 30 & 40 & 0 & -30 & 80 \end{pmatrix} \end{array}$$

$$[k^*]_2 = 10^3 \begin{array}{cc} & \begin{array}{cccccc} 4 & 5 & 6 & 7^* & 8^* & 9^* \end{array} \\ \begin{array}{c} 4 \\ 5 \\ 6 \\ 7^* \\ 8^* \\ 9^* \end{array} & \begin{pmatrix} 1036.923 & -1524.780 & 30.646 & -452.884 & 1787.474 & 30.646 \\ -1524.780 & 2307.582 & 20.431 & 643.585 & -2689.923 & 20.431 \\ 30.646 & 20.431 & 88.643 & -35.786 & -8.717 & 44.321 \\ -452.884 & 643.585 & -35.786 & 205.452 & -759.668 & -35.786 \\ 1787.474 & -2689.923 & -8.717 & -759.668 & 3139.053 & -8.717 \\ 30.646 & 20.431 & 44.321 & -35.786 & -8.717 & 88.643 \end{pmatrix} \end{array}$$





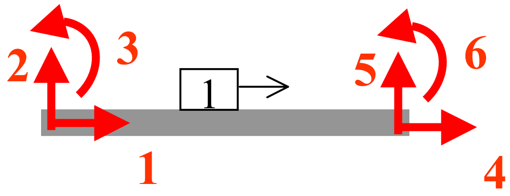
Global:

$$
\begin{bmatrix} Q_4 = -50 \\ Q_5 = 0 \\ Q_6 = -80 \\ Q_{7*} = 0 \\ Q_{9*} = 0 \end{bmatrix}
= 10^3
\begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 9 \end{array}
\begin{array}{c} \begin{array}{ccccc} 4 & 5 & 6 & 7* & 9* \end{array} \\
\begin{bmatrix}
4036.923 & -1524.780 & 30.646 & -452.884 & 30.646 \\
-1524.780 & 232.582 & -9.569 & 643.585 & -9.569 \\
30.646 & -9.569 & 168.643 & -35.786 & 44.321 \\
-452.884 & 643.585 & -35.786 & 205.452 & -35.786 \\
30.646 & 20.431 & 44.321 & -35.786 & 88.643
\end{bmatrix} \end{array}
\begin{bmatrix} D_4 \\ D_5 \\ D_6 \\ D_{7*} \\ D_{9*} \end{bmatrix}
+
\begin{bmatrix} 0 \\ 40 \\ -26.67 \\ 0 \\ 0 \end{bmatrix}
$$

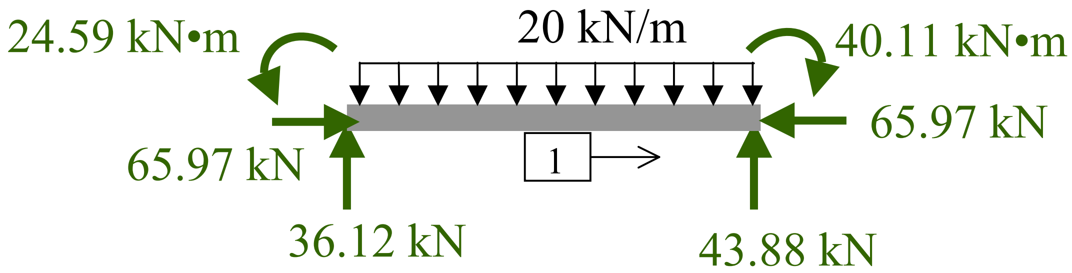
$$
\begin{bmatrix} D_4 \\ D_5 \\ D_6 \\ D_{7*} \\ D_{9*} \end{bmatrix}
=
\begin{bmatrix}
-2.199\text{x}10^{-5} & \text{m} \\
-3.095\text{x}10^{-4} & \text{m} \\
-2.840\text{x}10^{-4} & \text{rad} \\
0.979\text{x}10^{-3} & \text{m} \\
6.161\text{x}10^{-4} & \text{rad}
\end{bmatrix}
$$

Member 1:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = 10^3 \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 3000 & 0 & 0 & -3000 & 0 & 0 \\ 0 & 15 & 30 & 0 & -15 & 30 \\ 0 & 30 & 80 & 0 & -30 & 40 \\ -3000 & 0 & 0 & 3000 & 0 & 0 \\ 0 & -15 & -30 & 0 & 15 & -30 \\ 0 & 30 & 40 & 0 & -30 & 80 \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ D_4 \\ D_5 \\ D_6 \end{pmatrix} + \begin{pmatrix} 0 \\ 40 \\ 26.67 \\ 0 \\ 40 \\ -26.67 \end{pmatrix}$$



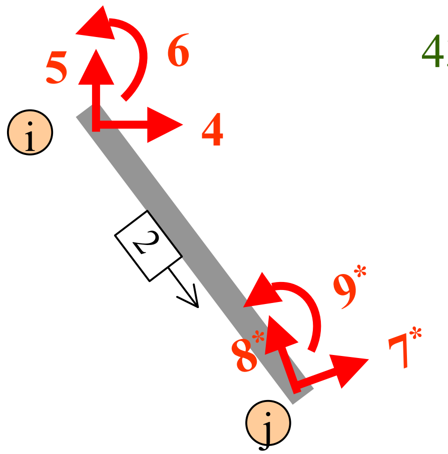
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{pmatrix} 65.97 & \text{kN} \\ 36.12 & \text{kN} \\ 24.59 & \text{kN}\cdot\text{m} \\ -65.97 & \text{kN} \\ 43.88 & \text{kN} \\ -40.11 & \text{kN}\cdot\text{m} \end{pmatrix}$$

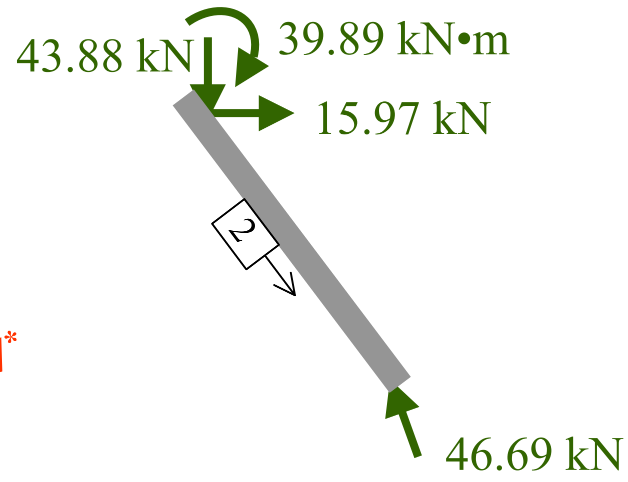


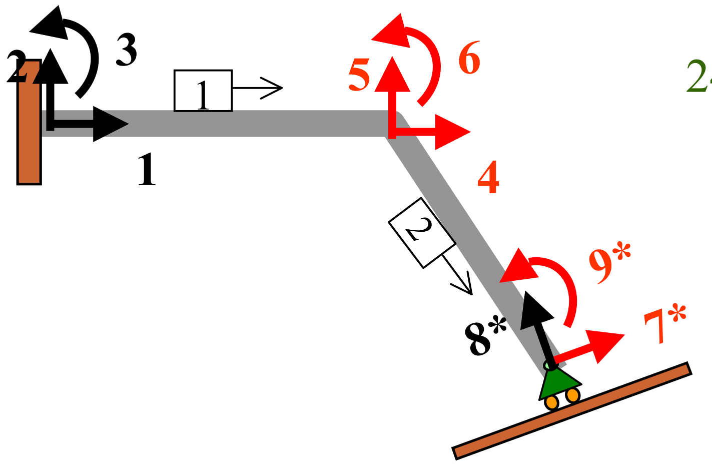
Member 2:

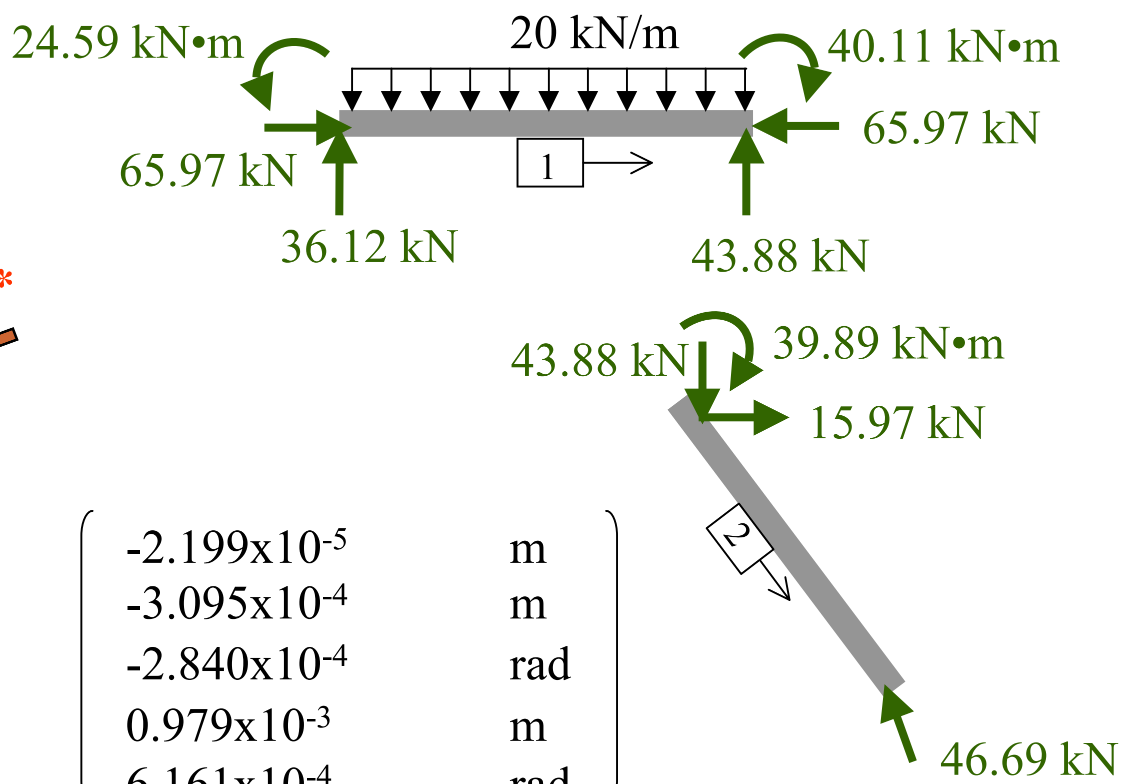
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{pmatrix} = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7* \\ 8* \\ 9* \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7* & 8* & 9* \end{array} \\ \begin{pmatrix} 1036.923 & -1508.14 & 30.57 & -455.21 & 1769.34 & 30.57 \\ -1508.14 & 2296.15 & 20.26 & 651.27 & -2678.93 & 20.26 \\ 30.57 & 20.26 & 88.64 & -35.73 & -8.84 & 44.32 \\ -455.21 & 651.27 & -35.73 & 210.67 & -769.1 & -35.73 \\ 1769.34 & -2678.93 & -8.84 & -769.1 & 3128.82 & -8.84 \\ 30.57 & 20.26 & 44.32 & -35.73 & -8.84 & 88.64 \end{pmatrix} \end{array} \begin{pmatrix} D_4 \\ D_5 \\ D_6 \\ D*_7 \\ 0 \\ D*_9 \end{pmatrix}$$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{pmatrix} = \begin{pmatrix} 15.97 & \text{kN} \\ -43.88 & \text{kN} \\ -39.89 & \text{kN•m} \\ 0 & \text{kN} \\ 46.69 & \text{kN} \\ 0 & \text{kN•m} \end{pmatrix}$$

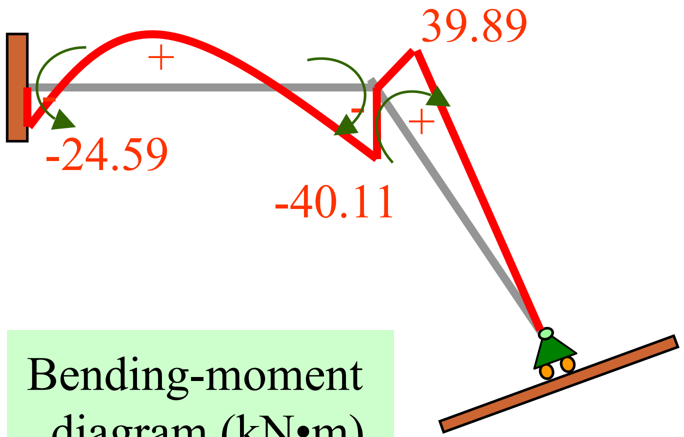




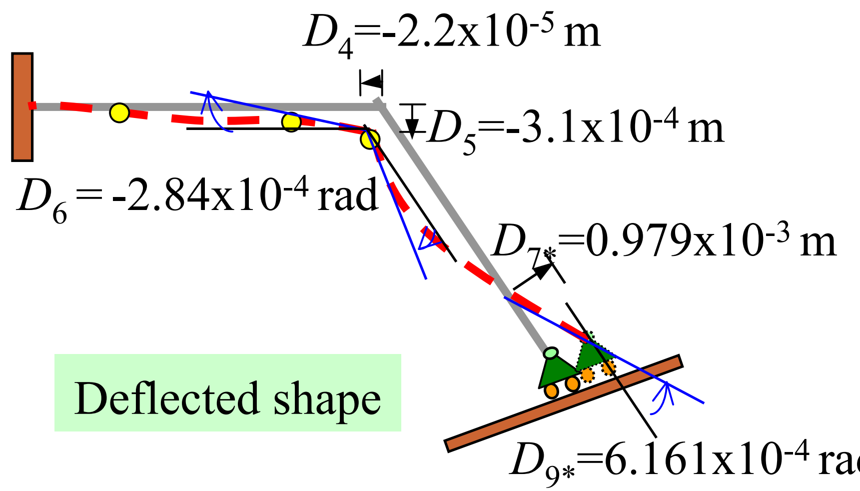




$$
\begin{pmatrix} D_4 \\ D_5 \\ D_6 \\ D_{7*} \\ D_{9*} \end{pmatrix} = \begin{pmatrix} -2.199\text{x}10^{-5} & \text{m} \\ -3.095\text{x}10^{-4} & \text{m} \\ -2.840\text{x}10^{-4} & \text{rad} \\ 0.979\text{x}10^{-3} & \text{m} \\ 6.161\text{x}10^{-4} & \text{rad} \end{pmatrix}
$$



Bending-moment diagram (kN•m)



Deflected shape