Bayes filtering : A deeper dive

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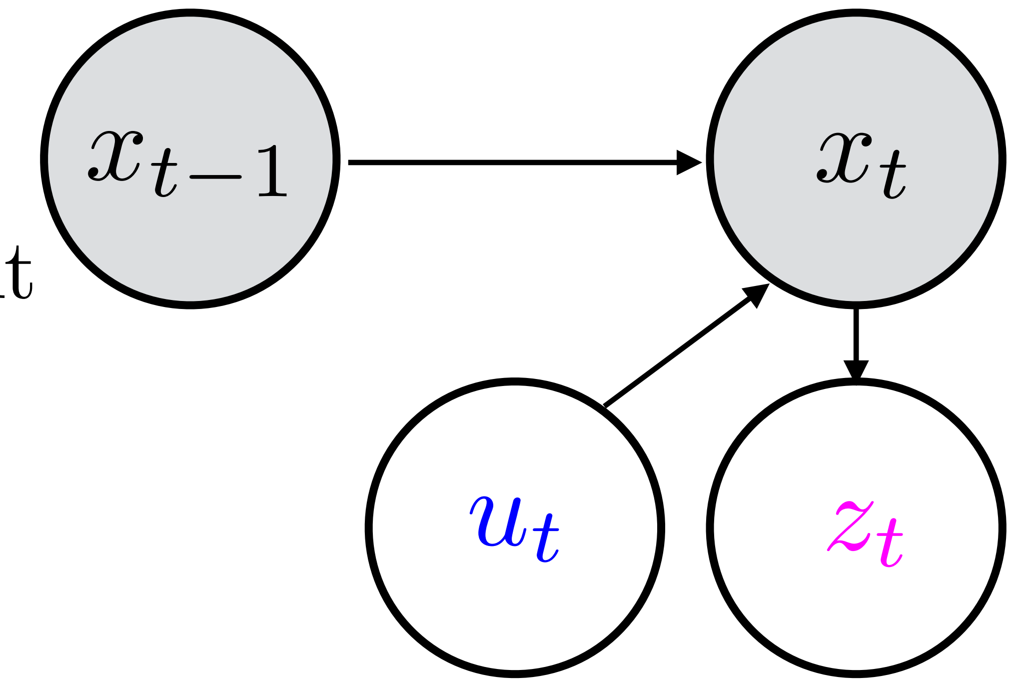
Recap: Key players in a Bayes filter

State

"Hidden stuff we want to know"

(everything needed to predict measurement / effect of action)



New state

Measurement

"Some information relevant to state"

Action

"Affects how state evolves"

Today's objective

1. Work through examples of Bayes filtering

2. Work through derivation

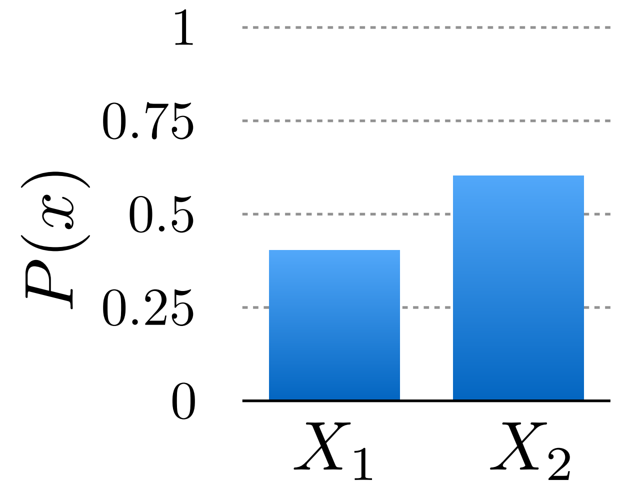
3. Question assumptions along the way

States and beliefs

State	Belief

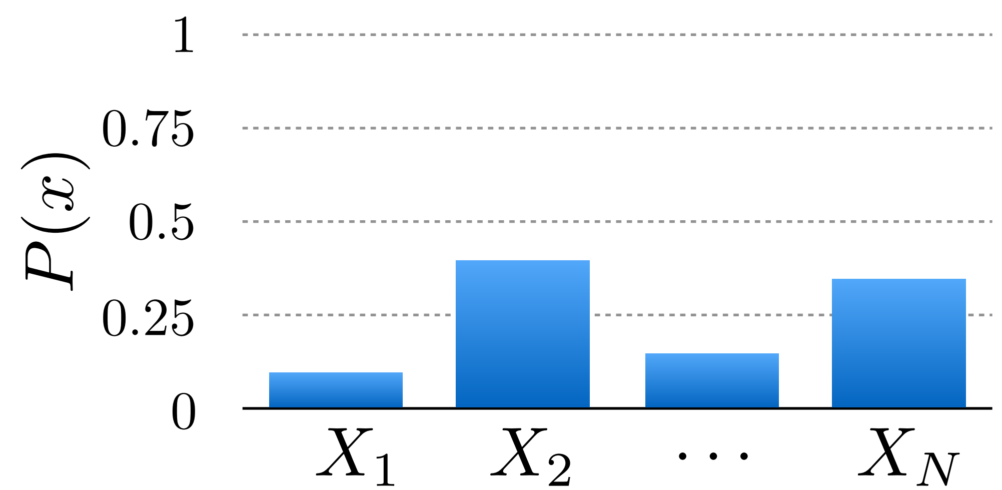
Discrete (Binary)

$$X = \{X_1, X_2\}$$



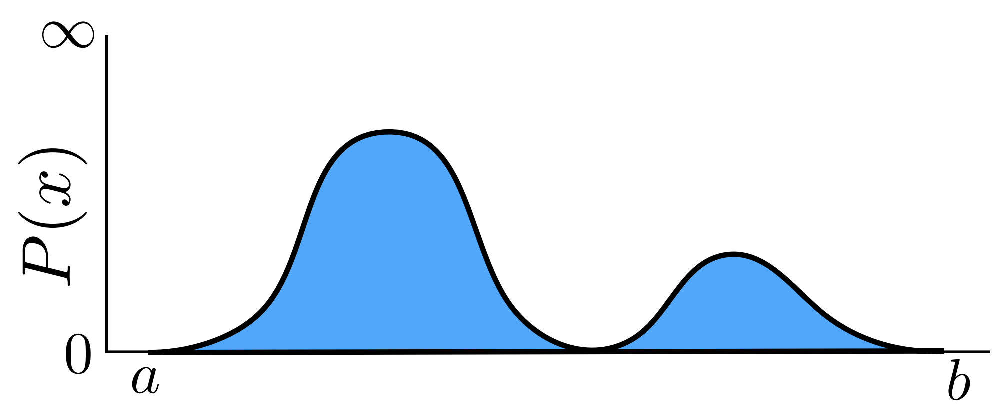
Discrete (More than 2)

$$X = \{X_1, X_2, \dots, X_N\}$$



Continuous

$$X = [a, b]$$



API of a general Bayes filter

Parameters of the Bayes filter:

Transition model: $P(x_t | x_{t-1}, u_t)$

Measurement model: $P(z_t | x_t)$

Input to the filter:

Old belief: $bel(x_{t-1})$

Action: u_t

Measurement: z_t

Output of the filter:

Updated belief: $bel(x_t)$

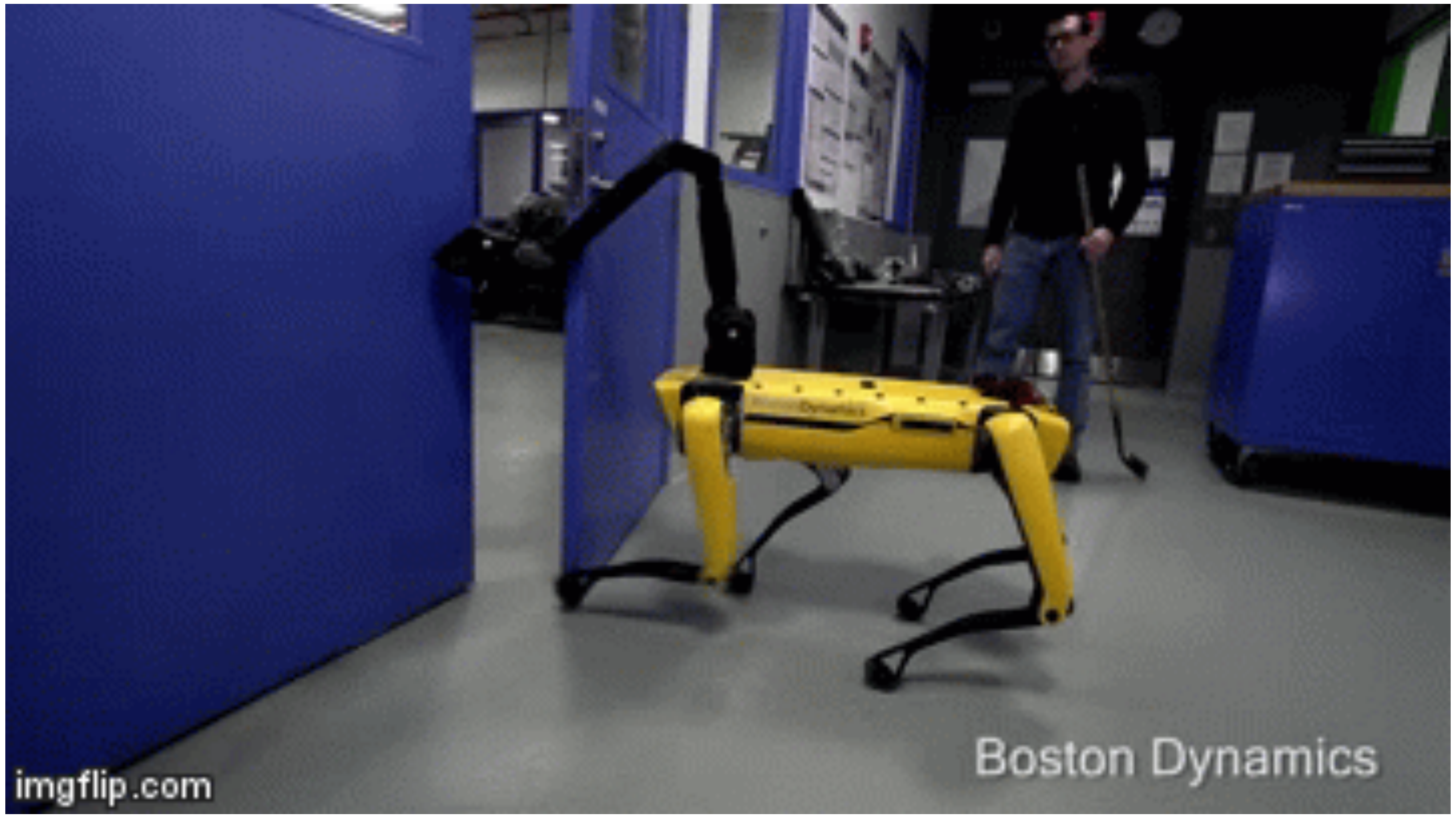
2 simple steps:

1. Predict belief after action

2. Correct belief after measurement

Discrete (Binary)

Example 1: Robot opening door



Example 1: Robot opening door

There are two states that we are tracking

X = { **O**pen, **C**losed}

Our robot can do two actions

A = { **P**ull, **L**eave}

We define a transition model (note: our robot is clumsy)

$$P(x_t | x_{t-1}, u_t)$$

$$P(\mathbf{O} \mid \mathbf{C}, \mathbf{P}) = 0.7 \qquad P(\mathbf{C} \mid \mathbf{C}, \mathbf{P}) = 0.3$$

.... and so on

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{\ \mathbf{O}\text{pen},\ \mathbf{C}\text{losed}\}$$

Our robot can do two actions

$$A = \{\ \color{blue}{\mathbf{P}\text{ull},\ \mathbf{L}\text{eave}}\color{black}\}$$

Rewrite the transition model as a matrix

$$\begin{bmatrix} P(x_t = \mathbf{O} | x_{t-1} = \mathbf{O}, \color{blue}{u_t}\color{black}) & P(x_t = \mathbf{O} | x_{t-1} = \mathbf{C}, \color{blue}{u_t}\color{black}) \\ P(x_t = \mathbf{C} | x_{t-1} = \mathbf{O}, \color{blue}{u_t}\color{black}) & P(x_t = \mathbf{C} | x_{t-1} = \mathbf{C}, \color{blue}{u_t}\color{black}) \end{bmatrix}$$

不管门

$$P(.|., \color{blue}{\mathbf{P}}\color{black}) = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \qquad P(.|., \color{blue}{\mathbf{L}}\color{black}) = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$$

close→close

Example 1: Robot opening door

There are two states that we are tracking

X = { **O**pen, **C**losed}

Our robot can do two actions

A = { **P**ull, **L**eave}

We have a door detector sensor. The sensor is kinda buggy!

Z = { **O**pen, **C**losed}

$$P(z_t|x_t)$$

.... let's use our matrix format

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

$$A = \{ \textbf{P}\text{ull}, \textbf{L}\text{eave}\}$$

$$Z = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

Rewrite the measurement model as a vector

$$\begin{bmatrix} P(z_t|\mathbf{O}) \\ P(z_t|\mathbf{C}) \end{bmatrix}$$

$$P(\mathbf{O}|.) = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix} \qquad P(\mathbf{C}|.) = \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix}$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{\, \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

$$A = \{\, \textbf{P}\text{ull}, \textbf{L}\text{eave}\}$$

$$Z = \{\, \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

Let's get ready to Bayes filter!

Example 1: Robot opening door

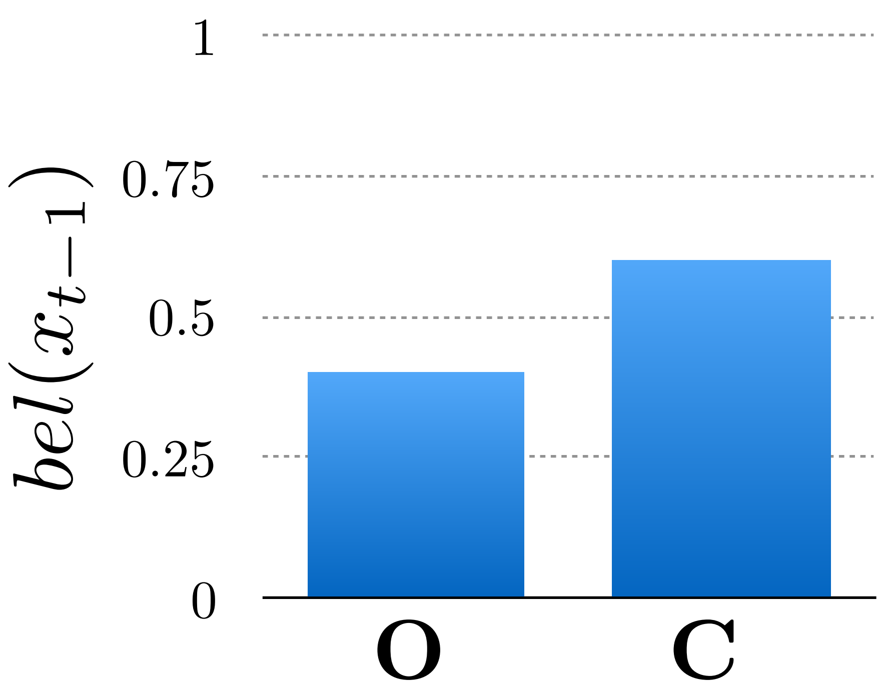
There are two states that we are tracking

$$X = \{\, \textbf{O}\text{pen},\ \textbf{C}\text{losed}\}$$

$$A = \{\, \textbf{P}\text{ull},\ \textbf{L}\text{eave}\}$$

$$Z = \{\, \textbf{O}\text{pen},\ \textbf{C}\text{losed}\}$$

Step 0. Start with the belief at time step t-1



$$bel(x_{t-1}) = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

Robot thinks the door is open with 0.4 probability

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

$$A = \{ \textbf{P}\text{ull}, \textbf{L}\text{eave}\}$$

$$Z = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

Robot executes action **Pull**

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

$$A = \{ \textbf{P}\text{ull}, \textbf{L}\text{eave}\}$$

$$Z = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \sum_{x_{t-1}} P(x_t | x_{t-1}, u_t) bel(x_{t-1})$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed} \}$$

$$A = \{ \textbf{P}\text{ull}, \textbf{L}\text{eave} \}$$

$$Z = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed} \}$$

Step 1: Prediction - push belief through dynamics given action

$$\underbrace{\begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix}}_{\overline{bel}(x_t)} = \begin{bmatrix} P(x_t = \mathbf{O} | x_{t-1} = \mathbf{O}, u_t) & P(x_t = \mathbf{O} | x_{t-1} = \mathbf{C}, u_t) \\ P(x_t = \mathbf{C} | x_{t-1} = \mathbf{O}, u_t) & P(x_t = \mathbf{C} | x_{t-1} = \mathbf{C}, u_t) \end{bmatrix} \underbrace{\begin{bmatrix} P(x_{t-1} = \mathbf{O}) \\ P(x_{t-1} = \mathbf{C}) \end{bmatrix}}_{bel(x_{t-1})}$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

$$A = \{ \textbf{P}\text{ull}, \textbf{L}\text{eave}\}$$

$$Z = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

Step 1: Prediction - push belief through dynamics given action

$$\underbrace{\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}}_{\overline{bel}(x_t)} = \underbrace{\begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix}}_{P(.|., \mathbf{P})} \underbrace{\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}}_{bel(x_{t-1})}$$

Robot thinks the door is open with 0.74 probability

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{\ \textbf{O}\text{pen},\ \textbf{C}\text{losed}\}$$

$$A = \{\ \textbf{P}\text{ull},\ \textbf{L}\text{eave}\}$$

$$Z = \{\ \textbf{O}\text{pen},\ \textbf{C}\text{losed}\}$$

Robot receives measurement

Closed

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{\ \textbf{Open},\ \textbf{Closed}\}$$

$$A = \{\ \textbf{Pull},\ \textbf{Leave}\}$$

$$Z = \{\ \textbf{Open},\ \textbf{Closed}\}$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t | x_t)\overline{bel}(x_t)$$

(normalize)

Example 1: Robot opening door

There are two states that we are tracking

X = { **O**pen, **C**losed}

A = { **P**ull, **L**eave}

Z = { **O**pen, **C**losed}

Step 2: Correction - apply Bayes rule given measurement

$$\begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix} = \eta \begin{bmatrix} P(z_t | \mathbf{O}) \\ P(z_t | \mathbf{C}) \end{bmatrix} * \begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix}$$

$bel(x_t)$ $P(\mathbf{C}|.)$ $\overline{bel}(x_t)$

element wise

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{\ \mathbf{O}\text{pen},\ \mathbf{C}\text{losed}\}$$

$$A = \{\ \mathbf{P}\text{ull},\ \mathbf{L}\text{eave}\}$$

$$Z = \{\ \mathbf{O}\text{pen},\ \mathbf{C}\text{losed}\}$$

Step 2: Correction - apply Bayes rule given measurement

$$\underbrace{\begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix}}_{bel(x_t)} = \eta \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \underbrace{\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}}_{\overline{bel}(x_t)}$$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

$$A = \{ \textbf{P}\text{ull}, \textbf{L}\text{eave}\}$$

$$Z = \{ \textbf{O}\text{pen}, \textbf{C}\text{losed}\}$$

Step 2: Correction - apply Bayes rule given measurement

$$\begin{bmatrix} P(x_t = \mathbf{O}) \\ P(x_t = \mathbf{C}) \end{bmatrix} = \eta \begin{bmatrix} 0.4 \\ 0.8 \end{bmatrix} \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \eta \begin{bmatrix} 0.296 \\ 0.208 \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$$

$bel(x_t)$ $\overline{bel}(x_t)$

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{\ \textbf{O}\text{pen},\ \textbf{C}\text{losed}\}$$

$$A = \{\ \textbf{P}\text{ull},\ \textbf{L}\text{eave}\}$$

$$Z = \{\ \textbf{O}\text{pen},\ \textbf{C}\text{losed}\}$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \begin{bmatrix} 0.58 \\ 0.42 \end{bmatrix}$$

Robot thinks the door is open with 0.58 probability

Example 1: Robot opening door

There are two states that we are tracking

$$X = \{\ \mathbf{O}\text{pen},\ \mathbf{C}\text{losed}\}$$

$$A = \{\ \mathbf{P}\text{ull},\ \mathbf{L}\text{eave}\}$$

$$Z = \{\ \mathbf{O}\text{pen},\ \mathbf{C}\text{losed}\}$$

Let's summarize

Robot thought the door is open with 0.4 probability

Robot executed Pull action.
Robot thinks the door is open with 0.74 probability

Robot got Closed measurement.
Robot thinks the door is open with 0.58 probability

Continuous (Non-parametric)

Bayes filter in a nutshell

Step 0. Start with the belief at time step t-1

$$bel(x_{t-1})$$

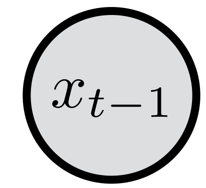
Step 1: Prediction - push belief through dynamics given action

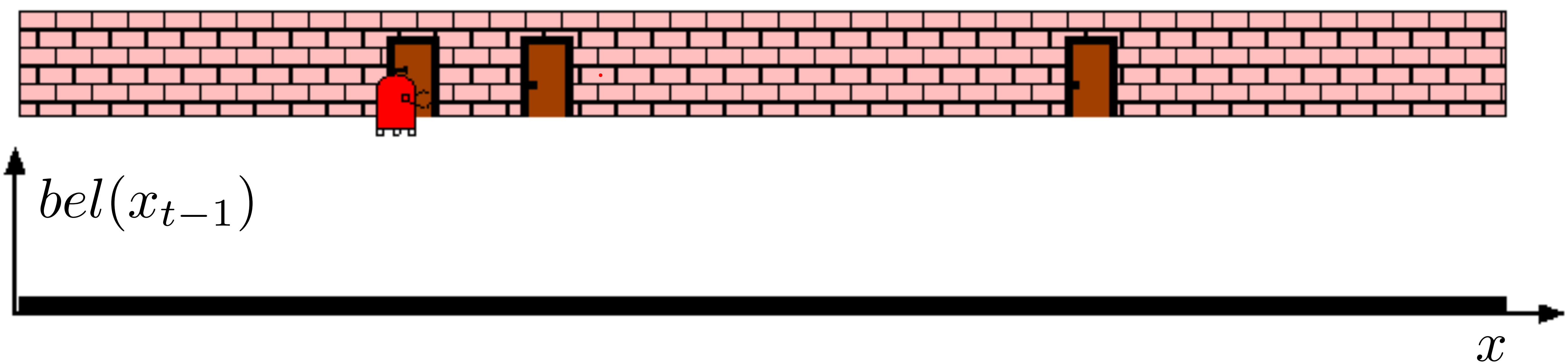
$$\overline{bel}(x_t) = \int P(x_t|u_t, x_{t-1})bel(x_{t-1})dx_{t-1}$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$

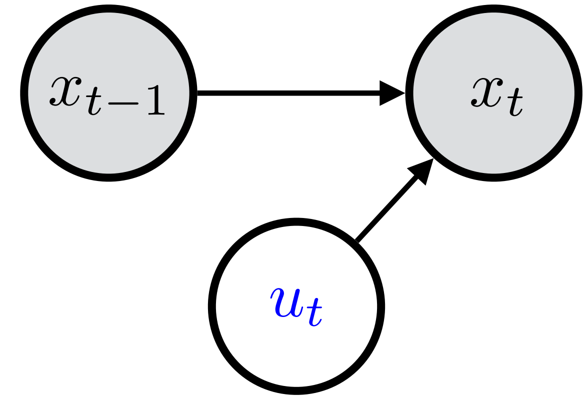
Robot lost in a 1-D hallway



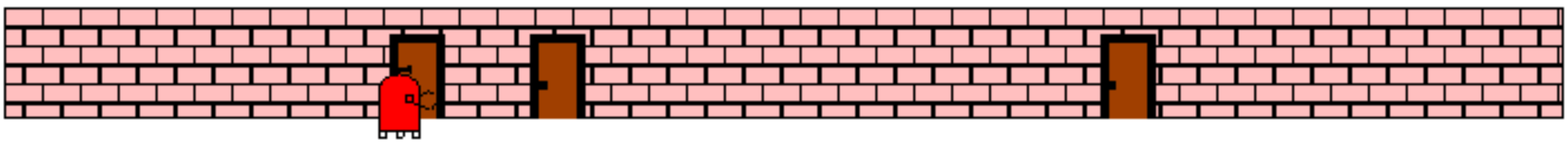


Action at time t: NOP

$$\textcolor{blue}{u_t} = \text{NOP}$$



$$P(x_t|\textcolor{blue}{u_t}, x_{t-1}) = \begin{cases} 1 & x_t = x_{t-1} \\ 0 & \end{cases}$$



$$\overline{bel}(x_t) = \int P(x_t|\textcolor{blue}{u_t}, x_{t-1}) bel(x_{t-1}) dx_{t-1} = bel(x_{t-1})$$

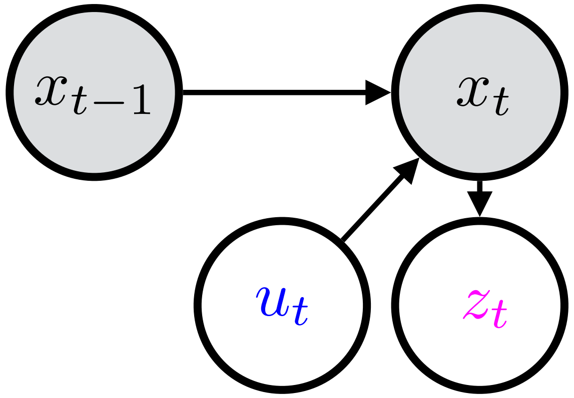
x

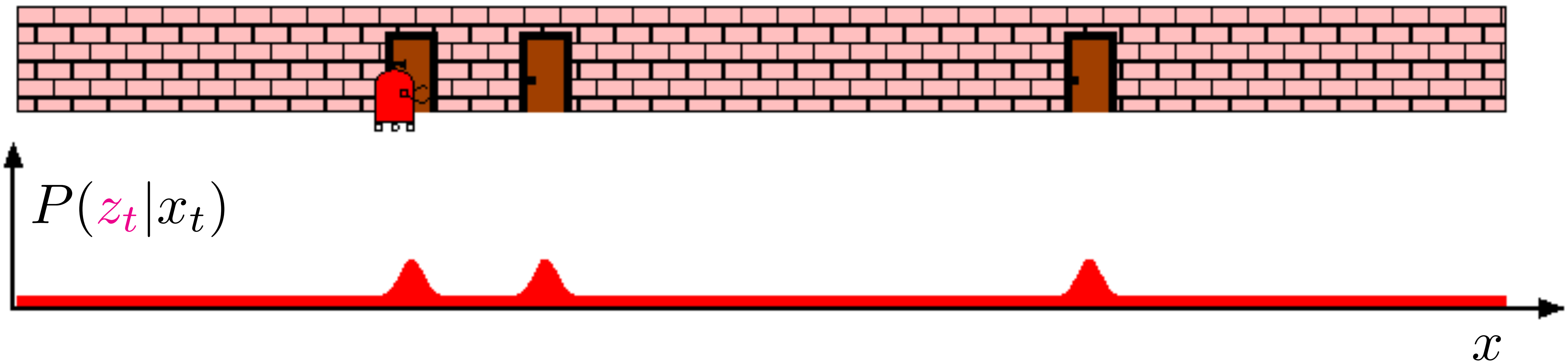
NOP action implies belief remains the same!

Measurement at time t: “Door”

$z_t = \text{Door}$

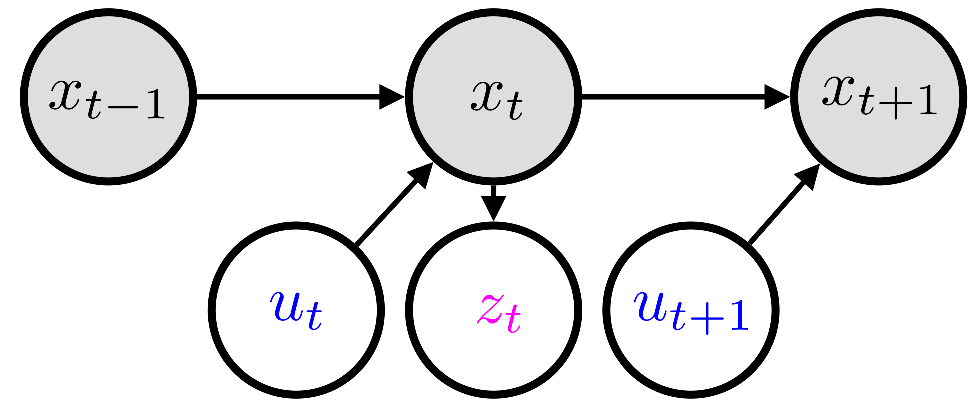
$P(z_t|x_t) = \mathcal{N}(\text{door centre}, 0.75m)$



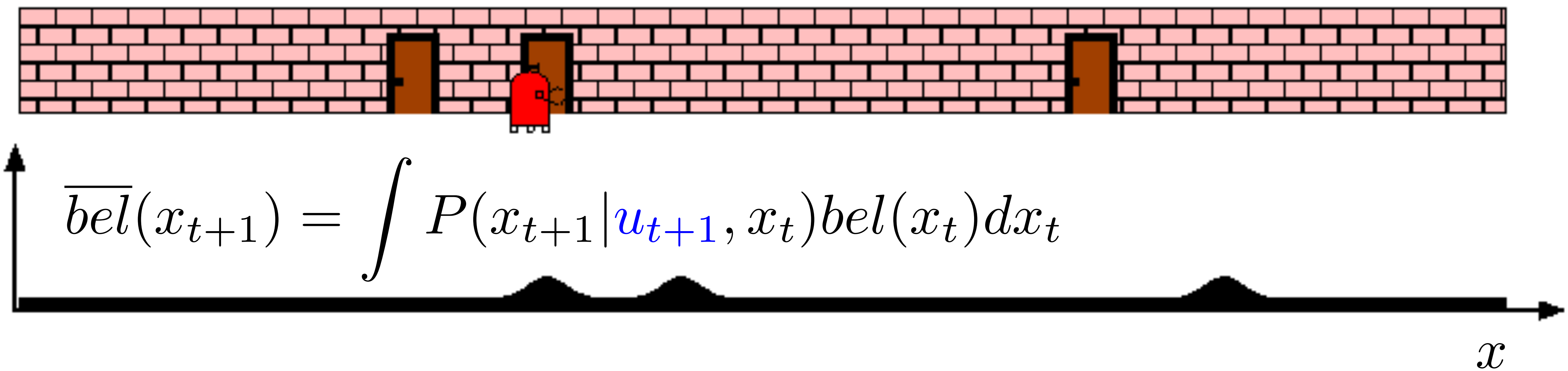


Action at time t+1: Move 3m right

$u_{t+1} = \text{3m right}$



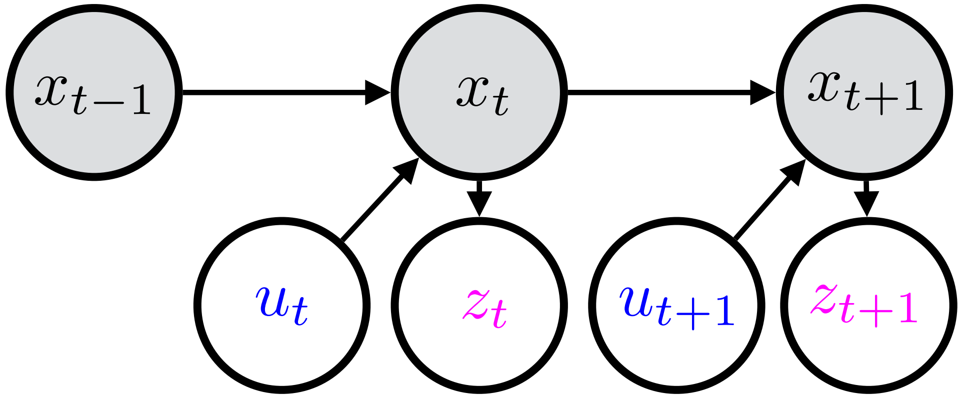
$$P(x_{t+1} | u_{t+1}, x_t) = \mathcal{N}(x_t + u_{t+1}, 0.25m)$$



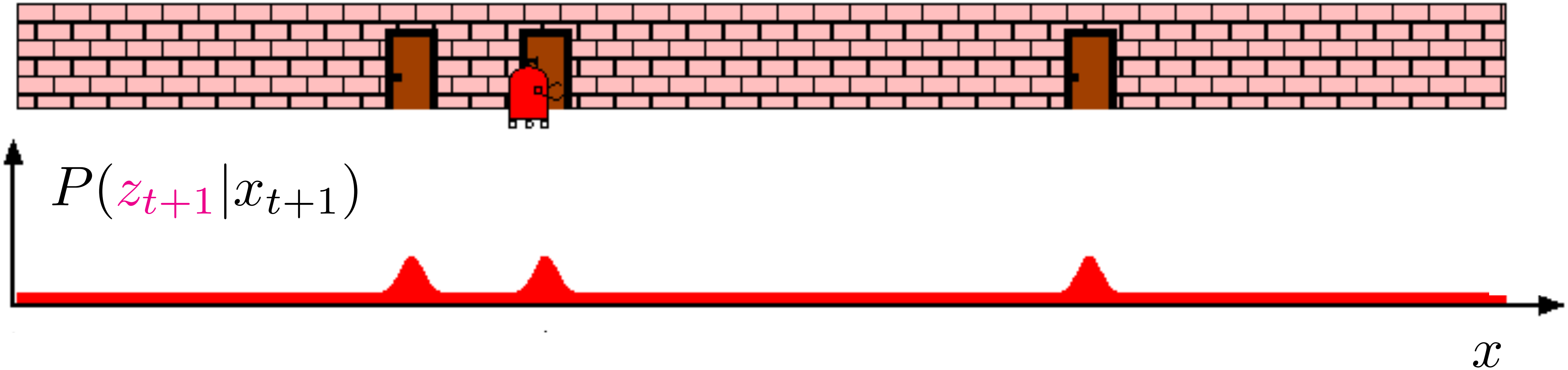
$$\overline{bel}(x_{t+1}) = \int P(x_{t+1} | u_{t+1}, x_t) bel(x_t) dx_t$$

Measurement at time t+1: “Door”

$z_{t+1} = \text{Door}$

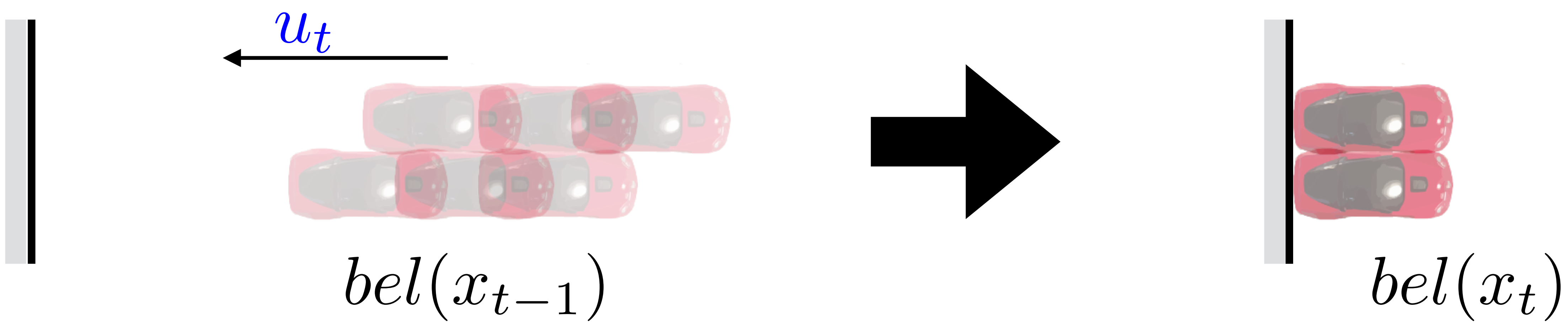


$P(z_{t+1}|x_{t+1}) = \mathcal{N}(\text{door centre}, 0.75m)$



Questions

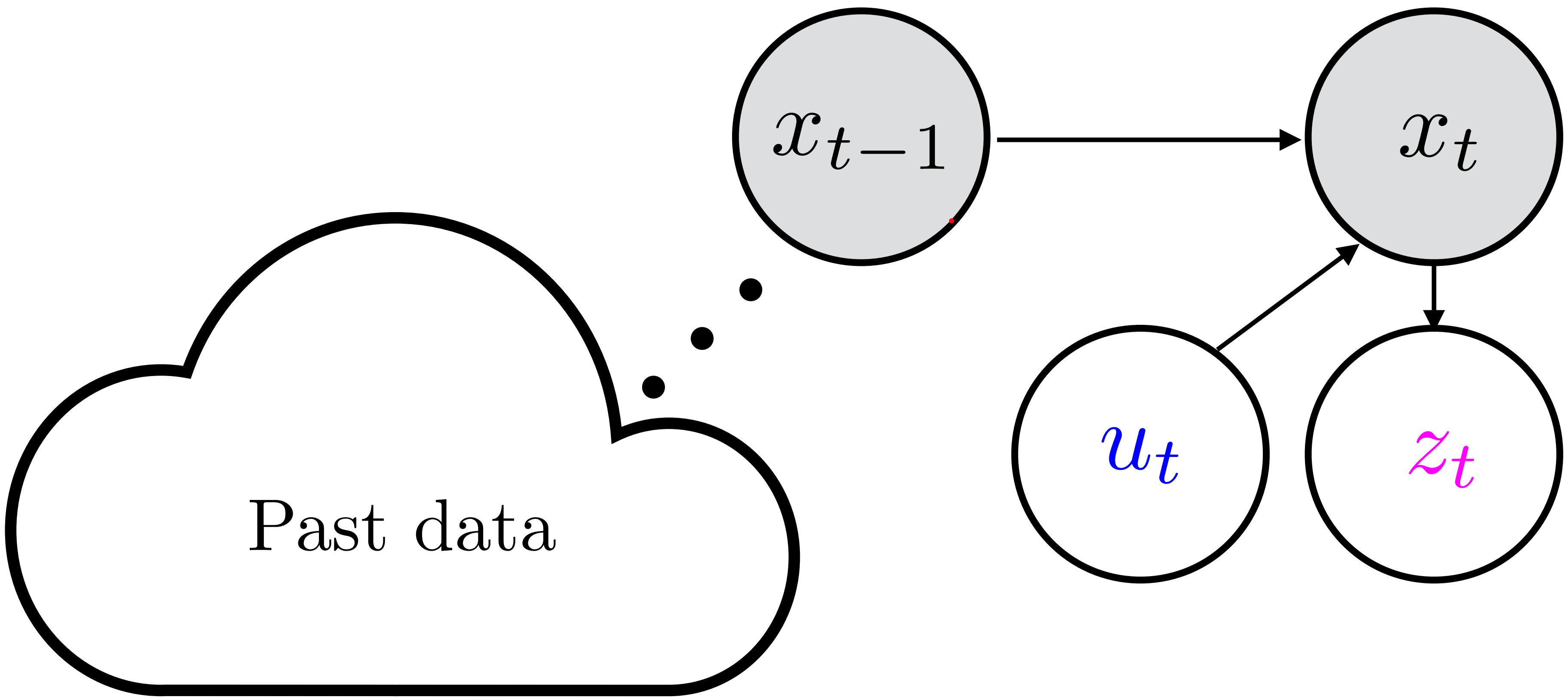
Do actions always increase uncertainty?



Do measurements always reduce uncertainty?

(What happens when you reach into your bag and don't find your keys? Example of a negative measurement)

Bayes derivation



Bayes derivation

$$bel(x_t) = P(x_t | z_{1:t-1}, u_{1:t-1}, z_t, u_t)$$

(Bayes) $$= \eta P(z_t | x_t, z_{1:t-1}, u_{1:t-1}, u_t) \; P(x_t | z_{1:t-1}, u_{1:t-1}, u_t)$$

(Markov) $$= \eta P(z_t | x_t) \; P(x_t | z_{1:t-1}, u_{1:t-1}, u_t)$$

$$= \eta P(z_t | x_t) \; \overline{bel}(x_t)$$

Bayes derivation

$$\overline{bel}(x_t) = P(x_t | z_{1:t-1}, u_{1:t-1}, u_t)$$

(Total prob.) $$= \int P(x_t | x_{t-1}, z_{1:t-1}, u_{1:t-1}, u_t) \; P(x_{t-1} | z_{1:t-1}, u_{1:t-1}, u_t) dx_{t-1}$$

(Markov) $$= \int P(x_t | x_{t-1}, u_t) \; P(x_{t-1} | z_{1:t-1}, u_{1:t-1}, u_t) dx_{t-1}$$

(Cond. indep) $$= \int P(x_t | x_{t-1}, u_t) \; P(x_{t-1} | z_{1:t-1}, u_{1:t-1}) dx_{t-1}$$

$$= \int P(x_t | x_{t-1}, u_t) \; bel(x_{t-1}) dx_{t-1}$$

After thoughts ...

Question: When is cond. independence not true?

$$= \int P(x_t | x_{t-1}, u_t) \; P(x_{t-1} | z_{1:t-1}, u_{1:t-1}, u_t) dx_{t-1}$$

(Cond. indep) $$= \int P(x_t | x_{t-1}, u_t) \; P(x_{t-1} | z_{1:t-1}, u_{1:t-1}) dx_{t-1}$$

i.e. when can you tell something about the past based on future data?

E.g. Motion capture data of a human.
Human knows the true state and generate control actions accordingly.

Bayes filter in a single line

$P(x_t|x_{t-1}, u_t)$

Motion model

$P(z_t|x_t)$

Measurement model

$$bel(x_t) = \eta P(z_t|x_t) \int P(x_t|x_{t-1}, u_t)\, bel(x_{t-1}) dx_{t-1}$$

Note that order does not really matter -
we can flip measurement and control.

Asynchronous streaming version of Bayes

Input : Datapoint d, Current belief $bel(x)$

Output : Updated belief $bel^+(x)$

Process :

If d is measurement z **then**

for all x

$$bel^+(x) = P(z|x)bel(x)$$

$$bel^+(x) = \texttt{Normalize}(bel^+(x))$$

Else if d is control u **then**

for all x

$$bel^+(x) = \sum_{x_{old}} P(x|x_{old}, u)bel(x_{old})$$

Return $bel^+(x)$

Things to keep in mind...

1. Bayes filter can be overconfident

Once belief collapses to 0/1 only motion model can shake it loose

2. Too many measurements will collapse belief

3. Correlated incorrect measurements are dangerous

Bayes filter is a powerful tool

Localization Mapping SLAM POMDP