Problem 8 Solution: Optimal Search Trees

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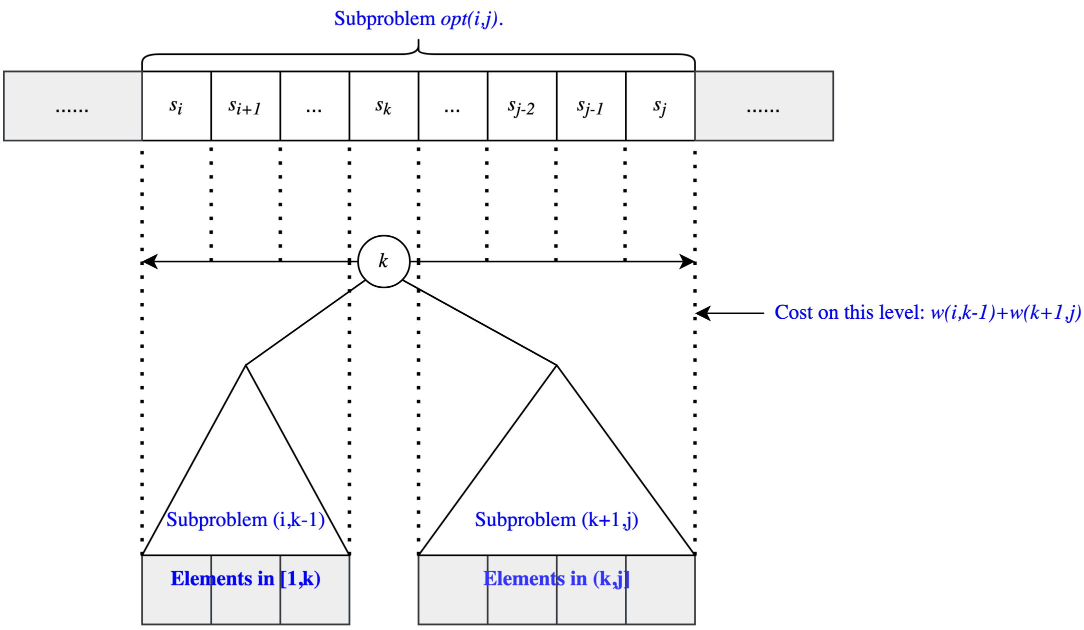
There were two parts and an extra credit problem to this problem. The first part asked you to come up with an $O(n^2)$-time dynamic programming scheme to solve the optimal binary search trees problem, and the second part asked you to solve the analogously defined "optimal ternary search trees" problem by generalizing your dynamic programming scheme from Part 1.

1 Solution to First Part

Let $\texttt{opt}(i, j)$ denote the optimal OBST cost for keys in range $[i, j]$. Then we can notice that

$$\texttt{opt}(i, j) = \min_{i \le k \le j} \left[\texttt{opt}(i, k-1) + w(i, k-1) + \texttt{opt}(k+1, j) + w(k+1, j) \right]$$

where the weight function $w(i, j) = \sum_{k=i}^{j} F_i$ is the total access frequency of the keys from i to j. The reason for this relation's correctness can be summed up in this diagram:



We're essentially just looping through all possible roots of the OBST on $[i, j]$ and asking what the best possible cost would be. If we select k as the root, then the best we can do is to make the left subtree the OBST on keys $[i, k-1]$, and the right subtree the OBST on keys $[k+1, j]$. In the process, we've "demoted" each of the nodes in the child subtrees, and thus we have increased our cost by the total cost of nodes in either of the two subtrees (which is where the $w(i, j)$ terms come from).

To turn this relation into an actual program that calculates the answer, we would first note that we know the base cases whenever $i \geq j$. Now, we have all the pre-requisite values for calculating $\texttt{opt}(i, j)$ if $j - i \leq 2$. After that, we will have all the pre-requisite values for whenever $j - i \leq 4$, and so on.

Unfortunately, even if we use prefix sums of $w(\cdot)$ to be able to retrieve $w(i, j)$ efficiently, the above algorithm still still takes $\Theta(n^3)$-time (memoized) in the worst case. However, observe the weight function satisfies the quadrangle inequality

$$w(i, j) + w(i', j') \leq w(i, j') + w(i', j), \text{ whenever } i \leq i' \leq j \leq j'$$

How does this help? There is a sequence of two lemmas in the original Yao paper and on pages 76-78 of this reference (which we would have encouraged you to Google during the competition) that ultimately prove that the calculation of all the elements on any 'diagonal' of the DP table (i.e. where $j - i$ is held constant) can be calculated in $O(n)$ steps, given that every diagonal below it has been calculated. Central to this is storing of an auxiliary *index table* K_B, where

$$K_B(i, j) = \max\{t \mid w(i, t) + \texttt{opt}(i, t-1) + \texttt{opt}(t, j) = \texttt{opt}(i, j)\}$$

That is, $K_B(i, j)$ is the *maximum* index that we can choose as the root of the OBST on keys $[i, j]$. It turns out that

Lemma 1: Suppose $w(i, j)$ satisfies the quadrangle inequality and is monotone (i.e. $w(i, j) \leq w(i', j')$ whenever $[i, j] \subseteq [i', j']$). Then if `opt` satisfies the recurrence at the beginning of this section, it follows that `opt` also satisfies the quadrangle inequality.

This in turn is used to show that

Lemma 2: If `opt` satisfies the quadrangle inequality, then $K_B(i, j-1) \leq K_B(i, j) \leq K_B(i+1, j)$.

And while lemma 2 may on its face seem unhelpful, it actually tells us two important things: First, the elements on the diagonal of the DP table are increasing as we increase the row index. That is, if we fix $d = j - i = j' - i' \geq 0$, then

$$K_B(i, j) \leq K_B(i', j'), \text{ if } i \leq i'$$

Furthermore, our search for the optimal root for keys $[i, j]$ need only go from $K_B(i, j-1)$ to $K_B(i+1, j)$. Note that these two values are on the $(d-1)$th diagonal of the DP matrix, so this suggests that like before, we should first calculate the values along the zeroth (main) diagonal, then the values along the first, and so on. All in all, the cost of calculating all elements on the dth diagonal (assuming keys are 1-indexed) is

$$\sum_{i=1}^{n-d} \left[K_B(i+1, i+d) - K_B(i, i+d-1)\right] = K_B(n-d+1, n) - K_B(1, d) \leq n$$

And since there are only n total diagonals, the algorithm runs in $O(n^2)$ time.

Proofs of the lemmas can be found in the original Yao paper. We provide two implementations of the optimal BST solution in C++: `OBST_Slow.cpp` which implements the $O(n^3)$-time dynamic programming scheme, and `OBST_Fast.cpp` which implements the quadrangle inequality-based speedup.

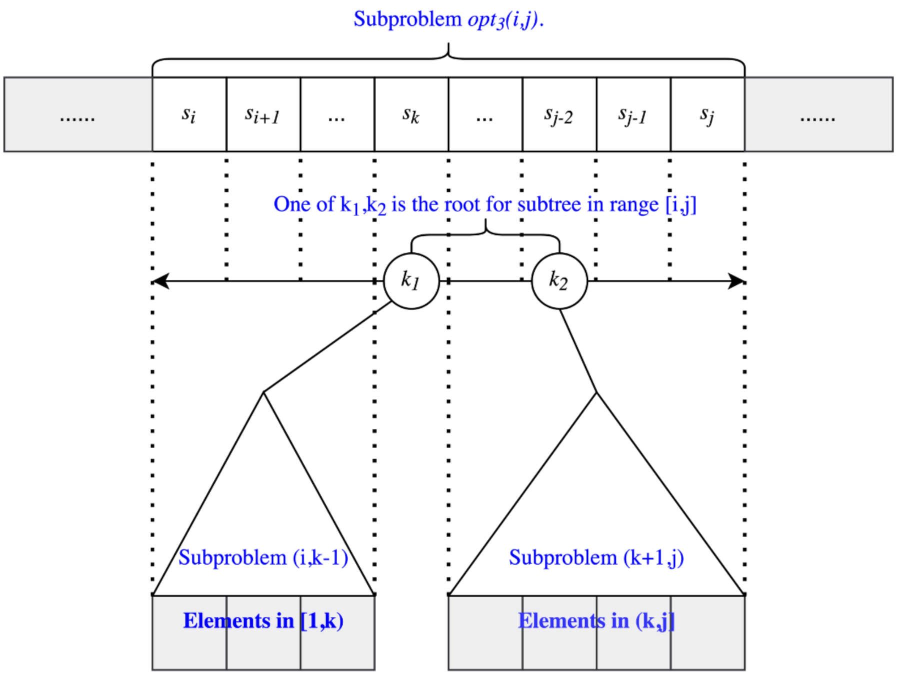
2 Solution to Second Part

Inspired by the naive dynamic programming recurrence from the first part, one might first think of a $O(n^4)$-time dynamic program $\texttt{opt}(i, j)$, which likewise stores the optimal cost for keys in range $i \leq k \leq j$, as follows:

$$\begin{aligned}\texttt{opt}(i, j) = \min_{i \leq k_1 \leq k_2 \leq j} \Big[&\texttt{opt}(i, k_1 - 1) + \texttt{opt}(k_1 + 1, k_2) + \texttt{opt}(k_2 + 1, j) + w(i, k_1 - 1) + w(k_1 + 1, j), \\ &\texttt{opt}(i, k_1) + \texttt{opt}(k_1 + 1, k_2 - 1) + \texttt{opt}(k_2 + 1, j) + w(i, k_2 - 1) + w(k_2 + 1, j)\Big]\end{aligned}$$

See the diagram on the next page for a visualization of this recurrence. Again, we're essentially just trying to brute-force guess the two points that are going to divide the range $[i, j]$. But since we're only allowed to store one element in every TST node, we check which of these two points is better-suited to be the root. The first argument of the minimum above is the best possible TST we can get with k_1 as the root, and the second is the best possible we can get with k_2 as the root. See the OBST solution for intuition on why/how the $w(\cdot)$ figure in as we've written.

Observe also that the above recurrence works in $O(n^4)$ because of the double nested loop computation induced by two indices k_1, k_2 at each recursion level, so again it fails to meet the running time requirements.



However, there is room for improvement, and the answer lies in something you might remember from COS 226: 2-3 trees. Instead of trying to find two pivots k_1 and k_2, let's just look for one pivot, which we'll call k^*. We'll then force k^* to only have two children, like in the OBST problem. The crucial point: one of the children of k^* in this BST configuration will be a "dummy node" and won't actually feature in the actual TST. The actual children of k^* in the TST will be its non-dummy node child in the BST configuration and the children of the dummy node.

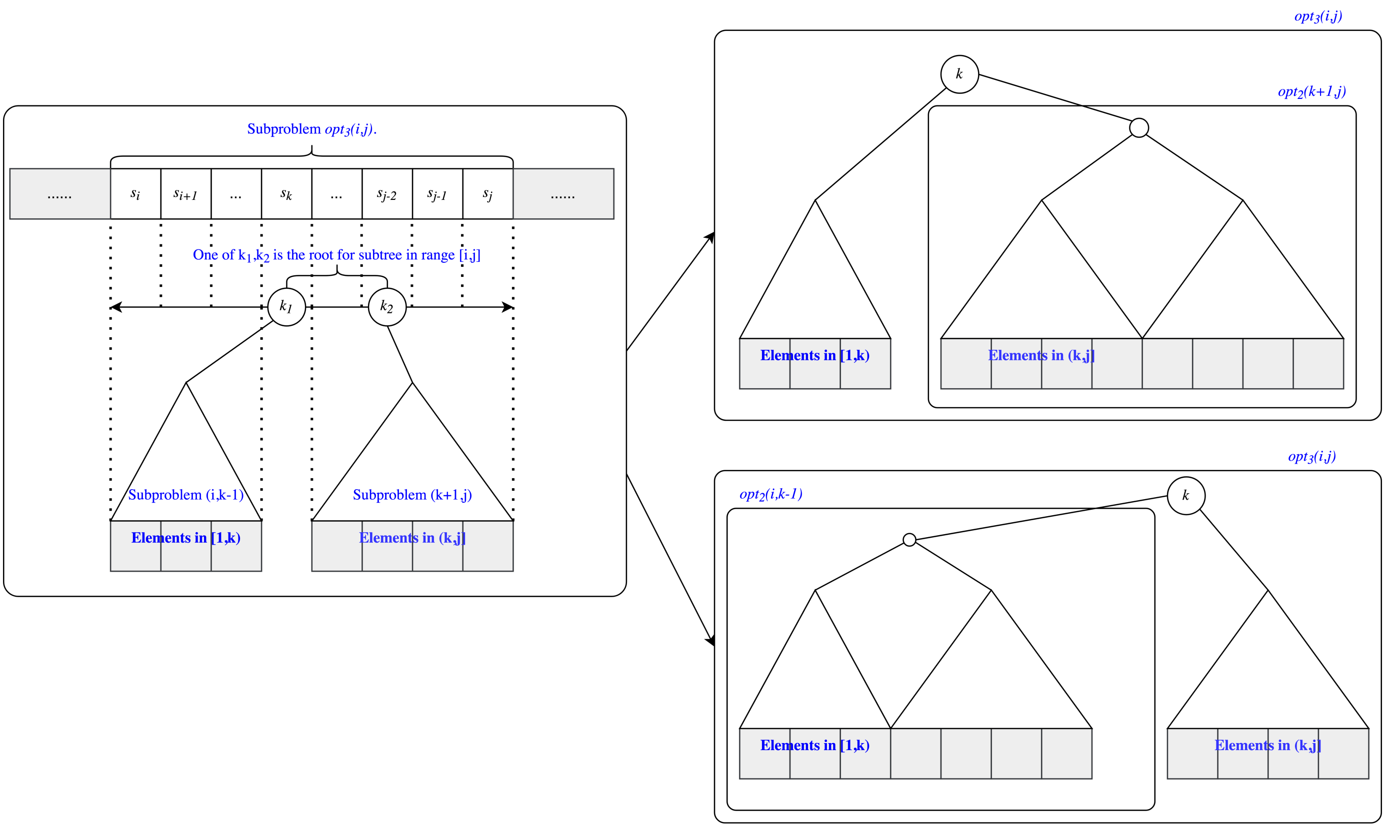
The only thing, then, is we must mandate is that the dummy nodes only have two children. This leads us to define two quantities: $\texttt{opt}_3(i, j)$, which is the minimum weight of a TST we can construct on keys in $[i, j]$, and $\texttt{opt}_2(i, j)$, which is the minimum weight of a TST we can construct on keys in $[i, j]$ *if the root is forced to be a binary dummy node (i.e. it only has two children and no actual key is allowed to be stored in the root)*. Then we may write

$$\texttt{opt}_2(i,j) = \min_{i \le k \le j} \Big[\texttt{opt}_3(i,k-1) + \texttt{opt}_3(k,j) \Big]$$

and because of the previous discussion,

$$\begin{aligned}\texttt{opt}_3(i,j) = \min_{i \le k \le j} \Big[\min \Big(& \texttt{opt}_3(i,k-1) + w(i,k-1) + \texttt{opt}_2(k+1,j) + w(k+1,j), \\ & \texttt{opt}_2(i,k-1) + w(i,k-1) + \texttt{opt}_3(k+1,j) + w(k+1,j) \Big) \Big]\end{aligned}$$

The two cases in the expression for $\texttt{opt}_3$ are essentially determining whether the root of the left subtree or the root of the right subtree should be the dummy node child. See the diagram below for a visualization:



And again, we see that we should calculate both $\texttt{opt}_2(i, j)$ and $\texttt{opt}_3(i, j)$ in order of $j - i$ (i.e. we start from the "diagonal" and move up), because both $\texttt{opt}_2(i, j)$ and $\texttt{opt}_3(i, j)$ only depends on $\texttt{opt}_3(i', j')$ and $\texttt{opt}_2(i', j')$, where $j' - i' = j - i - 1$.

All in all, we have decreased the runtime to $O(n^3)$ at the expense of storing another $n \times n$ matrix for $\texttt{opt}_2$. We provide two implementations of the optimal TST solution in C++: `OTST_Slow.cpp`, which implements the naive $O(n^4)$-time dynamic programming recurrence, and `OTST_Fast.cpp`, which implements the $O(n^3)$-time mutual recursive solution.

Plaudits:

- Congratulations to Antonio Molina-Lovett (grad), Yuping Luo (grad)/Xiaoqi Chen (grad)/Dingli Yu (grad), and Kaifeng Lyu (grad)/Shunyu Yao (grad)/Qinshi Wang (grad) for being the only teams to solve both parts of the problem!
- Congratulations to Yuping Luo (grad)/Xiaoqi Chen (grad)/Dingli Yu (grad) for being the first to submit a correct solution to the BST part of the problem, at a speedy 33 minutes.
- Congratulations to Antonio Molina-Lovett (grad) for being the first to submit a correct solution to the TST part of the problem, at one hour and thirteen minutes.