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MATH-112 Homework 2

Problem 2

(a) Let (X, d) be a metric space. Prove that X and $\emptyset$ are both open and closed.

(b) For $x, y \in \mathbb{R}$, define:

$$d_1(x,y) = |x-y|^3, \quad d_2(x,y) = |x^2 - y^2|, \quad d_3(x,y) = \frac{|x-y|}{1+|x-y|}.$$

Determine for each of these functions whether it defines a metric on $\mathbb{R}$.

(2.a): Below is a proof of each without relying on complements (which halves the problem):

(i) For every point $x \in X$ and epsilon-neighborhood $N(x, \varepsilon)$ of x, it must be that $N(x,\varepsilon) \subseteq X$ since X contains all points, meaning $X = \mathring{X}$.

(ii) Since X is the entire set it must contain all of its limit points, meaning $X = \overline{X}$.

(iii) Since $\emptyset$ contains no non-interior points, it is vacuously true that $\emptyset = \mathring{\emptyset}$.

(iv) Since $\emptyset$ contains no points, for any $x \in X$ and $y \in N(x, \varepsilon)$, we have $y \notin \emptyset$. Thus, the emptyset has no limit points, meaning $\emptyset = \overline{\emptyset}$.

We are done.

$\square$

(2.b): Consider each function:

(i) Let $x = 0,\ y = 1,\ z = 2$. Then:

$$d_1(x,z) = |0-2|^3 = 8 \not\leq 2 = |0-1|^3 + |1-2|^3 = d_1(x,y) + d_1(y,z),$$

failing the triangle inequality. Thus, d_1 **is not** a metric on $\mathbb{R}$.

(ii) Let $x = 1,\ y = -1$. Then:

$$d_2(x,y) = |(1)^2 - (-1)^2| = 0,$$

but $x \neq y$. Thus, d_2 **is not** a metric on $\mathbb{R}$.

(iii) First, since $|x-y| \geq 0,\ 1 + |x-y| \geq 1$, we have $d_3(x,y) \geq 0$ for all x, y. Also, we see:

$$d_3(x,y) = 0 \iff \frac{|x-y|}{1+|x-y|} = 0 \iff |x-y| = 0 \iff x = y,$$

since $|x-y|$ is a metric on $\mathbb{R}$. Thus, the first property of a metric holds for d_3. Now, since $|x-y|$ is a metric on $\mathbb{R}$, we know $|x-y| = |y-x|$, giving us:

$$\begin{aligned} d_3(x,y) &= \frac{|x-y|}{1+|x-y|} \\ &= \frac{|y-x|}{1+|y-x|} \\ &= d_3(y,x). \end{aligned}$$

Thus, the second property of a metric holds for d_3. Finally, since $|x-y|$ is a metric on $\mathbb{R}$, we know:

$$|x-z| \leq |x-y|+|y-z| \iff \frac{1}{|x-z|} \geq \frac{1}{|x-y|+|y-z|} \iff 1-\frac{1}{|x-z|} \leq 1-\frac{1}{|x-y|+|y-z|}.$$

We now observe:

$$\begin{aligned}
d_3(x,z) &= \frac{|x-z|}{1+|x-z|} \\
&= 1-\frac{1}{1+|x-z|} \\
&\leq 1-\frac{1}{1+|x-y|+|y-z|} \\
&= \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\
&= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\
&\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \\
&= d_3(x,y)+d_3(y,z),
\end{aligned}$$

which gives us:

$$d_3(x,z) \leq d_3(x,y)+d_3(y,z).$$

Thus, the third (and final) property of a metric holds for d_3, meaning d_3 **is** a metric on $\mathbb{R}$.

We are done.

$\square$