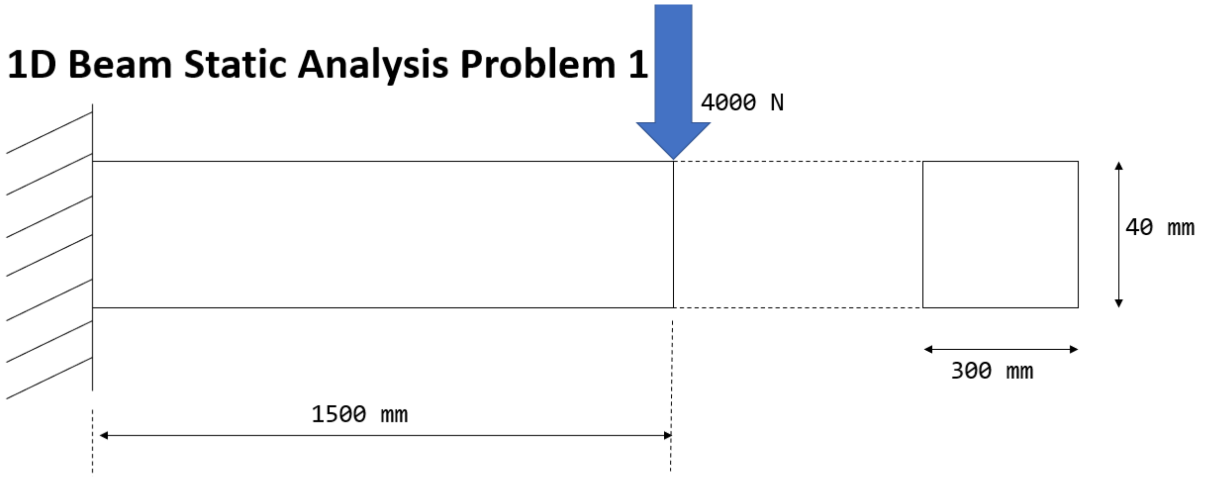
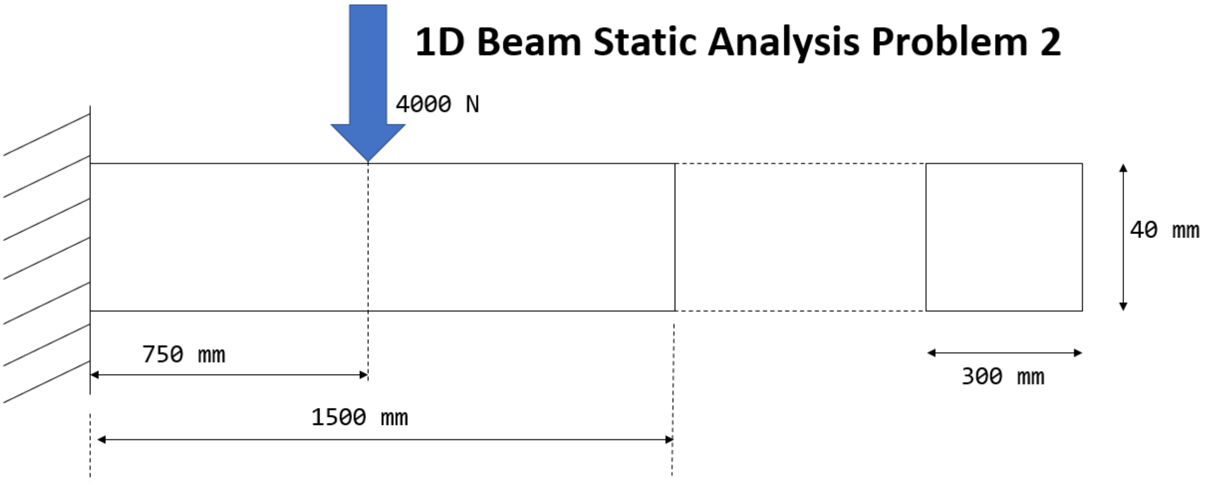
1D Beam Finite Element Method (FEM) Analysis

- This project contains Python code for performing Finite Element Method (FEM) analysis on 1D beam elements.
- The code includes implementations for Static-Linear, and modal analysis.
- Theoretical solutions for these analyses were derived based on established principles and then used to verify the accuracy of the FEM code.
- The Sympy library from Python was employed for symbolic mathematics to aid in the derivation and verification of theoretical solutions.
- This project demonstrates proficiency in FEM, numerical analysis, and symbolic computation using Python.

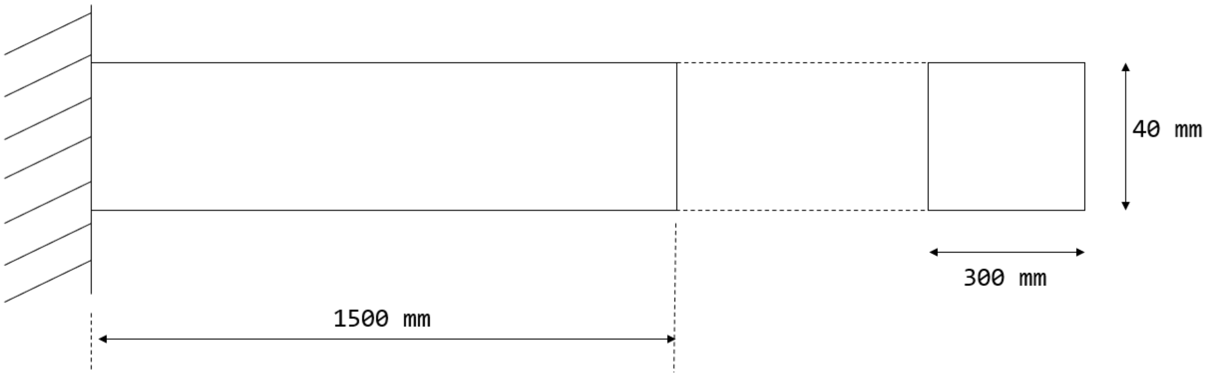
1D Beam Static Analysis Problem 1



1D Beam Static Analysis Problem 2



1D Beam Modal Analysis Problem



Derivation of Local Stiffness and Mass Matrix for FEM

$$\mathbf{M}\ddot{\bar{\mathbf{u}}}(\mathrm{t}) + \mathbf{K}_{\mathrm{e}}\bar{\mathbf{u}}(\mathrm{t}) = \mathbf{f}(\mathrm{t})$$

$$\mathbf{M} = \int_{\mathrm{V}} \mathbf{\Psi}^{\top}(\mathbf{x})\rho\mathbf{\Psi}(\mathbf{x})\mathrm{dV}$$

$$\mathbf{K}_{\mathrm{e}} = \int_{\mathrm{V}} \mathbf{B}_{\mathrm{e}}^{\top}(\mathbf{x})\mathbf{S}_{\mathrm{e}}(\mathrm{E},\nu)\mathbf{B}_{\mathrm{e}}(\mathbf{x})\mathrm{dV}$$

For 1D Bar the shape functions matrix is, $\mathbf{\Psi}(\mathrm{x}) =$

$$\left[1 - 3\left(\frac{x}{\mathrm{l_e}}\right)^2 + 2\left(\frac{x}{\mathrm{l_e}}\right)^3 \quad \left(\frac{x}{\mathrm{l_e}} - 2\left(\frac{x}{\mathrm{l_e}}\right)^2 + \left(\frac{x}{\mathrm{l_e}}\right)^3\right)\mathrm{l_e} \quad 3\left(\frac{x}{\mathrm{l_e}}\right)^2 - 2\left(\frac{x}{\mathrm{l_e}}\right)^3 \quad \left(-\left(\frac{x}{\mathrm{l_e}}\right)^2 + \left(\frac{x}{\mathrm{l_e}}\right)^3\right)\mathrm{l_e}\right]$$

$$\mathbf{B}_{\mathrm{e}} = \frac{\partial^2\mathbf{\Psi}(\mathrm{x})}{\partial \mathrm{x}^2}$$

$$\bar{\boldsymbol{M}} = \int_{x=0}^{L} [\mathbf{\Psi}^{\top}(x)\rho\mathbf{\Psi}(x)]_{6\times6}Adx$$

$$\bar{\boldsymbol{K}}_{\mathrm{e}} = \int_{x=0}^{le}\int_{A} [\boldsymbol{B}_{\mathrm{e}}^{\top}(x,y)E\boldsymbol{B}_{\mathrm{e}}(x,y)]_{6\times6}dAdx$$

$$\epsilon_{\mathrm{xx}} = \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}} = \mathbf{B}_{\mathrm{e}}\bar{\mathbf{u}}$$

$$\sigma_{\mathrm{xx}} = \mathrm{E}\,\epsilon_{\mathrm{xx}} = \mathrm{E}\,\frac{\partial^2\mathrm{u_y}}{\partial \mathrm{x}^2}\,\mathrm{y} = \mathrm{E}\,\mathbf{B}_{\mathrm{e}}\bar{\mathbf{u}}$$

Deriving Actual Solutions for Static-Linear Beam Analysis

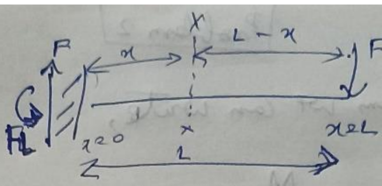
Actual Stress on the beam

$\sigma_{xx} = \frac{M}{I} y$

Now, to calculate tip deflection on Beam for Problem 1

Problem 1

For a beam we can write,



$$EI \frac{d^2u}{dx^2} = M$$

$$M = F(L-x)$$

For Fixed Free boundary condition

$u(0) = 0$

$u'(0) = 0$

$$EI\, u'' = F(L-x)$$

$$EI\, u' = F\left(Lx - \frac{x^2}{2}\right) + C_1$$

$u'(0) = 0$

$$EI(0) = F(0-0) + C_1 \Rightarrow \boxed{C_1 = 0}$$

$$EI\, u = F\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + C_2$$

$u(0) = 0$

$$EI(0) = F(0-0) + C_2 \Rightarrow \boxed{C_2 = 0}$$

$$u(x) = \frac{F}{EI}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right)$$

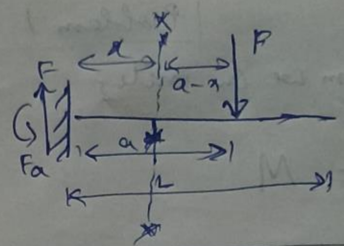
At tip, $u(L) = u$, $\boxed{u = \frac{FL^3}{3EI}}$

Now, to calculate tip deflection on Beam for Problem 2

$\boxed{\text{Problem 2}}$

For a beam we can write,

$$EI\frac{d^2u}{dx^2} = M$$



$$\boxed{\begin{array}{l}\text{The Boundary condition for fixed-free one} \\ u(0) = 0 \\ u'(0) = 0\end{array}}$$

For case of $0 \le x \le a$

$$M = F(a-x)$$

$$EI\,u'' = F(a-x)$$

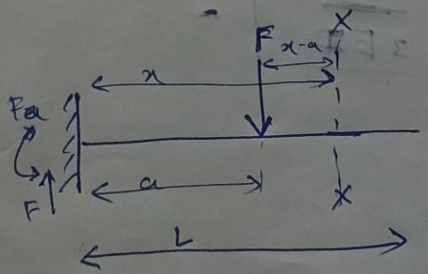
$$EI\,u' = F\left(ax - \frac{x^2}{2}\right) + C_1 \Rightarrow EI(0) = F(0+0) + C_1 \Rightarrow C_1 = 0$$

$$EI\,u = F\left(a\frac{x^2}{2} - \frac{x^3}{6}\right) + C_2 \cancel{\Rightarrow EI(0) = F(0+0) + C_2 \Rightarrow} C_1$$

$$EI(0) = F(0-0) + C_2 \Rightarrow C_2 = 0 \qquad \boxed{u'(a) = \frac{Fa^2}{2EI}}$$

$$u(x) = \frac{F}{EI}\left(a\frac{x^2}{2} - \frac{x^3}{6}\right) \Rightarrow \boxed{u(a) = \frac{Fa^3}{3EI}} \qquad \boxed{0 \le x \le a}$$

For case of $a < x \le L$



$$M = Fa - Fx + F(x-a) = 0$$

$$M = 0$$

$$EI\,u'' = 0$$

$$EI\,u' = C_3 \Rightarrow EI\,u'(a) = C_3 \Rightarrow EI\left(\frac{Fa^2}{2EI}\right) = C_3 \Rightarrow \boxed{C_3 = \frac{Fa^2}{2}}$$

$$EI\,u' = \frac{Fa^2}{2}$$

$$EI\,u = \frac{Fa^2x}{2} + C_4 \Rightarrow EI\,u(a) = C_4 \Rightarrow EI\left(\frac{Fa^3}{3EI}\right) = \frac{Fa^2a}{2} + C_4$$

$\cancel{C_4} =$

$$\boxed{C_4 = -\frac{Fa^3}{6}}$$

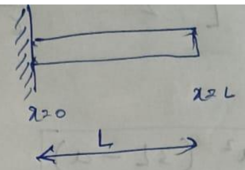
$$U(x) = \frac{\cancel{F}}{EI}\left[\frac{Fa^2}{2}x - \frac{Fa^3}{6}\right] = \frac{Fa^2}{6EI}(3x - a)$$

$U(L) = U$

$$\boxed{U = \frac{Fa^2}{6EI}(3L - a)}$$

Deriving equation of Natural Frequencies in Beam

The Strong form of beam,



$$EI\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2 u}{\partial t^2} = 0$$

$$\boxed{c^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0} \quad \boxed{c^2 = \frac{EI}{\rho A}}$$

Let solution be, $u(x,t) = U(x)\cos(\omega_n t)$

$$c^2\frac{\partial^4}{\partial x^4}[U(x)\cos(\omega_n t)] + \frac{\partial^2}{\partial t^2}[U(x)\cos(\omega_n t)] = 0$$

$$c^2 U^{iv}\cancel{\cos(\omega_n t)} + U(-\omega_n^2\cancel{\cos(\omega_n t)}) = 0$$

$$U^{iv} - \frac{\omega_n^2}{c^2}U = 0 \qquad \beta^4 = \frac{\omega_n^2}{c^2}$$

$$U^{iv} - \beta^4 U = 0$$

$$(D^4 - \beta^4)U = 0 \Rightarrow (D^4 - \beta^4) = 0 \Rightarrow \underbrace{(D^2 - \beta^2)}_{\text{Hyperbolic}}\underbrace{(D^2 + \beta^2)}_{\text{Trignometric}} = 0$$

$$U(x) = A\sinh(\beta x) + B\cosh(\beta x) + C\sin(\beta x) + D\cos(\beta x)$$

$$\boxed{u(x,t) = [A\sinh(\beta x) + B\cosh(\beta x) + C\sin(\beta x) + D\cos(\beta x)]\cos(\omega_n t)}$$

$$\beta^4 = \frac{\omega_n^2}{c^2} = \frac{\omega_n^2}{EI/\rho A} \Rightarrow \boxed{\omega_n = \beta^2\sqrt{\frac{EI}{\rho A}}}$$

The boundary conditions for Fixed - Free are

$u = 0$, $\frac{\partial u}{\partial x} = 0$; At $x = 0$

$\frac{\partial^2 u}{\partial x^2} = 0$, $\frac{\partial^3 u}{\partial x^3} = 0$; At $x = L$

$$u = A\cosh(\beta x) + B\sinh(\beta x) + C\cos(\beta x) + D\sin(\beta x)$$

$$\frac{\partial u}{\partial x} = A\beta\sinh(\beta x) + B\beta\cosh(\beta x) - C\beta\sin(\beta x) + D\beta\cos(\beta x)$$

$$\frac{\partial^2 u}{\partial x^2} = A\beta^2\cosh(\beta x) + B\beta^2\sinh(\beta x) - C\beta^2\cos(\beta x) - D\beta^2\sin(\beta x)$$

$$\frac{\partial^3 u}{\partial x^3} = A\beta^3\sinh(\beta x) + B\beta^2\cosh(\beta x) + C\beta^3\sin(\beta x) - D\beta^3\cos(\beta x)$$

$$u(0) = 0 \Rightarrow 0 = A(1) + B(0) + C(1) + D(0) \Rightarrow \boxed{C = -A}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \Rightarrow 0 = A(0) + B(1) - C(0) + D(1) \Rightarrow \boxed{D = -B}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0 \Rightarrow 0 = A\cosh(\beta L) + B\sinh(\beta L) + A\cos(\beta L) + B\sin(\beta L)$$

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{x=L} = 0 \Rightarrow 0 = A\sinh(\beta L) + B\cosh(\beta L) - A\sin(\beta L) + B\cos(\beta L)$$

$$0 = [\cosh(\beta L) + \cos(\beta L)]A + [\sinh(\beta L) + \sin(\beta L)]B$$

$$0 = [\sinh(\beta L) - \sin(\beta L)]A + [\cosh(\beta L) + \cos(\beta L)]B$$

$$\begin{bmatrix} \cosh(\beta L) + \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \\ \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) + \cos(\beta L) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = 0$$

$$\begin{vmatrix} \cosh(\beta L) + \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \\ \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) + \cos(\beta L) \end{vmatrix} = 0$$

$$\underbrace{\sin^2(\beta L) + \cos^2(\beta L)}_{1} + \underbrace{\cosh^2(\beta L) - \sinh^2(\beta L)}_{1} + 2\cosh(\beta L)\cos(\beta L) = 0$$

$$\boxed{\cos(\beta L)\cosh(\beta L) = -1}$$

This roots (β) can be obtained by using Taylor's series expansion