# Probabilistic Models

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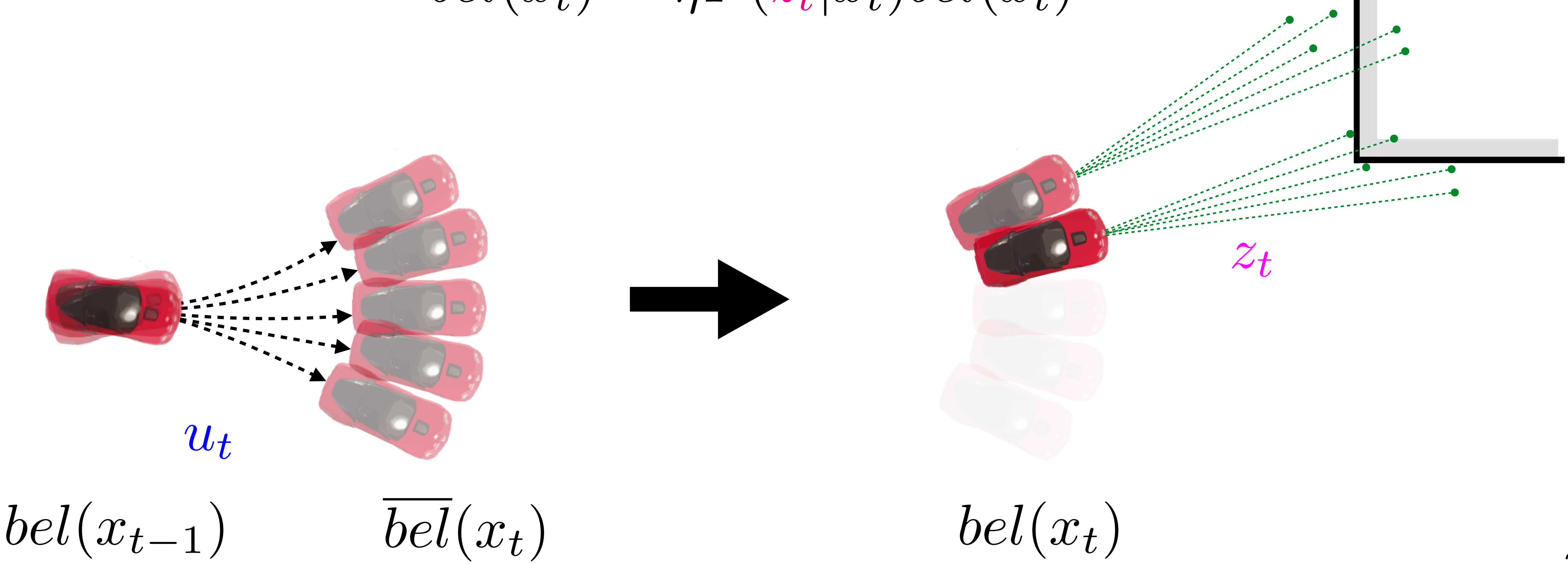
# Bayes filter in a nutshell

Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \int P(x_t|u_t, x_{t-1})bel(x_{t-1})dx_{t-1}$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$



# Today's objective

1. Briefly discuss different paradigms of Bayes filtering

2. Probabilistic motion model

3. (If time remaining) Having fun with 1-D Kalman filter!

# Bayes filter is a powerful tool

Localization Mapping SLAM POMDP

# Casting different tasks as Bayes filtering

| Tasks | State | Action | Measurement |
| --- | --- | --- | --- |
| Localization | | | |
| Mapping | | | |
| SLAM | | | |
| Pursuit-Evasion | | | |

# Casting different tasks as Bayes filtering

| Tasks | State | Action | Measurement |
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| Pursuit-Evasion | Pose of target | Guess where target can move | Camera image |

# Assembling Bayes filter

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Tasks

Localization

P(pose | data)

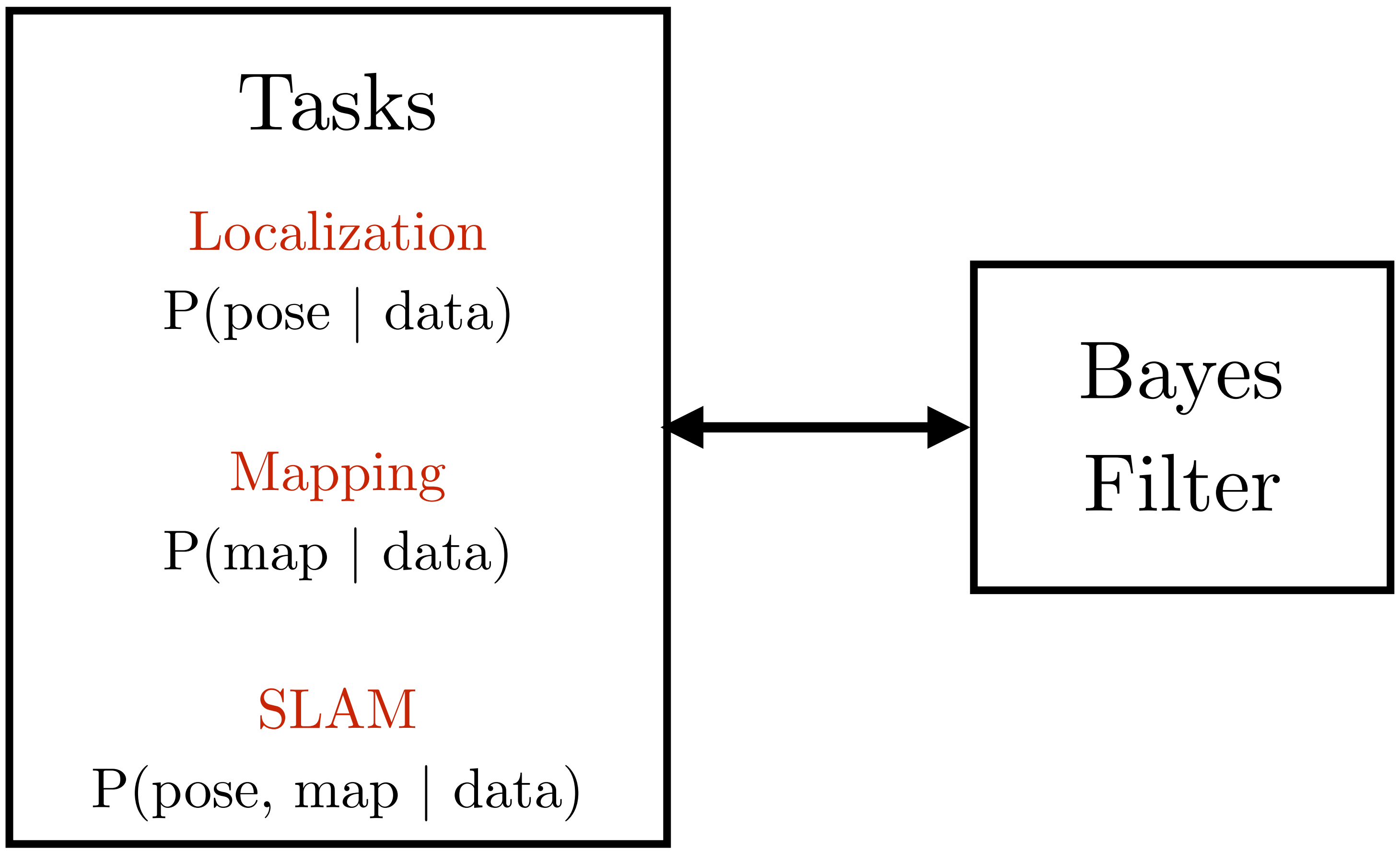
Mapping

P(map | data)

SLAM

P(pose, map | data)

# Assembling Bayes filter



# Assembling Bayes filter

Tasks

Localization

P(pose | data)

Mapping

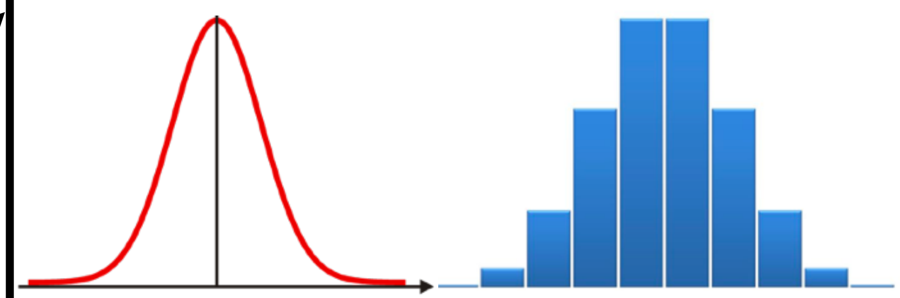
P(map | data)

SLAM

P(pose, map | data)

Bayes
Filter

Belief Representations



# Assembling Bayes filter

**Tasks**

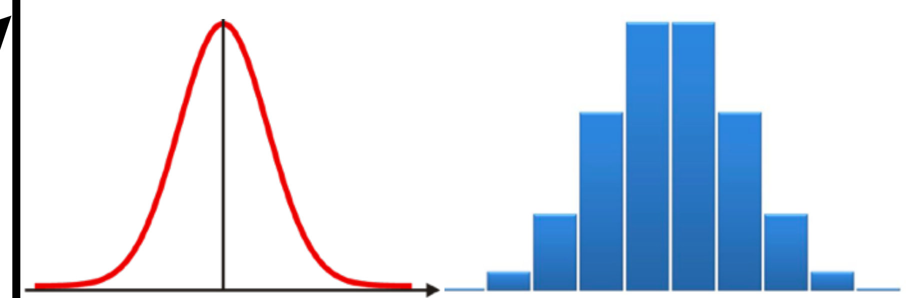
Localization
P(pose | data)

Mapping
P(map | data)

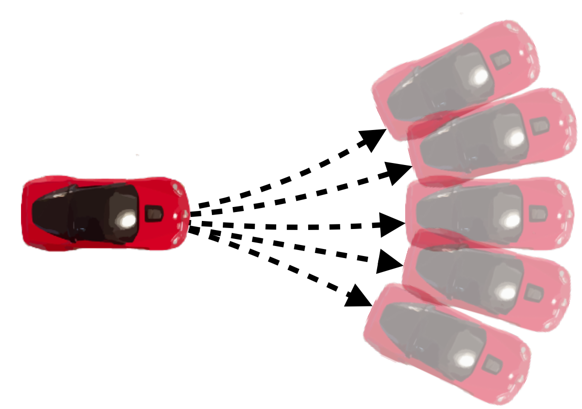
SLAM
P(pose, map | data)

**Bayes Filter**

**Belief Representations**



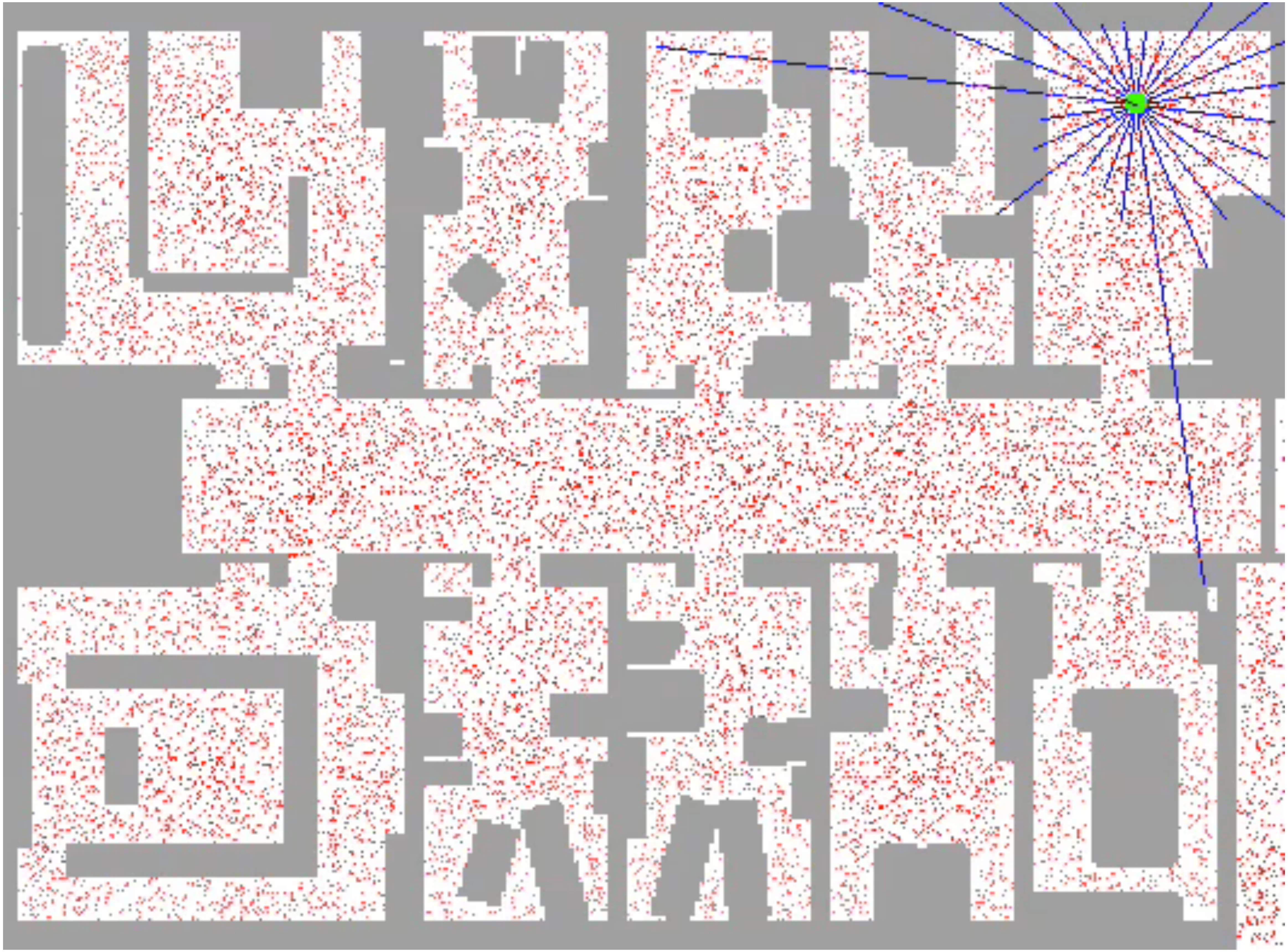
**Probabilistic Models**



# Tasks that we will cover

| Tasks | Belief Representation | Probabilistic Models |
|---|---|---|
| Localization<br>P(pose \| data)<br>(Week 3) | Gaussian / Particles | Motion model<br>Measurement model |
| Mapping<br>P(map \| data)<br>(Week 4) | Discrete (binary) | Inverse measurement model |
| SLAM<br>P(pose, map \| data)<br>(Week 4) | Particles+Gaussian<br>(pose, landmarks) | Motion model,<br>measurement model,<br>correspondence model |

# What is localization?



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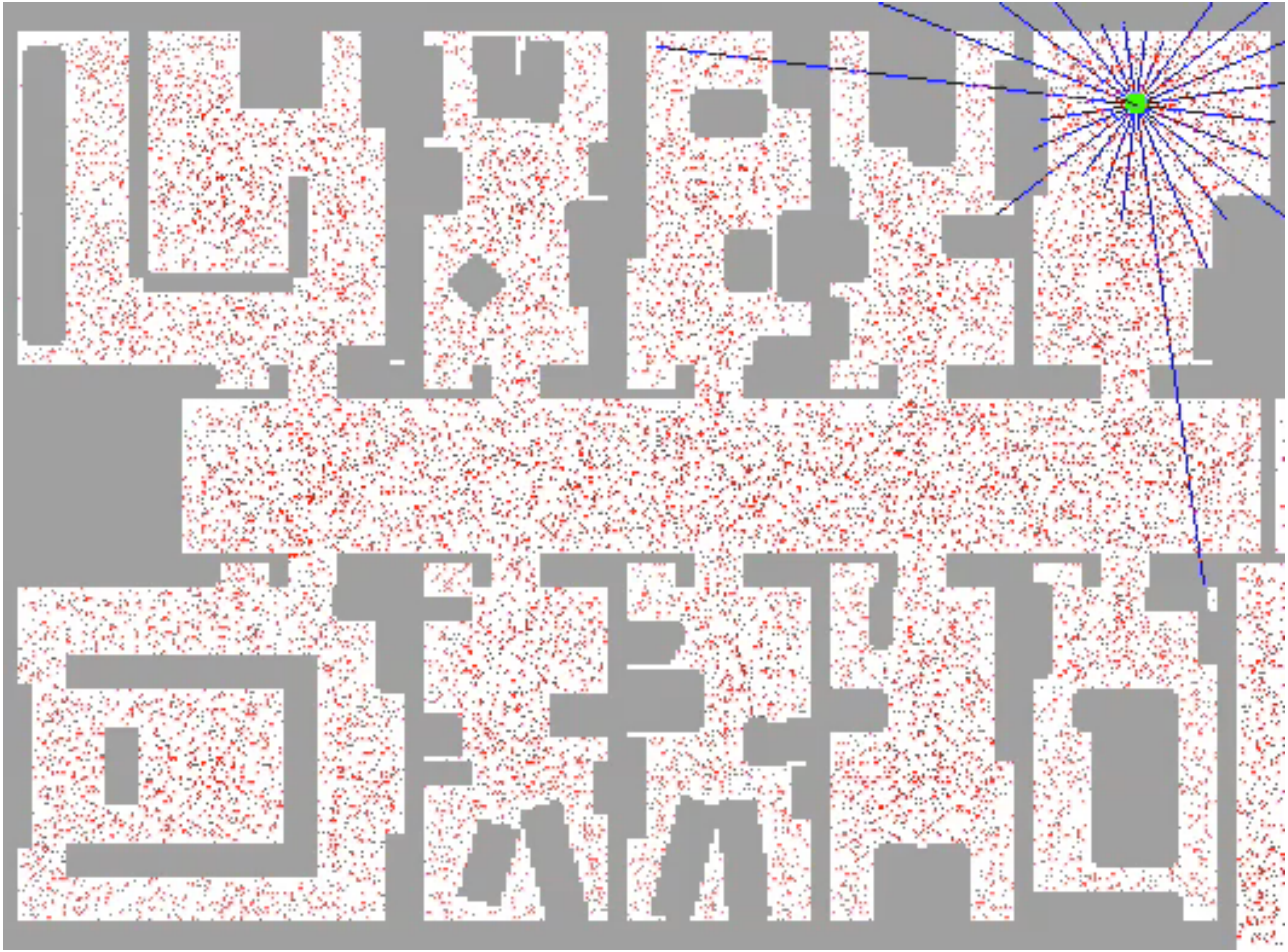
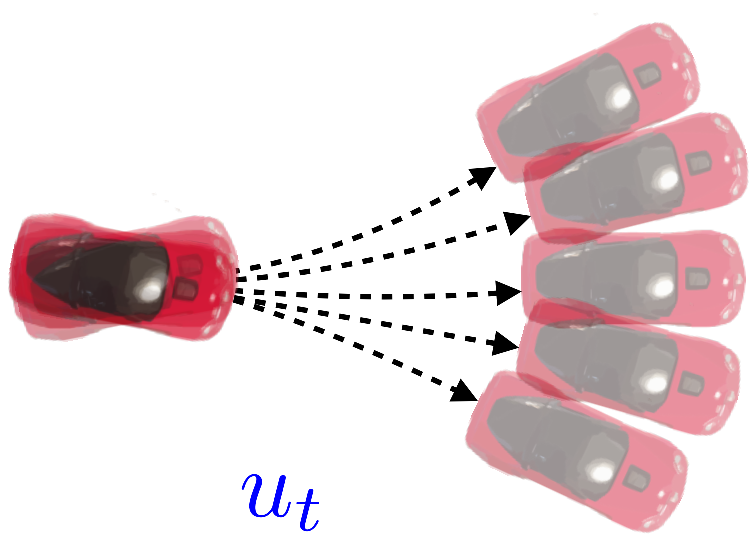


# Probabilistic models in localization

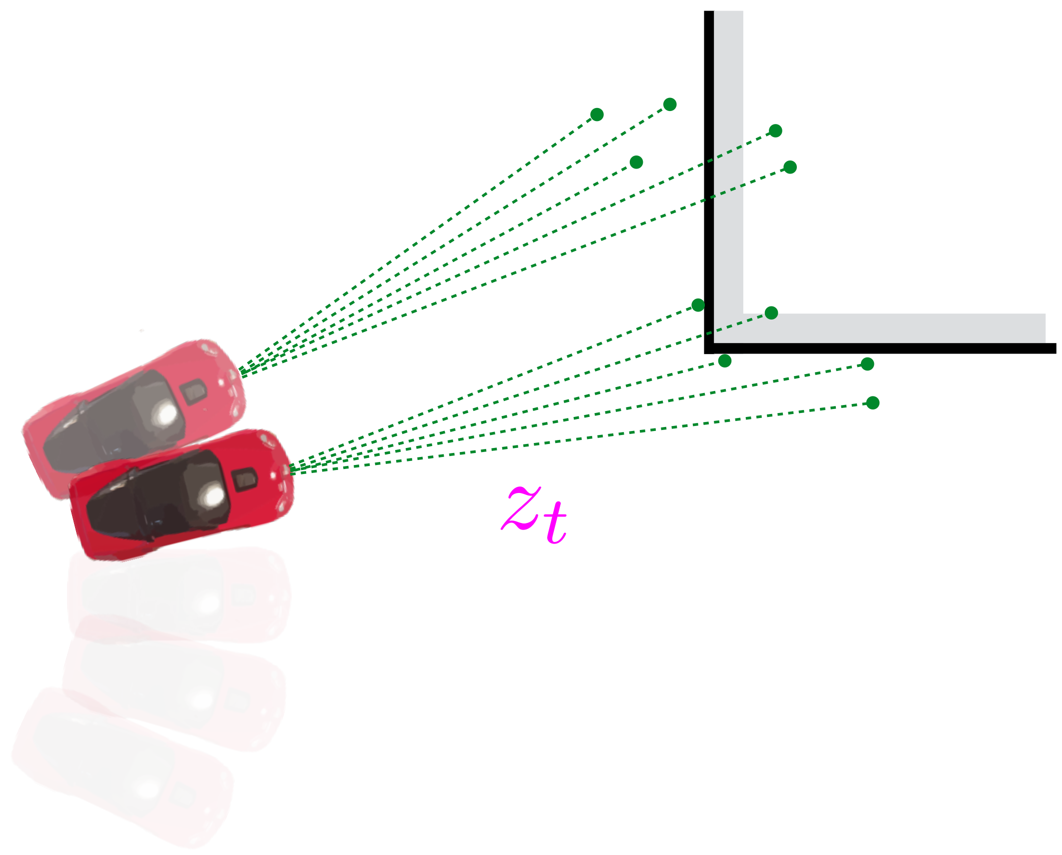
Motion model

$$P(x_t | u_t, x_{t-1})$$




Measurement model

$$P(z_t | x_t)$$



# How do we think about models?

# Three questions you should ask

1. Why is the model probabilistic?

2. What defines a good model?

3. What model should I use for my robot?

# Motion Model

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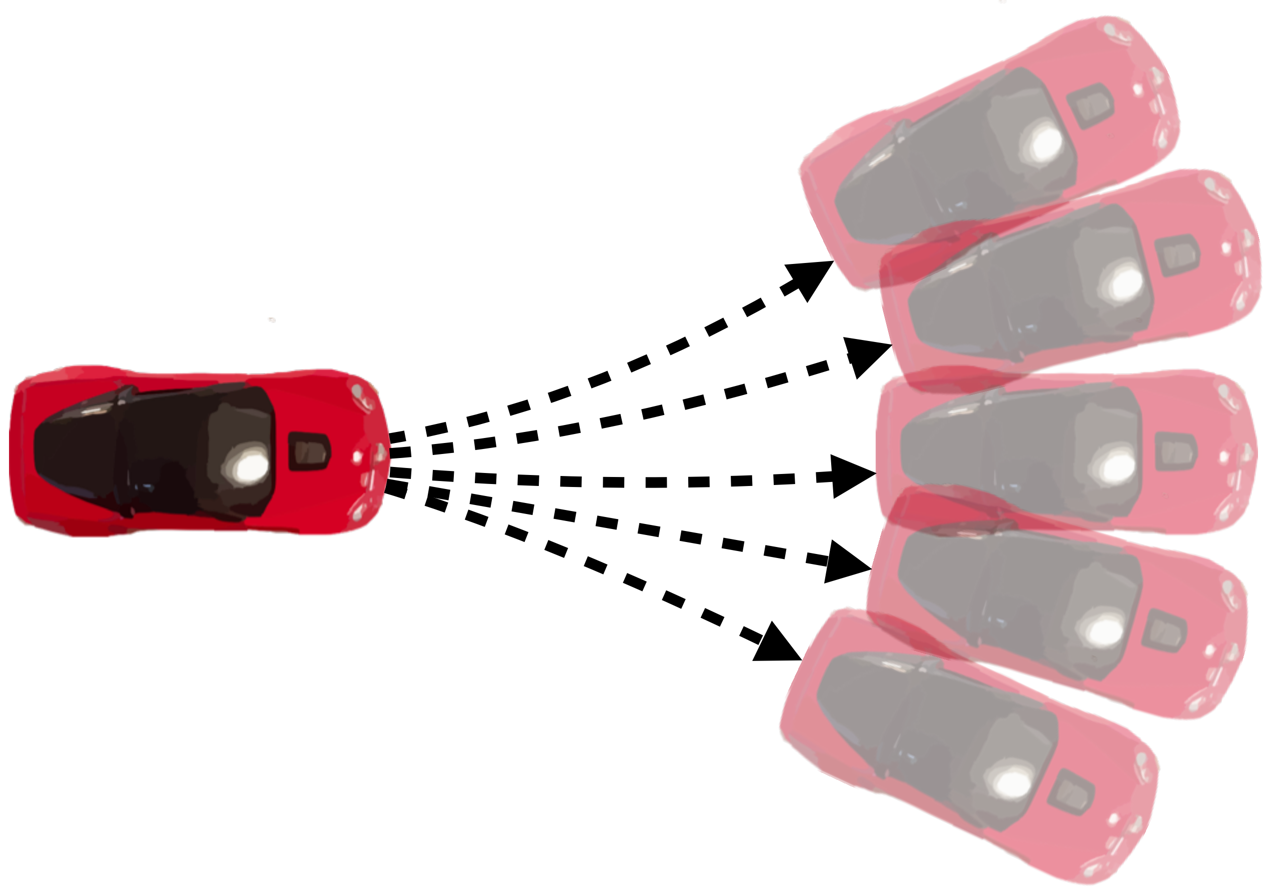
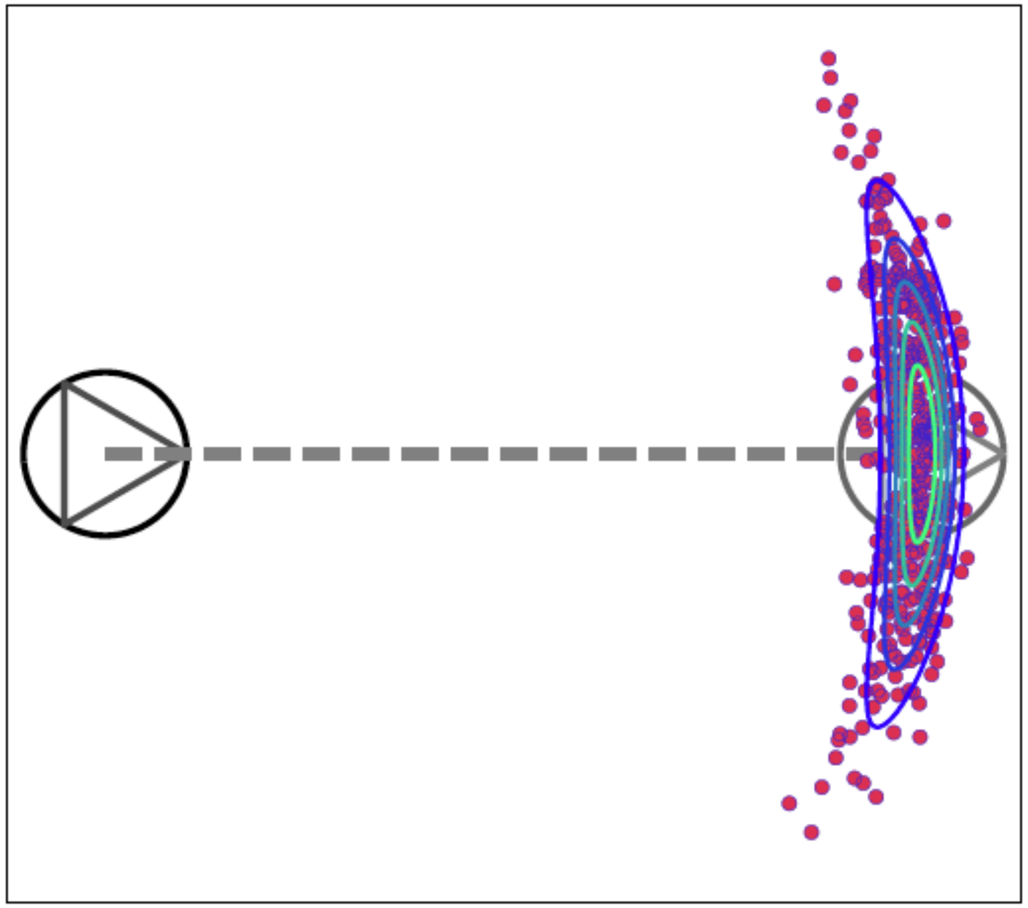


# Spectrum of motion models

(Redbull simulator)


vs



Highest fidelity models of everything

Simple model with lots of noise

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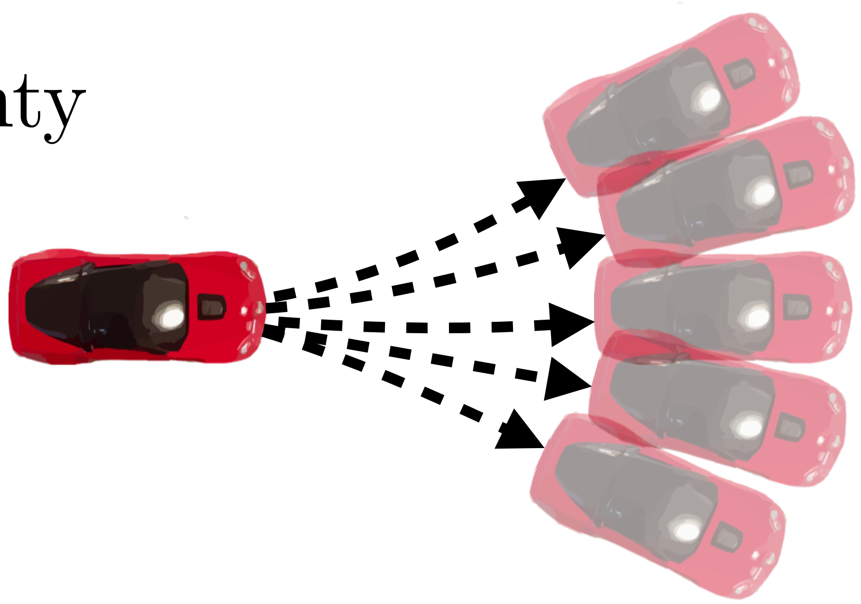
In theory - try to accurately model uncertainty



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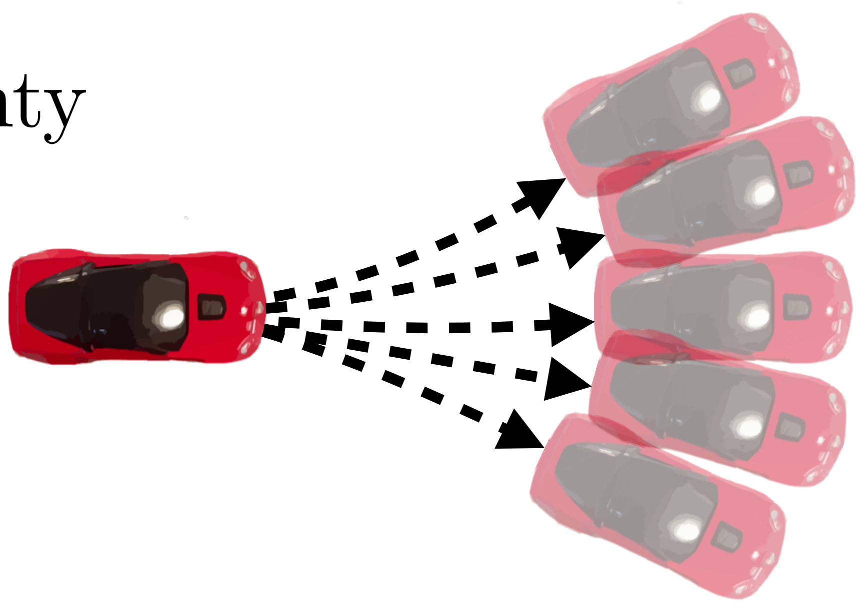

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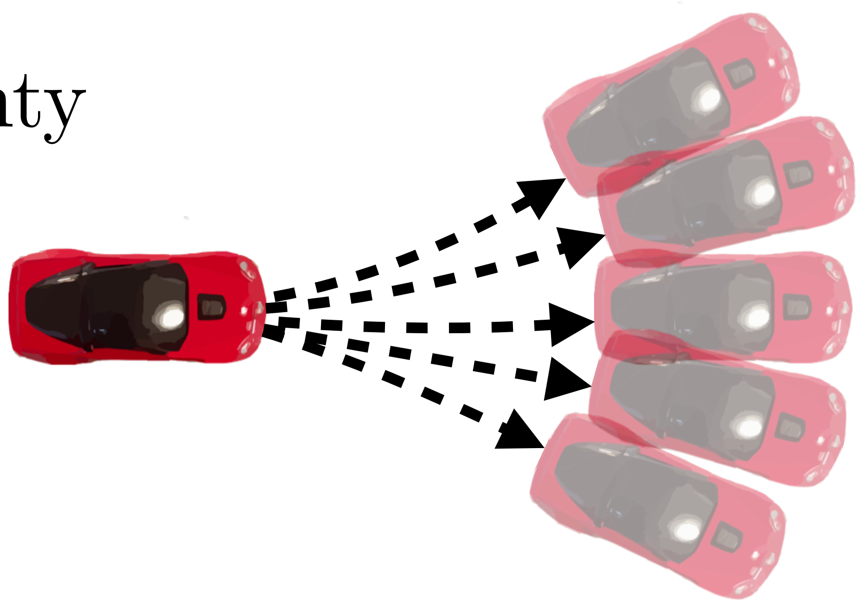

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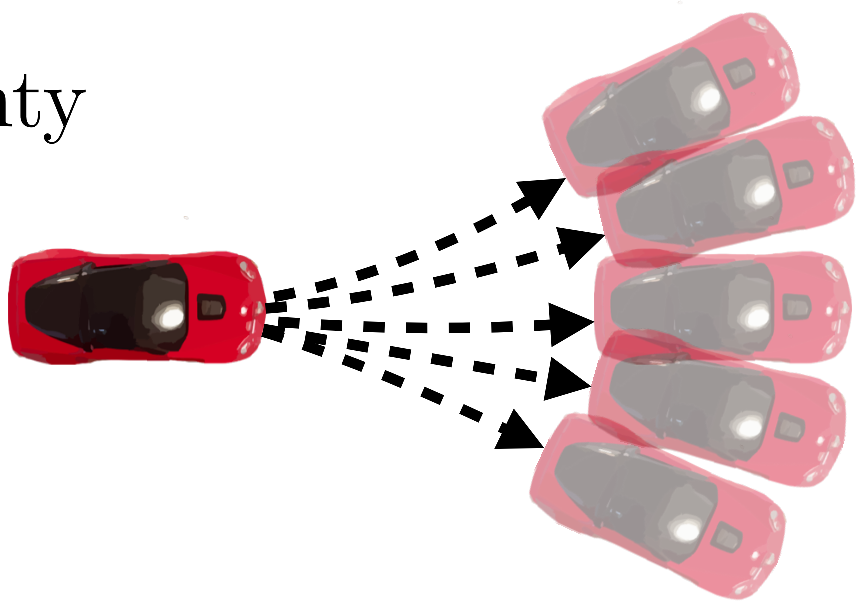

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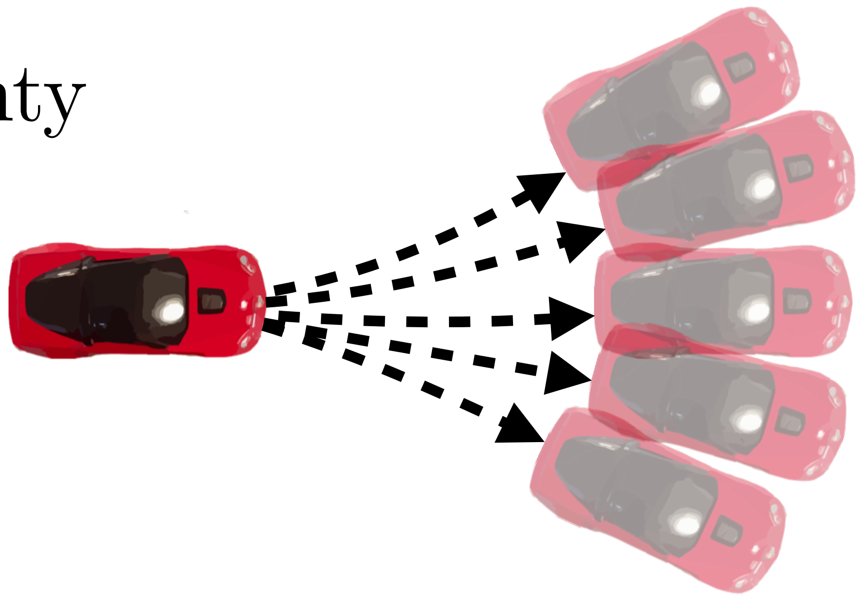

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1. We need something that is computationally cheap

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2. We need just enough stochasticity to explain any measurements we may see

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3. We need a model that can deal with unknown unknown

(No matter what the model, we need to overestimate uncertainty)

Key Idea:

Simple model + Stochasticity

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# A simple model: Kinematic car

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Assume wheels rolls on hard, flat, horizontal ground without slipping

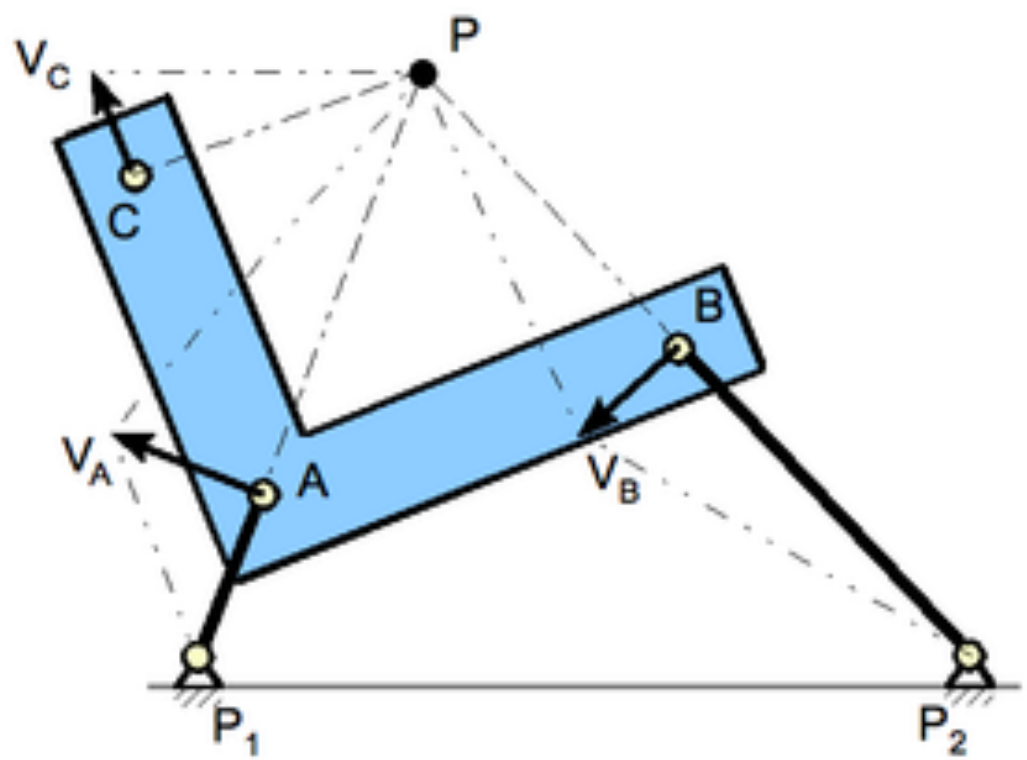
# Motion model

$$P(x_t | u_t, x_{t-1})$$

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \begin{matrix} \text{(x-coord)} \\ \text{(y-coord)} \\ \text{(heading)} \end{matrix}$$

$$\begin{bmatrix} V \\ \delta \end{bmatrix} \begin{matrix} \text{(speed)} \\ \text{(steering angle)} \end{matrix}$$

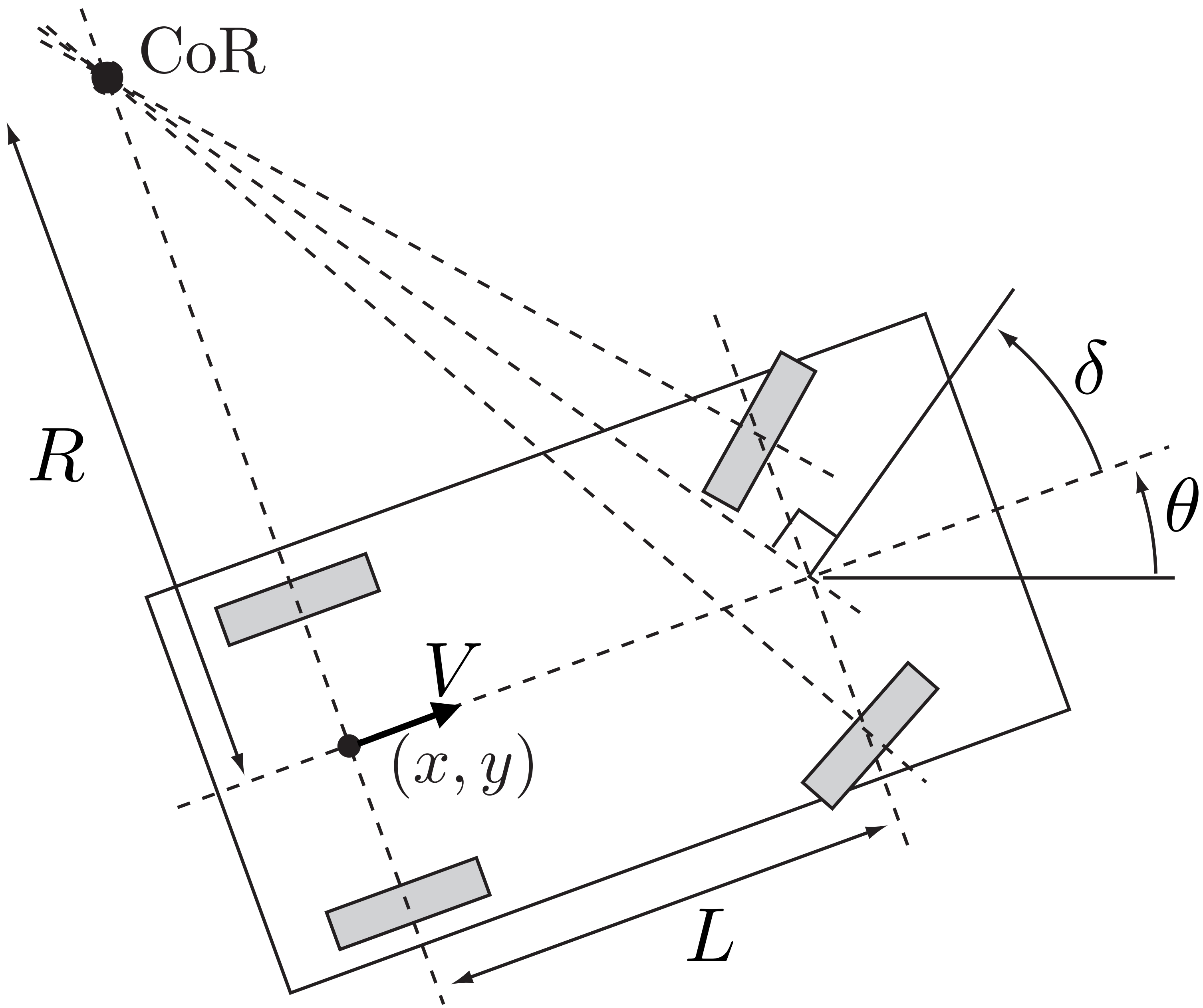
# Instant centre of rotation (CoR)




A rigid body undergoing rotation and translation can be viewed as pure rotation about a instant centre of rotation.

https://en.wikipedia.org/wiki/Instant_centre_of_rotation

# Equations of motion for rear axel



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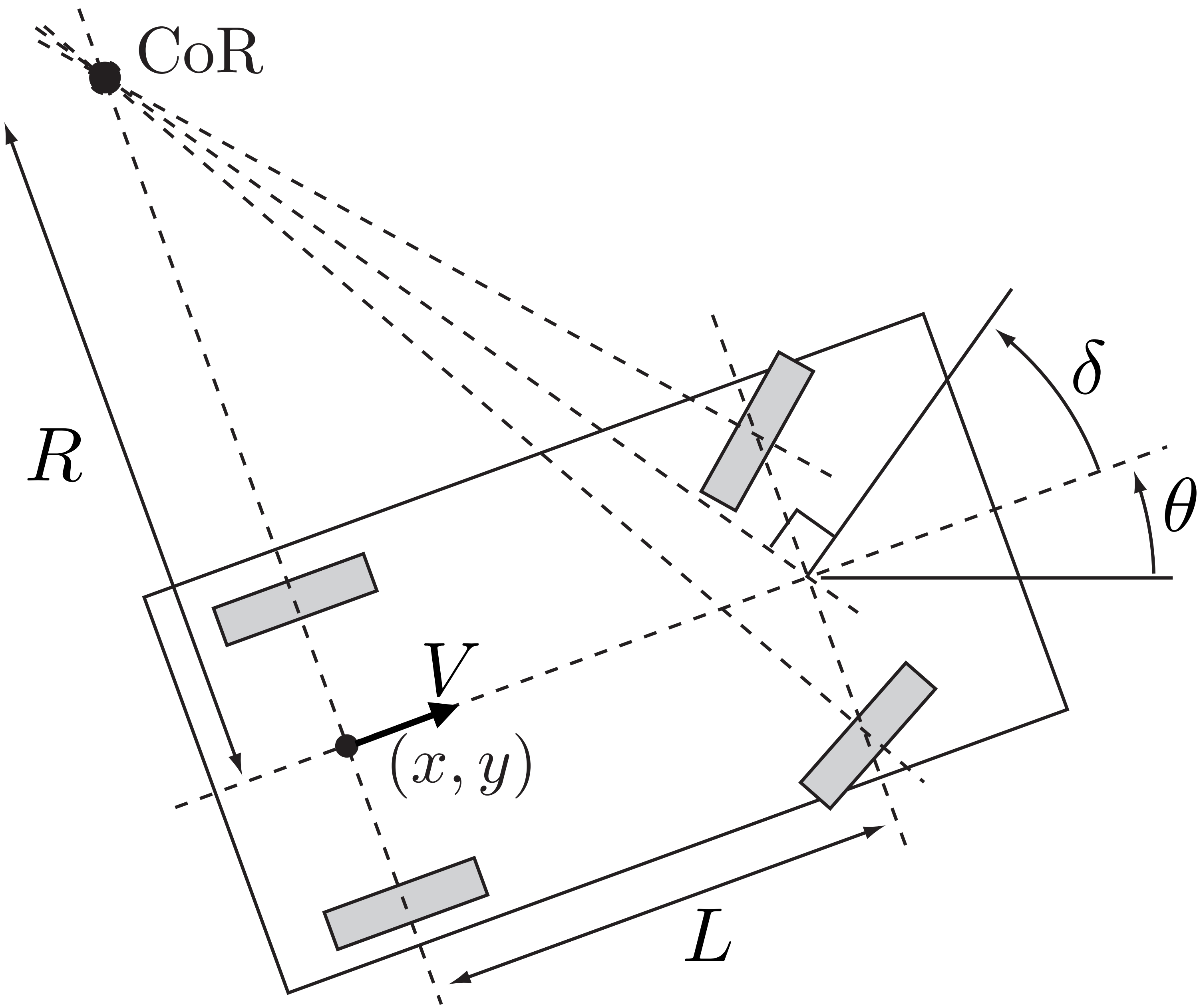


$$\tan\delta = \frac{L}{R}$$

$$R = \frac{L}{\tan\delta}$$

$$\omega = \frac{V}{R} = \frac{V\tan\delta}{L}$$

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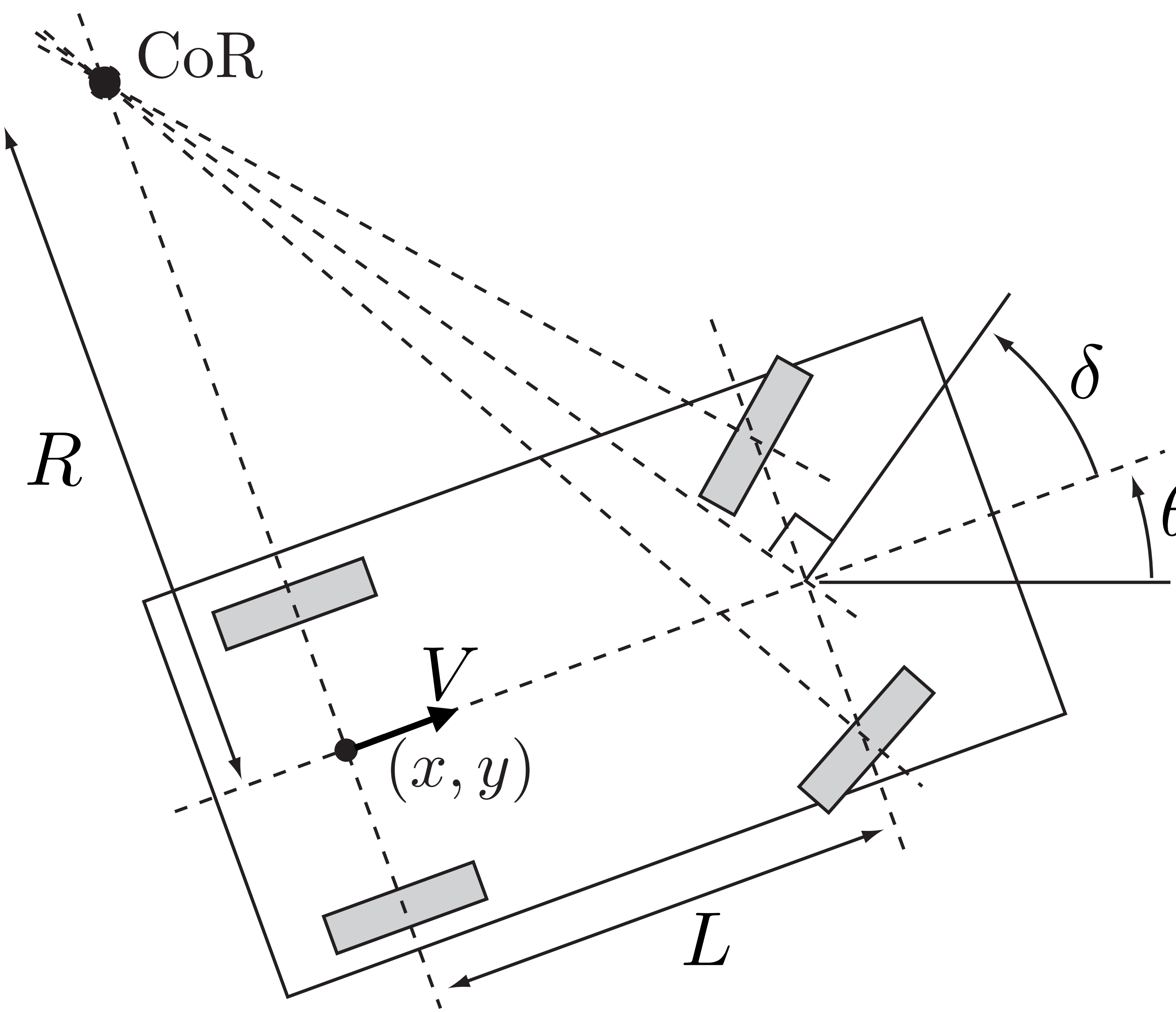


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$$R = \frac{L}{\tan\delta}$$

$$\omega = \frac{V}{R} = \frac{V\tan\delta}{L}$$

$$\dot{x} = V\cos(\theta)$$

$$\dot{y} = V\sin(\theta)$$

$$\dot{\theta} = \omega = \frac{V\tan\delta}{L}$$

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$$y_{t+1} - y_t = \frac{L}{\tan \delta} (-\cos \theta_{t+1} + \cos \theta_t)$$

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What are the sources of ~~noise~~ stochasticity?

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# Stochasticity

1. Control signal error

$$\hat{V} \sim \mathcal{N}(V, \sigma_v^2) \qquad \hat{\delta} \sim \mathcal{N}(\delta, \sigma_\delta^2)$$

2. Unknown physics parameters

$$\hat{L} \sim \mathcal{N}(L, \sigma_L^2)$$

3. Incorrect physics

$$\hat{x} \sim \mathcal{N}(x, \sigma_x^2) \qquad \hat{y} \sim \mathcal{N}(y, \sigma_y^2) \qquad \hat{\theta} \sim \mathcal{N}(\theta, \sigma_\theta^2)$$

# Questions

1. Can you derive the equations of motion for front axel?

2. Can you derive the equations of motion for centre of mass?