1D Bar Finite Element Method (FEM) Analysis

- This project contains Python code for performing Finite Element Method (FEM) analysis on 1D bar elements with Non-Linear Constitutive Behaviour.

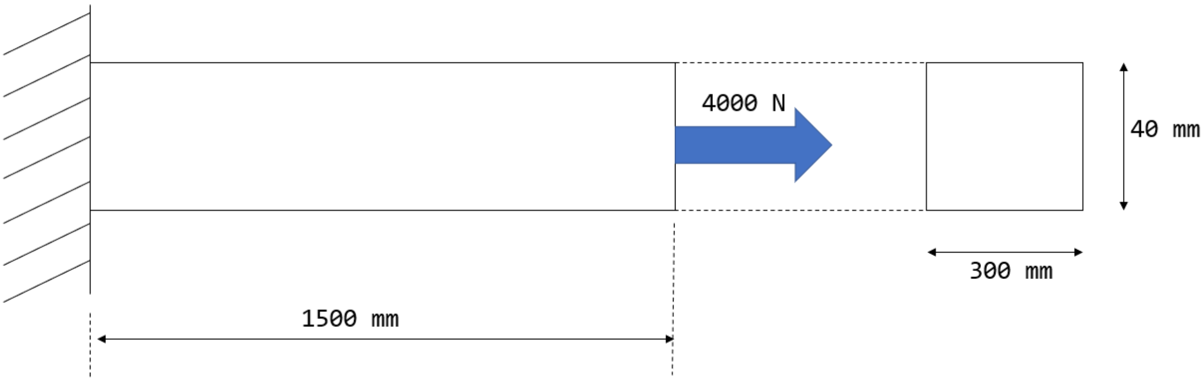
- The code includes implementations for Static Non-Linear for problems with Linear and Non-Linear Constitutive Behaviour.

- When Non-Linear FEM code is used to solve Linear Problem, the solution is obtained in 2 iterations only.

- For a regular Non-Linear Problem, the solution is obtained in multiple steps.

- The Non-Linearity effect can be observed only if, the E_0 is small or F is large.

1D Bar Static Analysis Problem



The Non-Linearity considered,

$\sigma(\varepsilon) = E_0\,\varepsilon + E_1\,\varepsilon^3$

The strong form equation of 1D bar can be written as,

$A\frac{d}{dx}(\hat{\sigma}(\varepsilon)) + b = 0, [0, l_e]$

Were,

$\varepsilon = \frac{du}{dx}$

If no body forces are considered, b = 0

$A\frac{d}{dx}(\hat{\sigma}(\varepsilon)) = 0, [0, l_e]$

The boundary conditions are as described below:

$$u = 0, \qquad x = 0$$
$$A\hat{\sigma}\left(\frac{du}{dx}\right) = F, \quad x = l_e$$

The equation can be converted into weak form by multiplying with a weight function and integrating over the entire domain as,

$$A\int_0^l \frac{d}{dx}(\hat{\sigma}(\varepsilon))w dx = 0$$

Integrating by parts,

$$A\int_0^l \frac{d}{dx}(\hat{\sigma}(\varepsilon))\frac{dw}{dx}dx = Pw(l)$$

Now, given $\hat{\sigma}(\varepsilon)$ is a non-linear function in ε, it should be linearized to obtain the solution. To linearize, Newton Raphson method is used where $u^{k+1} = u^k + \Delta u^k$ is considered and using Taylor series expansion

$$\hat{\sigma}\left(\frac{du^{k+1}}{dx}\right) = \hat{\sigma}\left(\frac{d(u^k+\Delta u^k)}{dx}\right) = \hat{\sigma}\left(\frac{d(u^k)}{dx}\right) + \hat{\sigma}'\left(\frac{d(u^k)}{dx}\right)\frac{d\Delta u^k}{dx}$$

Now,

$$A\int_0^l \left(\hat{\sigma}\left(\frac{d(u^k)}{dx}\right) + \hat{\sigma}'\left(\frac{d(u^k)}{dx}\right)\frac{d\Delta u^k}{dx}\right)\frac{dw}{dx}dx = Pw(l)$$

$$A\int_0^l \hat{\sigma}'\left(\frac{d(u^k)}{dx}\right)\frac{d\Delta u^k}{dx}\frac{dw}{dx}dx = Pw(l) - A\int_0^l \hat{\sigma}\left(\frac{d(u^k)}{dx}\right)\frac{dw}{dx}dx$$

The above equation is used to obtain Δu^k which is the in kth step and is added to u^k to get the displacement for the $(k+1)^{th}$ iteration. This equation is solved till the residue which is the RHS of equation becomes less than a tolerance value. The value of $\hat{\sigma}'$ and $\hat{\sigma}$ are obtained based on the assumed Constitutive mode (in this case σ(ε) = E_0 ε + E_1 ε^3) for each iteration.

According to FEA approximation, the values of displacements can be considered as sum of displacements at nodes multiplied by weight functions as shown below,

$$u^k(x) = \sum_{i=1}^N u_i^k\varphi_i(x) \quad = \quad \sum_i u_i^k\varphi_i(x)$$
$$w^k(x) = \sum_{j=1}^N \varphi_j(x) \quad = \quad \sum_j \varphi_j(x)$$
$$\Delta u^k(x) = \sum_{m=1}^N \Delta u_m^k\varphi_m(x) \quad = \quad \sum_m \Delta u_m^k\varphi_m(x)$$

Now the equation becomes,

$$A\int_0^l \hat{\sigma}'\left(\frac{d(\sum_i u_i^k \varphi_i(x))}{dx}\right)\frac{d(\sum_m \Delta u_m^k \varphi_m(x))}{dx}\frac{d(\sum_j \varphi_j(x))}{dx}dx = P \text{ - } A\int_0^l \hat{\sigma}\left(\frac{d(\sum_i u_i^k \varphi_i(x))}{dx}\right)\frac{d(\sum_j \varphi_j(x))}{dx}dx$$

$$\left[\sum_m \sum_j A\int_0^l \hat{\sigma}'\left(\sum_i \frac{d\varphi_i}{dx}u_i^k\right)\frac{d\varphi_m}{dx}\frac{d\varphi_j}{dx}dx\right]\{\Delta u_m^k\} = \{P\} \text{ - } \left\{\sum_j A\int_0^l \hat{\sigma}\left(\sum_i \frac{d\varphi_i}{dx}u_i^k\right)\frac{d\varphi_j}{dx}dx\right\}$$

Substituting the values of shape functions for each node, the equation simplifies to,

If i<N:

$$A\left(\int_{X_{i-1}}^{X_i} \sigma'\left(\frac{u_i-u_{i-1}}{X_i-X_{i-1}}\right)\varphi_i'\varphi_{i-1}'dx\right)\Delta u_{i-1} + A\left(\int_{X_i}^{X_{i+1}} \sigma'\left(\frac{u_{i+1}-u_i}{X_{i+1}-X_i}\right)\varphi_i'\varphi_{i+1}'dx\right)\Delta u_{i+1}$$
$$+\left(A\int_{X_{i-1}}^{X_i} \sigma'\left(\frac{u_i-u_{i-1}}{X_i-X_{i-1}}\right)\varphi_i'^2 dx + A\int_{X_i}^{X_{i+1}} \sigma'\left(\frac{u_{i+1}-u_i}{X_{i+1}-X_i}\right)\varphi_i'^2 dx\right)\Delta u_i$$
$$= -A\int_{X_{i-1}}^{X_i} \sigma\left(\frac{u_i-u_{i-1}}{X_i-X_{i-1}}\right)\varphi_i' dx - A\int_{X_i}^{X_i} \sigma\left(\frac{u_{i+1}-u_i}{X_{i+1}-X_i}\right)\varphi_i' dx + Fv_i$$

If i=1 or i=N, i.e at the end points of the bar

These equations can be analytically solved, as ϕ is a linear function of X. Solving it accordingly the equation simplifies to, where $h_i = X_{i+1} - X_i$

$$A\,\sigma'\left(\frac{u_i-u_{i-1}}{h_{i-1}}\right)\left(\frac{-1}{h_{i-1}}\right)\Delta u_{i-1} + A\,\sigma'\left(\frac{u_{i+1}-u_i}{h_i}\right)\left(\frac{-1}{h_i}\right)\Delta u_{i+1}$$
$$+A\,\sigma'\left(\frac{u_i-u_{i-1}}{h_{i-1}}\right)\left(\frac{1}{h_{i-1}}\right)\Delta u_i + A\,\sigma'\left(\frac{u_{i+1}-u_i}{h_i}\right)\left(\frac{1}{h_i}\right)\Delta u_i$$
$$= A\,\sigma\left(\frac{u_i-u_{i-1}}{h_{i-1}}\right) + A\,\sigma\left(\frac{u_{i+1}-u_i}{h_i}\right) + Fv_i$$