PRINCETON COS CON

Problem 2 Solution: Lord of the Lanternflies

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We were very pleased with the level of effort shown on this problem! Many of you were able to deduce that $f(n) = n$, though it was significantly trickier to prove it. One way of doing is to show that n was both an upper bound (i.e. $f(n) \leq n$) and a lower bound (i.e. $f(n) \geq n$).

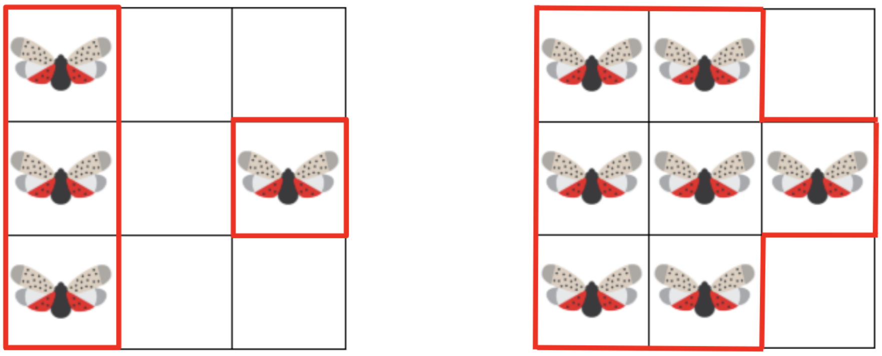
Upper Bound (5 points): The easier direction is undoubtedly showing that $f(n) \leq n$. One method is to describe an explicit construction of n elements that fills the board. A rigorous proof of why a diagonal arrangement populates the board could use induction as follows:

Base Case: For a 1×1 array, placing one bug on the only grid square fills the grid.

Inductive Step: By way of induction, we will assume that there exists some arrangement A of n flies on an $n \times n$ grid such that after some finite number T of timesteps, the grid will be filled. To construct an arrangement for an $(n+1) \times (n+1)$, board we first place A on the $n \times n$ grid with corners $(0, 0)$ and $(n-1, n-1)$ (i.e. the bottom left $n \times n$ corner). We then introduce one more fly on the square (n, n). After T timesteps, we know by our inductive assumption that all squares (i, j) with $i < n$ and $j < n$ will have a fly. That is, the bottom left $n \times n$ corner will be completely full.

So what about the part of the grid not in the bottom left $n \times n$ corner? Well, at timestep $T + 1$, we know that squares $(n, n-1)$ and $(n-1, n)$ will be filled, as if they weren't filled in the first T timesteps then they will be filled now as there are flies on two adjacent squares (n, n) and $(n-1, n-1)$. Similarly, after $T + 2$ timesteps we note that as squares $(n-1, n)$ and $(n-2, n-1)$ have flies then $(n-2, n)$ will have a fly (and similarly for $(n, n-2)$). We continue this until $T + n$ timesteps have passed and the whole board will be filled. This completes the proof of the inductive step and thus the proof of the upper bound. $\square$

Lower Bound (10 points): This part was more tricky. Many of you found the elegant proof we initially had in mind, which focuses on the *perimeter* of an arrangement. Specifically, define the perimeter to be the number of edges in the board that are adjacent to exactly one fly (i.e. have a fly in the square on one side, but no fly in the square on the other side). With this definition, we see that each fly is adjacent to at most 4 edges. Note also that if you start with k flies the initial perimeter is at most $4k$ (question: when is it exactly $4k$?), and that the final perimeter, if the board eventually gets filled, is $4n$.



Examples of edges on the perimeter (outlined in red) for two different board configurations.

What happens to the perimeter of the board's current configuration as time moves forward? When a new fly is born, it means that there were at least two cardinally adjacent squares with flies already on them. You can think of these flies that existed before as the "parents" of the newly spawned lanternfly. Now note that the edges that those parents were adjacent to go from being adjacent to one fly (just the parent) to two flies and are thus removed from the perimeter. As each fly is adjacent to at most 4 edges. This means that at least 2 edges were removed from the perimeter, and as such, at most 2 edges were added. Thus the perimeter cannot increase as flies are born!

Thus, if we want the final perimeter to be $4n$, the initial perimeter must be at least $4n$. The bare minimum number of flies needed for this, as we established earlier, is n, which shows that $f(n) \geq n$. $\square$

Remarks

- *Alternative Proofs*[1]: We also received several correct proofs that used slightly different tactics to show a lower bound. They began by noticing that after flies stopped reproducing that the clumps of flies had to be in the form of rectangles. Then, they defined a constant very similar to our perimeter (some, for example, defined it to be the sum of the dimensions of disjoint rectangles on the board which is just half the perimeter). Finally, they realized that when two rectangles "merged" or when a fly was added adjacent to a rectangle, the perimeter could not increase. This also works, but admittedly it's not as pretty as the solution above.

[1] Don't worry, these are more trustworthy than alternative facts.

- For partial credit on the lower bound, many of you would have had a correct inductive proof if one or two additional lemmas were proven. Most of these lemmas were variants of the following statement:

  Suppose we are given a board B with an empty row in the initial state, and removing that row yields two sub-boards b_1 and b_2. Then if B will eventually fill, so will b_1 and b_2.

  If you were proof would have worked or mostly worked with some variant of this lemma, I probably gave you either 1 or 3 points of partial credit for the lower bound depending on how rigorous the rest of the proof was. I encourage to see if you can prove this lemma now (it does, in fact, hold).

- I hope you all had fun with this problem (and for those of you who have yet to take COS 240/340, you will likely encounter more problems like this in the future). What I hope you take away is that 1) you should often look for an invariant (i.e. property that doesn't change) or a monovariant (i.e. property that either doesn't increase or doesn't decrease) when solving these sorts of problems; 2) to try to shift your perspective when one technique doesn't seem to be yielding results; and 3) to think about looking at the local changes in a property to simplify analysis.