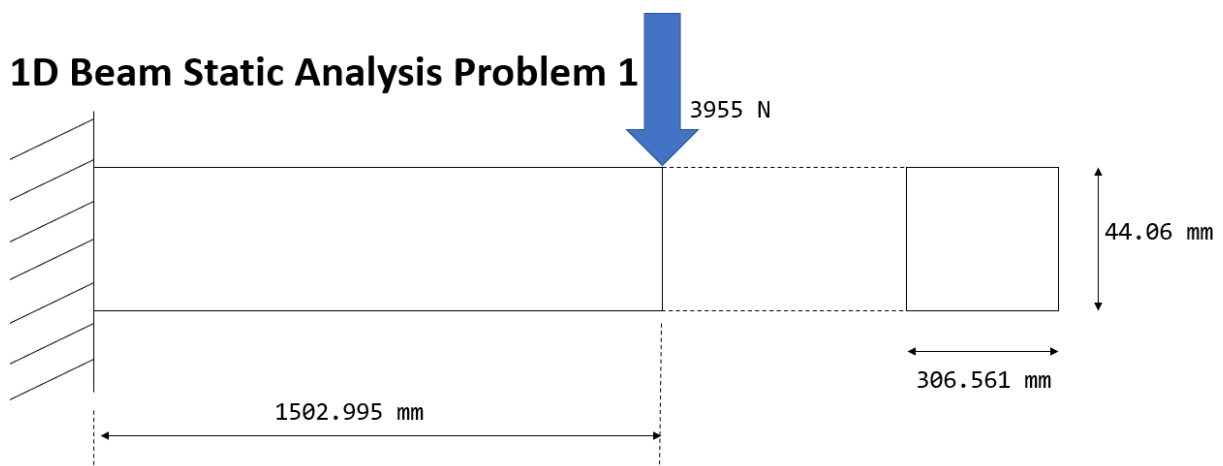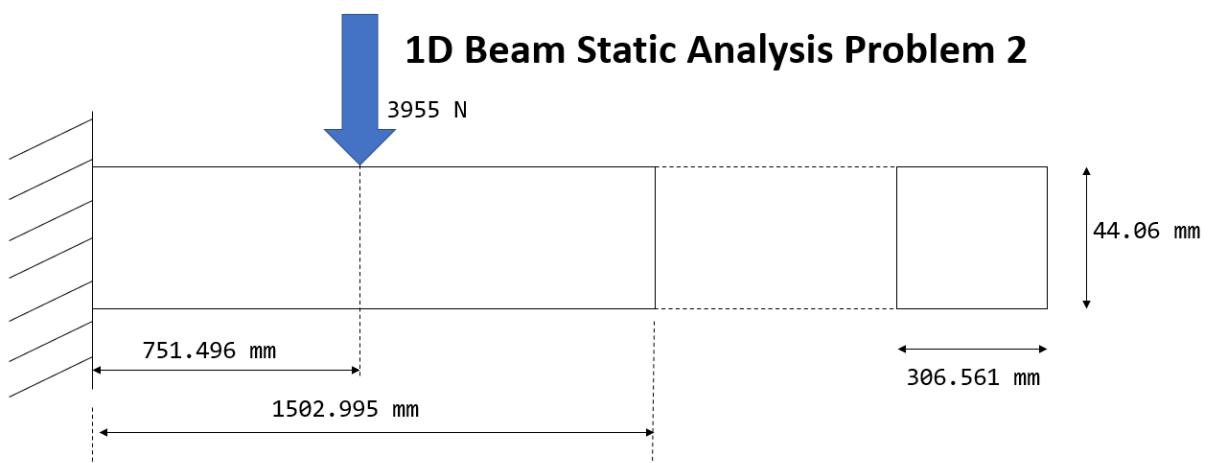# 1D Beam Finite Element Method (FEM) Analysis

- This project contains Python code for performing Finite Element Method (FEM) analysis on 1D beam elements.

- The code includes implementations for Static-Linear, and modal analysis.

- Theoretical solutions for these analyses were derived based on established principles and then used to verify the accuracy of the FEM code.

- The Sympy library from Python was employed for symbolic mathematics to aid in the derivation and verification of theoretical solutions.

- This project demonstrates proficiency in FEM, numerical analysis, and symbolic computation using Python.
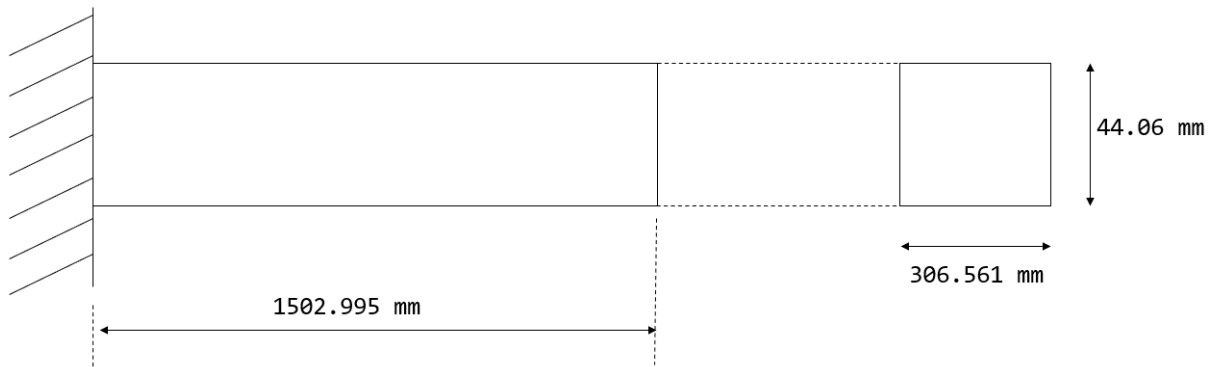
1D Beam Static Analysis Problem 1

3955 N

44.06 mm

306.561 mm

1502.995 mm

1D Beam Static Analysis Problem 2

3955 N

44.06 mm

306.561 mm

751.496 mm

1502.995 mm

1D Beam Modal Analysis Problem

1502.995 mm 44.06 mm 306.561 mm

Derivation of Local Stiffness and Mass Matrix for FEM

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}_e\bar{\mathbf{u}}(t) = \mathbf{f}(t)$$

$$\mathbf{M} = \int_V \mathbf{\Psi}^\top(\mathbf{x})\rho\mathbf{\Psi}(\mathbf{x})dV$$

$$\mathbf{K}_e = \int_V \mathbf{B}_e^\top(\mathbf{x})\mathbf{S}_e(E,\nu)\mathbf{B}_e(\mathbf{x})dV$$

For 1D Bar the shape functions matrix is, $\mathbf{\Psi}(x) =$

$$\left[1 - 3\left(\tfrac{x}{l_e}\right)^2 + 2\left(\tfrac{x}{l_e}\right)^3 \quad \left(\tfrac{x}{l_e} - 2\left(\tfrac{x}{l_e}\right)^2 + \left(\tfrac{x}{l_e}\right)^3\right)l_e \quad 3\left(\tfrac{x}{l_e}\right)^2 - 2\left(\tfrac{x}{l_e}\right)^3 \quad \left(-\left(\tfrac{x}{l_e}\right)^2 + \left(\tfrac{x}{l_e}\right)^3\right)l_e\right]$$

$$\mathbf{B}_e = \frac{\partial^2\mathbf{\Psi}(x)}{\partial x^2}$$

$$\bar{M} = \int_{x=0}^L [\mathbf{\Psi}^\top(x)\rho\mathbf{\Psi}(x)]_{6\times6}Adx$$

$$\bar{K}_e = \int_{x=0}^{le}\int_A [\boldsymbol{B}_e^\top(x,y)E\boldsymbol{B}_e(x,y)]_{6\times6}dAdx$$

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = \mathbf{B}_e\bar{\mathbf{u}}$$

$$\sigma_{xx} = E\,\epsilon_{xx} = E\,\frac{\partial^2 u_y}{\partial x^2}\,y = E\,\mathbf{B}_e\bar{\mathbf{u}}$$

Deriving Actual Solutions for Static-Linear Beam Analysis

Actual Stress on the beam

$$\sigma_{xx} = \frac{M}{I} y$$

Now, to calculate tip deflection on Beam for Problem 1

Problem 1

For a beam we can write,

$$EI \frac{d^2 u}{dx^2} = M$$

$$EI \, u''' = F(L - x)$$

$$EI \, u' = F\left(Lx - \frac{x^2}{2}\right) + C_1$$

$$u'(0) = 0$$

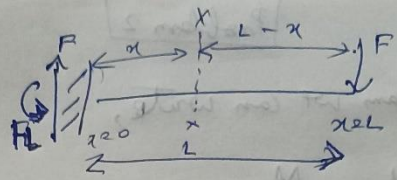$$EI(0) = F(0 - 0) + C_1 \Rightarrow \boxed{C_1 = 0}$$

$$EI \, u = F\left(L \frac{x^2}{2} - \frac{x^3}{6}\right) + C_2$$

$$u(0) = 0$$

$$EI(0) = F\left(0 - 0\right) + C_2 \Rightarrow \boxed{C_2 = 0}$$

$$u(x) = \frac{F}{EI}\left(L \frac{x^2}{2} - \frac{x^3}{6}\right)$$

At tip,

$$u(L) = u \qquad \boxed{u = \frac{FL^3}{3EI}}$$

$$M = F(L - x)$$

For Fixed Free boundary condition

$$u(0) = 0$$

$$u'(0) = 0$$

Now, to calculate tip deflection on Beam for Problem 2

Problem 2

For a beam we can write,

$$EI \frac{d^2 u}{dx^2} = M$$

For Case of $0 \le x \le a$

$$M = F(a-x)$$

$$EI\, u'' = F(a-x)$$

$$EI\, u' = F\left(ax - \frac{x^2}{2}\right) + C_1 \Rightarrow EI(0) = F(0 + 0) + C_1 \Rightarrow C_1 = 0$$

$$EI\, u = \left\{ F\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) + C_2 \Rightarrow EI(0) = F(0 + 0) + C_2 \Rightarrow C_1 \right.$$

$$EI(0) = F(0-0) + C_2 \Rightarrow C_2 = 0 \qquad \boxed{u'(a) = \frac{Fa^2}{2EI}}$$

$$U(x) = \frac{F}{EI}\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) \Rightarrow \boxed{u(a) = \frac{Fa^3}{3EI}} \qquad \boxed{0 \le x \le a}$$

For Case of $a \le x \le L$

$$M = Fa - Fx + F(x-a) = 0$$

$$M = 0$$

$$EI\, u'' = 0$$

$$EI\, u' = C_3 \Rightarrow EI\, u'(a) = C_3 \Rightarrow EI\left(\frac{Fa^2}{2EI}\right) = C_3 \Rightarrow \boxed{C_3 = \frac{Fa^2}{2}}$$

$$EI\, u' = \frac{Fa^2}{2}$$

$$EI\, u = \frac{Fa^2 x}{2} + C_4 \Rightarrow EI\, u(a) = C_4 \Rightarrow EI\left(\frac{Fa^3}{3EI}\right) = \frac{Fa^2 a}{2} + C_4$$

$$C_4 =$$

$$\boxed{C_4 = -\frac{Fa^3}{6}}$$

$$U(x) = \frac{F}{EI}\left[\frac{Fa^2}{2}x - \frac{Fa^3}{6}\right] = \frac{Fa^2}{6EI}(3x - a)$$
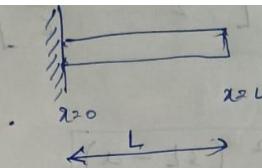
$$U(L) = u$$

$$\boxed{u = \frac{Fa^2}{6EI}(3L - a)}$$

Deriving equation of Natural Frequencies in Beam

The Strong form of beam,

$$EI\frac{\partial^4 u}{\partial x^4} + \rho A\frac{\partial^2 u}{\partial t^2} = 0$$

$$\boxed{c^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0} \qquad \boxed{c^2 = \frac{EI}{\rho A}}$$

Let solution be, $u(x,t)$ & $U(x,t) = U(x)\cos(w_n t)$

$$c^2\frac{\partial^4}{\partial x^4}[U(x)\cos(w_n t)] + \frac{\partial^2}{\partial t^2}[U(x)\cos(w_n t)] = 0$$

$$c^2 U^{IV}\cos(w_n t) + U(-w_n^2\cos(w_n t)) = 0$$

$$U^{IV} - \frac{w_n^2}{c^2}U = 0 \qquad \beta^4 = \frac{w_n^2}{c^2}$$

$$U^{IV} - \beta^4 U = 0$$

$$(D^4 - \beta^4)U = 0 \Rightarrow (D^4 - \beta^4) = 0 \Rightarrow \underbrace{(D^2 - \beta^2)}_{\text{Hyperbolic}}\underbrace{(D^2 + \beta^2)}_{\text{Trignometric}} = 0$$

$$U(x,t) = A\sinh(\beta x) + B\cosh(\beta x) + C\sin(\beta x) + D\cos(\beta x)$$

$$\boxed{U(x,t) = [A\sinh(\beta x) + B\cosh(\beta x) + C\sin(\beta x) + D\cos(\beta x)]\cos(w_n t)}$$

$$\beta^4 = \frac{w_n^2}{c^2} = \frac{w_n^2}{EI/\rho A} \Rightarrow \boxed{w_n = \beta^2\sqrt{\frac{EI}{\rho A}}}$$

The boundary conditions for Fixed - Free are

$$u = 0 ; \frac{\partial u}{\partial x} = 0 ; \text{ At } x = 0$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \frac{\partial^3 u}{\partial x^3} = 0 ; \text{ At } x = L$$

$$u = A\cosh(\beta x) + B\sinh(\beta x) + C\cos(\beta x) + D\sin(\beta x)$$

$$\frac{\partial u}{\partial x} = A\beta\sinh(\beta x) + B\beta\cosh(\beta x) - C\beta\sin(\beta x) + D\beta\cos(\beta x)$$

$$\frac{\partial^2 u}{\partial x^2} = A\beta^2\cosh(\beta x) + B\beta^2\sinh(\beta x) - C\beta^2\cos(\beta x) - D\beta^2\sin(\beta x)$$

$$\frac{\partial^3 u}{\partial x^3} = A\beta^3\sinh(\beta x) + B\beta^2\cosh(\beta x) + C\beta^3\sin(\beta x) - D\beta^3\cos(\beta x)$$

$u(0) = 0 \Rightarrow 0 = A(1) + B(0) + C(1) + D(0) \Rightarrow \boxed{C = -A}$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0 \Rightarrow 0 = A(0) + B(1) - C(0) + D(1) \Rightarrow \boxed{D = -B}$

$\left.\dfrac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0 \Rightarrow 0 = A\cosh(\beta L) + B\sinh(\beta L) + A\cos(\beta L) + B\sin(\beta L)$

$\left.\dfrac{\partial^3 u}{\partial x^3}\right|_{x=L} = 0 \Rightarrow 0 = A\sinh(\beta L) + B\cosh(\beta L) - A\sin(\beta L) + B\cos(\beta L)$

$0 = \left[\cosh(\beta L) + \cos(\beta L)\right] A + \left[\sinh(\beta L) + \sin(\beta L)\right] B = 0$

$0 = \left[\sinh(\beta L) - \sin(\beta L)\right] A + \left[\cosh(\beta L) + \cos(\beta L)\right] B$

$$\begin{bmatrix} \cosh(\beta L) + \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \\ \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) + \cos(\beta L) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = 0$$

$$\begin{vmatrix} \cosh(\beta L) + \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \\ \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) + \cos(\beta L) \end{vmatrix} = 0$$

$$\underbrace{\sin^2(\beta L) + \cos^2(\beta L)}_{1} + \underbrace{\cosh^2(\beta L) - \sinh^2(\beta L)}_{1} + 2\cosh(\beta L)\cos(\beta L) = 0$$

$$\boxed{\cos(\beta L)\cosh(\beta L) = -1}$$

This roots (β) can be obtained by using Taylor's series expansion