Problem 8: Invisible Devan [File Upload] (25 points)

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Problem Statement

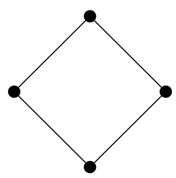
Exhausted from running ACM, Devan snatched two apples from the RoMa dining hall and ran away. President Eisgruber, upon hearing about the incident, decided to track him down.
Princeton's campus can be seen as an unweighted simple connected graph G with finitely many vertices, with Devan having the advantage of an invisibility cloak that hides his location from President Eisgruber. Devan first picks a vertex v in G to hide. Every minute, the following happens in order:

- President Eisgruber, selects **up to 3 vertices** of G and installs a tracker at each of the selected vertices.
- Devan, **after seeing the locations of the trackers**, either moves to an adjacent vertex of his choice or stays at his current location.
- Then, each tracker reports the minimum (graph-theoretic) distance from its location to where Devan is hiding. (After reporting, the tracker self-destructs and cannot be reused.)

We say President Eisgruber wins the game if he can in a finite number of minutes, based on the information provided by the trackers, deduce the exact vertex where Devan is hiding. Devan wins if he can evade Eisgruber's search.

Part 1 (5 points)

Suppose that President Eisgruber is further restricted to installing at most one tracker per turn (instead of three), and that G is the cycle graph C_4 (drawn below). Prove that Devan has a winning strategy.



Part 2 (5 points)

Let D be a positive integer. Let H be a finite simple graph with v vertices with the property that one can label the vertices of H as $h_1, h_2, \cdots, h_v$ such that for each $1 \leq i \leq v$, the number of neighbors of h_i in the set $\{h_1, \ldots, h_{i-1}\}$ is less or equal to D. Then, prove that H is $(D+1)$-colorable; in other words, one can assign each vertex in H a number in $\{1, 2, \cdots, D+1\}$ so that any two adjacent vertices have different colors.

(Hint: A greedy/inductive approach might help!)

Part 3 (15 points)

Let us return to the original setup where President Eisgruber could install up to three trackers. Suppose that President Eisgruber has a winning strategy. Prove that G must be 27-colorable; in other words, one can assign each vertex in G a number in $\{1, 2, \cdots, 27\}$ so that any two adjacent vertices have different colors.