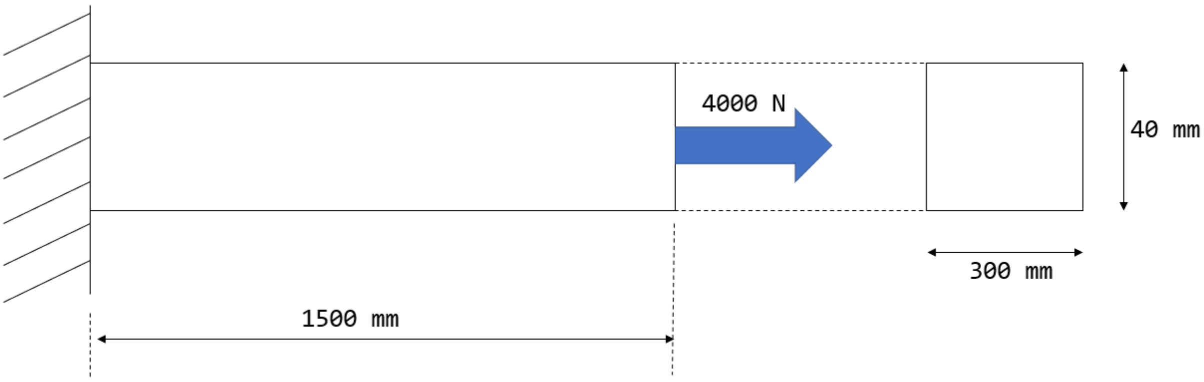
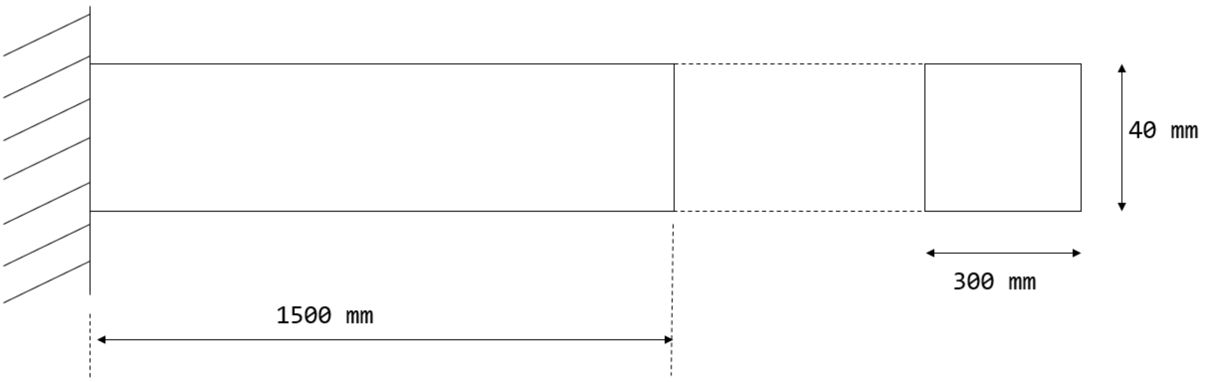
1D Bar Finite Element Method (FEM) Analysis

- This project contains Python code for performing Finite Element Method (FEM) analysis on 1D bar elements.
- The code includes implementations for Static-Linear, and modal analyses.
- Theoretical solutions for these analyses were derived based on established principles and then used to verify the accuracy of the FEM code.
- The Sympy library from Python was employed for symbolic mathematics to aid in the derivation and verification of theoretical solutions.
- This project demonstrates proficiency in FEM, numerical analysis, and symbolic computation using Python.

1D Bar Static Analysis Problem



1D Bar Modal Analysis Problem



Derivation of Local Stiffness and Mass Matrix for FEM

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}_e\bar{\mathbf{u}}(t) = \mathbf{f}(t)$$

$$\mathbf{M} = \int_V \boldsymbol{\Psi}^\top(\mathbf{x})\rho\boldsymbol{\Psi}(\mathbf{x})dV$$

$$\mathbf{K}_e = \int_V \mathbf{B}_e^\top(\mathbf{x})\mathbf{S}_e(E,\nu)\mathbf{B}_e(\mathbf{x})dV$$

For 1D Bar the shape functions matrix is, $\boldsymbol{\Psi}(x) = \begin{bmatrix} 1 - \frac{x}{l_e} & \frac{x}{l_e} \end{bmatrix}$

$$\mathbf{B}_e = \frac{\partial\boldsymbol{\Psi}(x)}{\partial x} = \begin{bmatrix} -\frac{1}{l_e} & \frac{1}{l_e} \end{bmatrix}$$

$$\bar{\mathbf{M}} = \int_{x=0}^{l_e} [\boldsymbol{\Psi}^\top(x)\rho\boldsymbol{\Psi}(x)]Adx$$

$$\bar{\mathbf{K}}_e = \int_{x=0}^{l_e} [\mathbf{B}_e^\top E\mathbf{B}_e]Adx$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \mathbf{B}_e\bar{\mathbf{u}}$$

$$\sigma_{xx} = E\,\epsilon_{xx} = E\,\frac{\partial u_x}{\partial x} = E\,\mathbf{B}_e\bar{\mathbf{u}}$$

Deriving Actual Solutions for Static-Linear Bar Analysis

$$\sigma_{xx} = E\,\epsilon_{xx}$$

$$\frac{F}{A} = E\,\frac{u}{L}$$

Actual Displacement at tip of the bar

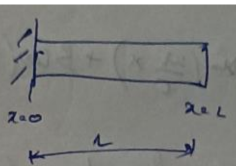
$$u = \frac{F}{EA}L$$

Actual Stress on the bar

$$\sigma_{xx} = \frac{F}{A}$$

Deriving equation of Natural Frequencies in Bar

The strong form of Bar,



$$EA \frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2}$$

$$\boxed{\boxed{c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}} \quad \boxed{c^2 = \frac{E}{\rho}}} \quad —(1)$$

To find solution, let's assume, $u(x,t) = U(x)\, g(t)$

$$c^2 \frac{\partial^2}{\partial x^2}[U(x)\, g(t)] = \frac{\partial^2}{\partial t^2}[U(x)\, g(t)]$$

$$\boxed{c^2 U'' g = U \ddot{g}} \; (2) \Rightarrow c^2 \frac{U''}{U} = \frac{\ddot{g}}{g} = \gamma \Rightarrow \ddot{g} - \gamma g = 0$$

$$\ddot{g} - \gamma g = 0$$

$$(D^2 - \gamma) g = 0 \Rightarrow D = \pm\sqrt{\gamma} \quad , \text{if } \gamma = -\omega_n^2,\; D = \pm j\omega_n$$

$$\boxed{g(t) = C \sin(\omega_n t) + D \cos(\omega_n t)}$$

Now, $u(x,t) = U(x)\,[C \sin(\omega_n t) + D \cos(\omega_n t)]$

Now, ~~(2) bring~~ use $g(t)$ in (2)

$$c^2 U'' [\cancel{C \sin(\omega_n t) + D\cos(\omega_n t)}] = U[(-\omega_n^2)(\cancel{C \sin(\omega_n t) + D \cos(\omega_n t)})]$$

$$c^2 U'' = -\omega_n^2 U \Rightarrow U'' + \frac{\omega_n^2}{c^2} U = 0 \Rightarrow U'' + \lambda^2 U = 0 \quad \text{where, } \lambda^2 = \frac{\omega_n^2}{c^2}$$

$$U'' + \lambda^2 U = 0 \Rightarrow (D^2 + \lambda^2) U = 0 \Rightarrow D = \pm j\lambda = \pm j \frac{\omega_n}{c}$$

$$\boxed{U(x) = A \sin\left(\frac{\omega_n}{c} x\right) + B \cos\left(\frac{\omega_n}{c} x\right)}$$

Now, $$\boxed{u(x,t) = \left[A \sin\left(\frac{\omega_n}{c} x\right) + B \cos\left(\frac{\omega_n}{c} x\right)\right]\left[C \sin(\omega_n t) + D \cos(\omega_n t)\right]}$$

$$u(x,t) = \left[A \sin\left(\frac{\omega_n}{c} x\right) + B \cos\left(\frac{\omega_n}{c} x\right)\right]\left[C \sin(\omega_n t) + D \cos(\omega_n t)\right]$$

For, Fixed-Free boundary Condition

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$$

$$\cancel{u(0,t) = 0 = [0 + B]\left[C \sin(\omega_n t) + D \cos(\omega_n t)\right]} \Rightarrow B = 0$$

$$u(0,t) = [0 + B]\left[C \sin(\omega_n t) + D \cos(\omega_n t)\right] \Rightarrow B = 0$$

$$u(x,t) = A \sin\left(\frac{\omega_n}{c} x\right)\left[C \sin(\omega_n t) + D \cos(\omega_n t)\right]$$

$$\frac{\partial u}{\partial x}(x,t) = A \frac{\omega_n}{c} \cos\left(\frac{\omega_n}{c} x\right)\left[C \sin(\omega_n t) + D \cos(\omega_n t)\right]$$

$$\frac{\partial u}{\partial x}(L,t) = 0 = A \frac{\omega_n}{c} \cos\left(\frac{\omega_n}{c} L\right)\left[C \sin(\omega_n t) + D \cos(\omega_n t)\right]$$

$$\cos\left(\frac{\omega_n}{c} L\right) = 0 \Rightarrow \frac{\omega_n L}{c} = (2n-1)\frac{\pi}{2}$$

$$\cancel{\omega_n = (2n-1)\frac{\pi}{2L} \sqrt{\frac{E}{\rho}}} \qquad \boxed{\omega_n = (2n-1)\frac{\pi}{2L}\sqrt{\frac{E}{\rho}}}$$

$$f_n = \frac{\omega_n}{2\pi} \Rightarrow \boxed{f_n = \frac{(2n-1)}{4L}\sqrt{\frac{E}{\rho}}}$$