Linear Quadratic Regulator

Sanjiban Choudhury

TAs: Matthew Rockett, Gilwoo Lee, Matt Schmittle

Different control laws

1. Bang-bang control
2. PID control
3. Pure-pursuit control
4. Lyapunov control
5. LQR
6. MPC

Recap of controllers

PID / Pure pursuit: Worked well, no provable guarantees

Lyapunov: Provable stability, convergence rate depends on gains

Table of controllers

	Control Law	Uses model	Stability Guarantee	Minimize Cost
PID	$u = K_p e + \ldots$	No	No	No
Pure Pursuit	$u = \tan^{-1}\left(\frac{2B\sin\alpha}{L}\right)$	Circular arcs	Yes - with assumptions	No
Lyapunov	$u = \tan^{-1}\left(-\frac{k_1 e_{ct} B}{\theta_e}\sin\theta_e - \frac{B}{V}k_2\theta_e\right)$	Non-linear	Yes	No

Is stability enough?

$$\lim_{t \to \infty} e(t) = 0$$

Is stability enough of a guarantee?

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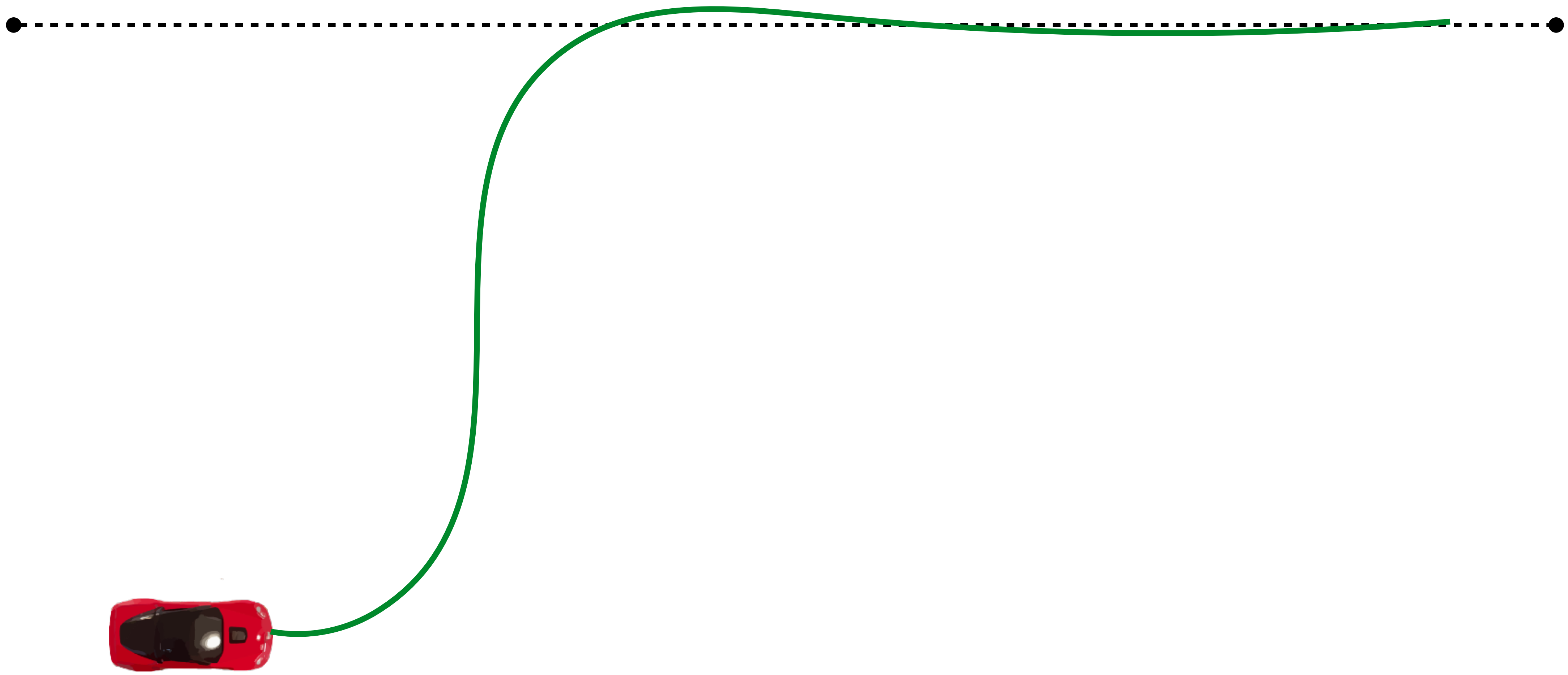

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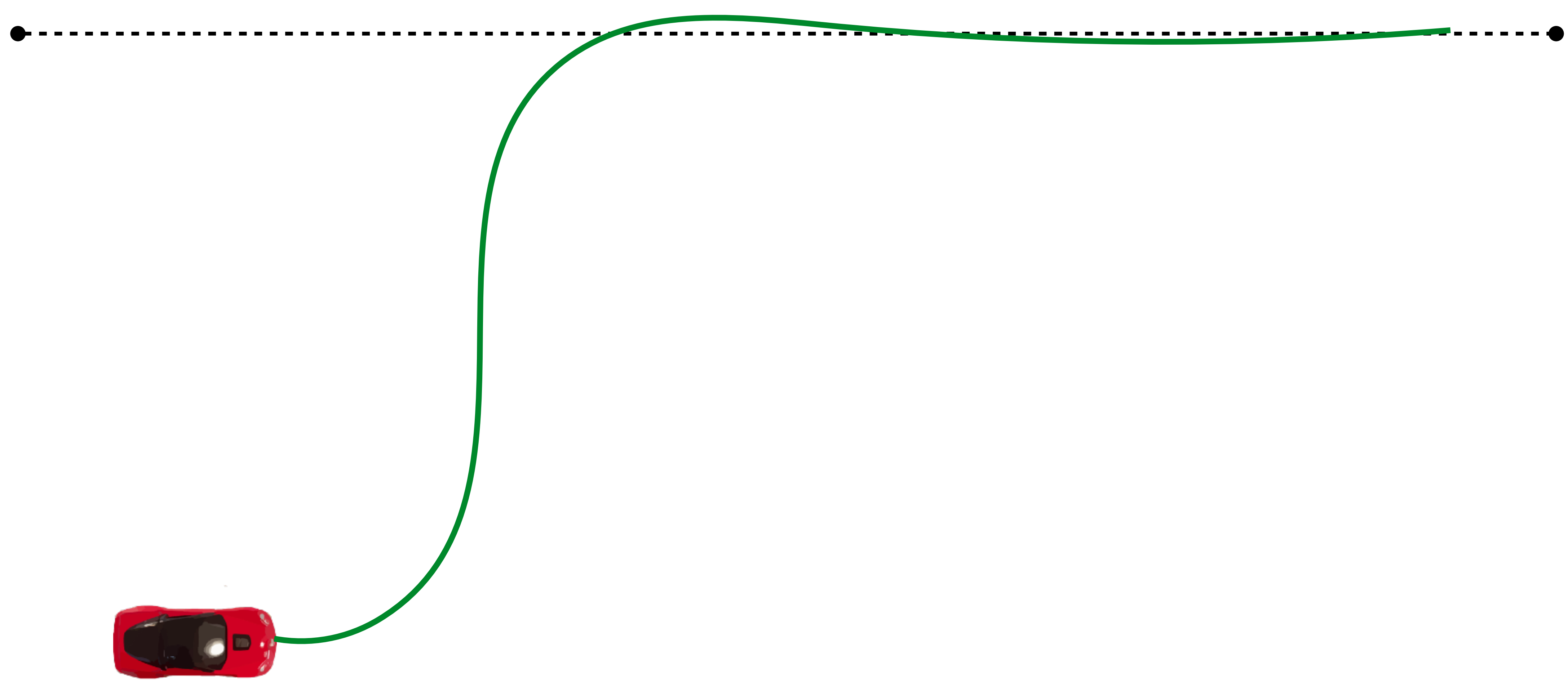
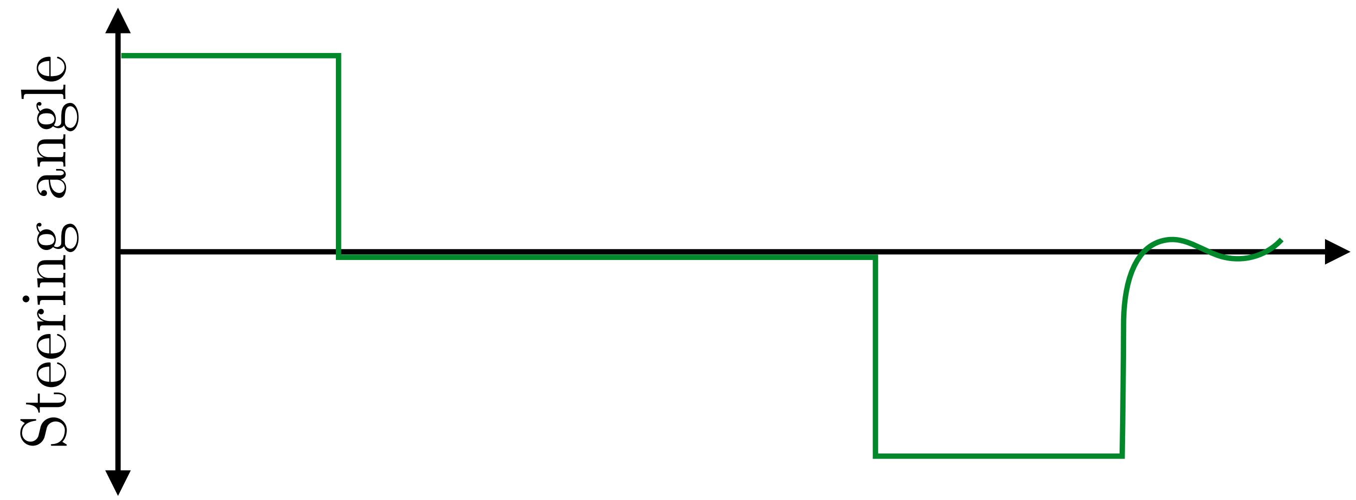


Control action changes abruptly - why is this bad?



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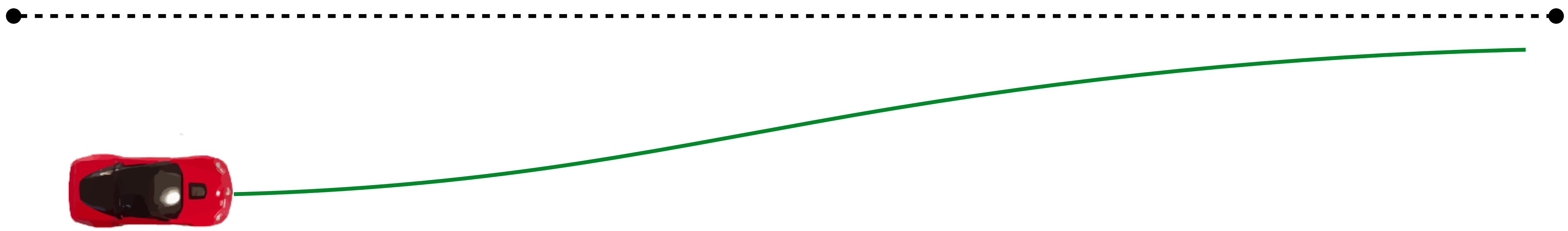

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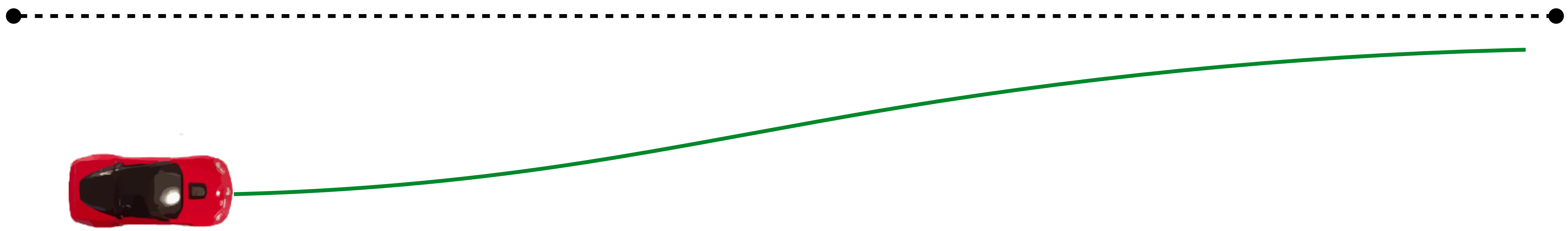


What if we just choose really small gains?

Stability guarantees that the error will go to zero ...
but can take arbitrary long time

Question:

How do we trade-off both driving error to zero

AND

keeping control action small?

Key Idea:

Turn the problem into an optimization

$$\min_{u(t)} \int_0^\infty \left(w_1 e(t)^2 + w_2 u(t)^2 dt \right)$$

Optimal Control

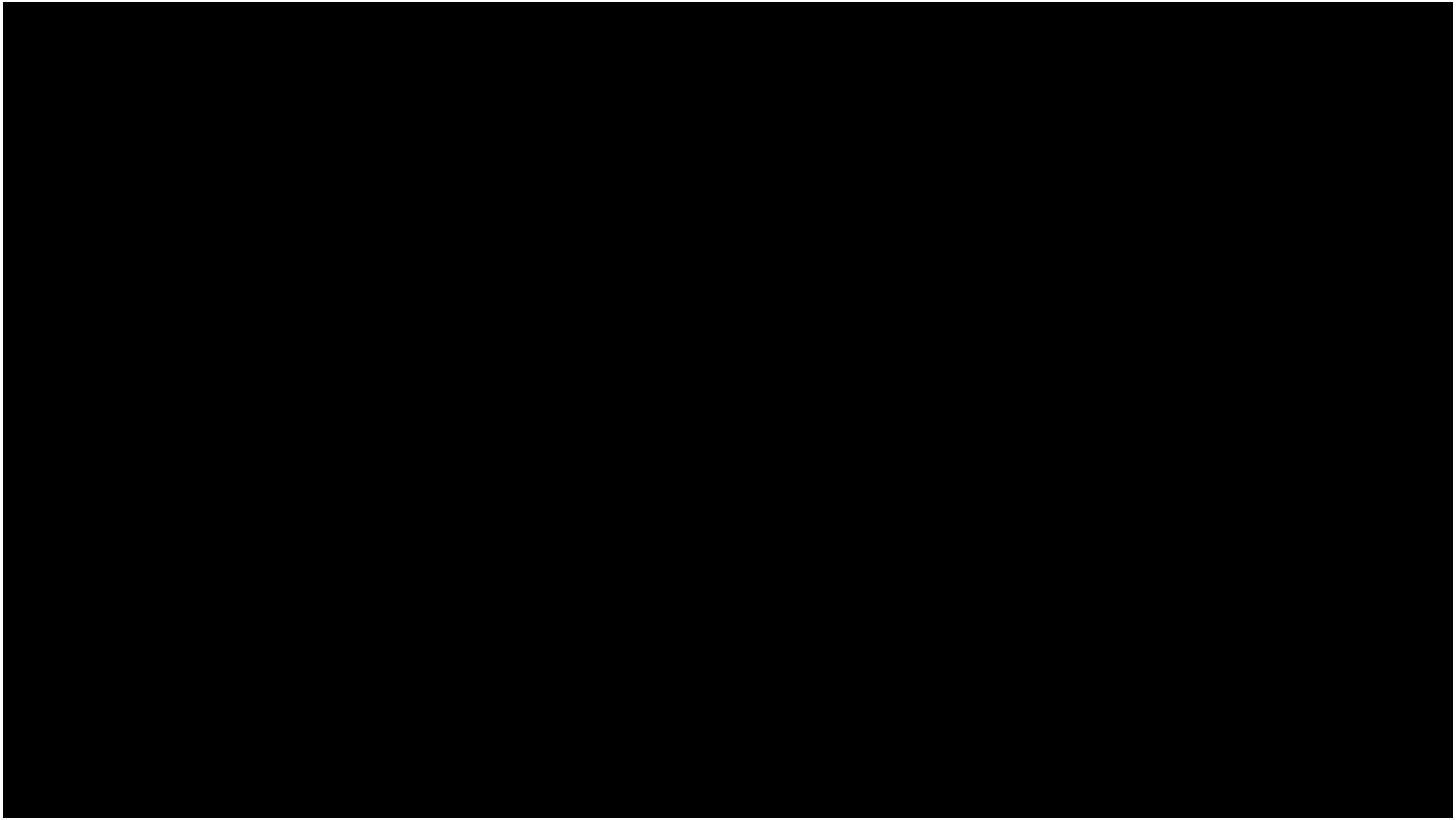
A fundamental framework:

Linear Quadratic Regulator

Trivia! :) (from http://www.uta.edu/utari/acs/history.htm)

In 1960 three major papers were published by R. Kalman and coworkers...
1. One of these [Kalman and Bertram 1960], publicized the vital work of Lyapunov in the time-domain control of nonlinear systems.
2. The next [Kalman 1960a] discussed the optimal control of systems, providing the design equations for the linear quadratic regulator (LQR).
3. The third paper [Kalman 1960b] discussed optimal filtering and estimation theory, providing the design equations for the discrete Kalman filter.

LQR flying RC helicopters



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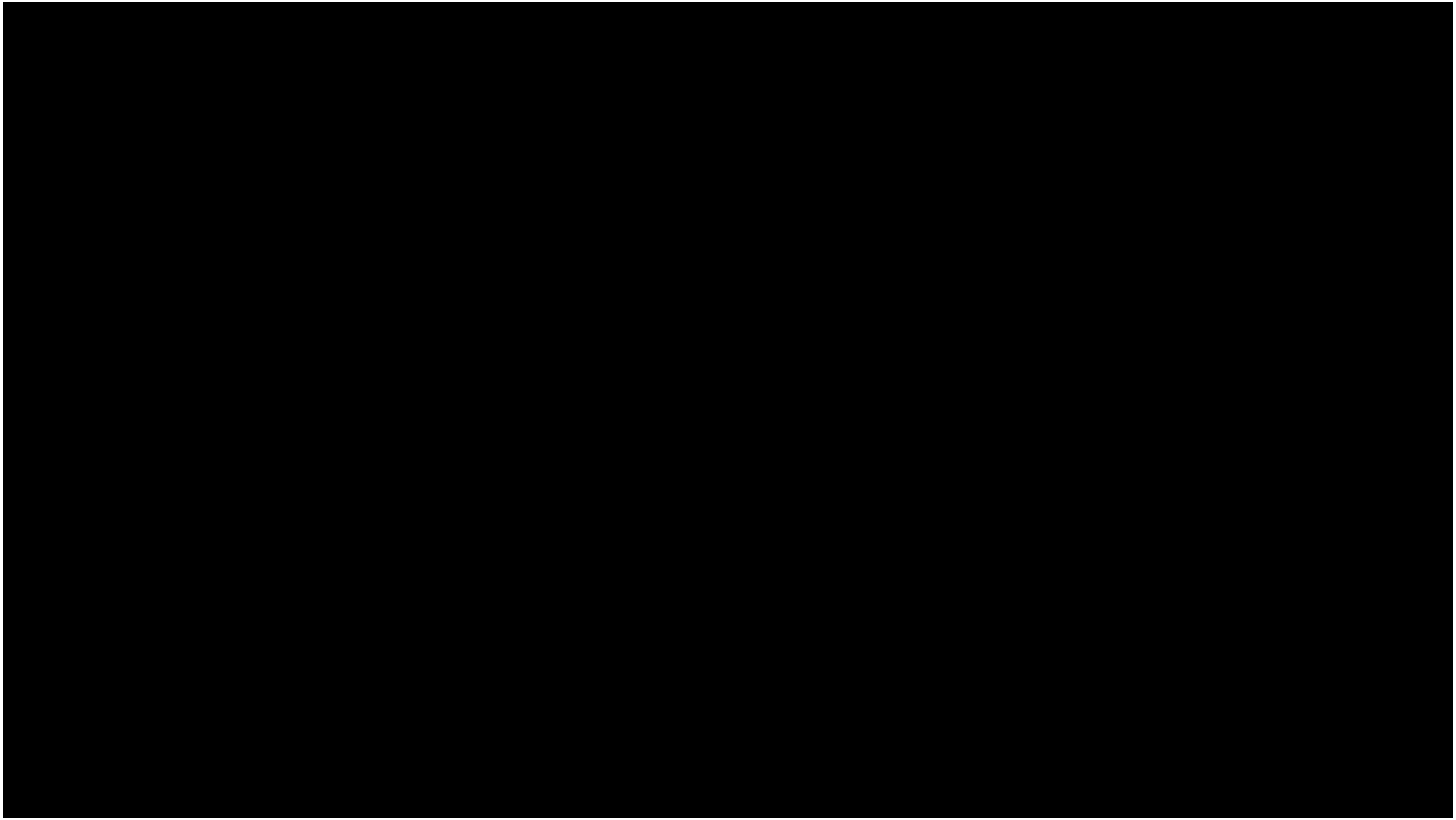

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A fundamental framework:

Linear Quadratic Regulator

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$$\begin{aligned} c(x_t, u_t) &= (x_t - x_{ref})^T Q (x_t - x_{ref}) + u_t^T R u_t \\ &= x_t^T Q x_t + u_t^T R u_t \end{aligned}$$

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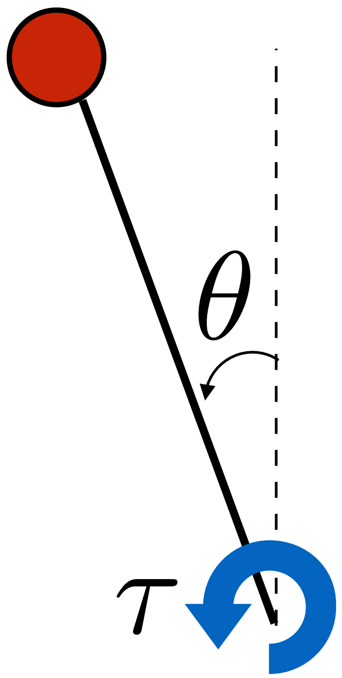
Goal: Compute control actions to minimize cumulative cost (value)

$$J = \sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

Example: Inverted Pendulum



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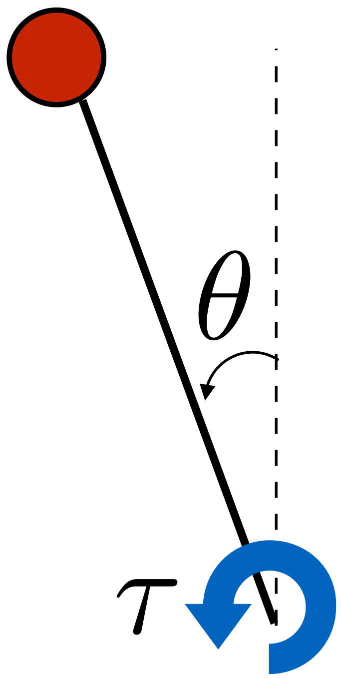

Equations of motion

$$ml^2\ddot{\theta} - mgl\sin\theta = \tau$$

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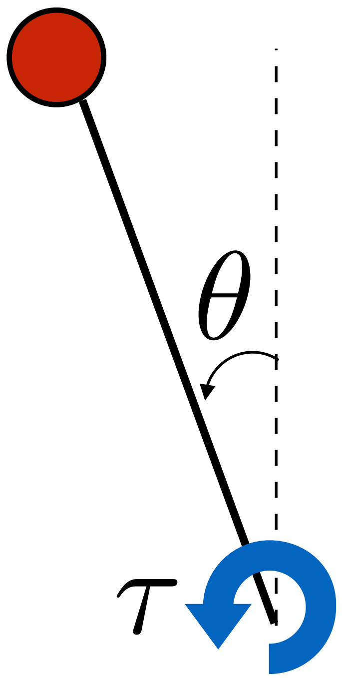

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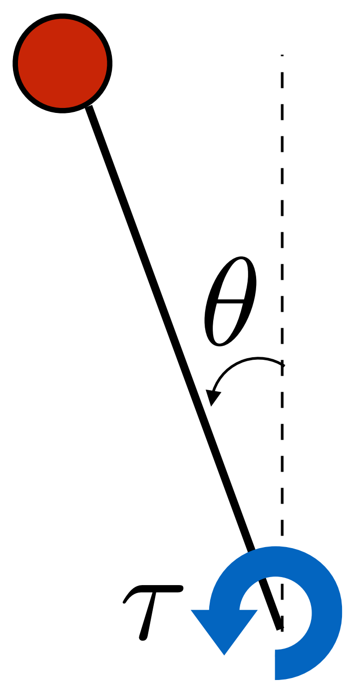

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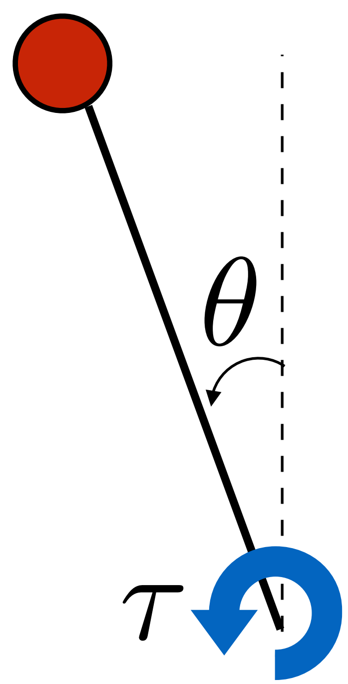

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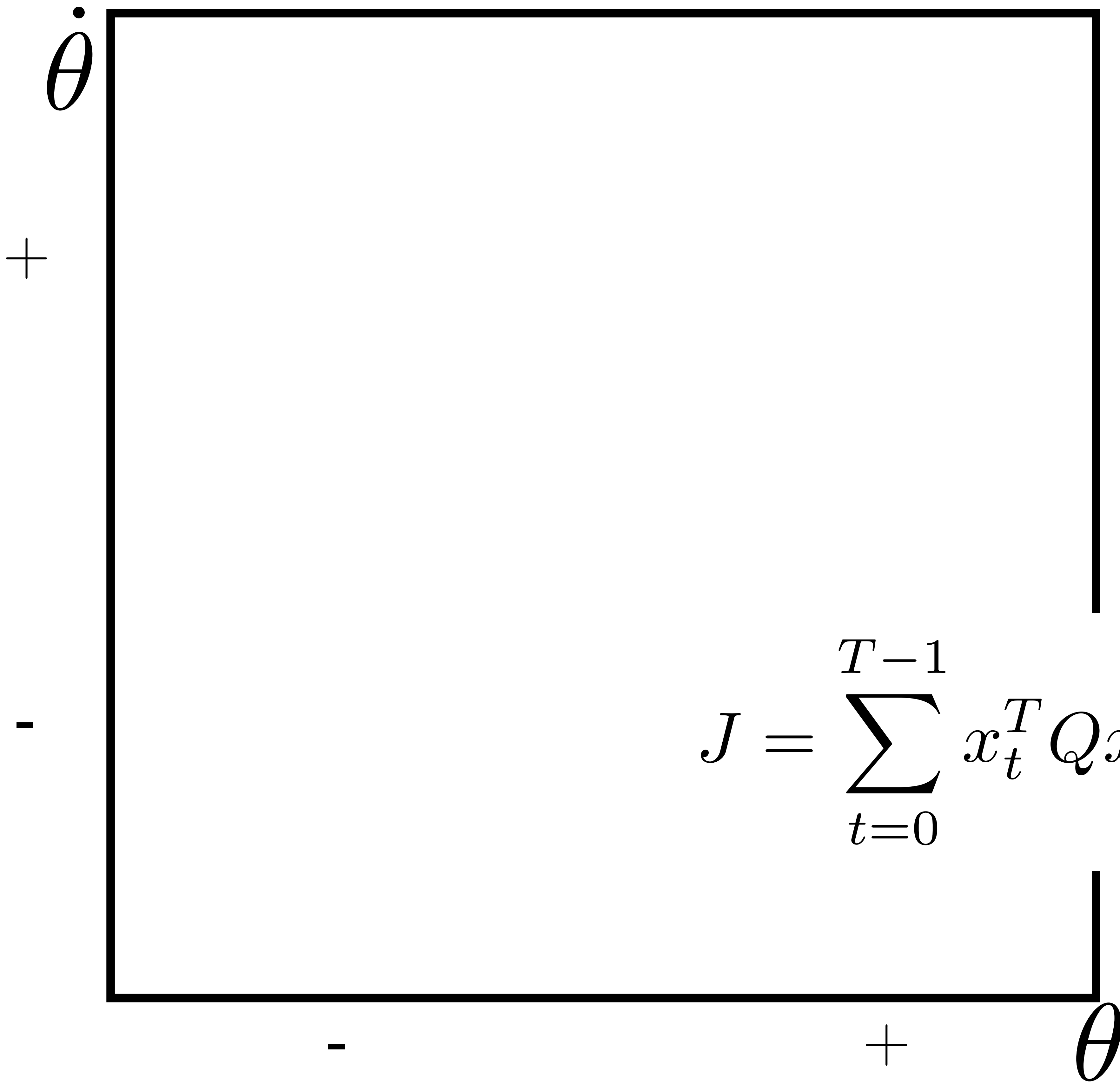
$$\approx \frac{g}{l}\theta + \frac{1}{ml^2}\tau$$

$$\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}_{t+1} = \underbrace{\begin{bmatrix}1 + \frac{1}{2}\frac{g}{l}\Delta t^2 & \Delta t \\ \frac{g}{l}\Delta t & 1\end{bmatrix}}_{A}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}_{t} + \underbrace{\begin{bmatrix}\frac{1}{2}\Delta t^2 \\ \Delta t\end{bmatrix}}_{B}\frac{\tau}{ml^2}$$

Get to (0,0) while minimizing cost



$$J = \sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t$$

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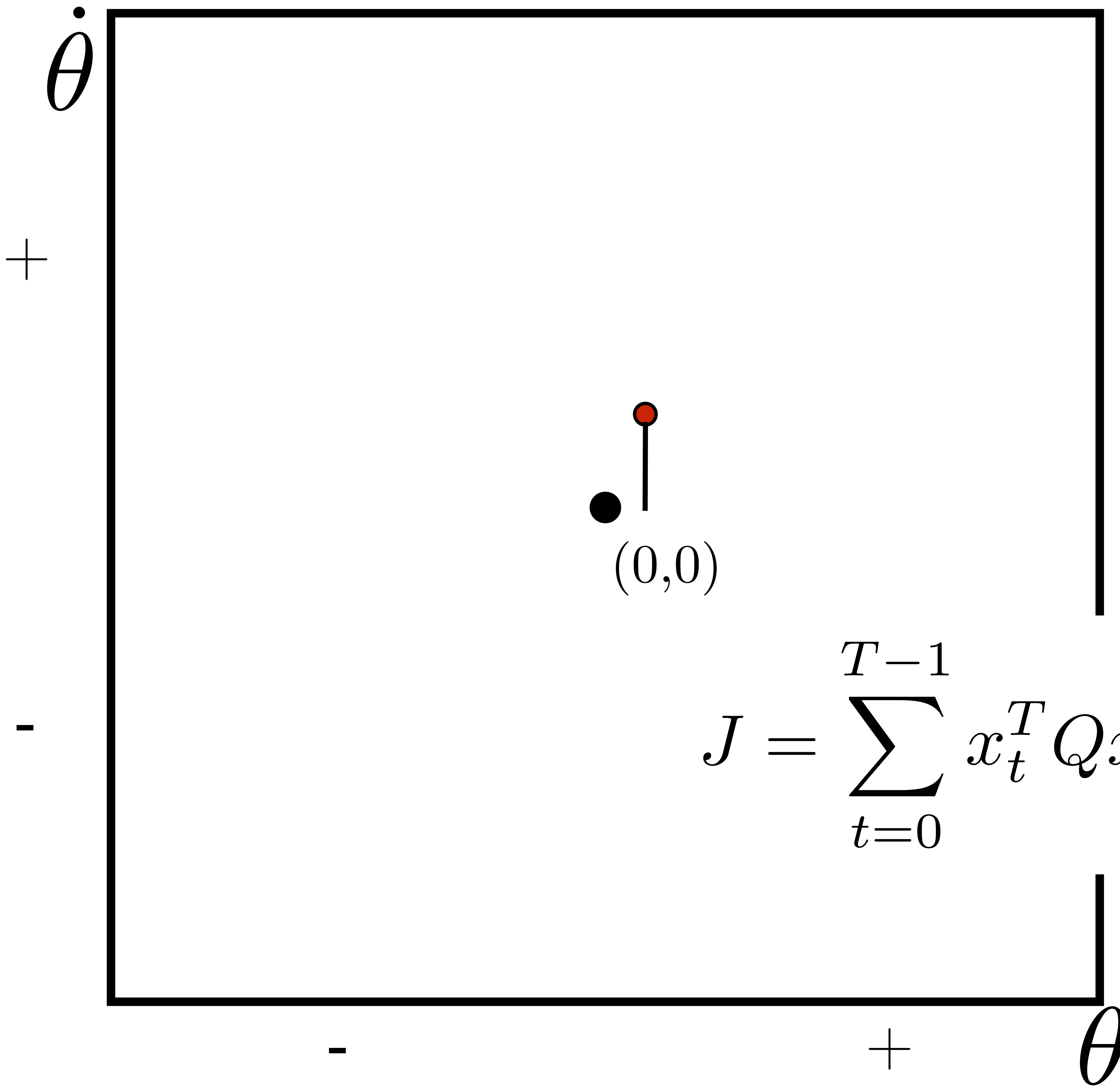

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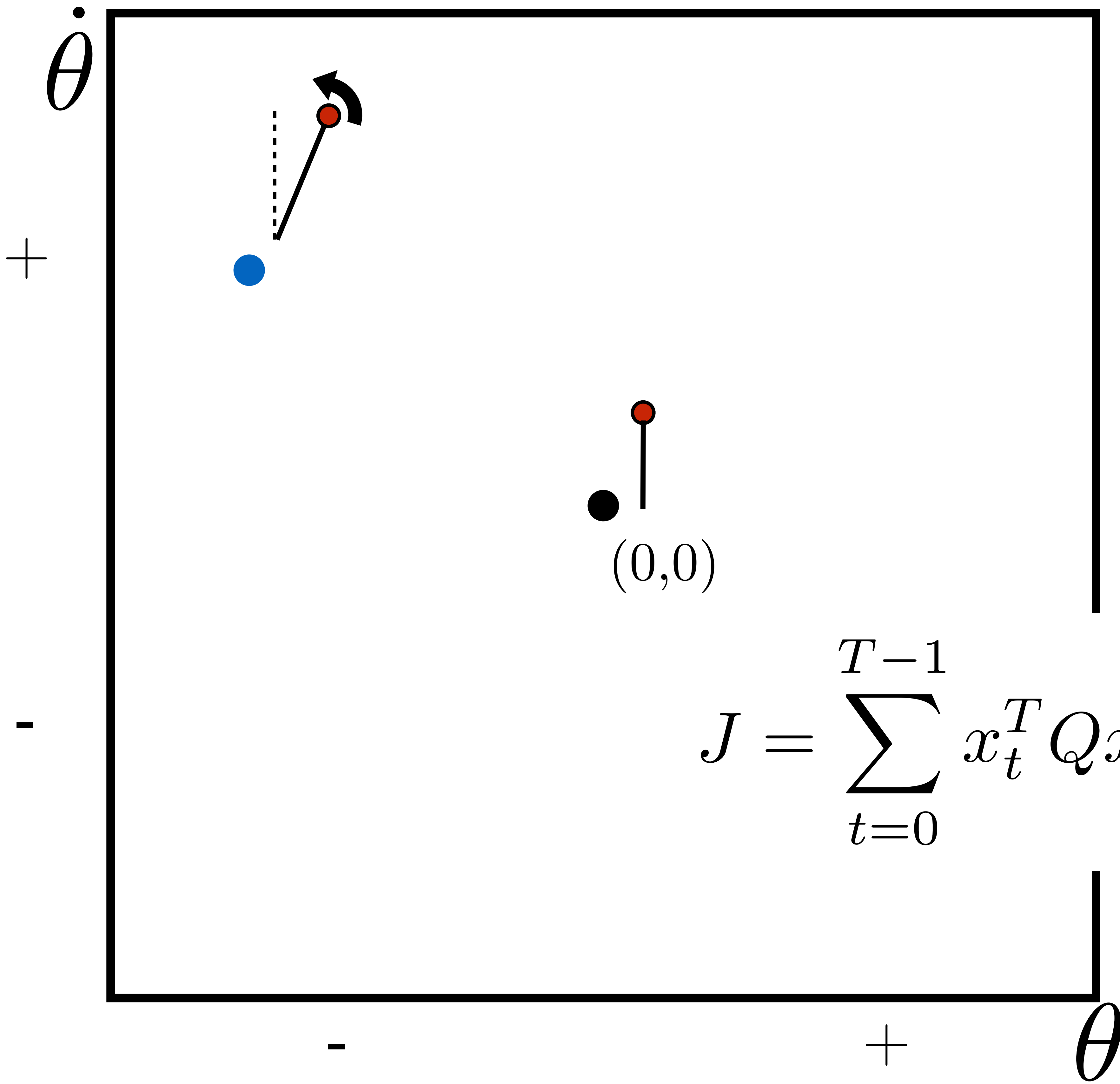

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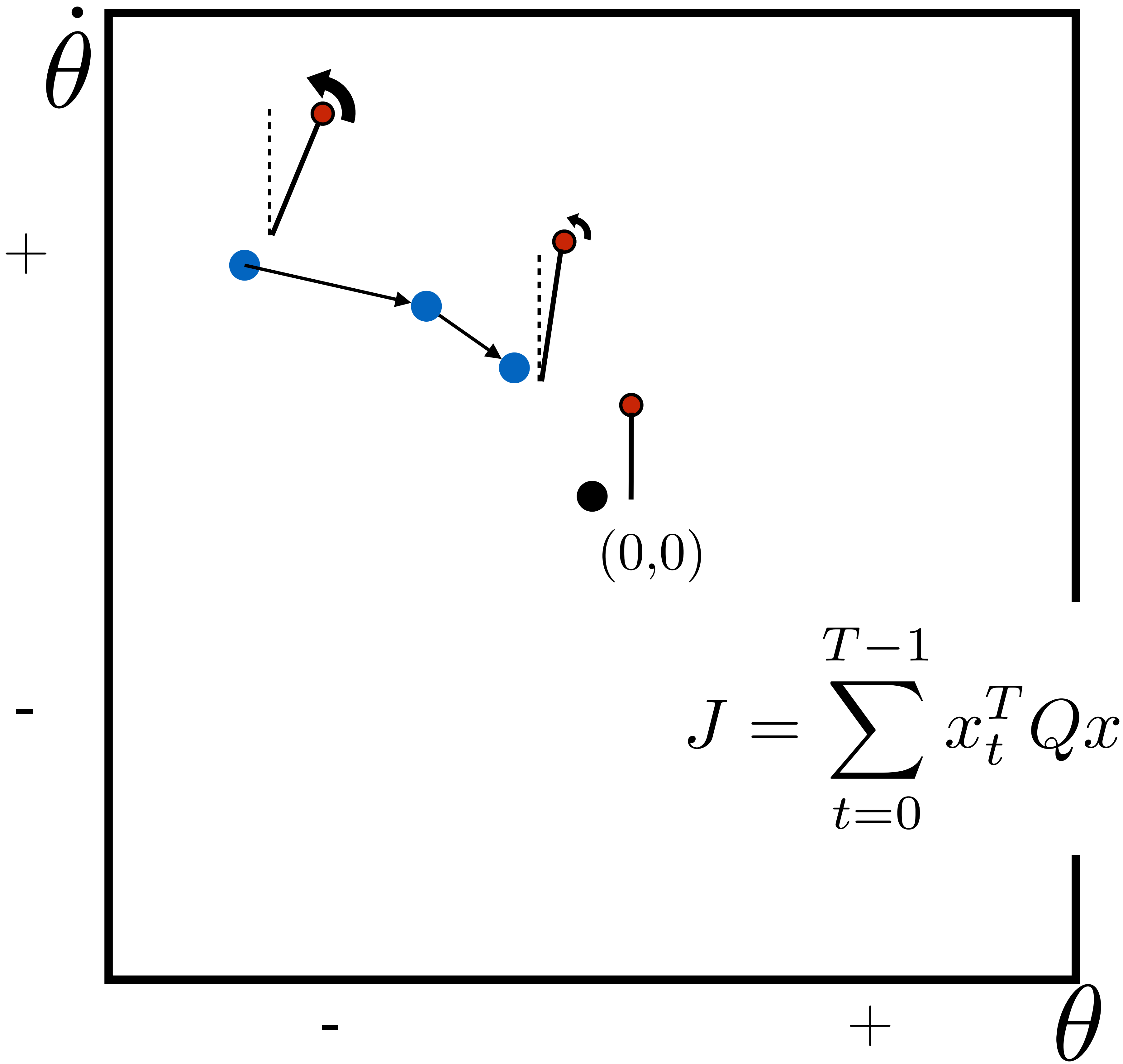

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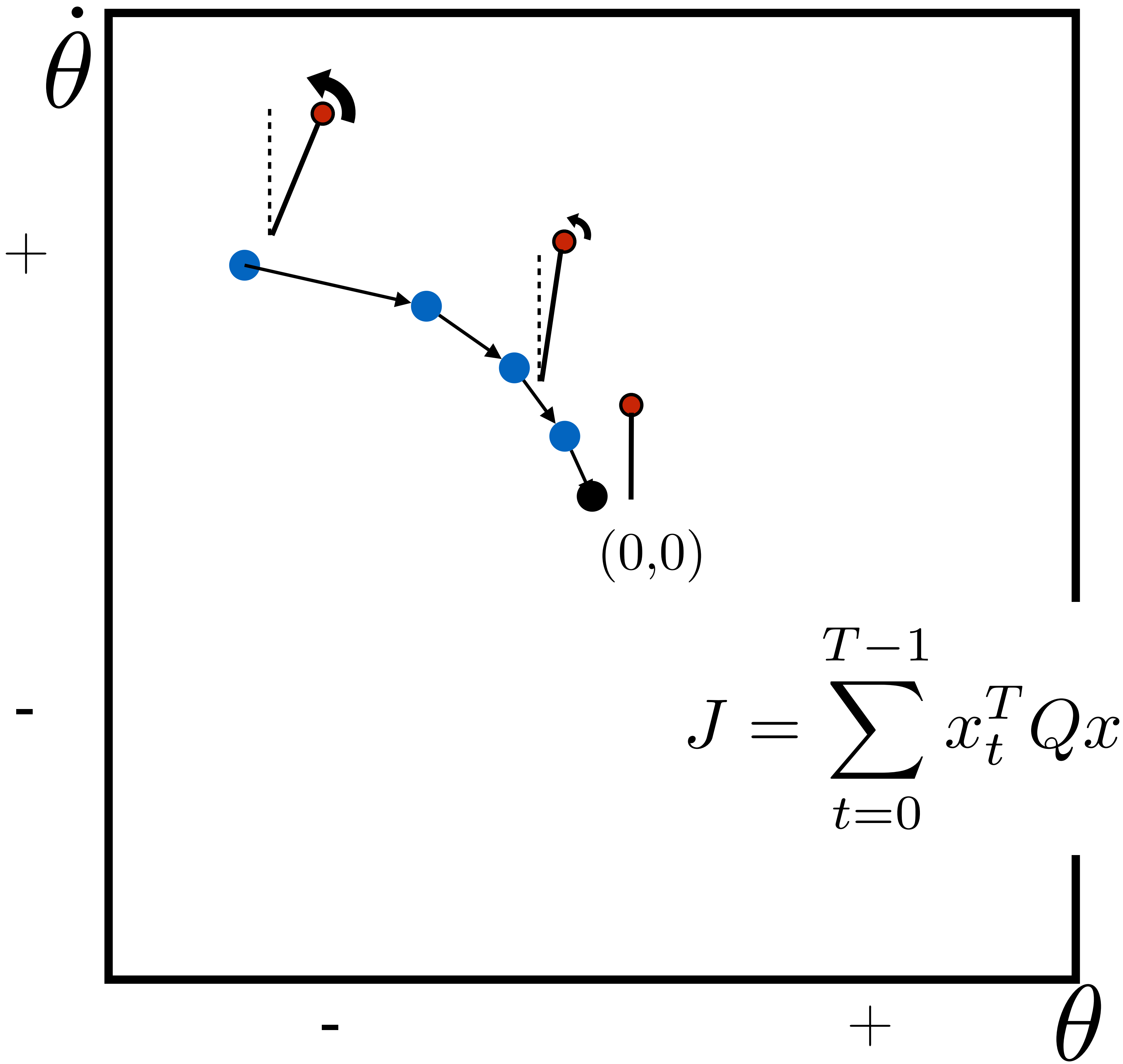

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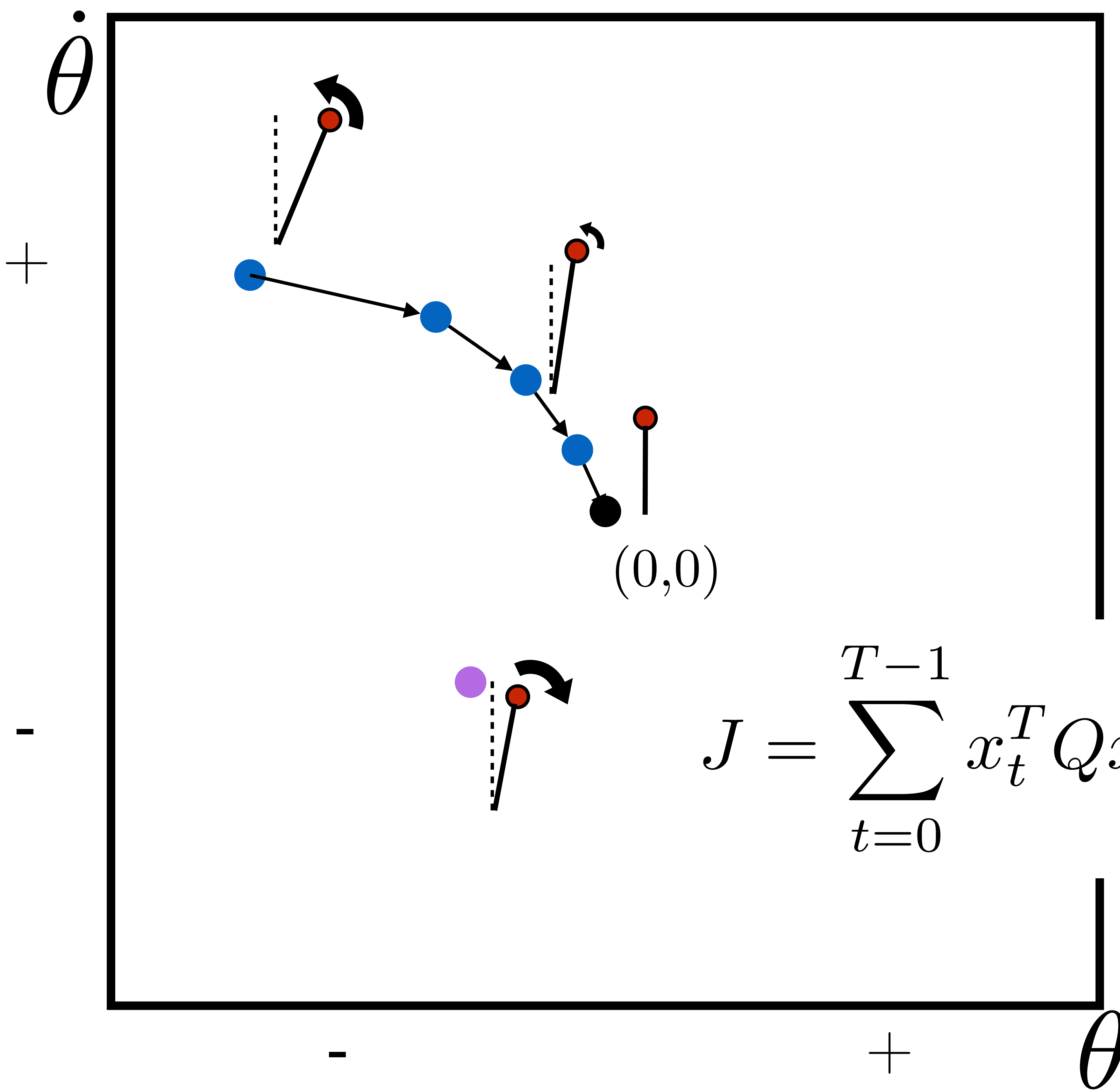

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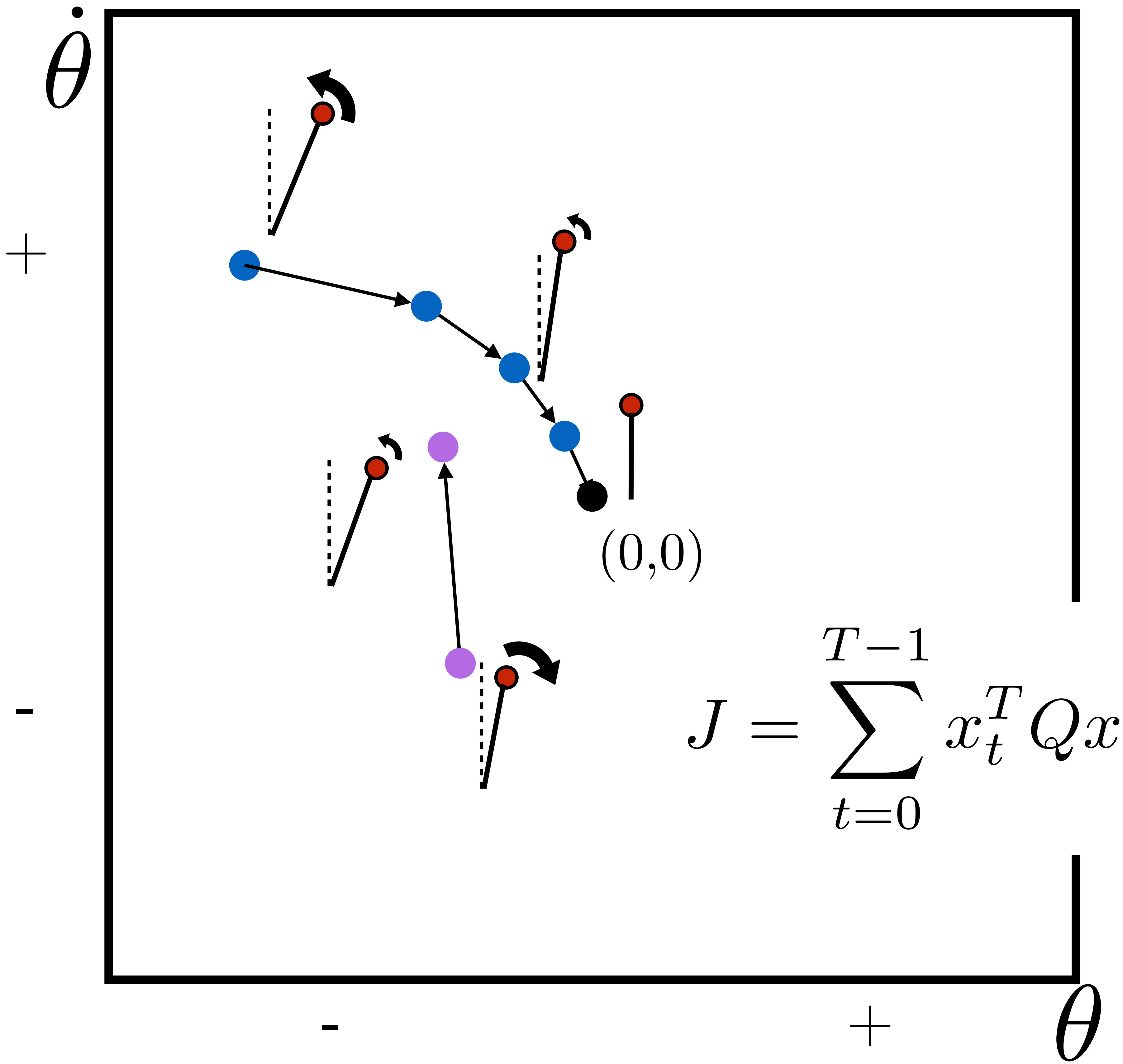

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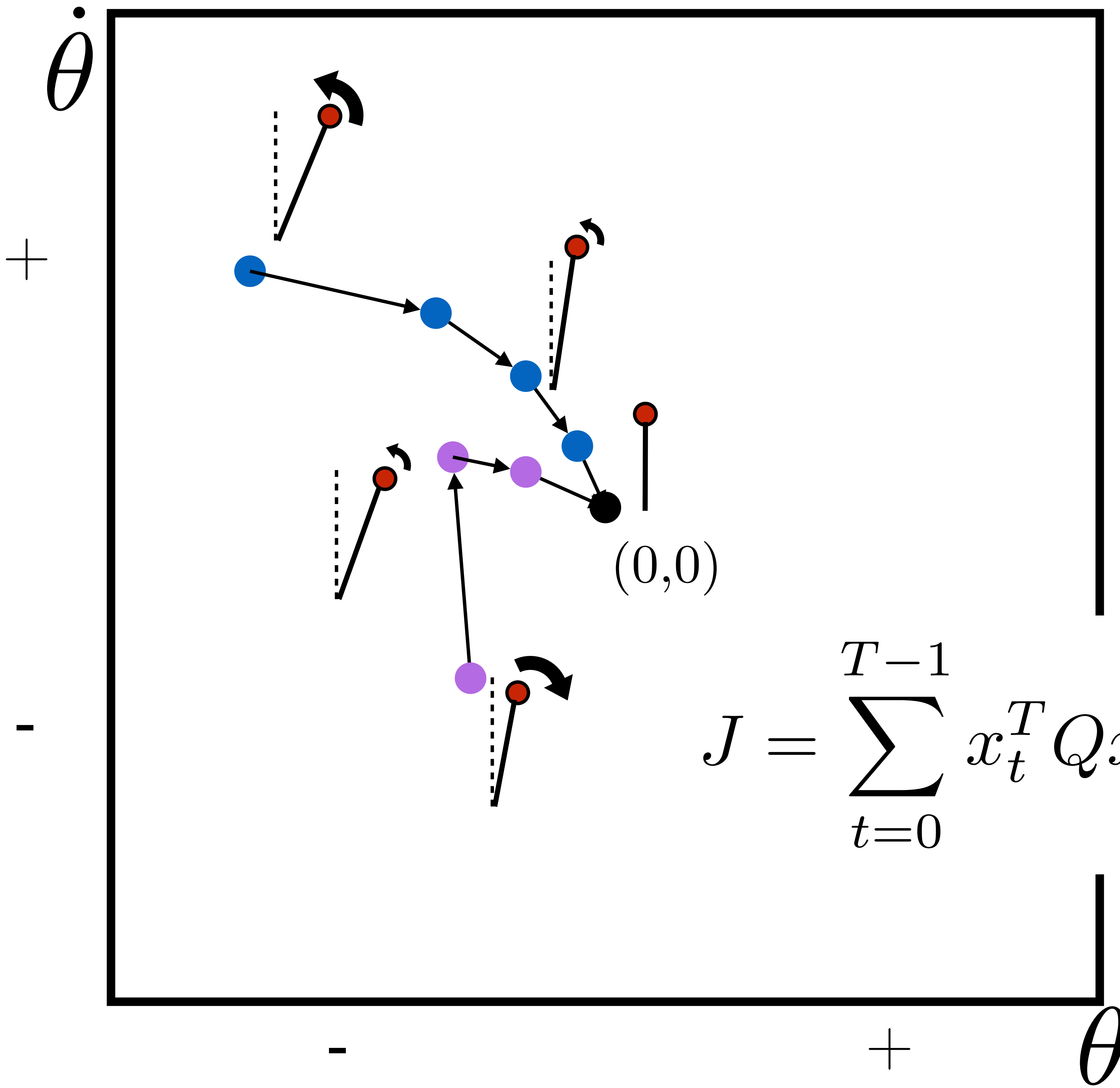

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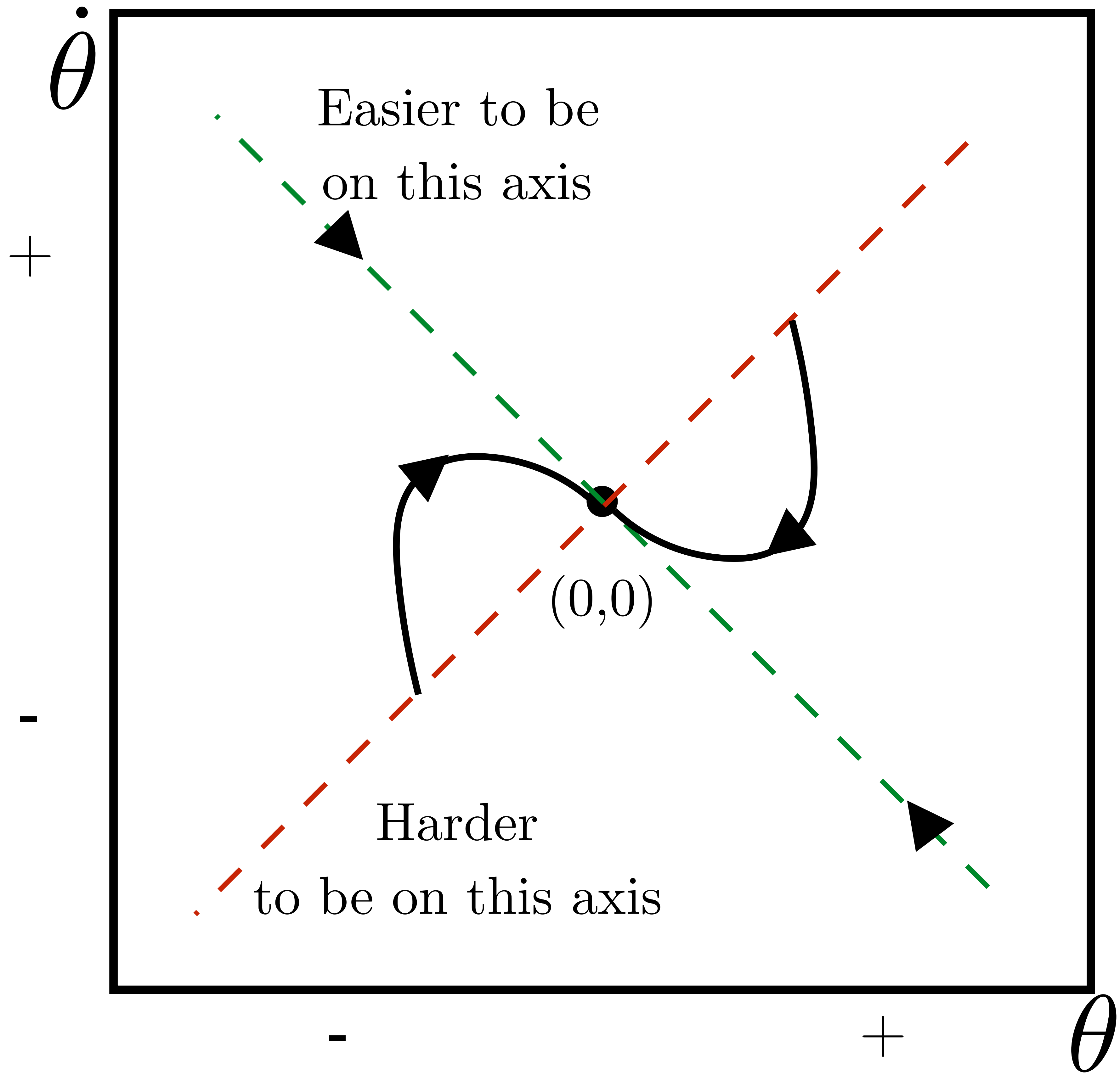
Observation: Cost-to-go is not uniform



How do we solve for controls?

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Dynamic programming to the rescue!

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Recall the Bellman function that relates value at consecutive time steps

$$J(x_t, t) = \min_{u_t} c(x_t, u_t) + J(x_{t+1}, t+1)$$

$$= \min_{u_t} x_t^T Q x_t + u_t^T R u_t + J(x_{t+1}, t+1)$$

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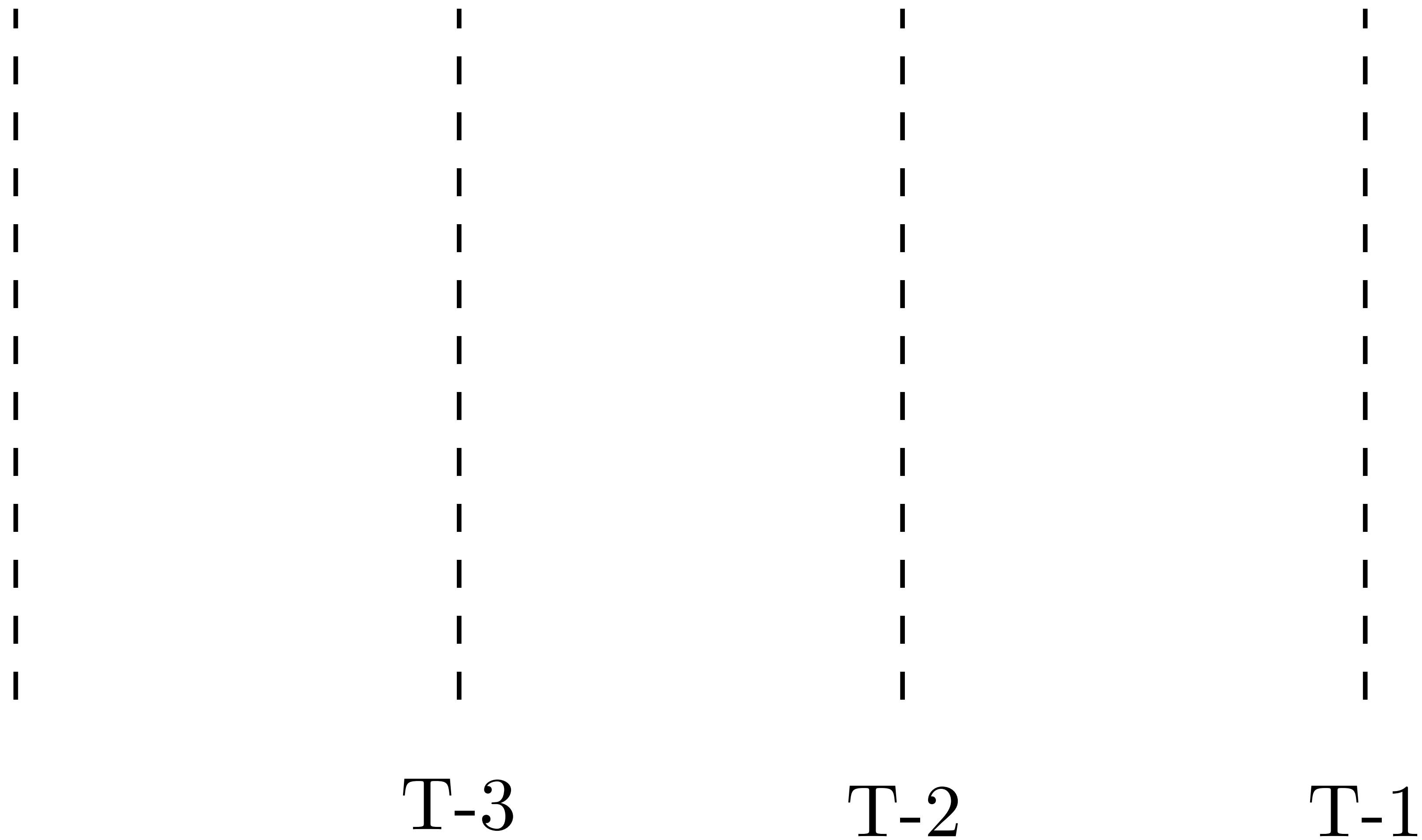

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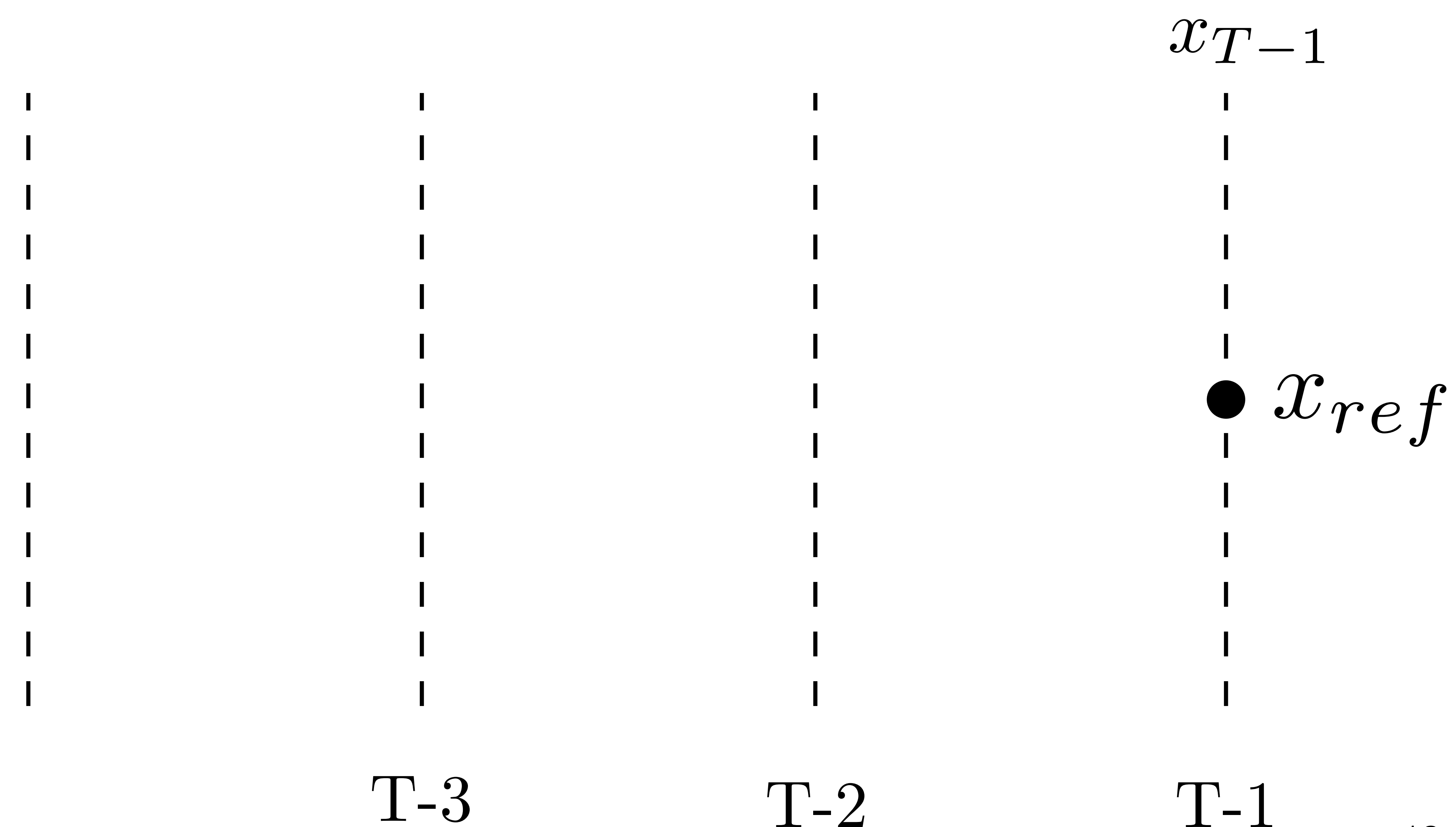

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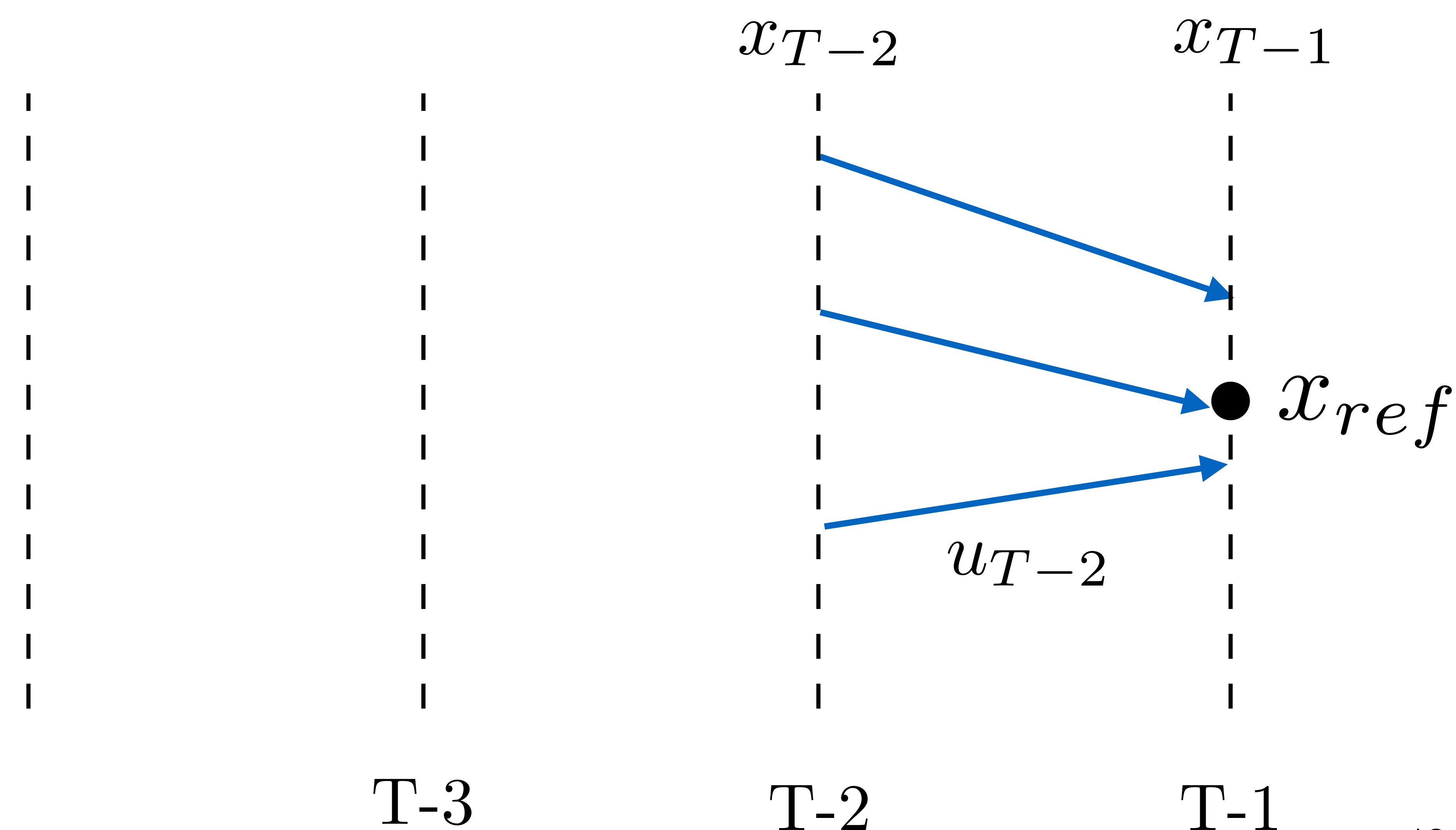

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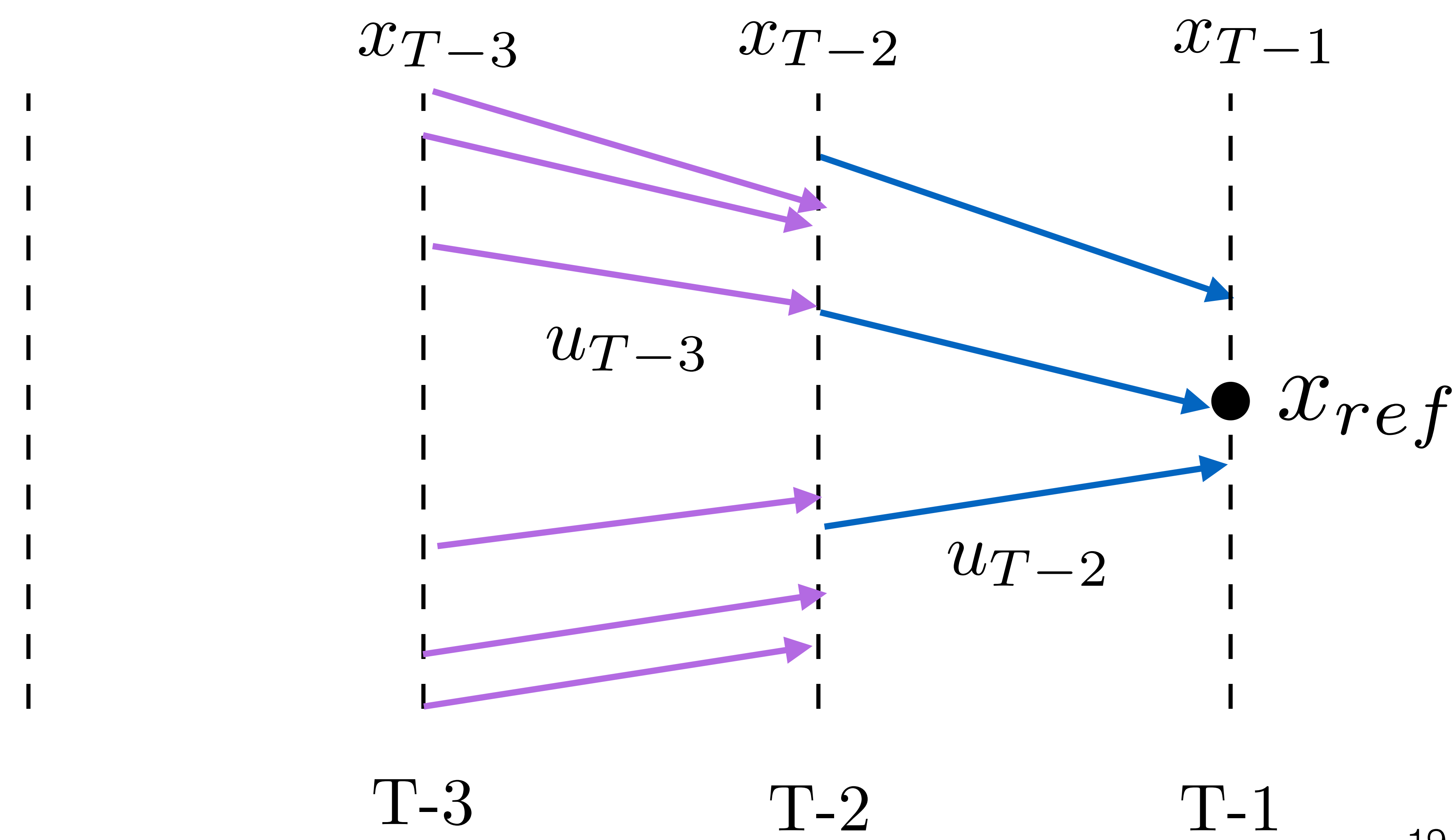
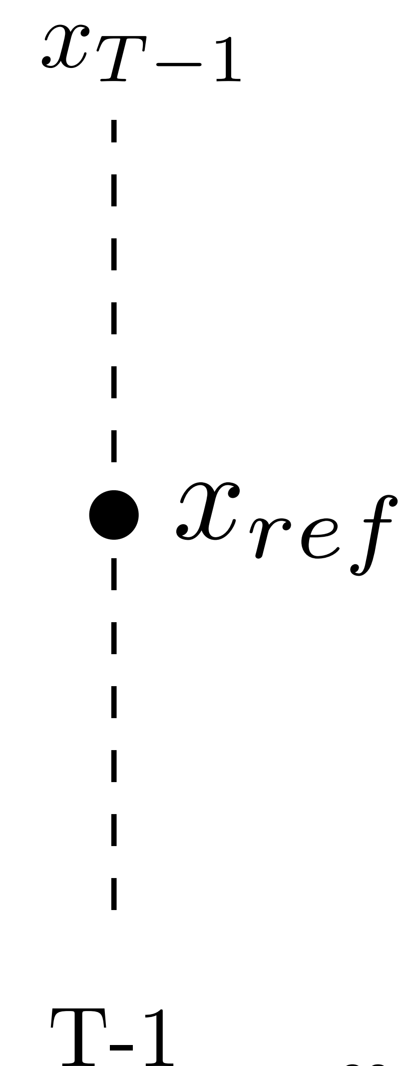


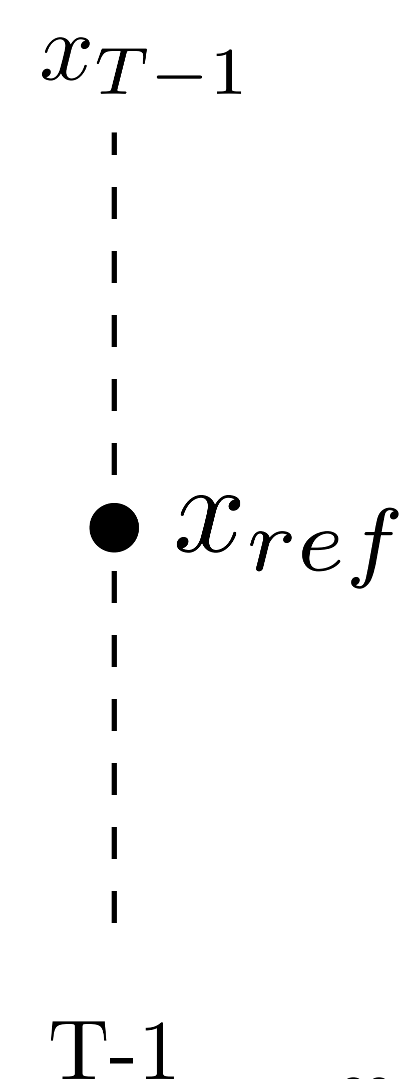
Last time step T-1



Last time step T-1

We have only 1 term in the cost function

$$J(x_{T-1}, u_{T-1}) = \min_{u_T} \; x_{T-1}^T Q x_{T-1} + u_{T-1}^T R u_{T-1}$$



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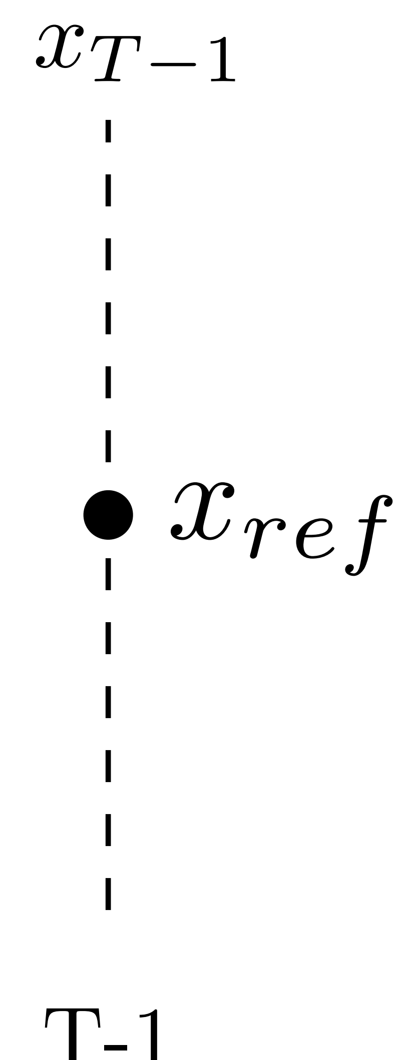

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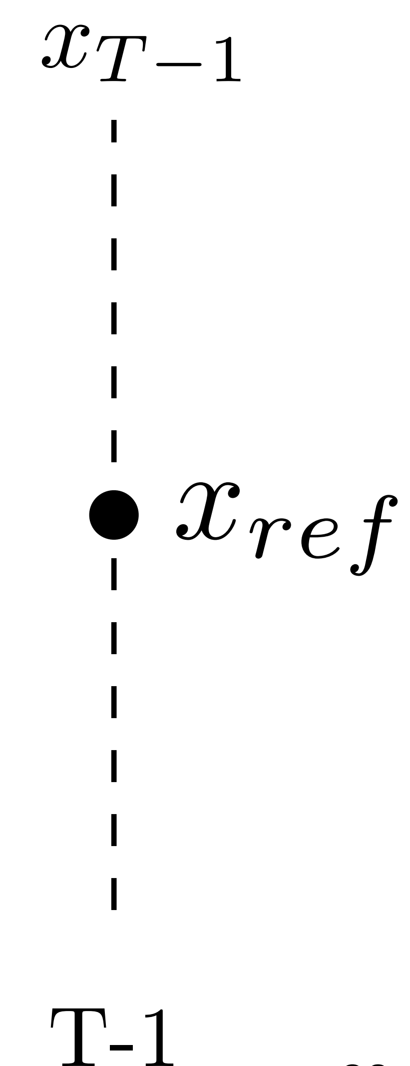
(this is a value matrix)



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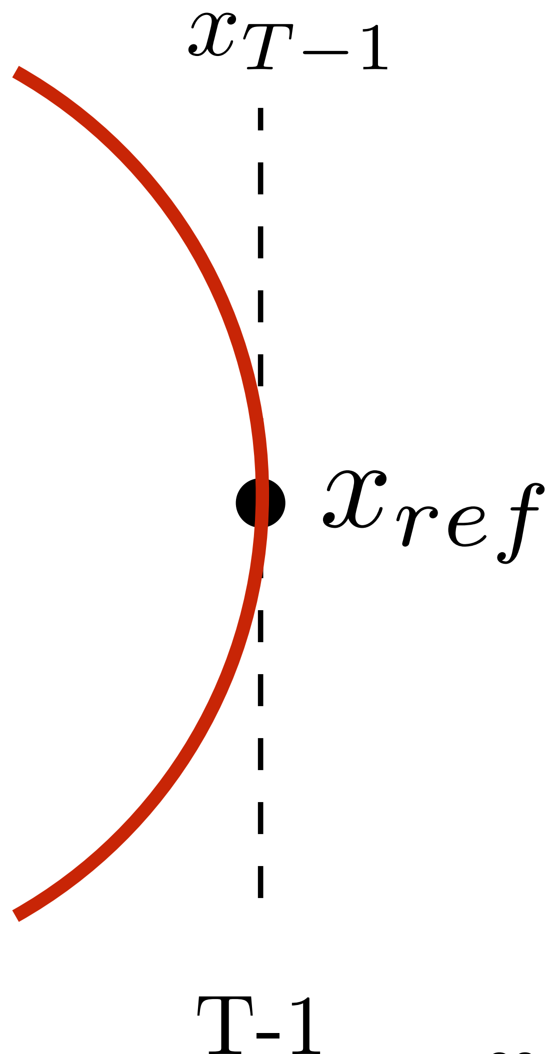
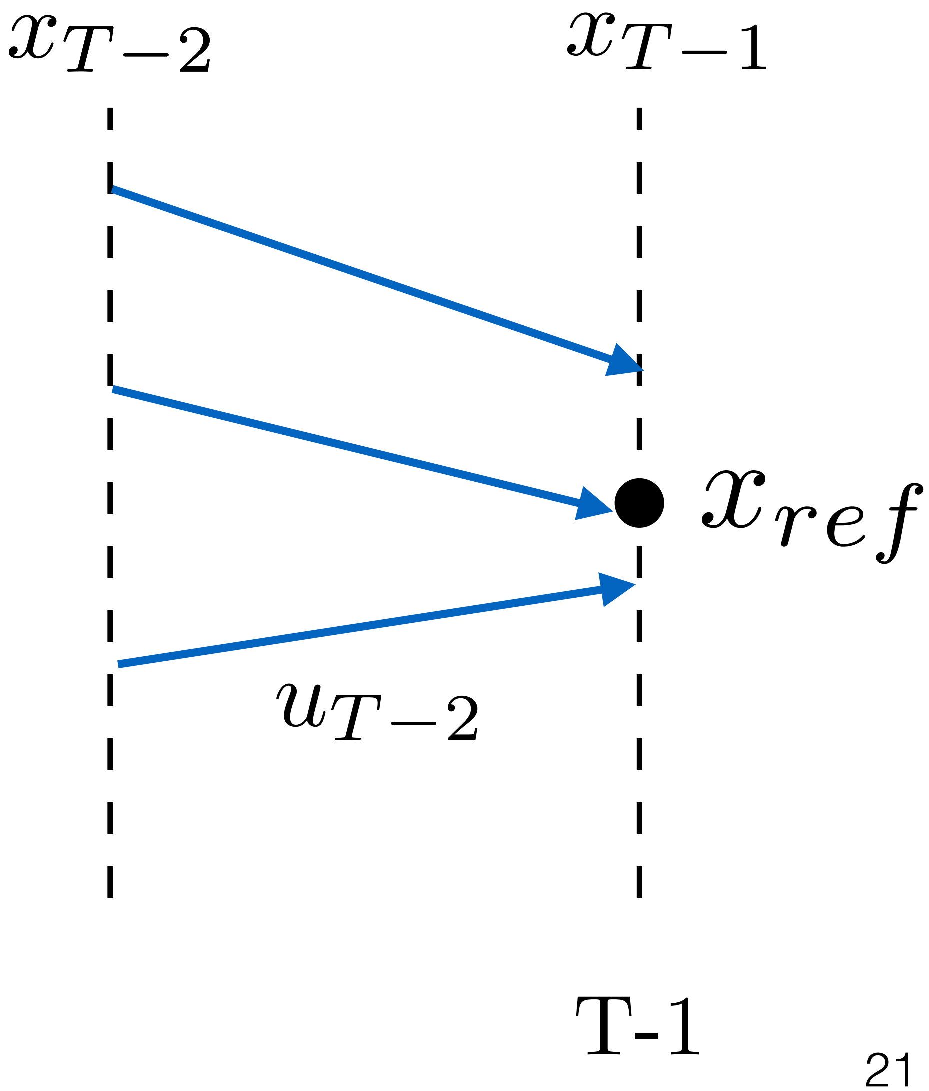
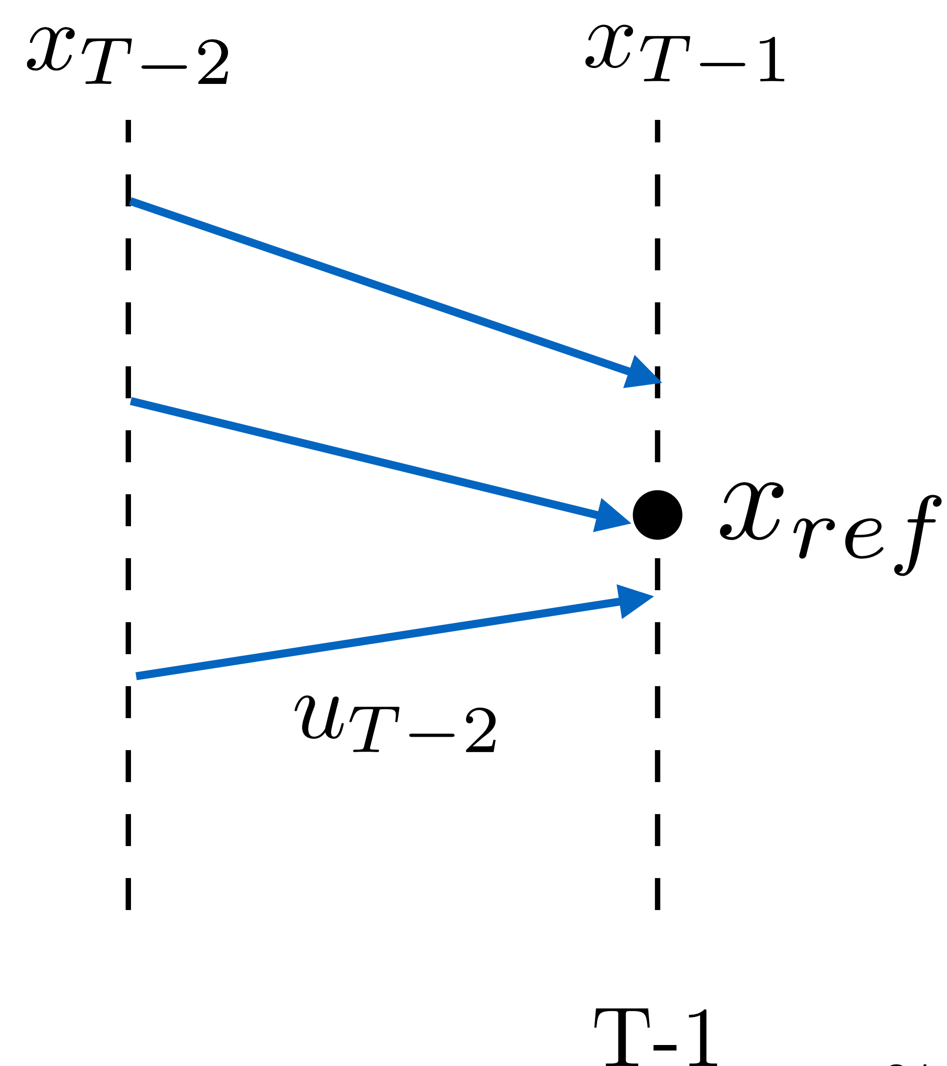


Last time step T-2



Last time step T-2

$$J(x_{T-2}, T-2) = \min_{u_{T-2}} c(x_{T-2}, u_{T-2}) + J(x_{T-1}, T-1)$$



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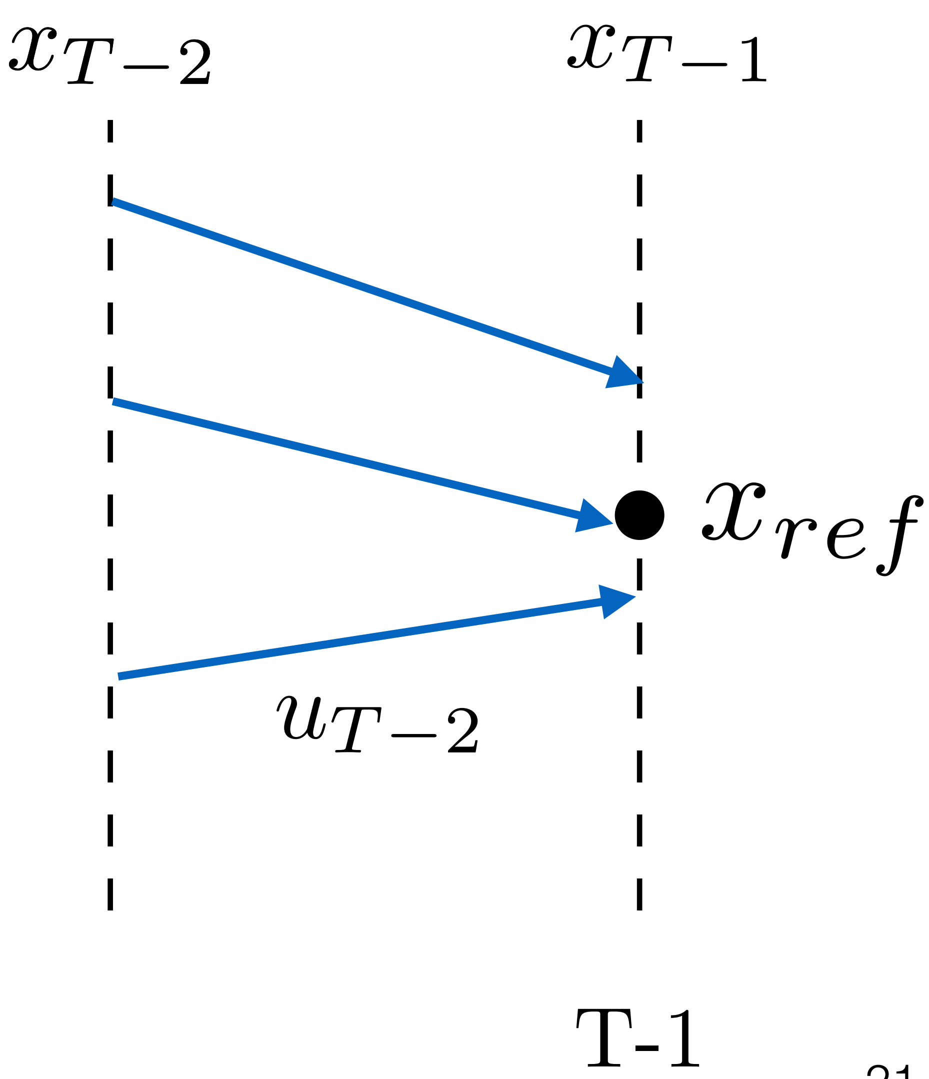

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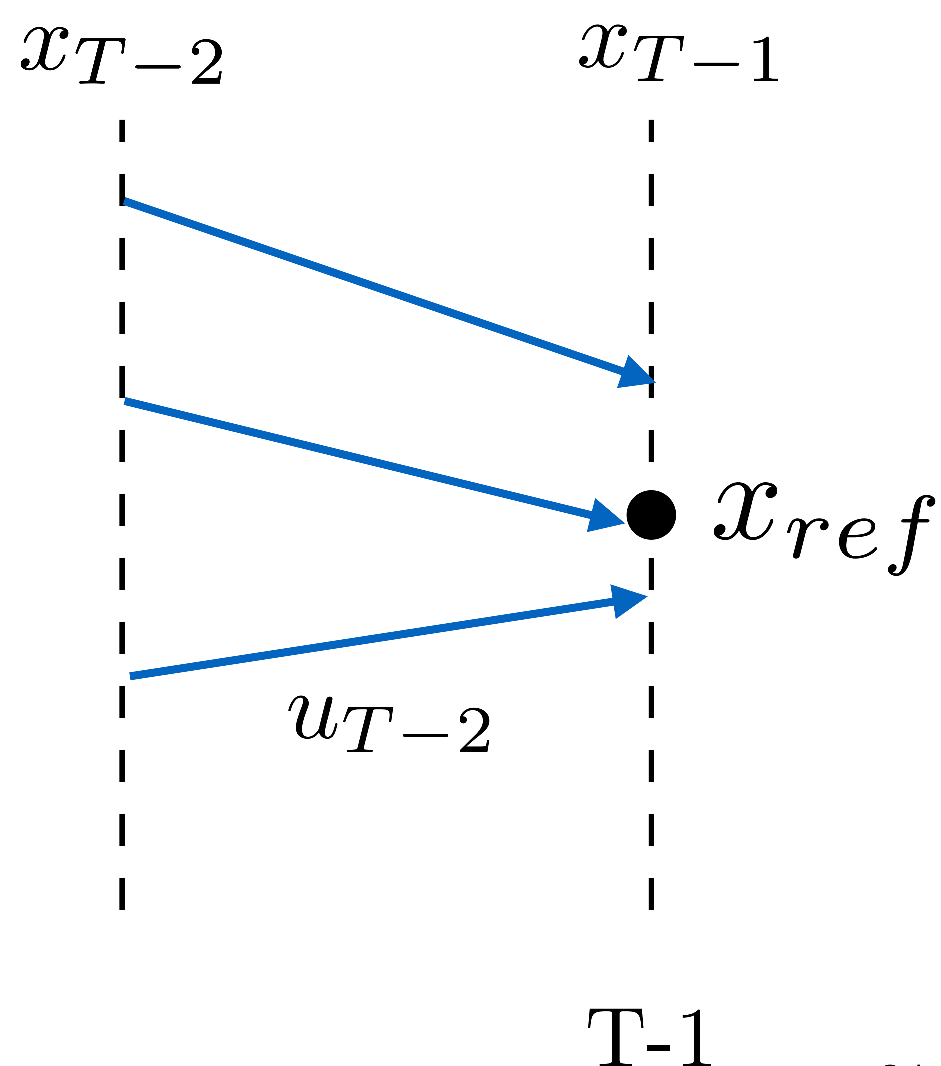
Solve for control at timestep T-1



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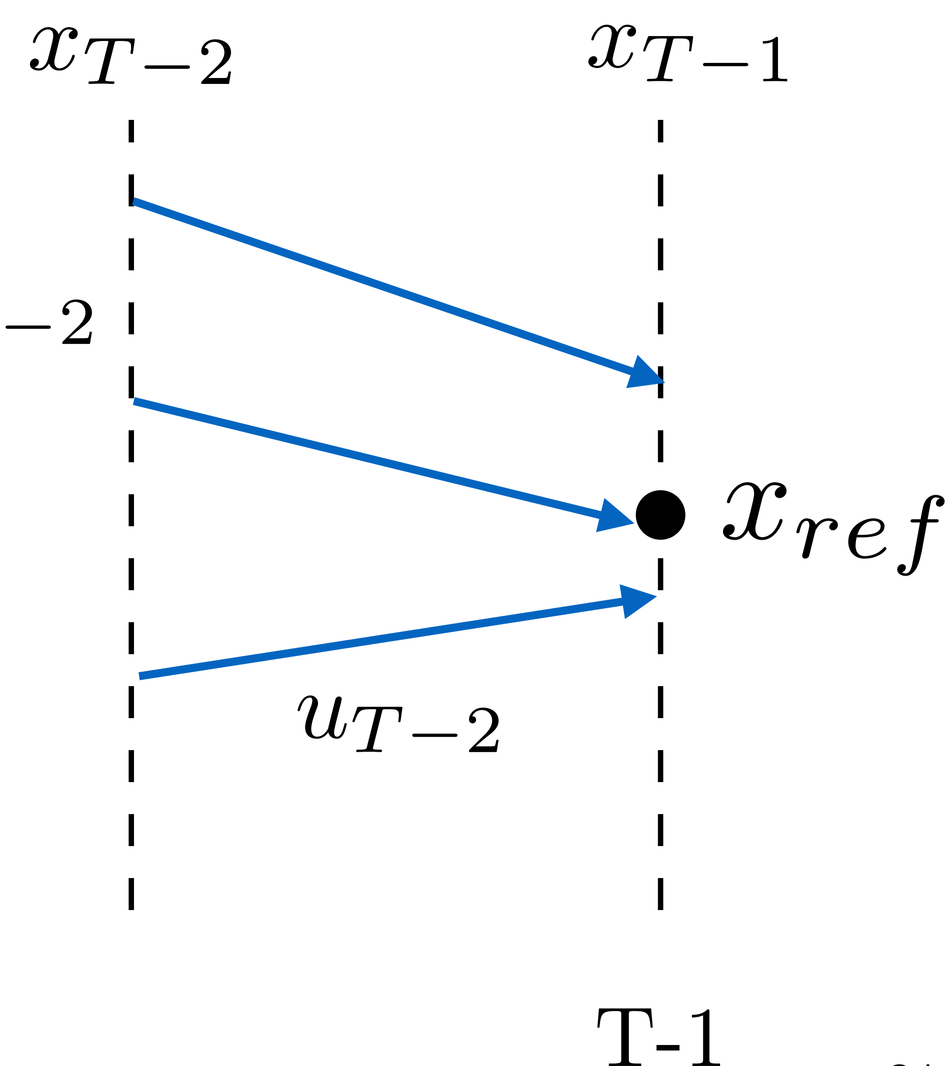
Solve for control at timestep T-1

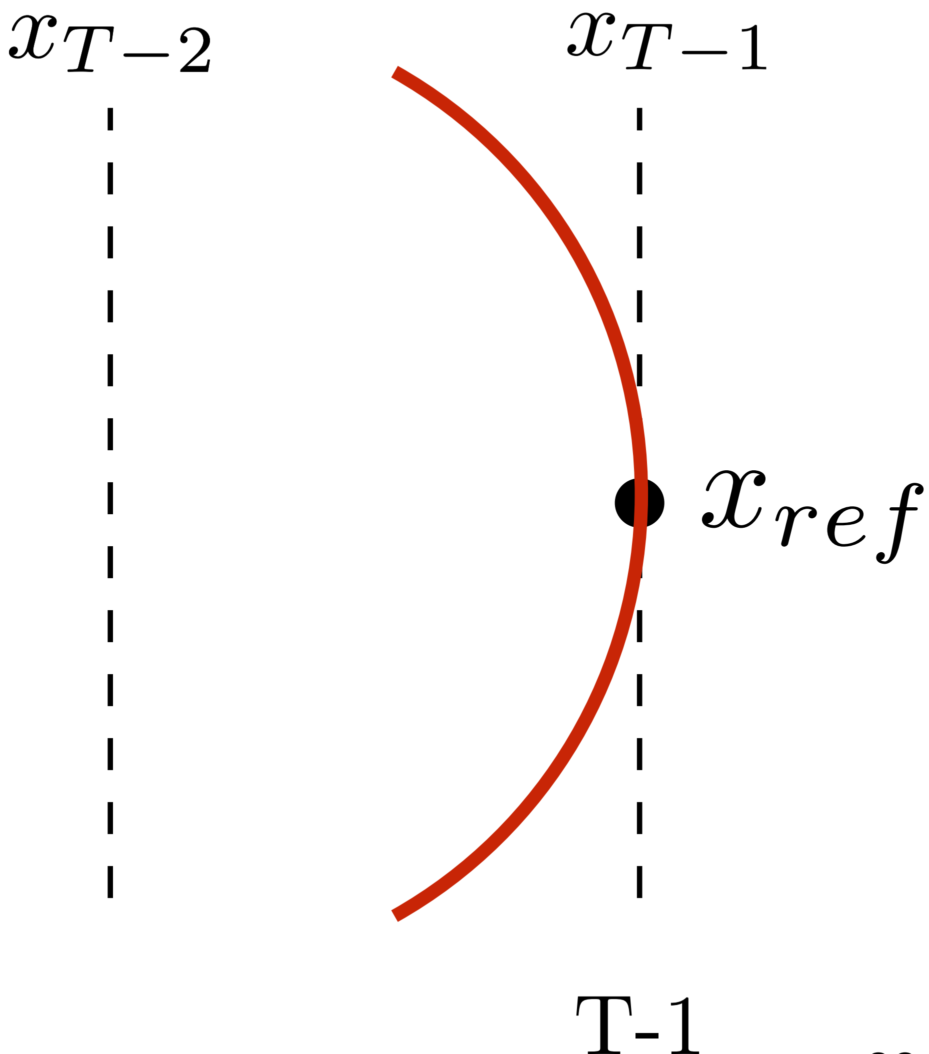
$$u_{T-2} = \underbrace{-(R + B^T V_{T-1} B)^{-1} B^T V_{T-1} A}_{K_{T-2}} x_{T-2}$$

Observation: Control law is linear!



Key insight: Value function is always quadratic

Plug back control in the value function (cumulative cost)



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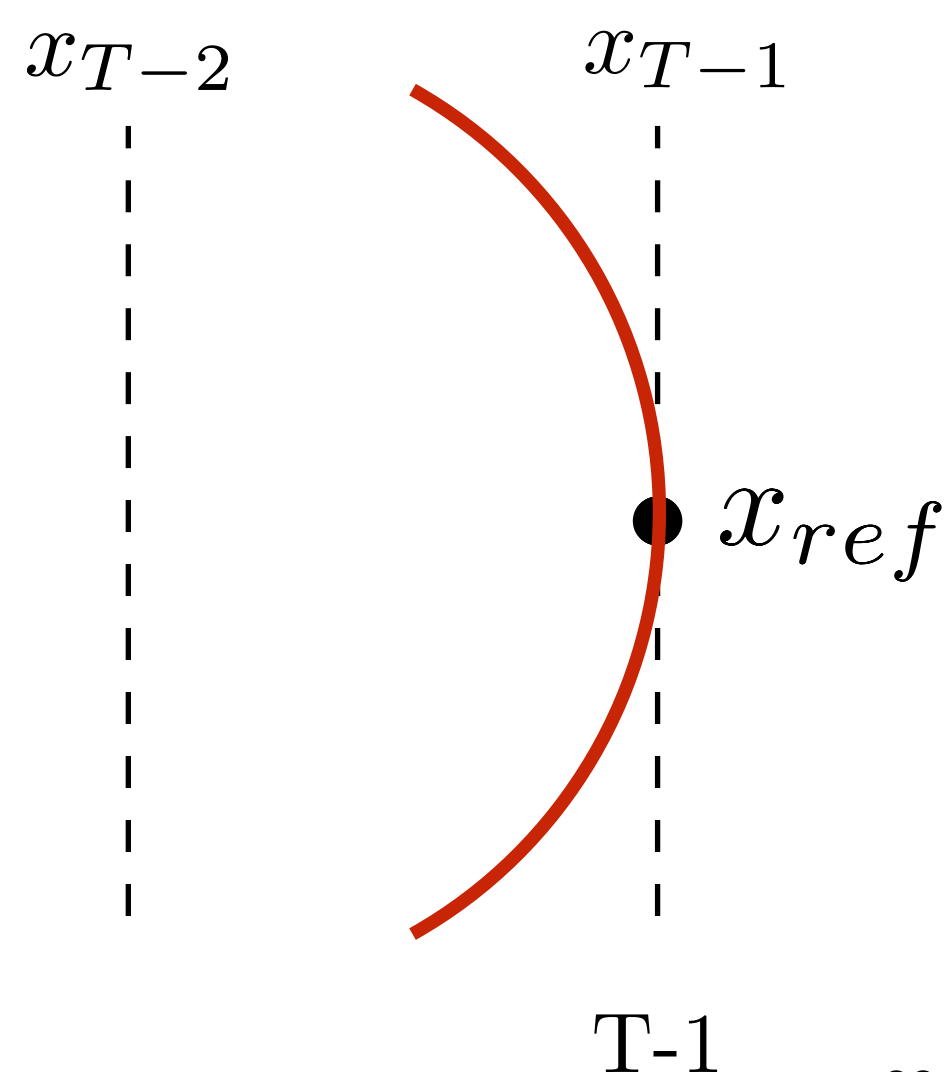
$$J(x_{T-2}, T-2) = x_{T-2}^T \underbrace{(Q + K_{T-2}^T R K_{T-2} + (A + BK_{T-2})^T V_{T-1} (A + BK_{T-2}))}_{V_{T-2}} x_{T-2}$$



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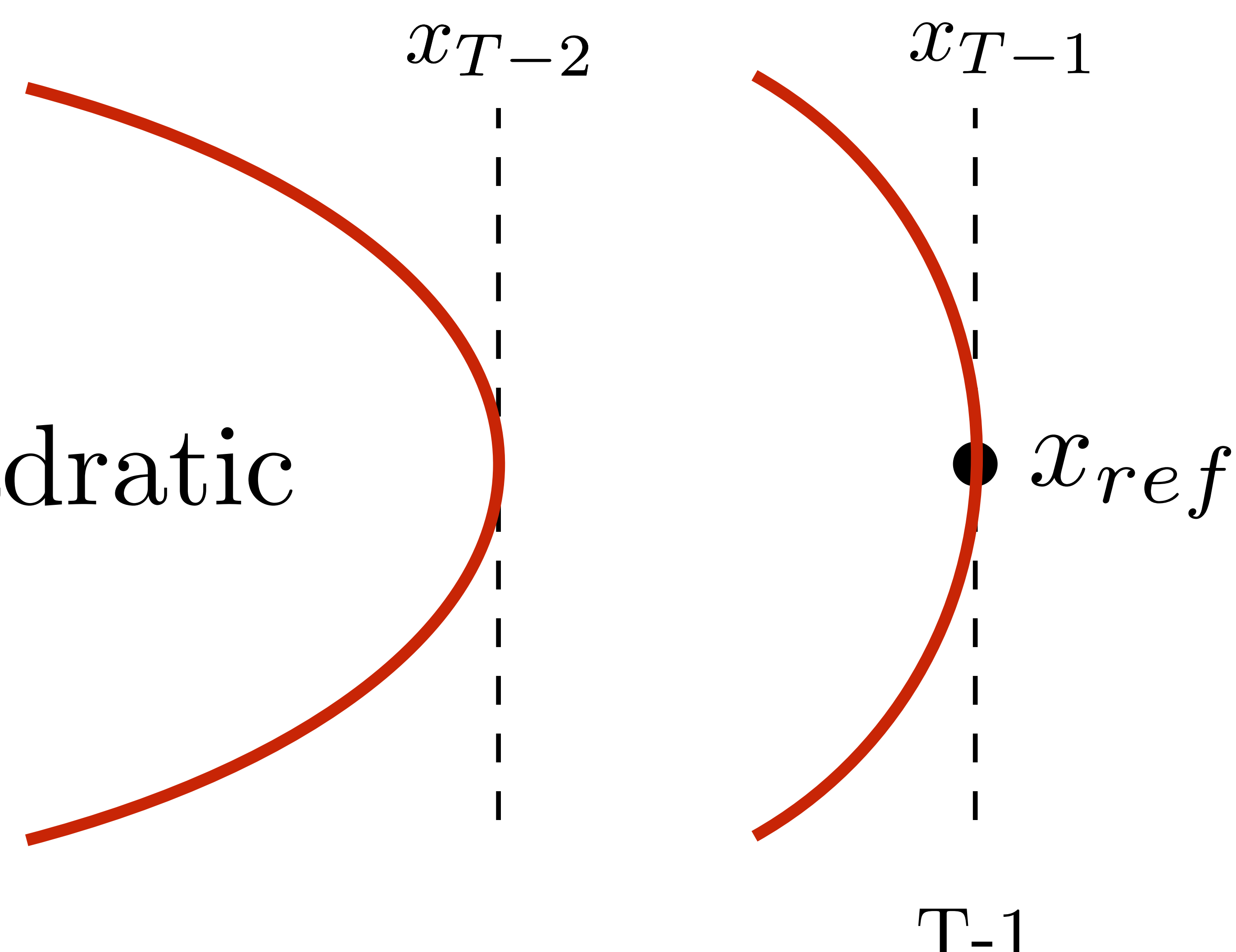
Quadratic remains quadratic



We can derive this relation at ALL time steps

$$K_t = -(R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A$$

$$V_t = Q + K_t^T R K_t + (A + BK_t)^T V_{t+1} (A + BK_t)$$

Current cost	Action cost	Closed loop dynamics	Future value func	Closed loop dynamics

The LQR algorithm

Algorithm `OptimalValue(`A, B, Q, R, t, T`)`

    if $t = T - 1$ **then**

        return Q

    end

    else

        $V_{t+1} =$ `OptimalValue(`$A, B, Q, R, t + 1, T$`)`

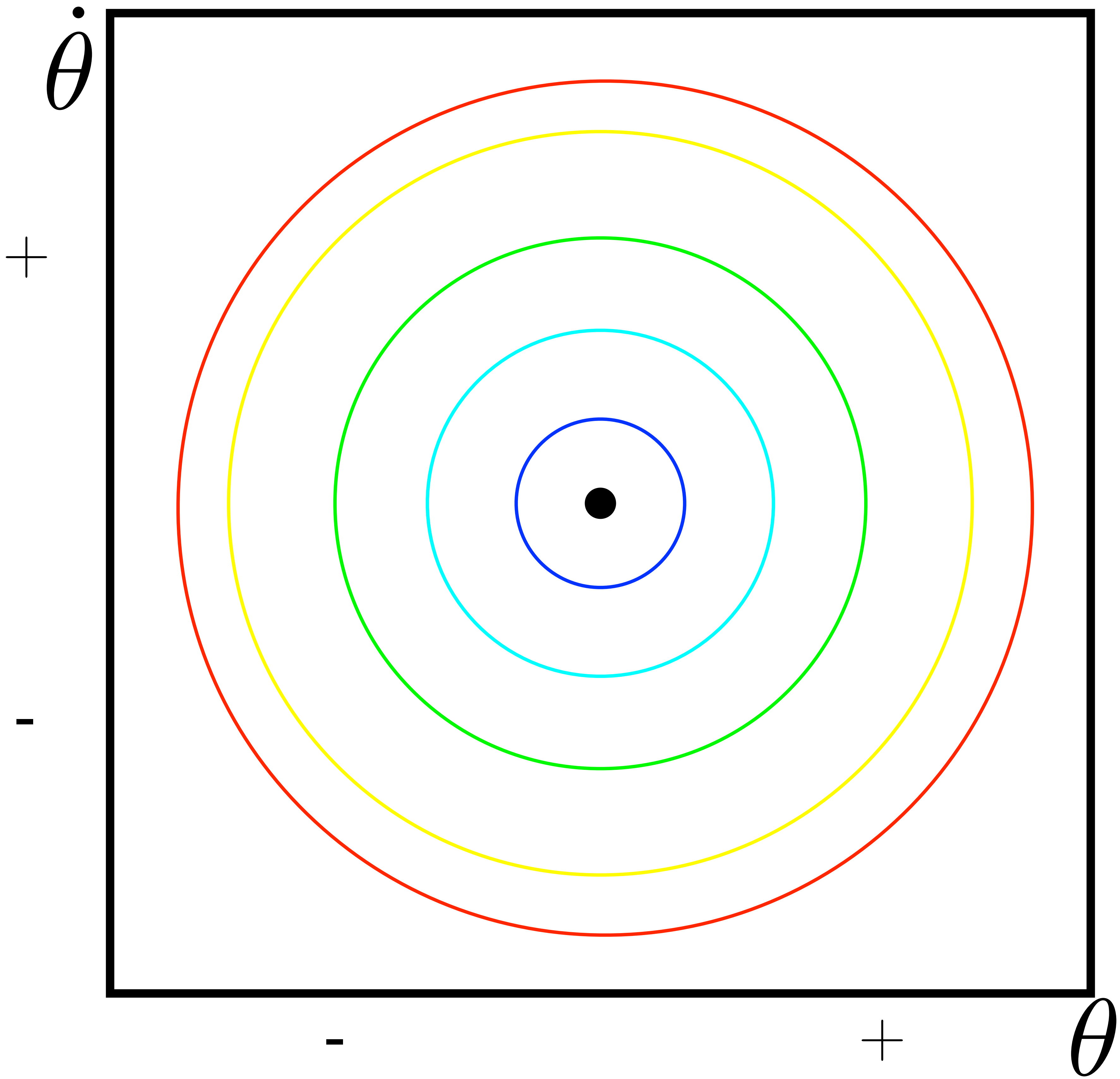
        $K_t = -(B^T V_{t+1} B + R)^{-1} B^T V_{t+1} A$

        return $V_t = Q + K_t^T R K_t + (A + BK_t)^T V_{t+1} (A + BK_t)$

    end

(Courtesy Drew Bagnell)

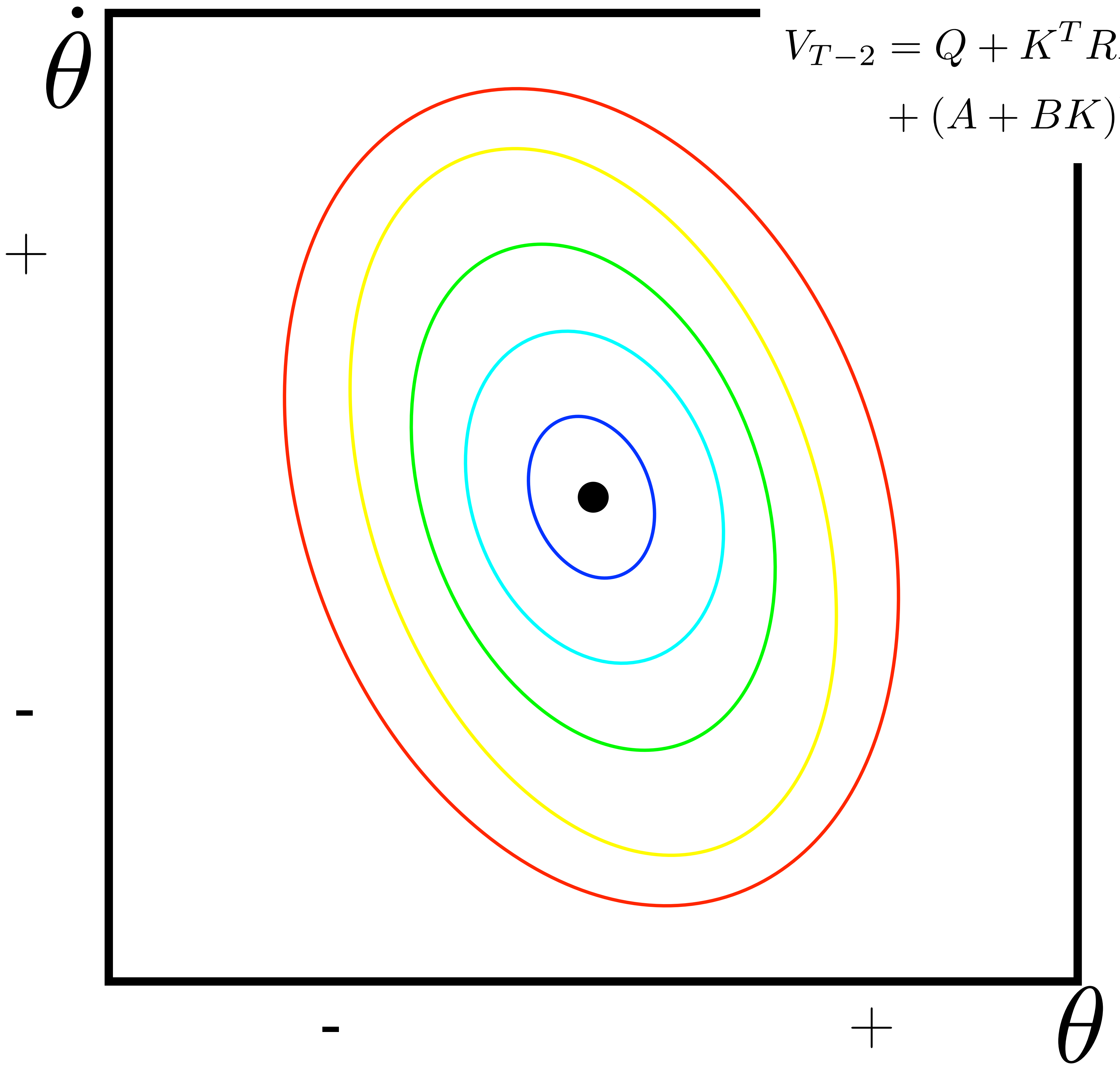
Contours of value function (T-1)



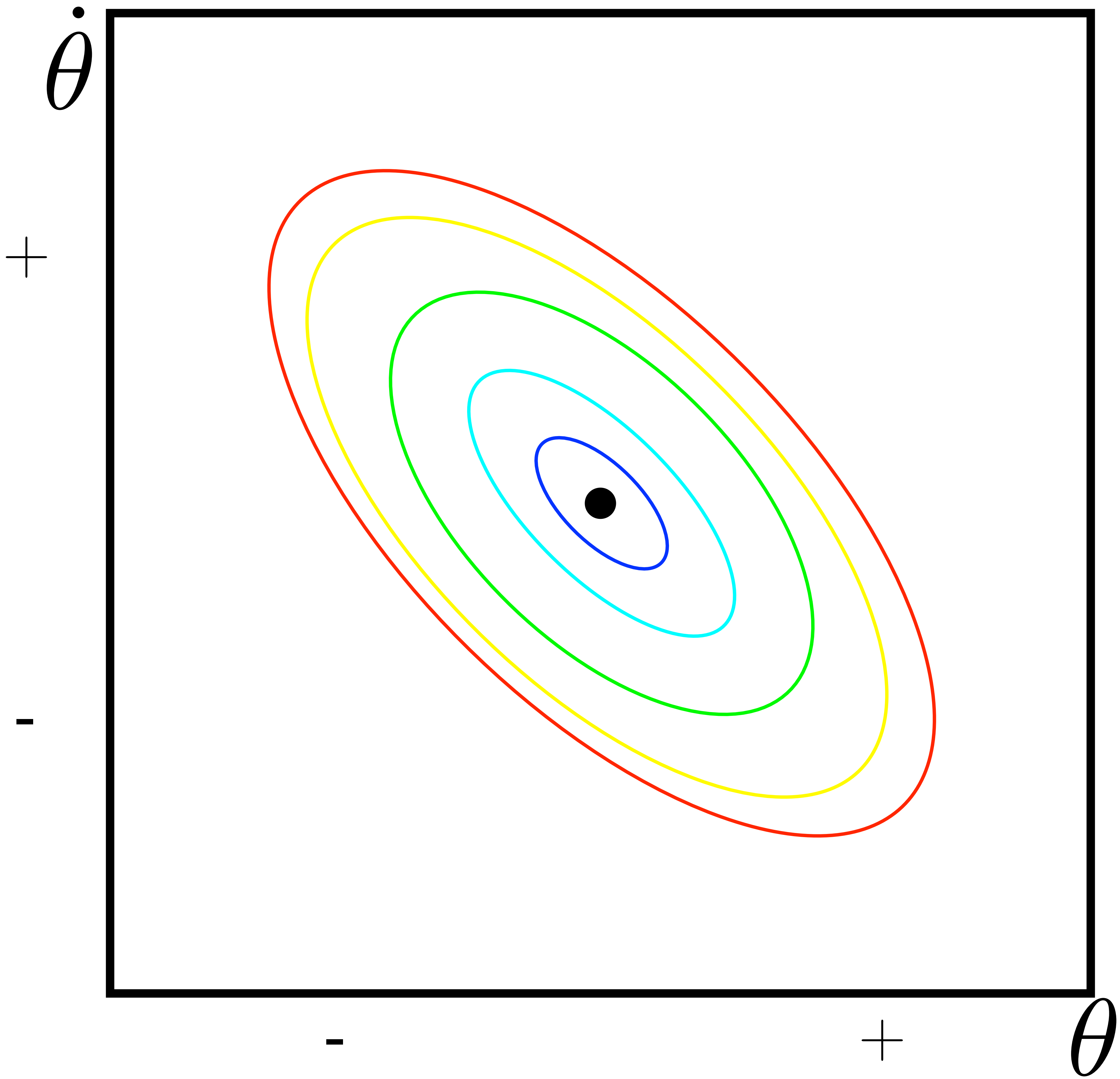
$$V_{T-1} = Q$$

Contours of value function (T-2)

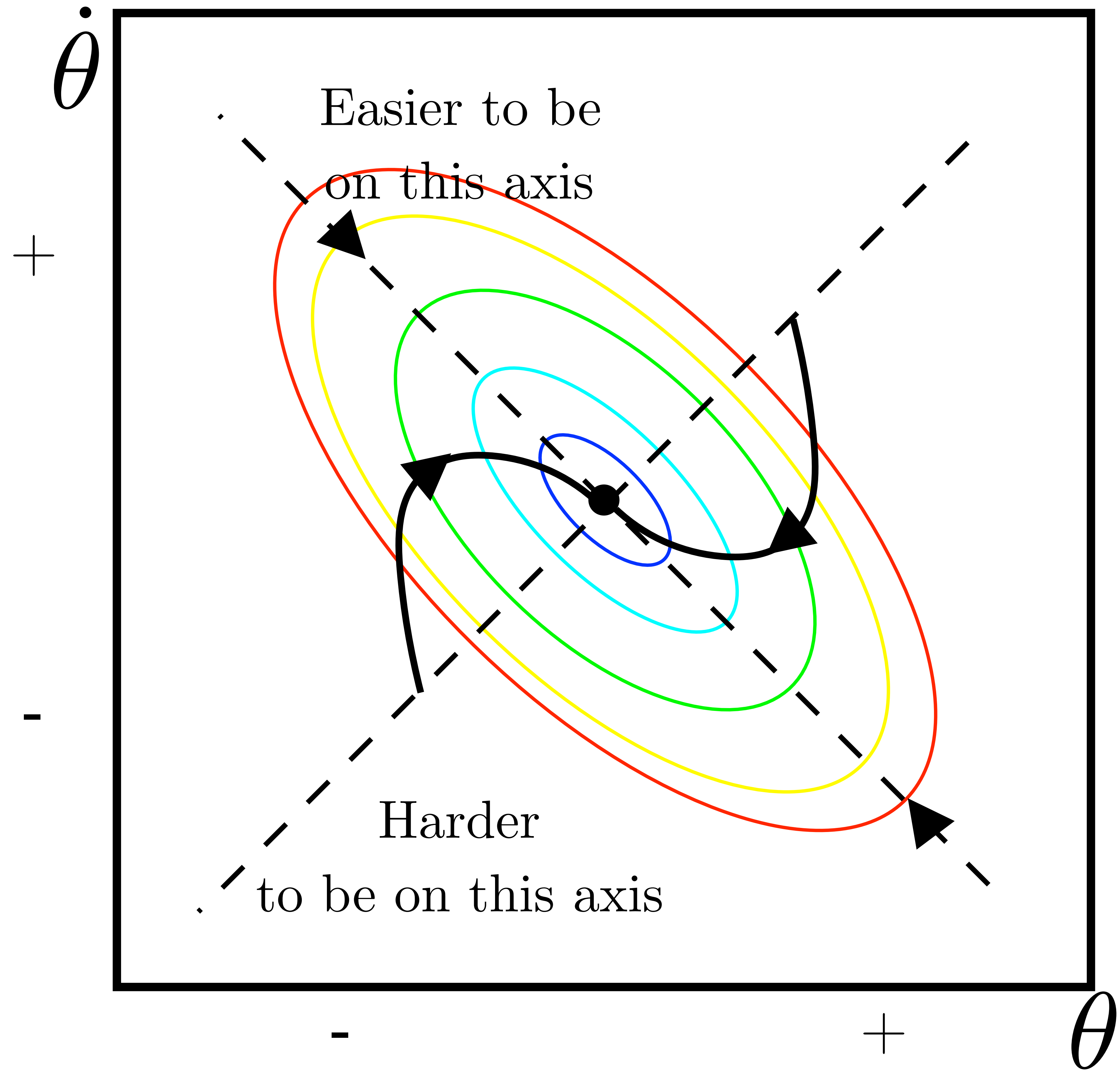
$$V_{T-2} = Q + K^T R K \\ + (A + BK)^T V_{T-1} (A + BK)$$



Contours of value function (many steps)



How does the value function evolve?



What if my time horizon is very very very large?

Convergence of value iteration

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Theorem: If the system is stabilizable, then the value V will converge

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$$K = -(R + B^T V B)^{-1} B^T V A$$

Discrete Algebraic Ricatti Equation (DARE)

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Discrete Algebraic Ricatti Equation (DARE)

How do I solve? Can iterate over V / use eigen value decomposition [1]

Type into MATLAB: `dare(A,B,Q,R)`

[1] Arnold, W.F., III and A.J. Laub, "Generalized Eigenproblem Algorithms and Software for Algebraic Riccati Equations," *Proc. IEEE®*, 72 (1984), pp. 1746-1754.

So, can this controller stabilize inverted pendulum for all angles?

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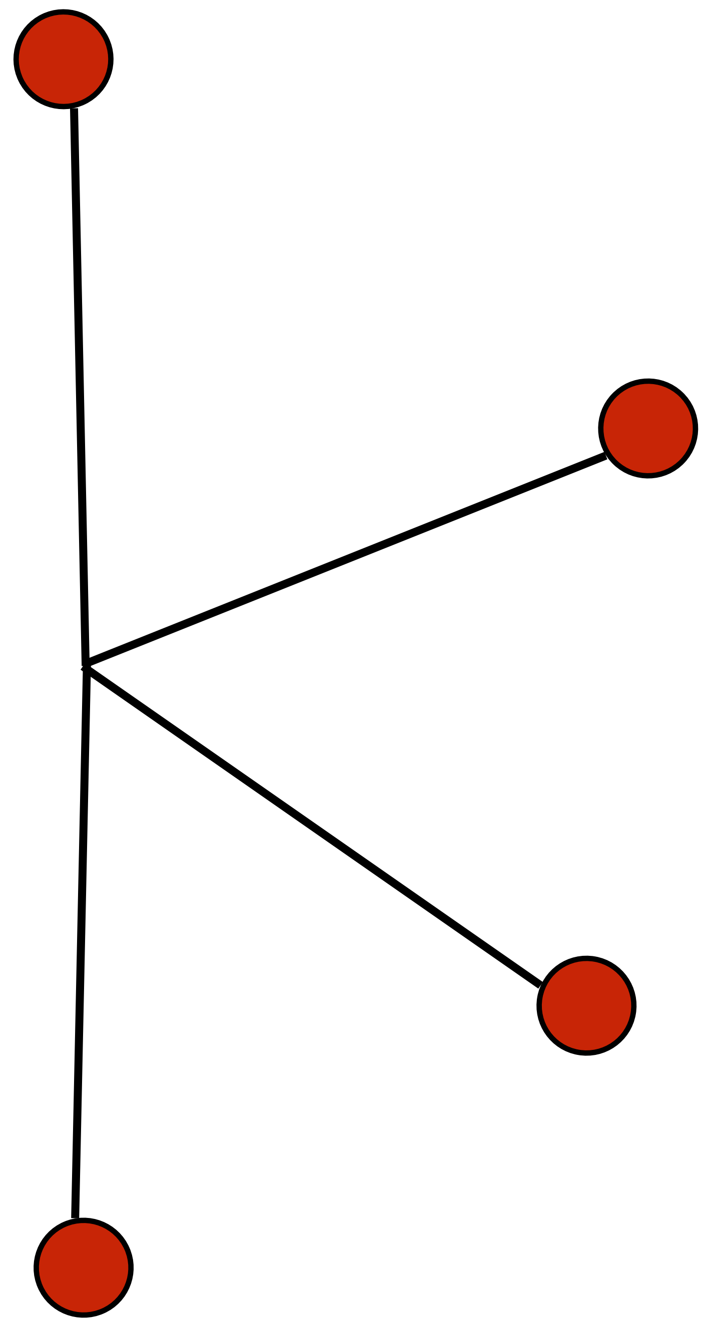
No!

Linearization error is too large when angle is large

Instead, can we use LQR to track reference trajectory?



Yes

But, first we need to talk about time-varying systems

LQR for Time-Varying Dynamical Systems

$$x_{t+1} = A_t x_t + B_t u_t$$

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Straight forward to get LQR equations

$$K_t = -(R_t + B_t^T V_{t+1} B_t)^{-1} B_t^T V_{t+1} A_t$$

$$V_t = Q_t + K_t^T R_t K_t + (A_t + B_t K_t)^T V_{t+1} (A_t + B_t K_t)$$

Why do we care about time-varying?

Ans: Linearization about a trajectory

Linearize about a time-varying trajectory

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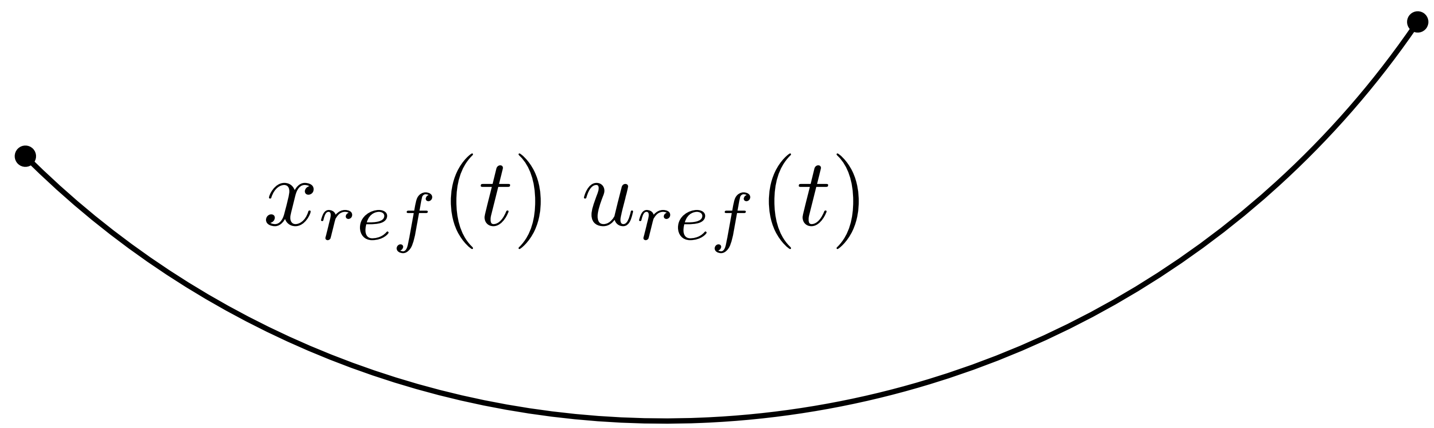
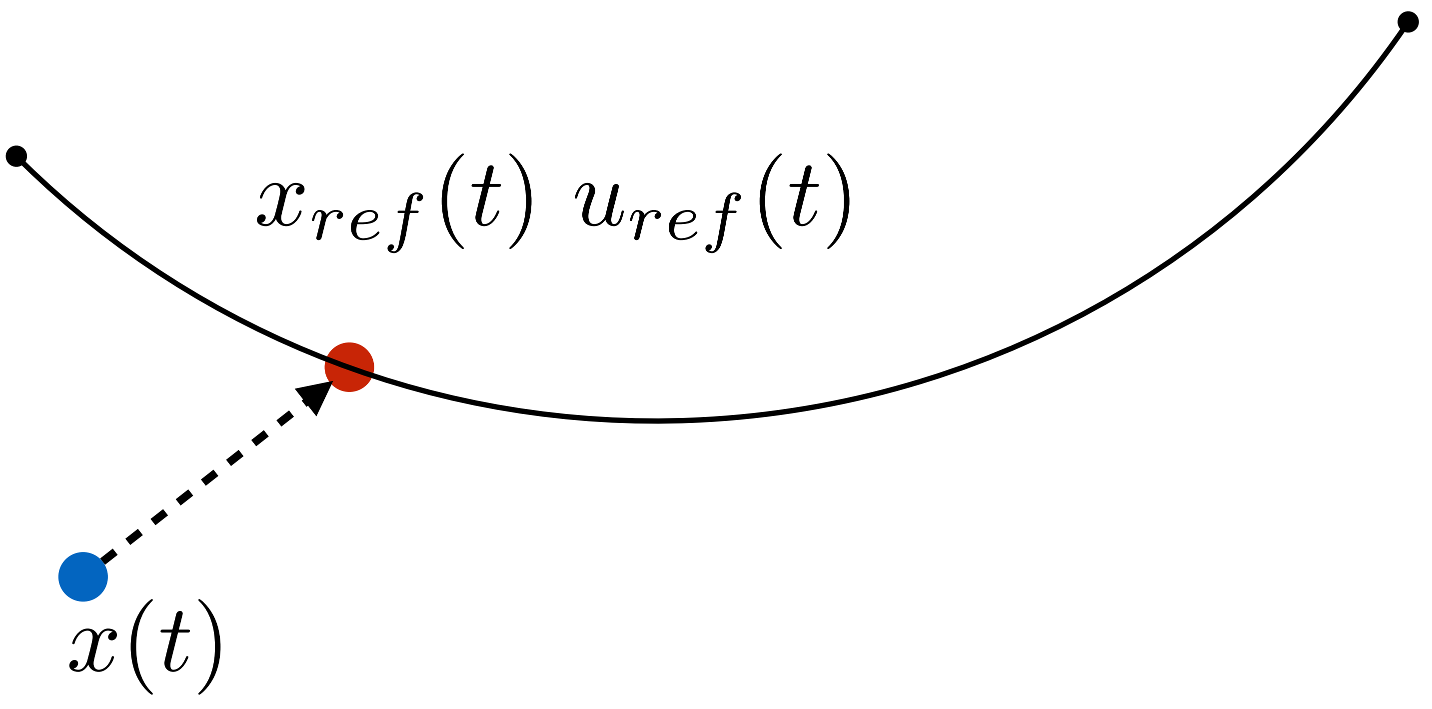


Linearize about a time-varying trajectory



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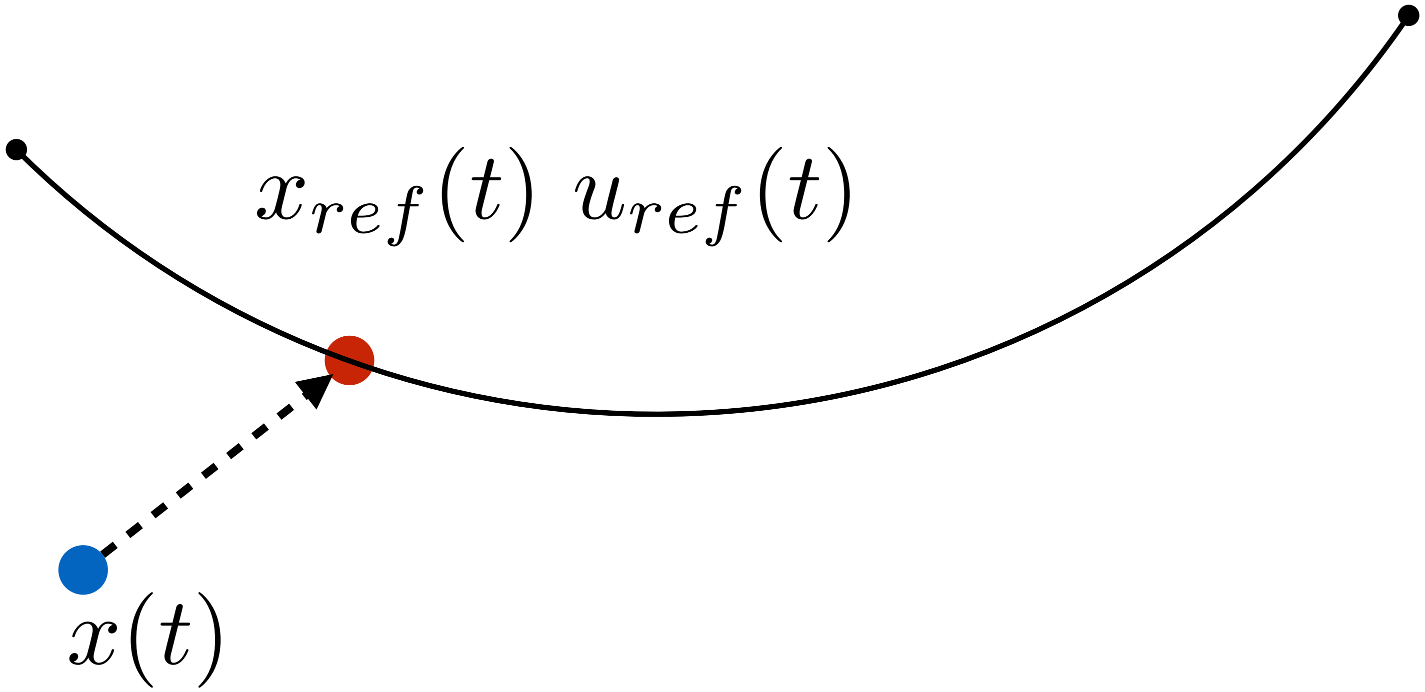
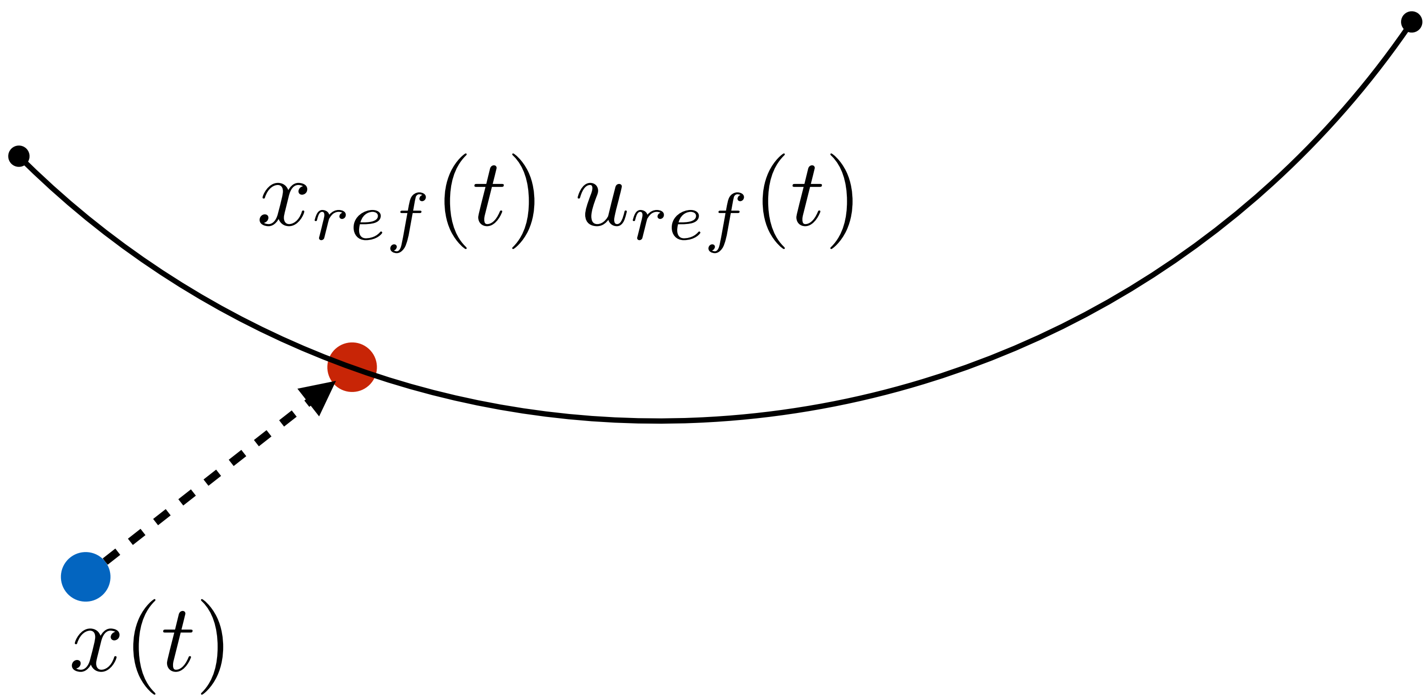


$$\dot{x} = f(x, u)$$

Linearize about a time-varying trajectory



$$\dot{x} = f(x, u)$$

$$A_t = \left. \frac{\partial f}{\partial x} \right|_{x_{ref}(t)}$$

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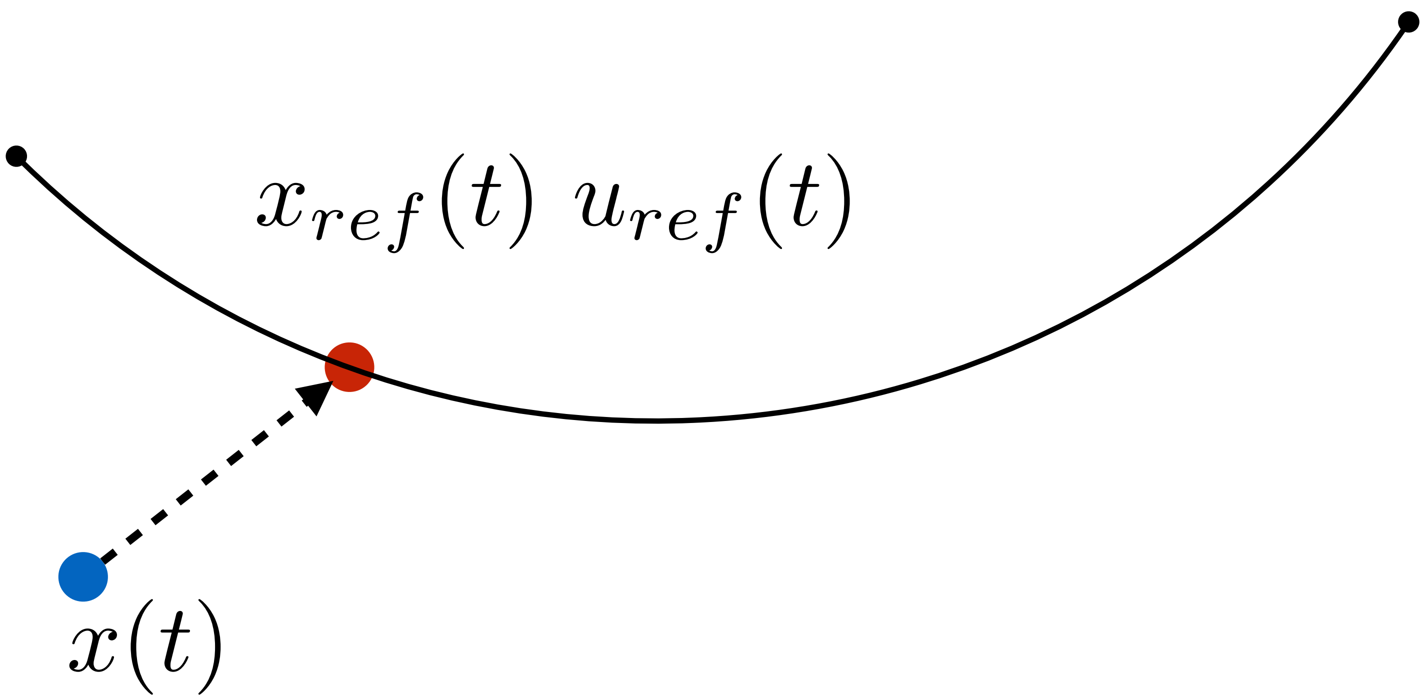

$$\dot{x} = f(x, u)$$

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$$B_t = \left.\frac{\partial f}{\partial u}\right|_{u_{ref}(t)}$$

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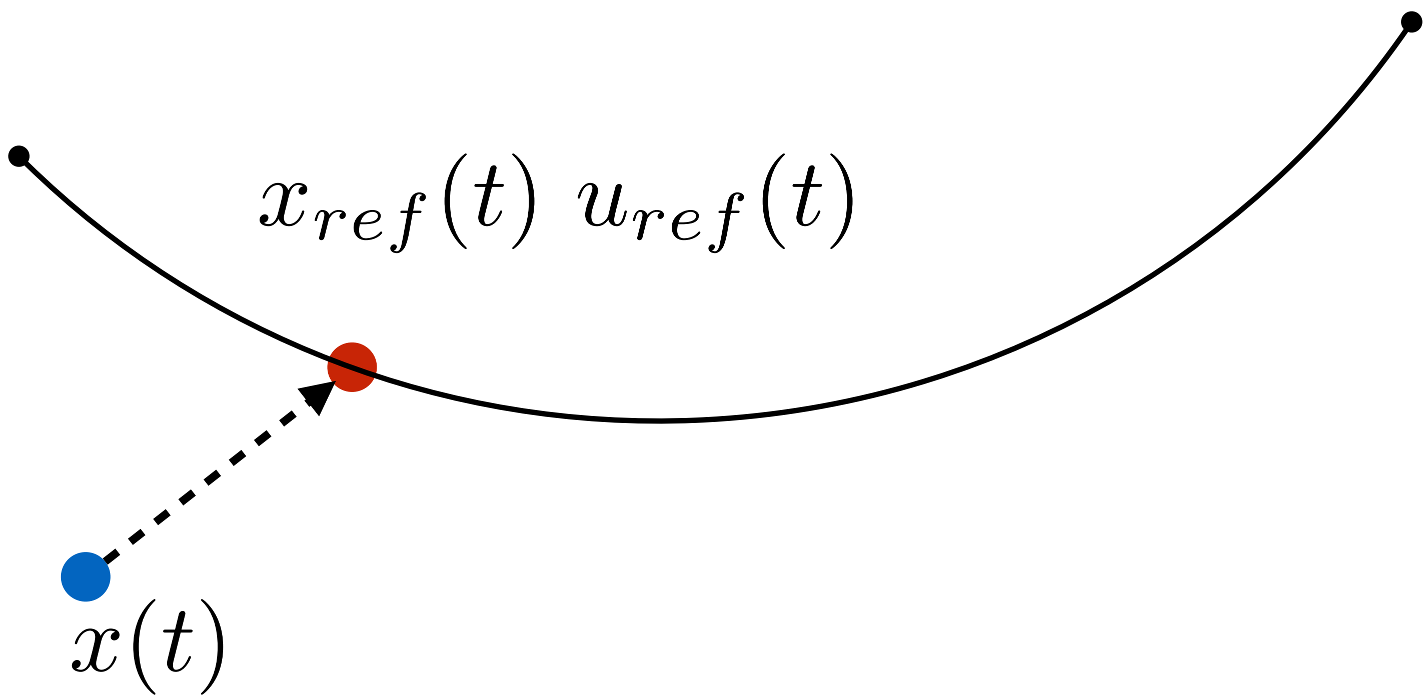

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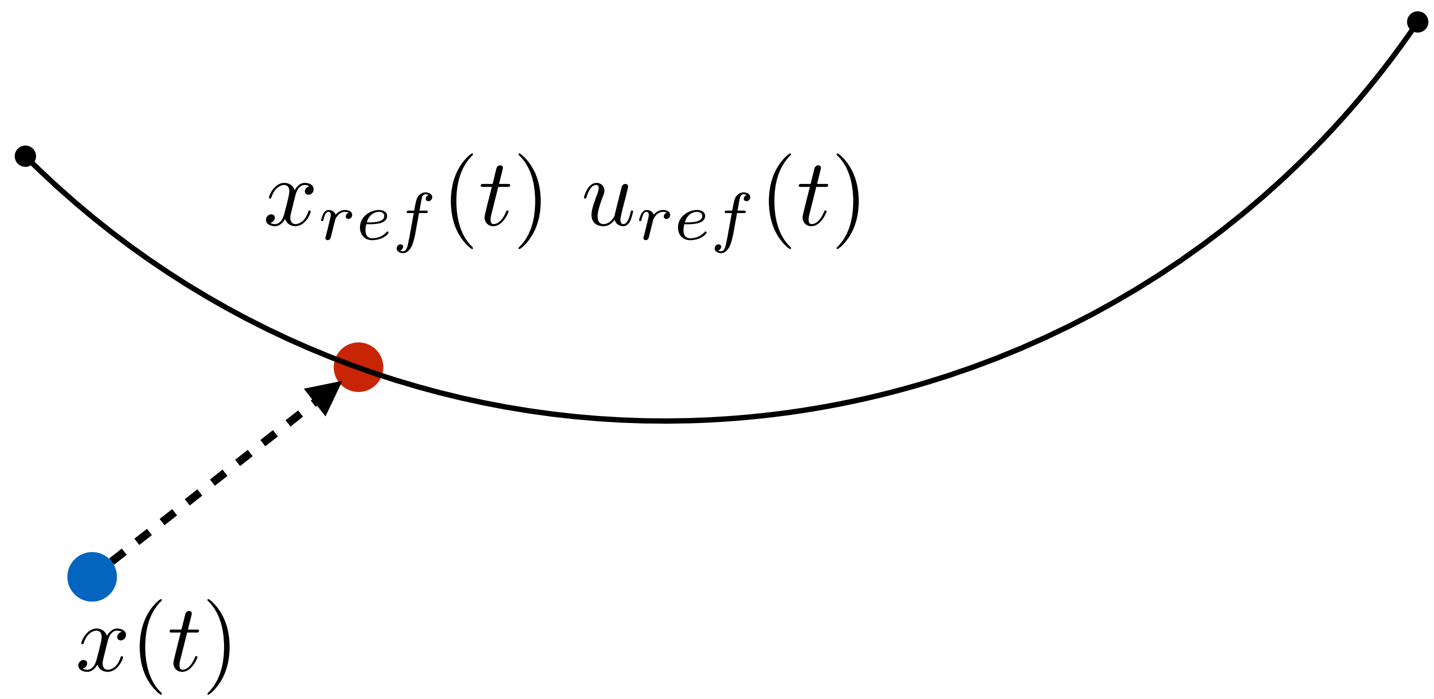
Linearize about a time-varying trajectory



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Making an affine system linear

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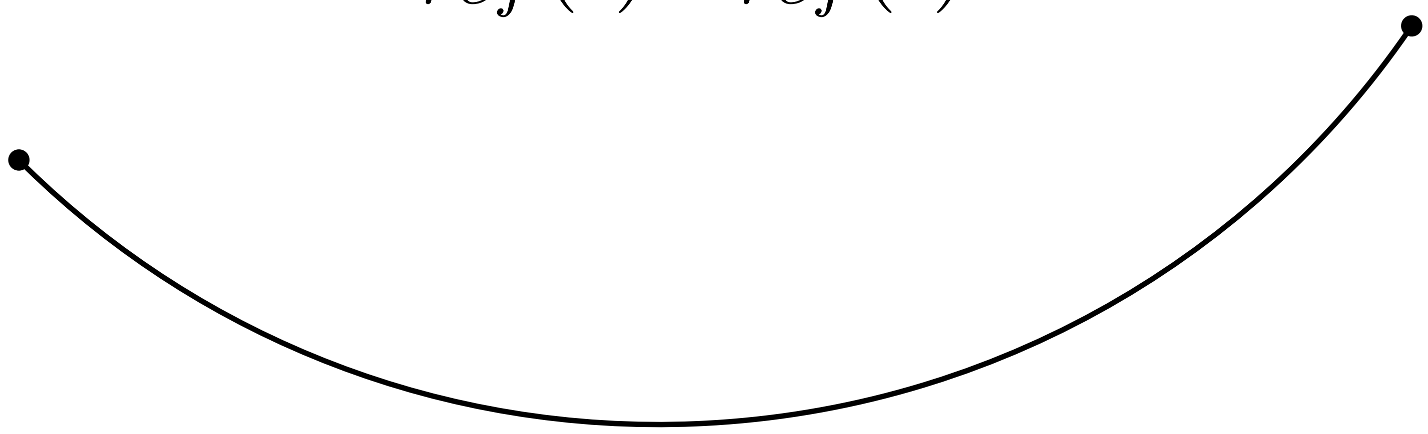
Similarly you can transform cost function

$$c(\tilde{x}_t, u_t) = \tilde{x}_t^T \tilde{Q}_t \tilde{x}_t + u_t^T R_t u_t$$

Shape of the value function changes along trajectory

$x_{ref}(t)$ $u_{ref}(t)$



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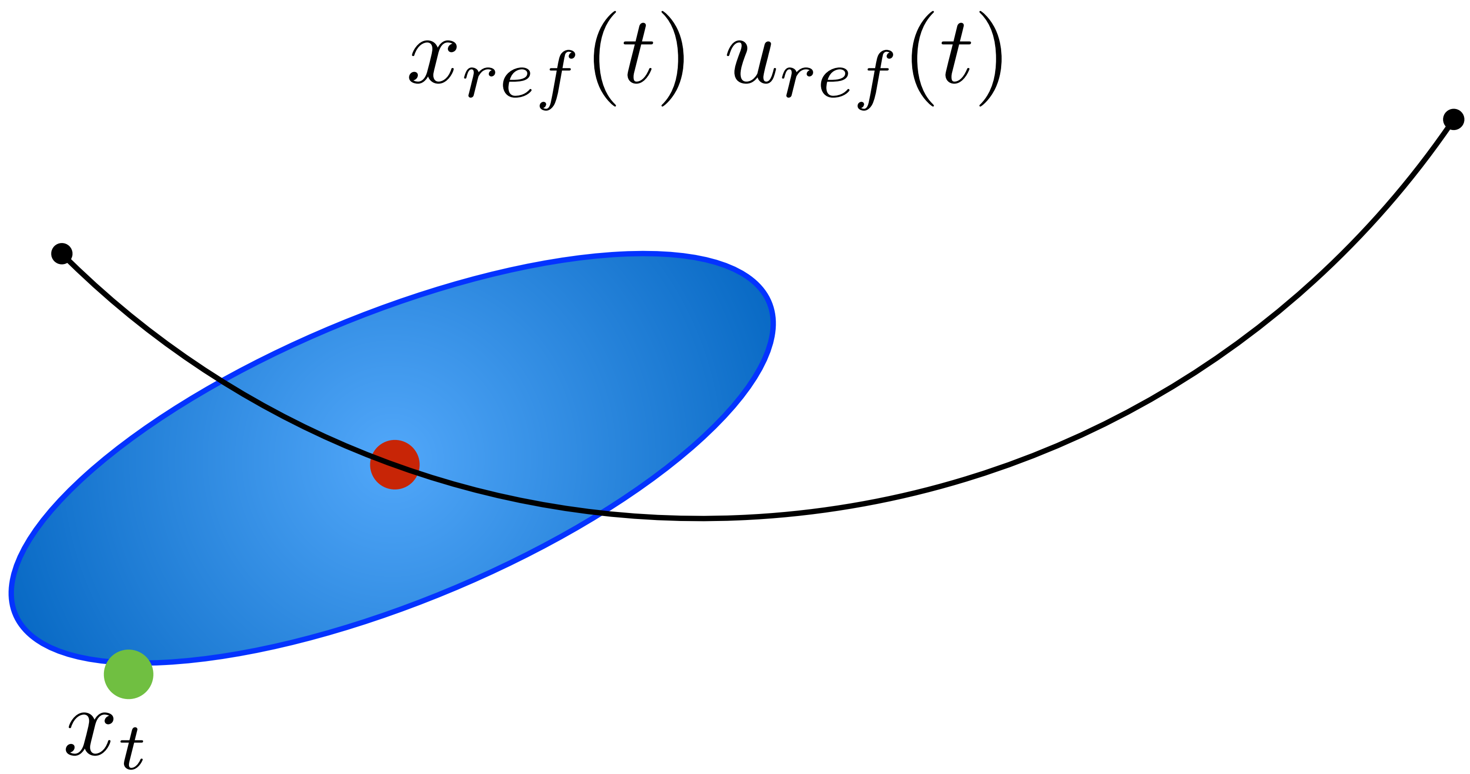

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$\tilde{R}_t$

$\tilde{K}_t, \tilde{V}_t$

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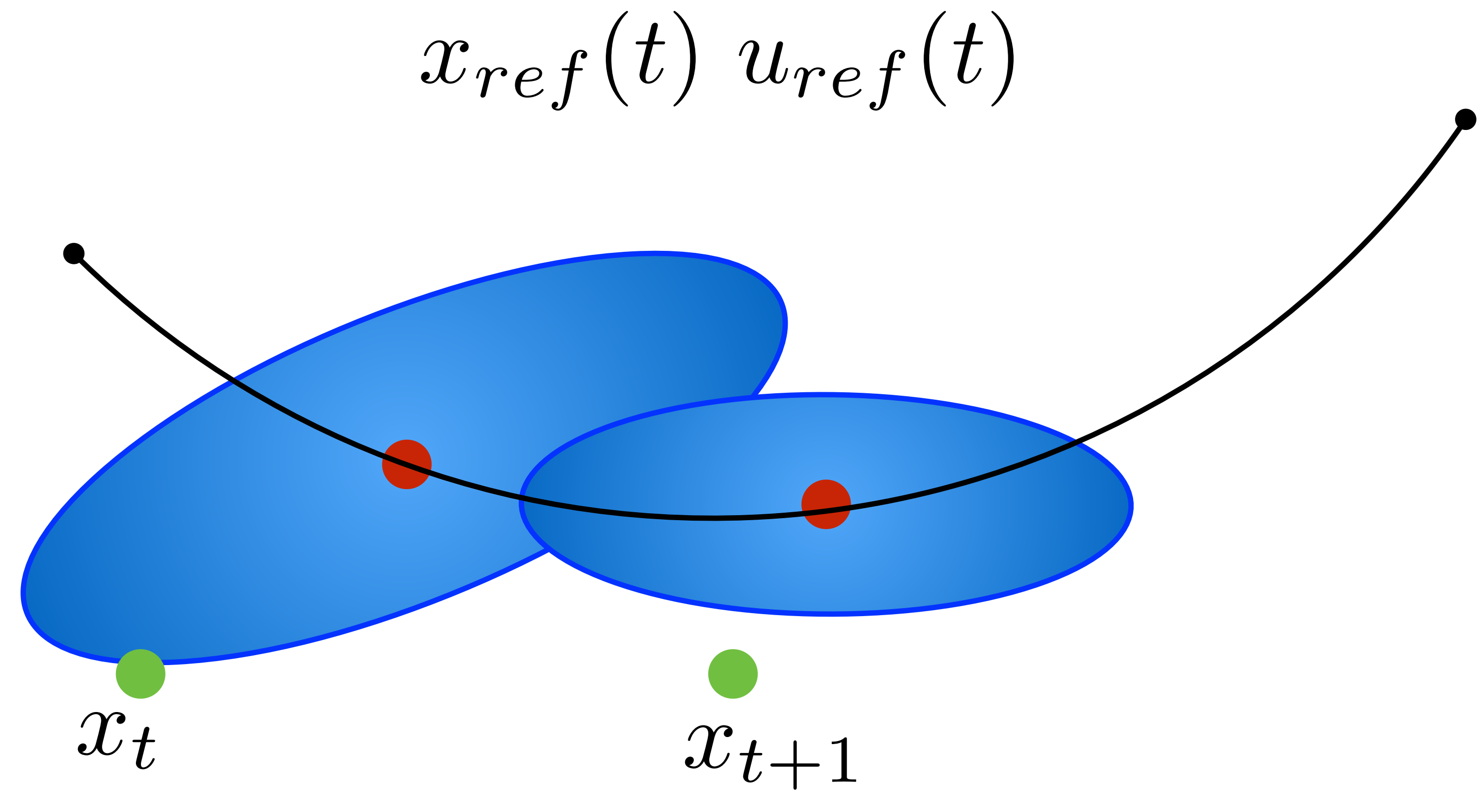

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$\vdots$

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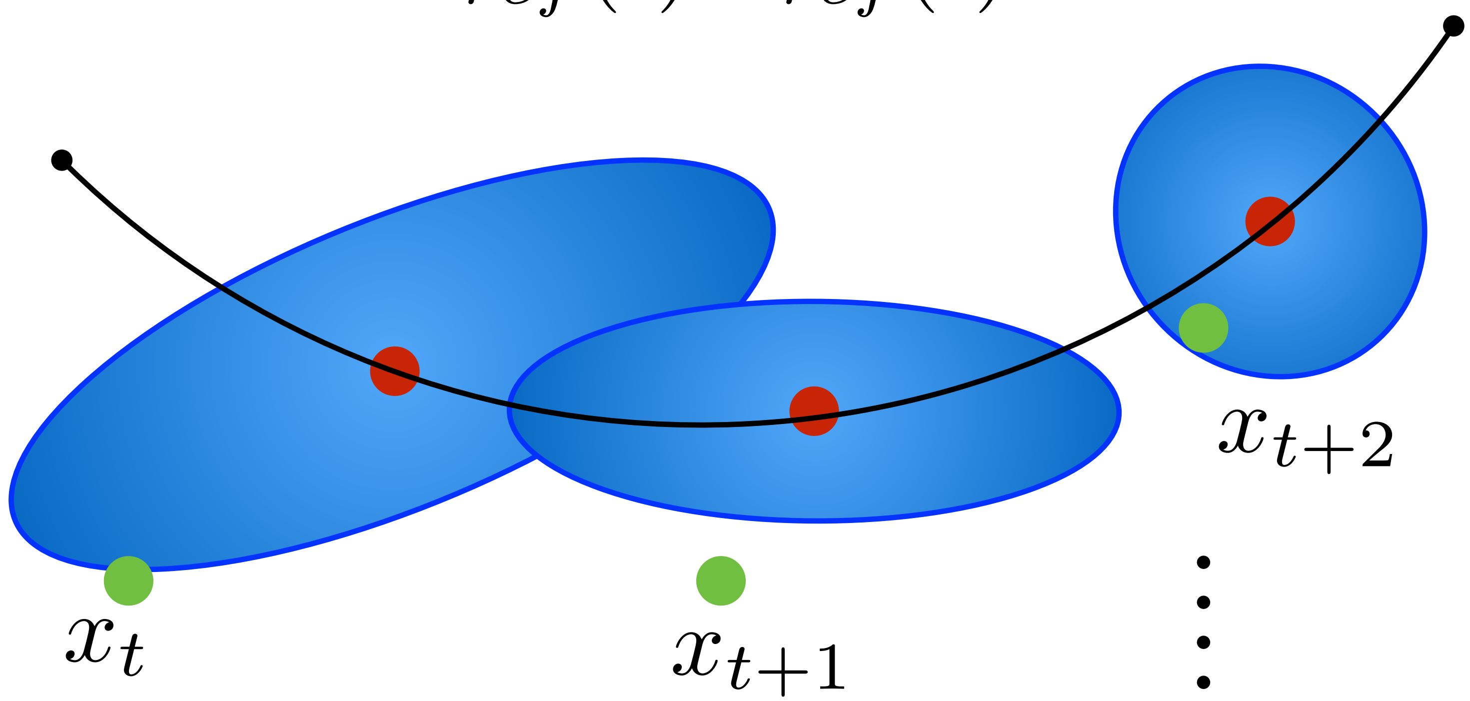

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4. Can we handle noisy dynamics?

   Yes! Gaussian noise does not change the answer

Trivia: Duality between control and estimation

R. Kalman "A new approach to linear filtering and prediction problems." (1960)

linear-quadratic regulator	**Kalman-Bucy filter**	
V	Σ	(state variance)
A	A^T	(dynamics)
B	H^T	(measurement)
R	DD^T	(dynamics noise)
Q	CC^T	(motion noise)
t	$t_f - t$	

(Table from E.Todorov "General duality between optimal control and estimation", CDC, 2008)