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MATH-112 Homework 1

Problem 2

(a) Let $p \in \mathbb{N}$ be a prime number. Let $q \in \mathbb{Q}$ be a rational number. Show that q can be written as $q = \frac{n}{m}$ where at least one of m and n is not divisible by p.

(b) Say that two integers $n, m \in \mathbb{Z}$ are **co-prime** if there is no prime number p such that p divides both n and m. Show that any rational number q can be written as $q = \frac{n}{m}$ for some integers n and m which are co-prime.

(2.a): Let $p \in \mathbb{N}$ be prime and $q \in \mathbb{Q}$ be rational. By definition, we may express:

$$q = \frac{a}{b},$$

for some integers a, b with $b > 0$. If $p \nmid a$ and/or $p \nmid b$, we are done. Now consider the case where $p \mid a$ and $p \mid b$. By the Fundemental Theorem of Arithmetic, we may factor:

$$a = p^{ord_p(a)}a', \quad b = p^{ord_p(b)}b',$$

where $p \nmid a', b'$ and $ord_p(a), ord_p(b) \geq 1$. Then:

$$\frac{a}{b} = \frac{p^{ord_p(a)}a'}{p^{ord_p(b)}b'} = p^{ord_p(a)-ord_p(b)}\frac{a'}{b'}.$$

Now, we reduce to casework.

(i) Consider $ord_p(a) = ord_p(b)$. Then:

$$\frac{a}{b} = \frac{a'}{b'},$$

where $p \nmid a', b'$, as desired.

(ii) Consider $ord_p(a) > ord_p(b)$. Then:

$$\frac{a}{b} = \frac{p^{ord_p(a)-ord_p(b)}a'}{b'} = \frac{a''}{b'},$$

where $p \nmid b'$, as desired.

(iii) Consider $ord_p(a) < ord_p(b)$. Then:

$$\frac{a}{b} = \frac{a'}{p^{ord_p(b)-ord_p(a)}b'} = \frac{a'}{b''},$$

where $p \nmid a'$, as desired.

We are done.

$\square$

(2.b): Let $q \in \mathbb{Q}$. By definition, we may express:

$$q = \frac{a}{b},$$

for some integers a, b with $b > 0$. If $(a, b) = 1$, we're done. Now consider $(a, b) = d > 1$. By the Fundemental Theorem of Arithmetic, we may factor:

$$a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} a', \quad b = p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k} b', \quad (a', b') = 1,$$

for some positive integers $k, e_i \coloneqq ord_{p_i}(a), f_i \coloneqq ord_{p_i}(b)$ and set $\{p_i\}$ of prime integers. Then $d \coloneqq \prod_{i=1}^{k} p_i^{\min(e_i, f_i)}$ and:

$$\frac{a}{b} = \frac{a' \prod_{e_i > f_i} p_i^{e_i - f_i}}{b' \prod_{e_j < f_j} p_j^{f_j - e_j}} = \frac{a''}{b''},$$

where $(a'', b'') = 1$, as desired. We are done.

$\square$