Having fun with 1-D Kalman Filter

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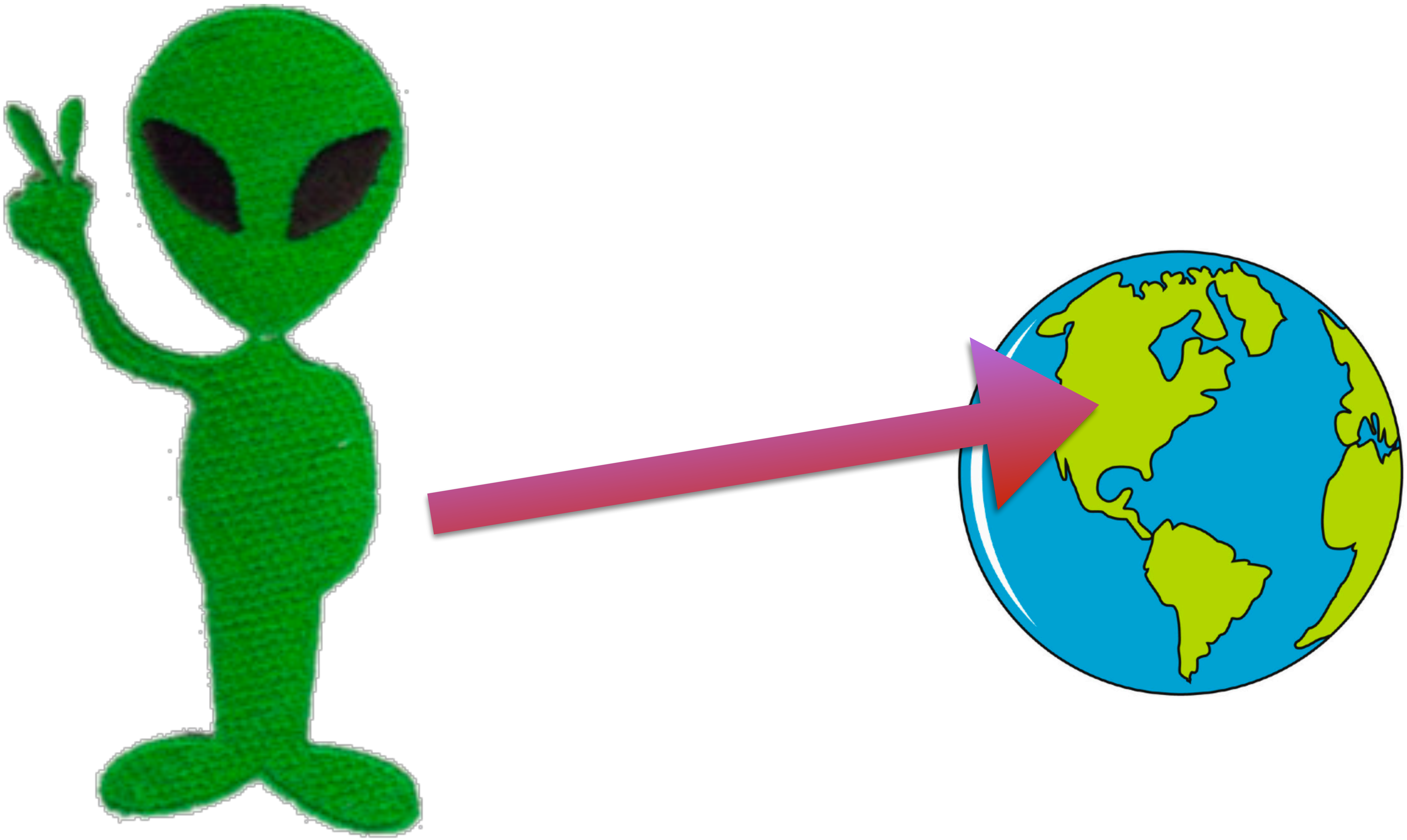
Suppose you are an alien beamed to earth ...

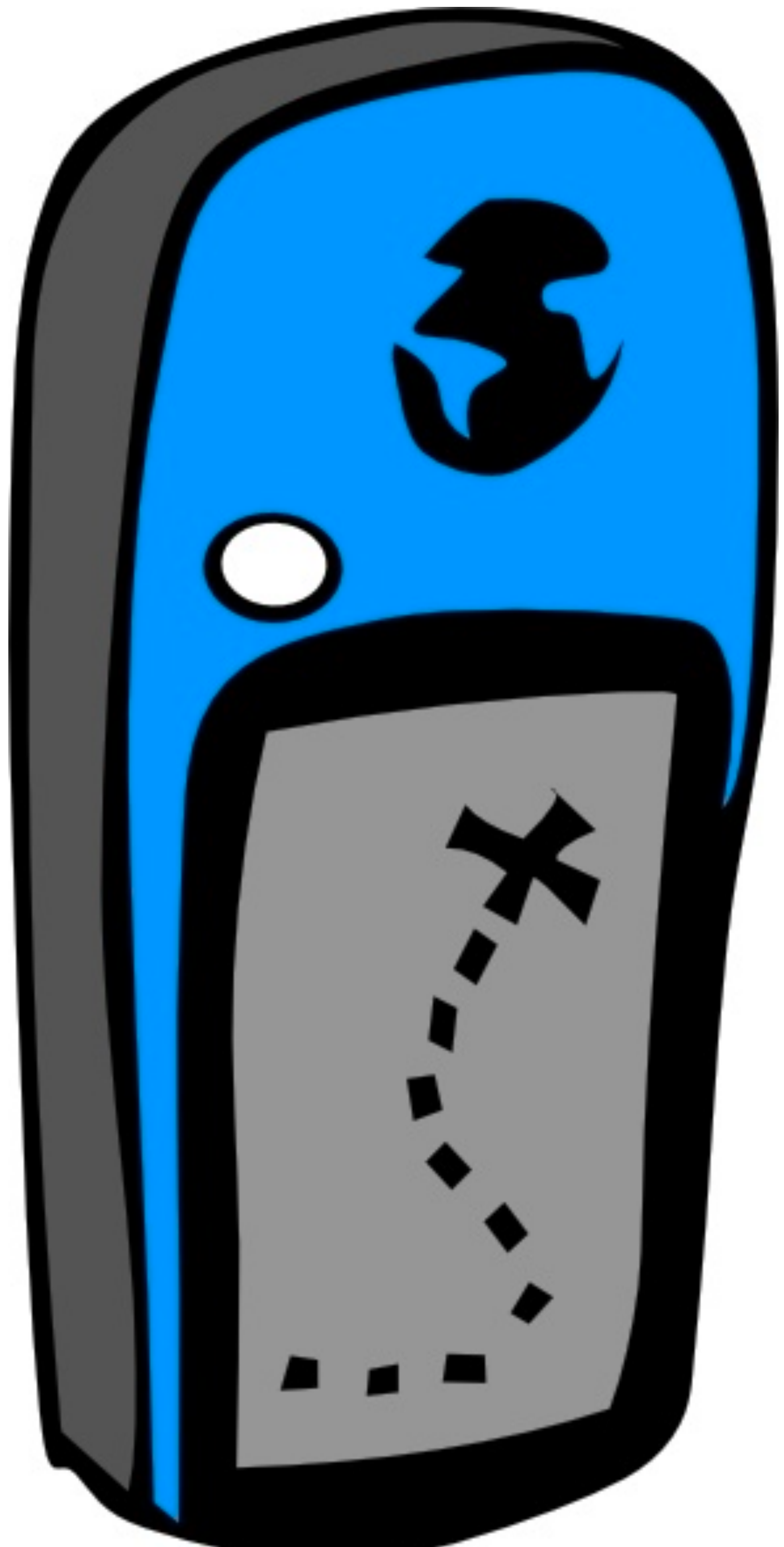


.. and you think you landed in UW

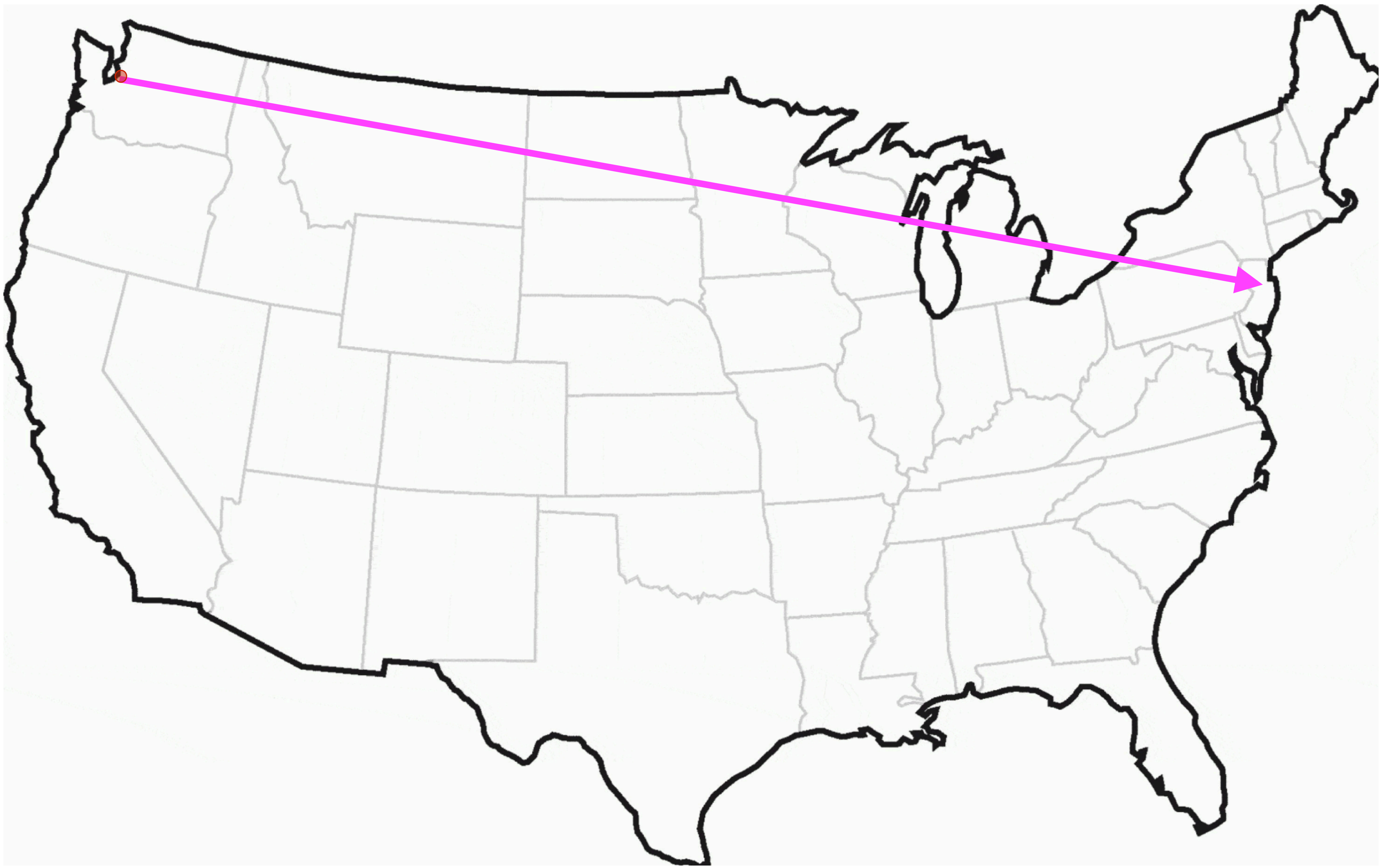
Current belief

$$bel(x_t)$$

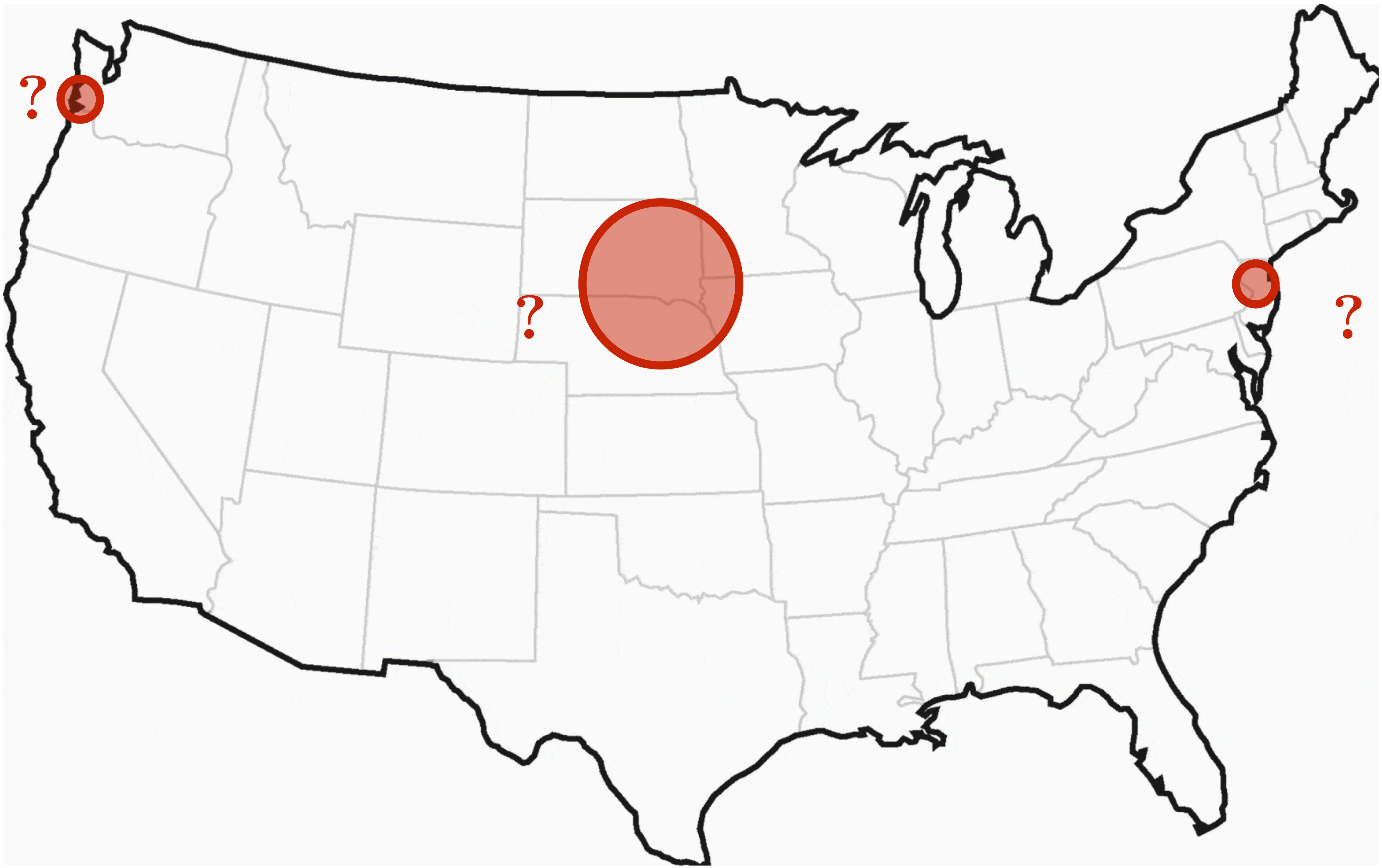
Eventually GPS measurement comes in...



… and says you are in New York



What should we set as our new belief?

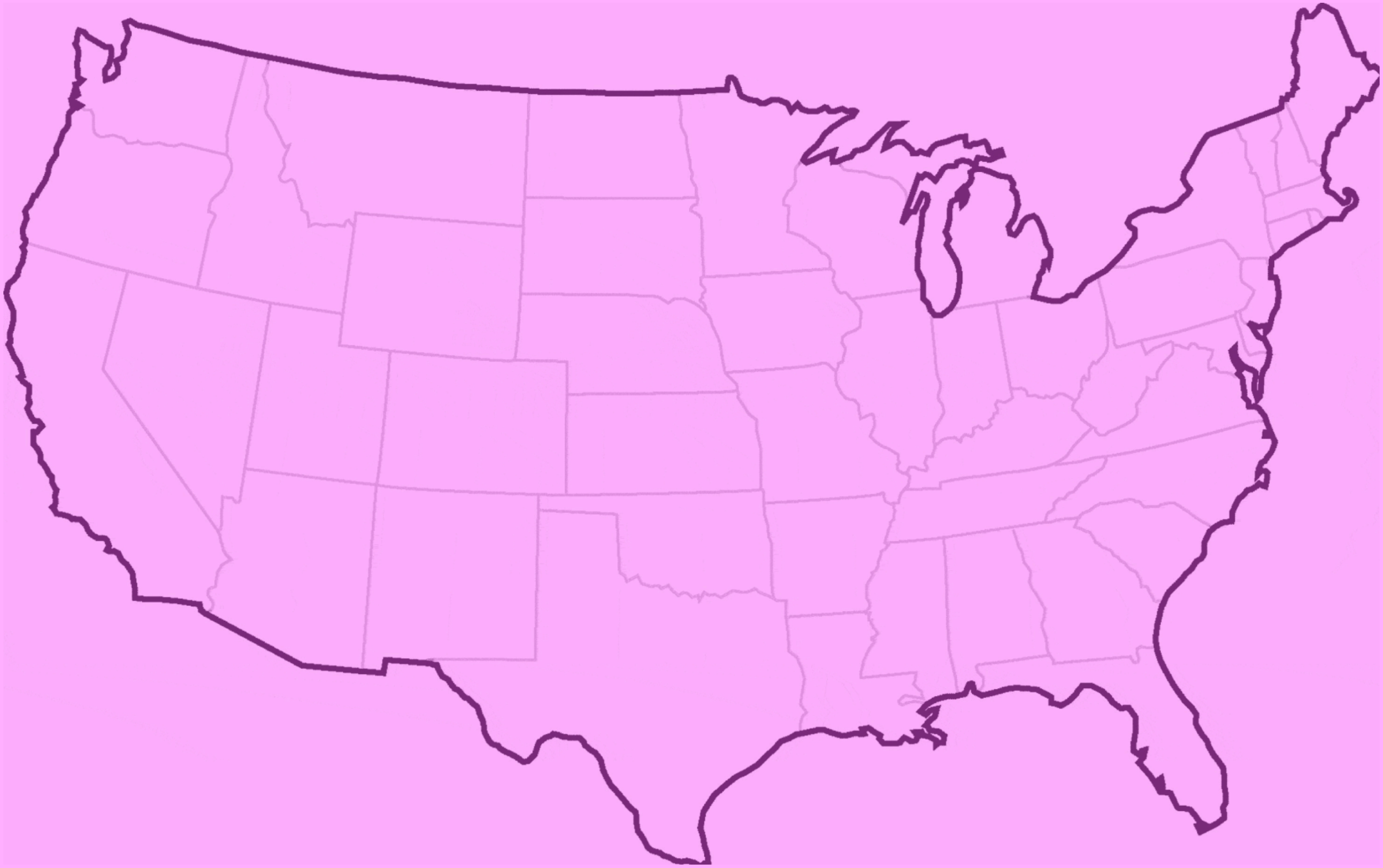


Depends on measurement uncertainty

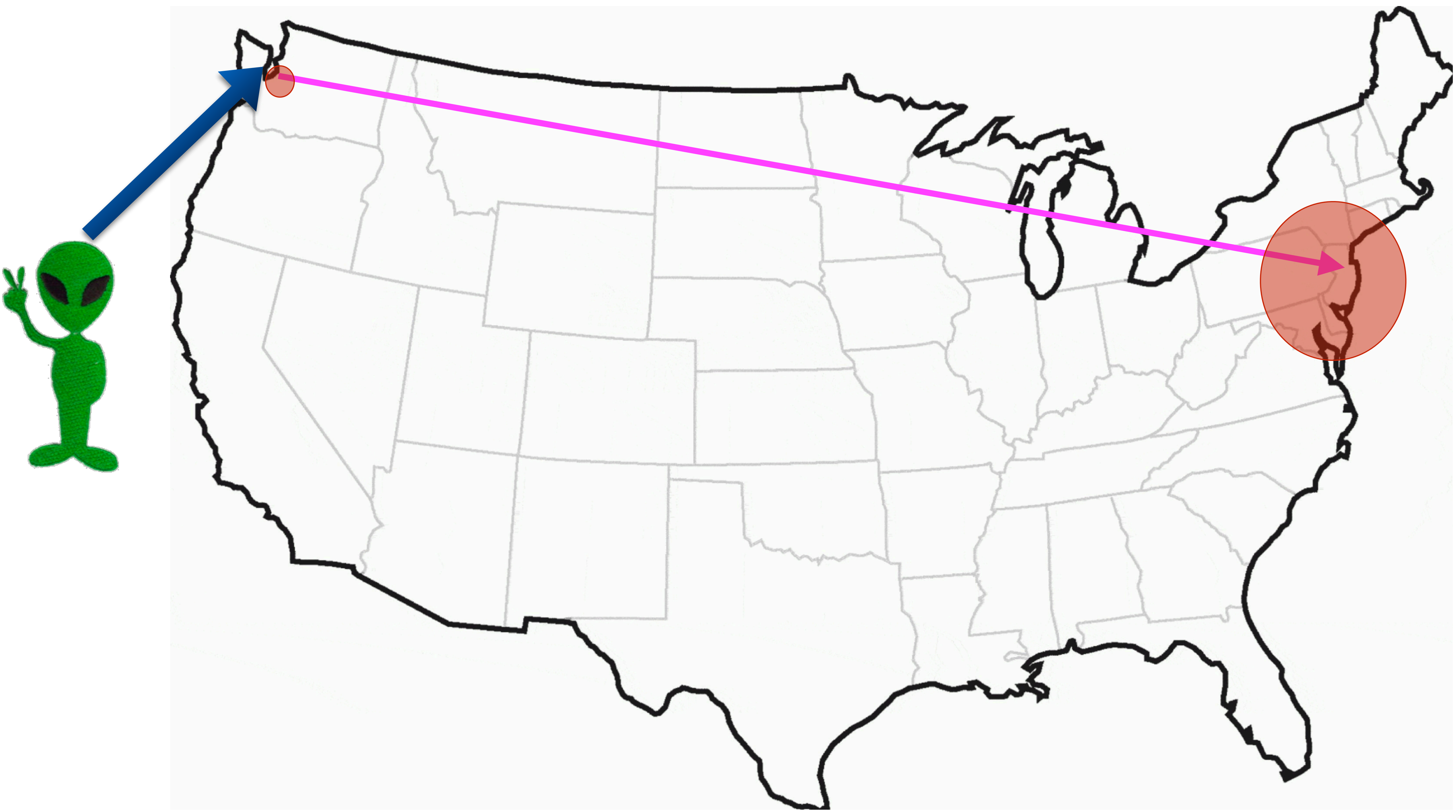
Case A: Measurement uncertainty same as state

Case B: Uncertainty is 100x state

Case C: Uncertainty is anywhere on earth



Recap of the scenario



What should we set as our new belief?

If we were to do Bayes filtering in our head ...

Measurement Uncertainty	**Updated belief**
Small (0.5 km)	
Medium (50 km)	
Large (Anywhere on earth)	

The Kalman Filter

(Bayes filter with
Gaussian beliefs and linear models)

1-D Kalman Filtering

Belief is a Gaussian

$$bel(x_t) = P(x_t) = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{(x_t - \mu_t)^2}{2\sigma_t^2}}$$

$$= \mathcal{N}(\mu_t, \sigma_t^2)$$

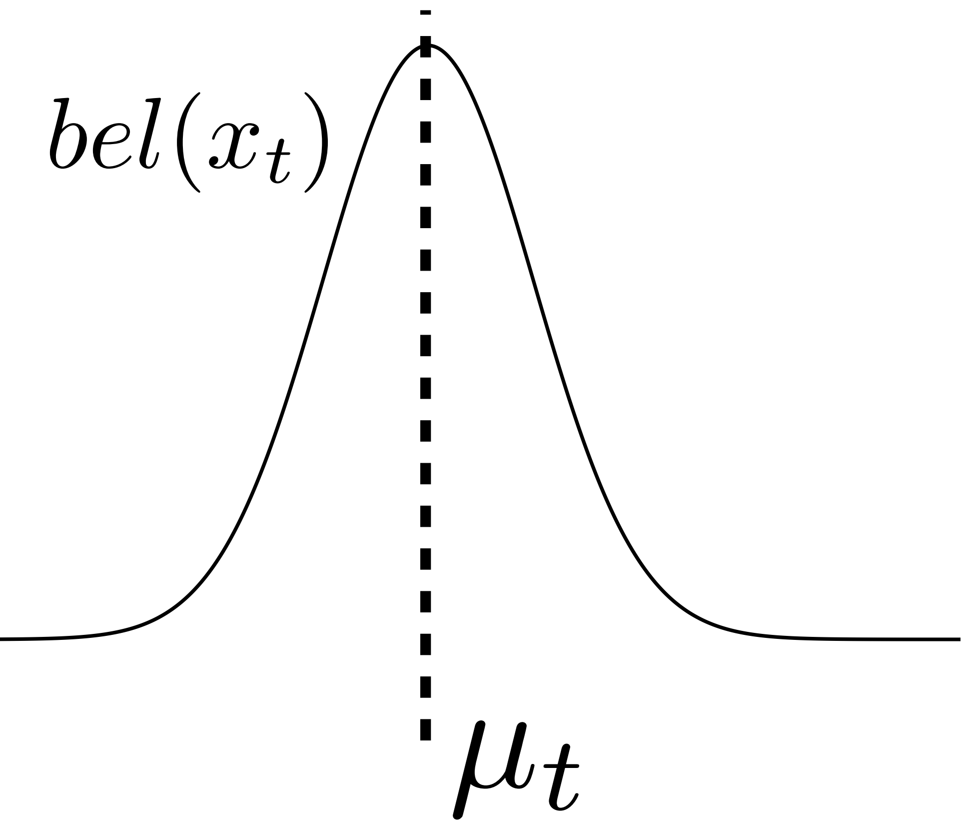
Motion model is linear with Gaussian noise

$$x_{t+1} = x_t + u_{t+1} + \mathcal{N}(0, \sigma_u^2)$$

Observation model is linear with Gaussian noise

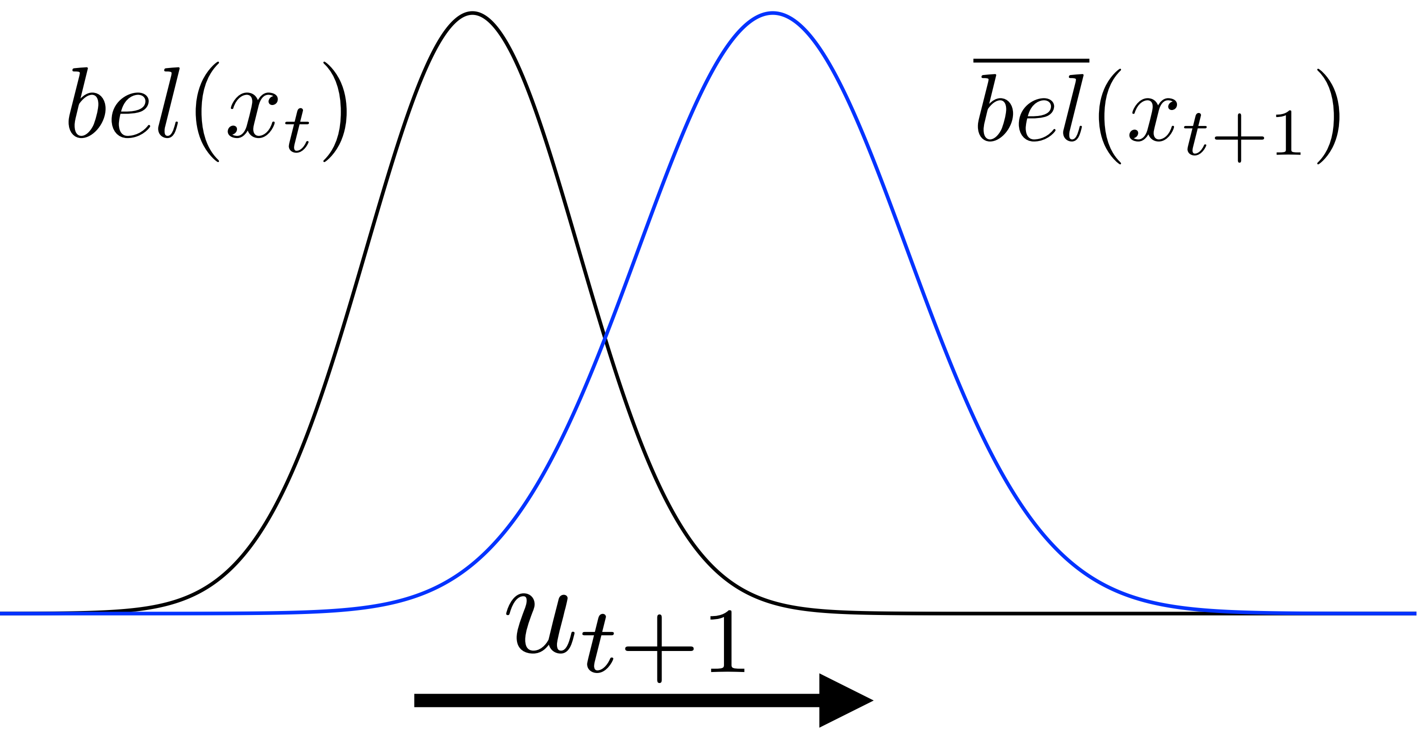
$$z_{t+1} = x_{t+1} + \mathcal{N}(0, \sigma_z^2)$$

Step 0: Start with belief at time t



$$bel(x_t) = \mathcal{N}(\mu_t, \sigma_t^2)$$

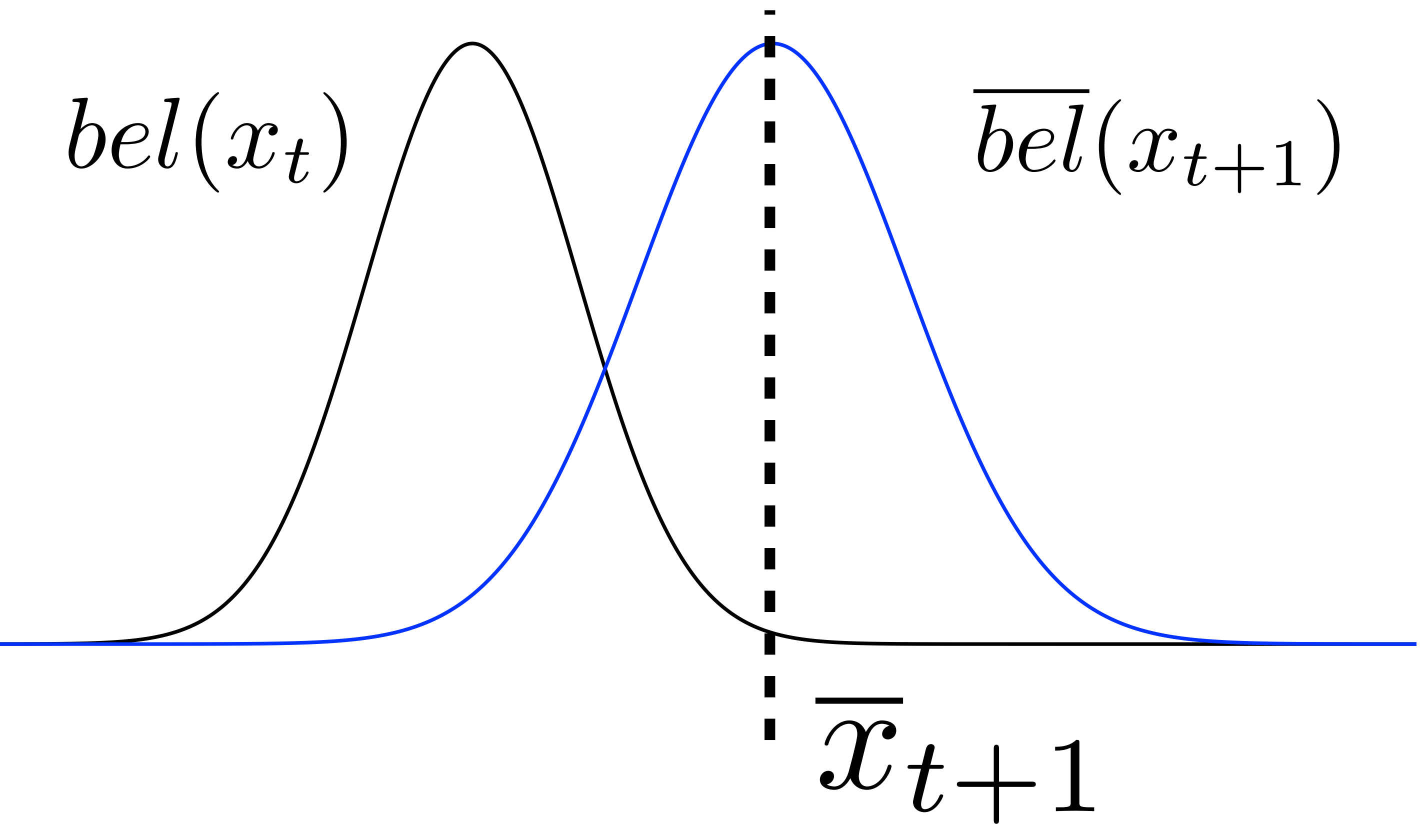
Execute control action



$$\overline{bel}(x_{t+1}) = \int_{-\infty}^{\infty} P(x_{t+1} | x_t, u_{t+1}) \; bel(x_t) \; dx_t$$

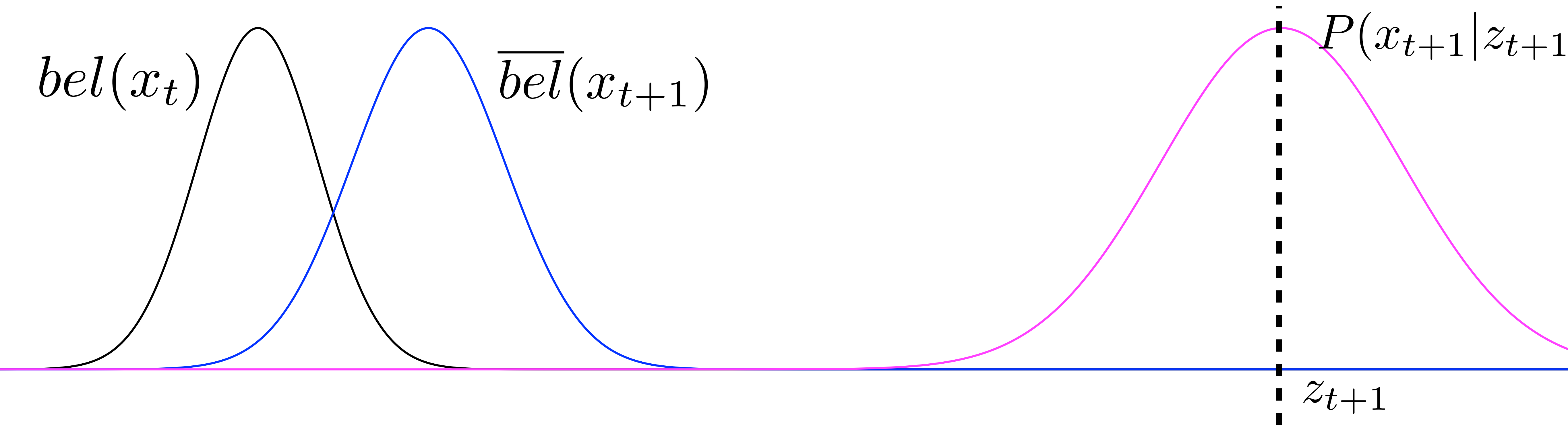
Gaussian Gaussian Gaussian

Step 1: Apply motion model

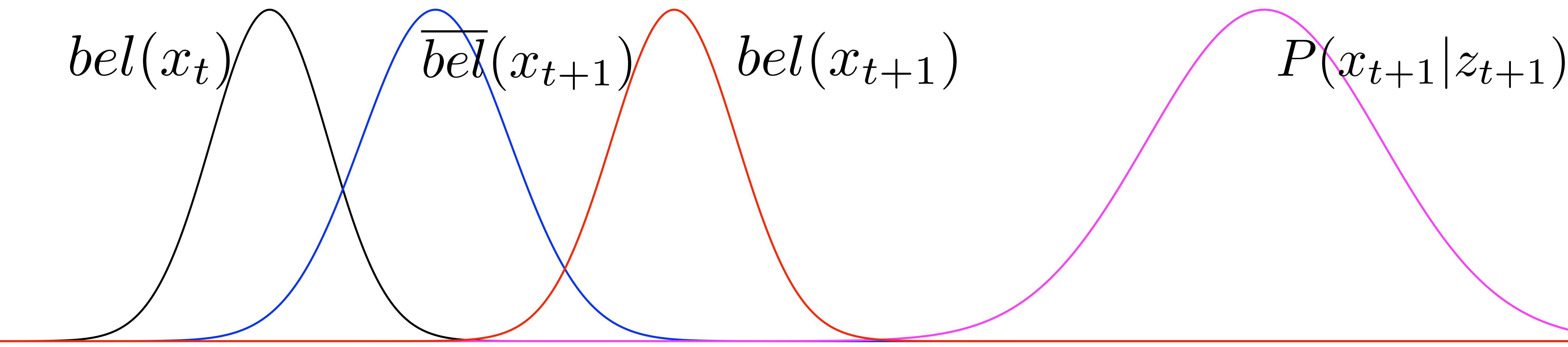


$$\overline{bel}(x_{t+1}) = \mathcal{N}(\mu_t + u_{t+1}, \sigma_t^2 + \sigma_u^2)$$
$$= \mathcal{N}(\overline{x}_{t+1}, \overline{\sigma}_{t+1}^2)$$

Receive a measurement



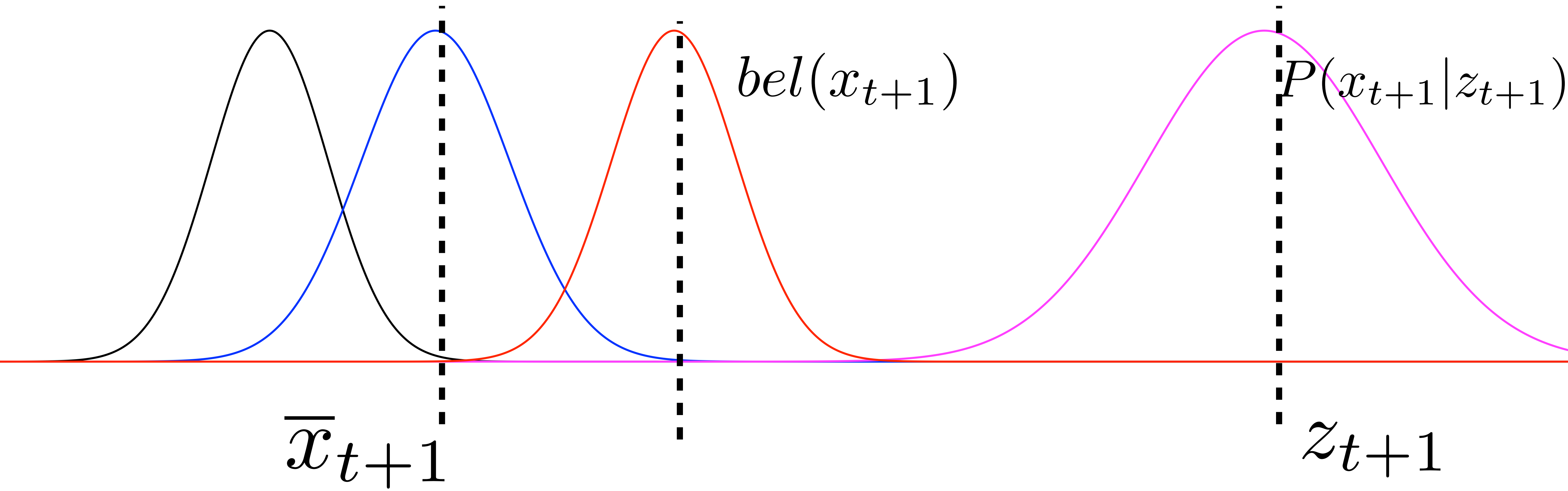
Step 2: Apply Bayes' rule



$$bel(x_{t+1}) = \eta \; P(z_{t+1}|x_{t+1}) \; \overline{bel}(x_{t+1})$$

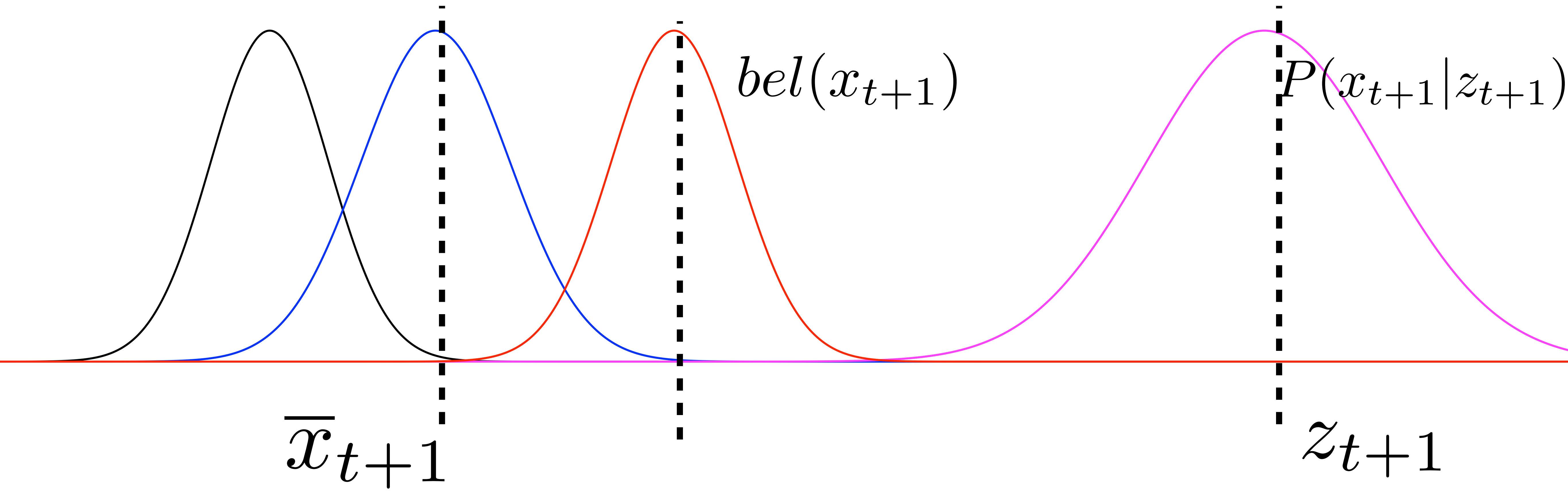
(Gaussian) (Gaussian) (Gaussian)

Updated belief also a Gaussian!



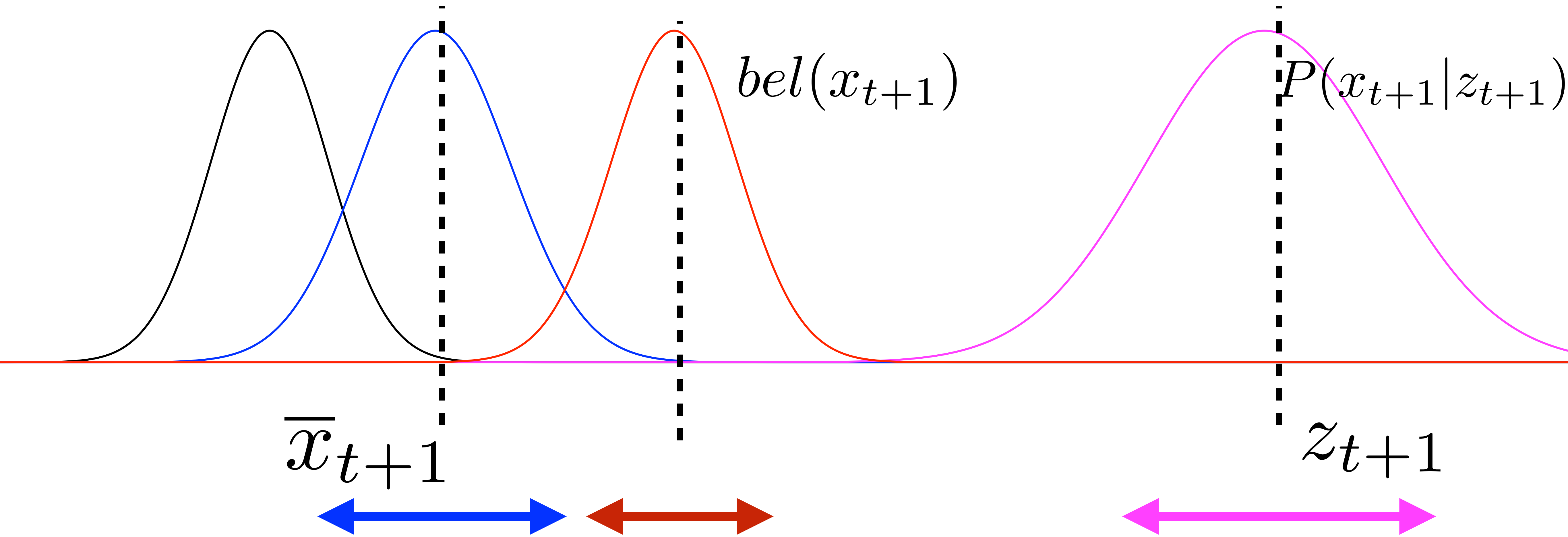
$$bel(x_{t+1}) = \mathcal{N}\left(\frac{\frac{1}{\overline{\sigma}^2_{t+1}}\overline{x}_{t+1} + \frac{1}{\sigma^2_z}z_{t+1}}{\frac{1}{\overline{\sigma}^2_{t+1}} + \frac{1}{\sigma^2_z}}, \frac{1}{\frac{1}{\overline{\sigma}^2_{t+1}} + \frac{1}{\sigma^2_z}}\right)$$

Linearly interpolate prediction and measurement



$$bel(x_{t+1}) = \mathcal{N}\left(\frac{\frac{1}{\overline{\sigma}^2_{t+1}}\overline{x}_{t+1} + \frac{1}{\sigma^2_z}z_{t+1}}{\frac{1}{\overline{\sigma}^2_{t+1}} + \frac{1}{\sigma^2_z}}, \frac{1}{\frac{1}{\overline{\sigma}^2_{t+1}} + \frac{1}{\sigma^2_z}}\right)$$

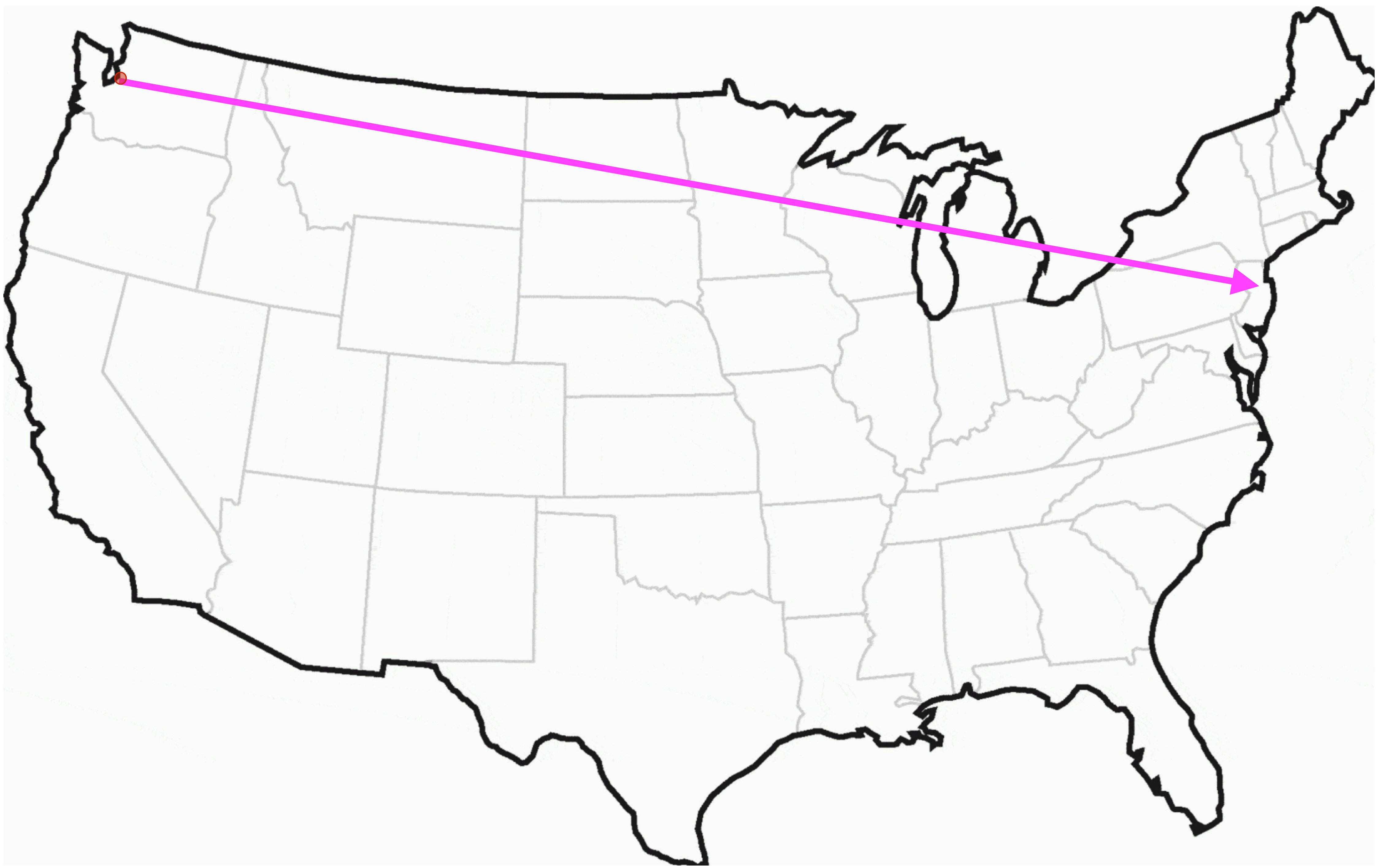
Problem: Variance ALWAYS decreases!



$$bel(x_{t+1}) = \mathcal{N}\left(\frac{\frac{1}{\overline{\sigma}^2_{t+1}}\overline{x}_t + \frac{1}{\sigma_z^2} z_t}{\frac{1}{\overline{\sigma}^2_{t+1}} + \frac{1}{\sigma_z^2}}, \frac{1}{\frac{1}{\overline{\sigma}^2_{t+1}} + \frac{1}{\sigma_z^2}} \right)$$

... no matter what the measurement values are!

Back to example ...



What should we set as our new belief?

Measurement Uncertainty	Our reasonable guess	Kalman Filter
Small (0.5 km)	Large (Anywhere on earth)	Small ($<$0.5km) (Centered at midpoint!)
Medium (50 km)	Large (Anywhere on earth)	Small ($<$0.5km) (Centered close to UW)
Large (Anywhere on earth)	Original belief (UW, 500m)	Original belief (0.5km) (Centered at UW)

What is broken ?!?

Is the
linear model
broken?

Is the
Gaussian
assumption
broken?

Is the
Bayes
filtering
broken?

Nothing is broken - Linear Gaussian model says the probability of event 1 and 2 is astronomically low.

Bayes filter in a nutshell

Step 1: Prediction - push belief through dynamics given action

$$\overline{bel}(x_t) = \int P(x_t|u_t, x_{t-1})bel(x_{t-1})dx_{t-1}$$

Step 2: Correction - apply Bayes rule given measurement

$$bel(x_t) = \eta P(z_t|x_t)\overline{bel}(x_t)$$