PRINCETON COS CON

Problem 4: Dean's Date Ultimate Frisbee (20 points) [File Upload]

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It's 5:01 PM on Dean's Date and it's finally time to relax! You call up all your friends for a fun game of Ultimate Frisbee on Poe Field. In Ultimate Frisbee, there is an $n \times n$ meter field, where there are two $n \times 10$ meter end-zones (contained inside the field, so the total play area is $n \times n$) at opposite ends.

Unfortunately, everyone is completely exhausted from their Dean's Date all-nighters, so after each player independently goes to a uniformly random location on the field, no one is going to be able to move. Everyone is able to perfectly throw the frisbee to anyone within a 10 meter radius.

To score a point, a series of passes needs to occur such that the frisbee starts with a player in one end-zone and ends up with a player in the other end-zone. Obviously, the game would not be very fun if it's impossible to score, so you want to make sure that you have enough players such that scoring is possible.

Formally:

- The field is $n \times n$ meters with 10 meter deep end-zones contained at the top and bottom of the field.
- Each player is modeled as a point, independently placed uniformly randomly on the field.
- Each player can pass/throw the frisbee up to 10 meters away.
- Scoring is possible if there is a series of passes which takes the frisbee from a player in the top end-zone to a player in the bottom end-zone.

Part 1 (3 points)

Suppose Poe Field is 100×100 meters. Prove that for every point on the field to be within a player's throwing range, you need at least 32 players on the field.

Part 2 (7 points)

Suppose Poe Field is 100×100 meters. Give an estimate (within a ± 3 error) for the minimum number of friends you should invite so that scoring is possible with probability at least $1/2$.

Bonus (10 points)

Your math major friends don't believe in computers and will only show up to the frisbee game if you can give them provable guarantees on the possibility of scoring.

Prove that for sufficiently large n, scoring is possible with probability at least $1 - e^{-n}$ on a field of size $n \times n$ meters so long as $m = n^2$ players show up.

You will get half credit (5 points) if you prove the claim for $m = \Omega(n^2 \log n)$.